Problems in geometric probability for intersections of balls

Doctoral thesis booklet

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Contents

Chapter 1

Introduction

The problems discussed in this thesis originate from topics in stochastic geometry, particularly Random Polytope Theory. The results belong to three broad categories: probability models in spindle convexity, estimation of the error terms of interesting functionals, and necessary differentiability conditions for approximation to convex bodies.

The dissertation begins with a brief history of Random Polytope Theory, which includes the earlier history of Silvester's four-point problem, Wendel's equality, and Rényi and Sulanke's results on random approximations to convex bodies. In the second chapter, a Wendel-type equality for the spindle convex case is proved. In the third chapter, finite series expansions are proved for formulas obtained by Fodor, Kevei, and Vígh [FKV14] for the expected number of vertices and missed area of uniform random disc-polygons in smooth convex discs. In the fourth chapter, the same asymptotic formulas of Fodor, Kevei, and Vígh [FKV14] are are proved under weaker differentiability conditions on the container. The fifth chapter contains a list of concluding remarks and possible research problems. Finally, two summaries of the main results contained in the dissertation are provided. One in the English language and another one in Hungarian, in the sixth and seventh chapters of the dissertation, respectively.

The mathematical contents of the dissertation are based on the following two joint publications of the author:

[FMV23] F. Fodor, P. N. A. Montenegro, and V. Vígh, *On Wendel's equality for intersections of balls*, Aequationes Math. **97** (2023), no. 2, 439–451, DOI 10.1007/s00010-022-00912-3. MR4563622

[FMP24] F. Fodor and N. A. Montenegro Pinzón, *Series expansions for random disc-polygons in smooth plane convex bodies*, J. Appl. Probab. **61** (2024), no. 4, Published online May 16, 2024, DOI 10.1017/jpr.2024.27.

We note that in this booklet numbering of the chapters, theorems and lemmas are the same as in the thesis. At each statement, we indicate the page number where it can be found in the dissertation.

Chapter 2

On Wendel's equality for intersections of balls

This chapter of the dissertation is based on the paper [FMPV23].

The aim of this chapter is to investigate a problem of Wendel [Wen62], that was originally stated for the classical convex case, in the spindle convex setting. Let n i.i.d. points be selected from $\mathbb{R}^d$ according to a probability measure that is symmetric about the origin (and does not assign positive probability to hyperplanes). What is the probability that their (classical) convex hull P_n (which is a random polytope) does not contain the origin?

Wendel proved that

$$\mathbb{P}(o \notin P_n) = 2^{-n+1} \sum_{k=0}^{d-1} \binom{n-1}{k}, \tag{2.1}$$

which is (surprisingly) independent of the distribution (but, of course, depends on the number of points n and the dimension of space d).

We will use the following concept of convexity which goes back to Mayer [May35], and possibly even longer. Let $R > 0$ be a positive real number and let x and y be points in the d-dimensional space, no more than distance $2R$ apart.The R-spindle determined by this pair of points is the intersection of all balls of radius R that contain x and y.

If the distance of x and y is more than $2R$, then the R-spindle (segment) they determine is the whole space, by definition. We say that a set X, with diameter less that or equal to $2R$ is R-spindle convex if given a pair of points x and y in X, the R-spindle they determine is also in X. The simplest example of an R-spindle convex body is the ball of radius R.

Let $X \subset \mathbb{R}^d$ be a set. If X is contained in a ball of radius R, then the (closed) R-spindle convex hull of X is defined as the intersection of radius R closed balls containing X. If X is not contained in a ball of radius R, then its R-spindle convex hull is the whole space $\mathbb{R}^d$.

The spindle convex analog of Wendel's question is as follows: Let $R > 0$ a real number and let $K \subseteq R^d$ be an o-symmetric convex body that is R-spindle convex, and $x_1, \ldots, x_n$ be *i.i.d* uniform random points from K. We denote the R-spindle convex hull of $x_1, \ldots, x_n$ by $K_n^R = [x_1, \ldots, x_n]_R$.

What is the probability that $0 \in K_n^R$ (or, equivalently, $0 \notin K_n^R$)? We study the special case when $K = rB^d$ with $0 < r \leq 1$ and assume that $R = 1$ (by scaling). We wish to determine the probability

$$P(d, r, n) := \mathbb{P}(o \in [x_1, \ldots, x_n]_1).$$

The main results of this chapter are stated in the following theorems:

Theorem 1 (p. 18 of the thesis, and [FMPV23]). *Let $K = rB^d$. Then*

$$P(d, r, 2) = \frac{\omega_{d-1}\omega_d}{(r^d\kappa_d)^2} \int_0^r \int_0^r \int_0^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2}\varphi \, d\varphi dr_2 dr_1,$$

where $\varphi(r_1, r_2) = \arcsin(r_1/2) + \arcsin(r_2/2)$. *In particular,*

$$P(2, 1, 2) = \frac{\sqrt{3}}{\pi} - \frac{1}{3} = 0.2179\ldots,$$
$$P(3, 1, 2) = \frac{1}{64}(23 + 12\sqrt{3}\pi - 8\pi^2) = 0.1459\ldots.$$

The symbols κ_d and ω_d denote the volume and surface area of the d-dimensional unit ball B^d.

Theorem 2 (p. 18 of the thesis, and [FMPV23]). *Let $K = rB^2$. Then*

$$P(2, 1, 3) = \frac{-84\pi^2 - 477 + 360\sqrt{3}\pi}{144\pi^2} = 0.4594\ldots.$$

Wagner and Welzl [WW01] proved that in the classical convex version of the problem o-symmetric distributions are extremal in the sense that if the distribution is absolutely continuous with respect to the Lebesgue measure, then the probability that the origin is not contained in the (classical) random polytope P_n is at least (2.1). It is natural to ask whether a similar result holds in the spindle convex version of the problem.

We consider the model in which $R = 1$ and $x_1, \dots, x_n$ are i.i.d. random points in $\mathbb{R}^d$ that are distributed according to the standard normal distribution with density function

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2}}, \ x \in \mathbb{R}^d.$$

We are interested in the following probability:

$$P_N(d, 1, n) := \mathbb{P}(o \in [x_1, \dots, x_n]_1),$$

that is, the probability that the Gaussian random polytope $[x_1, \dots, x_n]_1$ contains the origin.

Our main result in Section 2.4 of the dissertation is that we determine $P_N(2, 1, 2)$, see formula (2.6) on page 28, that is, the probability that the spindle spanned by two Gaussian random points in the plane contain the origin. The fact that we obtained different probabilities for the uniform and Gaussian distributions points out that no analog result to that of Wagner and Welzl [WW01] holds in the spindle convexity realm.

Chapter 3

Series expansions for random disc-polygons in smooth plane convex bodies

This chapter of the dissertation is based on the paper [FMP24].

The asymptotic formulas obtained by Rényi and Sulanke [RS63, RS64] for the expected number of vertices and missed area and permitere difference of (classical) uniform random polygons in a plane convex body K required that the boundary of the convex body K to be at least three times continuously differentiable with everywhere positive curvature. Their seminal results started, in fact, the investigations of the asymptotic properties of random polytopes, which turned out to be a very prolific theory.

The natural question arose: what is the effect of the smoothness of the boundary of K on these asymptotic formulas? By requiring a higher degree of differentiability, Reitzner [Rei01, Rei04] proved finite series expansions of the quantities considered by Rényi and Sulanke.

The aim of this chapter is to obtain a similar result in spindle convexity to the following theorem by Reitzner [Rei04]:

Let K be a convex body in $\mathbb{R}^d$ with $V(K) = 1$ whose boundary ∂K is C_+^{k+1} smooth for some integer $k \geq 2$. Then

$$\begin{aligned}&\mathbb{E}(V(K) - V(K_n))\\ &\qquad = c_2^{(d,d)}(K)n^{-\frac{2}{d+1}} + c_3^{(d,d)}(K)n^{-\frac{3}{d+1}} + \ldots + c_k^{(d,d)}(K)n^{-\frac{k}{d+1}} + O(n^{-\frac{k+1}{d+1}})\end{aligned}$$

as $n \to \infty$, where the constants $c_j^{(d,d)}(K)$, $j = 2, \ldots, k$ depend only on K and the dimension d.

In fact, the original theorem of Reitzner is much more general in the sense that it is not only about the volume but also concerns all intrinsic volumes $V_i(\cdot)$ as well. This also explains the double index in the constants, since in the general case these constants $c_j^{(i,d)}$ also depend on which intrinsic volume is considered. However, we only need the part of the theorem that is about the expectation of the missed volume.

Under the same conditions on the boundary, one can also obtain a series expansion for the expected number of vertices $\mathbb{E}(f_0(K_n))$

$$\mathbb{E}(f_0(K_n)) = d_2(K)n^{\frac{d-1}{d+1}} + d_3(K)n^{\frac{d-2}{d+1}} + \ldots + d_k(K)n^{\frac{d-k+1}{d+1}} + O(n^{\frac{d-k+2}{d+1}})$$

as $n \to \infty$.

Our objective is to obtain similar series expansion for the following asymptotic formulas proved by Fodor, Kevei and Vígh [FKV14]:

If ∂K is C^2_+ and $R > \max 1/\kappa(x)$ for all $x \in \partial K$, then

$$\mathbb{E}(f_0(K_n^R)) = z_1(K)n^{\frac{1}{3}} + o\left(n^{\frac{1}{3}}\right),$$

$$\mathbb{E}(A(K \setminus K_n^R)) = A(K)z_1(K)n^{-\frac{2}{3}} + o\left(n^{-\frac{2}{3}}\right),$$

as $n \to \infty$, where

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} .dx$$

The symbol $\kappa(x)$ is the curvature at the boundary point x and integration is with respect to arclength.

The two formulas are connected by an Efron-type identity [BE65], which states that

$$\mathbb{E}(f_0(K_n^R)) = n\frac{\mathbb{E}(A(K \setminus K_{n-1}^R))}{A(K)}.$$

The main results of this chapter are stated in the following theorems:

Theorem 5 (on p. 34 of the thesis, and [FMP24])**.** *Let $k \geq 2$, and let K be a convex disc with C^{k+1}_+ smooth boundary. Then for all $R > \max_{x\in\partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(f_0(K_n^R)) = z_1(K)n^{\frac{1}{3}} + \ldots + z_{k-1}(K)n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}})$$

as $n \to \infty$. All coefficients $z_1, \ldots, z_k$ can be determined explicitly. In particular,

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x) - \frac{1}{R}\right)^{\frac{1}{3}} dx,$$

$$z_2(K) = 0,$$

$$z_3(K) = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3A(K)}{2}}\int_{\partial K}\frac{\kappa''(x)}{3(\kappa(x) - \frac{1}{R})^{\frac{4}{3}}} + \frac{2R^2\kappa^2(x) + 7R\kappa(x) - 1}{2R^2(\kappa(x) - \frac{1}{R})^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x) - \frac{1}{R})^{\frac{7}{3}}}\, dx.$$

By the spindle convex version of Efron's identity we obtain the following corollary.

Theorem 6 (on p. 34 of the thesis, and [FMP24])**.** *Let $k \geq 2$ be an integer, and let K be a convex disc with C_+^{k+1} smooth boundary. Then for all $R > \max_{x\in\partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(A(K \setminus K_n^R)) = z_1'(K)n^{-\frac{2}{3}} + \ldots + z_{k-1}'(K)n^{-\frac{k}{3}} + O(n^{-\frac{k+1}{3}})$$

as $n \to \infty$, where $z_i'(K) = A(K)z_i(K)$ for $i = 1, \ldots, k$.

In the case when $K = B(R)$, that is, K is a radius R closed circular disc, the expected number of vertices behaves fundamentally differently from the previously discussed situation. It was proved in [FKV14] that

$$\mathbb{E}(f_0(B(R)_n^R)) = \frac{\pi^2}{2} + o(1).$$

as $n \to \infty$.

Applying the same method as before, we prove a finite expansion of the expected number of vertices $\mathbb{E}(f_0(B(R)_n^R))$ in terms of the powers of n.

Chapter 4

The less smooth case

In Chapter 4 of the dissertation we investigate the asymptotic formulas of Fodor, Kevei and Vígh [FKV14] under weaker differentiability conditions on the boundary of K.

In order to state the main result on this chapter, we need the following two definitions: A convex disc K has a rolling circle if there exists a real number $r_0 > 0$ with the property that any $x \in \partial K$ lies in some closed circular disc of radius r_0 contained in K. We say that a convex disc K slides freely in a circular disc $R_0 B^2$ of radius R_0 if for each $x \in \partial K$ there exists a $p \in \mathbb{R}^2$ such that $x \in \partial(R_0 B^2) + p$ and $K \subset R_0 B^2$.

On one hand, the rolling circle condition ensures that the boundary of K has a unique outer unit normal everywhere; it has no vertices. On the other hand, the condition that K slices freely in a circle guarantees that K is strictly convex; the boundary contains no segment.

Both conditions ensure that ∂K is C^1, but it may not be C^2. Since we do not require ∂K to be C^2, we need a notion of generalized second-order differentiability and curvature.

We say that ∂K is twice differentiable in the generalized sense if it can be approximated by a quadratic function in the following sense: Let $x \in \partial K$. If K is positioned in such a way that $x = o$ and $\mathbb{R}$ is a support line of K, then in a neighborhood of the origin o, ∂K is the graph of a convex function f defined on an open interval containing o satisfying

$$f(z) = \frac{\kappa(x)}{2} z^2 + o(z^2)$$

as $z \to 0$. If ∂K is differentiable in the generalized sense x, then we call $\kappa(x)$ the generalized curvature. The generalized curvature coincides with the usual curvature if ∂K is differentiable twice in the usual sense at x. The boundary of a convex disc is differentiable twice almost everywhere with respect to the arc-length according to a classical result of Alexandrov [Gru07]. For more information on generalized second-order differentiability, we refer to the paper by Schütt and Werner [SW].

The main result of the chapter is the following theorem:

Theorem 11 (on p. 51 of the thesis)**.** *Let K be a convex disc that has a rolling circle and slides freely in a circle of R_0. Then for any $R > R_0$, it holds that*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^R)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} dx,$$

$$\lim_{n\to\infty} \mathbb{E}(A(K \setminus K_n^R)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}} \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} dx.$$

The proof is based on the original argument of Fodor, Kevei and Vígh [FKV14] adapted to the less smooth setting.

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