Problems in geometric probability for intersections of balls

Ph.D. thesis

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Chapter 1

Introduction

1.1 Definitions and notations

We start this thesis by fixing some common notation used throughout the arguments. Some of these notations may be recalled in the individual chapters if necessary, in order to facilitate easy reading. We made an effort to unify notations. However, the reader may notice that some objects are sometimes denoted in slightly different ways in the individual chapters of this dissertation. The reason for this lies in the different setup and circumstances of the particular problem we investigate.

In this dissertation we work in d-dimensional Euclidean space $\mathbb{R}^d$, in particular, most frequently in the plane, that is, in $\mathbb{R}^2$. Points (and vectors) of space are denoted by lowercase letters while we use capitals for sets. The Euclidean distance of points $x, y \in \mathbb{R}^d$ is denoted by $|x - y|$. For a set $S \subset \mathbb{R}^d$, the boundary is denoted by the symbol ∂S. The origin of space is o. The unit radius origin centred closed ball is $B^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$, and its boundary, the origin centred unit sphere is $S^{d-1} = \partial B^d = \{x \in \mathbb{R}^d : |x| = 1\}$. We introduce special notation for the open ball, if necessary, in the individual sections where it is used. In general, for $r > 0$ and $c \in \mathbb{R}^d$, the radius r closed ball centred at the point c is

$$rB^d + c = \{x \in \mathbb{R}^d : |x - c| \leq r\}.$$

The volume (Lebesgue measure) of a (measurable) set $S \subset \mathbb{R}^d$ is denoted by $V_d(S)$. If there can be no confusion about the dimension of space, then may omit the subscript d

and write simply $V(S)$ for the volume. Furthermore, if $d = 2$ (in the plane) we use the notation $A(S) = V_2(S)$ for the area. In particular, the volume of B^d is denoted by

$$\kappa_d = V_d(B^d) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)},$$

where $\Gamma(\cdot)$ is Euler's gamma function defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} \, \mathrm{d}x, \qquad \text{for } x > 0.$$

If f is a measurable function in $\mathbb{R}^d$, then its integral with respect to volume (Lebesgue measure) is generally denoted by $\int_{\mathbb{R}^d} f(x) \, \mathrm{d}x$, or $\int_{\mathbb{R}^d} f \, \mathrm{d}\lambda$. More specialized notations for individual cases may be defined in certain sections.

The symbol ω_d denotes the surface volume (spherical Lebesgue measure) of the d-dimensional unit sphere. It is known (see, for example, Schneider [Sch14]) that

$$\omega_d = d\kappa_d = d\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}.$$

In the plane, when no confusion is possible, we may use the simplified notations $B = B^2$ or $S = S^1$. If this is the case, then it is indicated clearly at the beginning of such a use.

In this dissertation, we mostly deal with convex sets of $\mathbb{R}^d$. A set $S \subset \mathbb{R}^d$ is convex if for any $x, y \in K$, the closed segment $[x, y]$ with endpoints x and y is also contained in K, that is $[x, y] \subset K$. The theory of convex sets is a rich subject that has an enormous literature. We do not attempt here to give an introduction to the general properties of convex sets as this would be way beyond the scope of a dissertation. Instead, we refer the interested reader to consult the following two classical monographs of the field: Gruber [Gru07] and Schneider [Sch14].

By a d-dimensional convex body $K \subset \mathbb{R}^d$ we always mean a compact, convex set K that has non-empty interior (thus K is full dimensional). We note that the term convex body is also used in the literature without this last condition (the non-empty interior) in which case, for example, a closed segment would also be considered a convex body in the plane. However, in our setting the definition requiring non-empty interior is more appropriate, so for us, a segment in the plane is not a convex body (although it is a compact convex set). In particular, when $d = 2$ (in the plane) the term convex disc is often used to designate a convex body, similarly to the common usage of the words circle for 2-dimensional unit sphere and unit circular disc for B^2.

It is known that convex bodies are Lebesgue measurable. For $d = 2$, the boundary ∂K of a convex disc is rectifiable, meaning that it has finite arc-length. In some sections we will integrate functions along the boundary of a convex disc with respect to the arc-length (one can view this as the 1-dimensional Hausdorff measure on ∂K). Such integration of a measurable function f will be denoted by $\int_{\partial K} f(x)\,\mathrm{d}x$, where the fact that integration is with respect to arc-length is indicated in the domain of integration.

We introduce the more specialized notations in the individual chapters as the need arises. Some of these notations may not be uniform as they slightly vary between chapters as the different situations require.

1.2 Brief overview of history and results

In a nutshell, Stochastic Geometry studies properties of random geometric objects (points, lines, planes, etc.). The field of stochastic geometry can be traced back at least to three problems: Buffon's needle problem (1733), Sylvester's four-point problem (1864), and Bertrand's paradox (1888). The mathematical problems mentioned above are the first records of the use of geometric tools to calculate probabilities.

However, these three problems are also interesting on their own. In this brief historical overview, we focus on Sylvester's four-point problem because it can be considered as the origin of Random Polytope Theory. We suggest the interested reader to consult, for example, Mathai's book [Mat99] for a detailed discussion of the other two problems.

Sylvester published the following problem in the journal Educational Times [Syl64]: "*Show that the chance of four points forming the apices (vertices, in modern language) of a reentrant quadrilateral is* $1/4$ *if they be taken at random in an indefinite plane, but* $1/4 + e^2 + x^2$, *where* e *is a finite constant and* x *a variable quantity if they be limited by an area of any magnitude and of any form.*"

We should note that Sylvester does not mention any particular distribution in the problem. This omission has led researchers to assume that Sylvester formulated the problem in terms of the uniform distribution, cf. [Bár08].

The assumption of uniformity, although meaningful, nullifies the first part of the problem which asks to calculate probabilities in an indefinite space (the whole plane $\mathbb{R}^2$). The reason is that there is no uniform probability distribution on the entire space $\mathbb{R}^2$. Nonetheless, the problem can be formulated in terms of other distributions, such as the normal distribution, see, for example, [ES11].

Another natural reformulation of the original question is when one selects the random points from a container, for example, a compact convex set. This makes it possible to choose the points according to the uniform distribution. It quickly turned out that the probability that the points are (or are not) in convex position does depend on the shape of the container.

Thus, Sylvester's problem can be formulated both in terms of a particular density function, as well as a particular convex domain (e.g. circle, triangle, etc.). We advise the interested reader to consult Pfiefer's paper [Pfi89] for a more detailed history of the earlier formulations of the problem.

Sylvester's original question restricts the dimension and the number of points. However, the problem can be generalized to d-dimensional space as follows: what is the probability that $d+2$ independent and identically distributed (i.i.d.) random points selected according to the uniform probability distribution from a convex body K in d-dimensional Euclidean space are in a convex position? Results on this question have been obtained only for a quite restricted set of bodies. For a more in-depth treatment of the problem, we advise the reader to consult the surveys [Rei10], [Sch17].

Another offspring of Sylvester's problem is about the maximal and minimal probabilities. This variational type problem, instead of asking for the probability in a specific domain, seeks to find the domain in which it is maximal or minimal. Blaschke (1917, 1923) [Bla17, Bla23] proved in the plane that the probability that four i.i.d. random points are in non-convex position is minimal for the circle and maximal for the triangle.

Blaschke's result has been partially extended to d-dimensional space by Groemer [Gro73] who proved that the d-dimensional ball gives the minimum probability that $d+2$ i.i.d. random points are in non-convex position. However, Blaschke's argument for the triangle does not generalize to higher dimensions. It is conjectured that the regular simplex in $\mathbb{R}^d$ provides the maximum in this sense. This latter conjecture, often called the Simplex Conjecture, is still open, and it is one of the major unsolved problems in geometric probability and convex geometry.

In the 1960s, three classical papers by Rényi and Sulanke [RS63, RS64, RS68] opened a new line of research about the convex hull of random points. They laid the foundations of the asymptotic analysis of approximations of convex bodies by random polytopes. In particular, one of the models Rényi and Sulanke studied is in which a random polygon is generated as the convex hull of n i.i.d. random points chosen in a fixed convex disc K with a sufficiently smooth boundary according to the uniform probability distribution. Objects of interes include the expected number of edges, the missed area (the difference between the areas of the convex disc K and the random polygon), and the perimeter difference, as $n \to \infty$.

In their original paper [RS63], Rényi and Sulanke calculated the expected number of edges and the missed area independently. Nonetheless, a classical result due to Efron [BE65] (which had been unknown in the times of the papers [RS63, RS64]) allows one to obtain one quantity from the other: the expected number of edges (vertices) from the missing area, and vice versa.

Due to the importance of Efron's equality and its simplicity, we provide a short proof: Let K_n be the convex hull of n i.i.d. uniform random points $x_1, \ldots, x_n$ from a convex body $K \subset \mathbb{R}^d$, and let $f_0(K_n)$ be the number of vertices of K_n. The symbol $\mathbb{P}(\cdot)$ denotes the probability of an event, and $\mathbb{E}(\cdot)$ the expectation of a random variable in this model. Efron's equality states that

$$\mathbb{E} f_0(K_n) = n(V_d(K) - \mathbb{E} V_d(K_{n-1})).$$

The proof is the following simple sequence of equalities:

$$\begin{aligned}
\mathbb{E} f_0(K_n) &= \sum_{i=1}^{n} \mathbb{P}(x_i \text{ is a vertex of } K_n) \\
&= n\mathbb{P}(x_n \text{ is a vertex of } K_n) \\
&= n\mathbb{P}(x_n \notin [x_1, ..., x_{n-1}]) \\
&= n\mathbb{P}(x \notin K_{n-1}) \\
&= n((V_d(K) - \mathbb{E} V_d(K_{n-1})).
\end{aligned}$$

The first equality is due to the independence of the random points. The second equality holds because a particular random point is a vertex of K_n if and only if it is not contained in the convex hull of the other points, which happens exactly when it is in the missed part $K \setminus K_n$ of K by K_n.

A possible research path in random polytope theory concerns reformulations of the problems listed above using other notions of convexity. This subject has been investigated intensively in the last decade. One of the most frequently considered alternative to the usual (linear) concept of convexity is the so-called spindle convexity.

The notion of spindle convexity was (probably) first introduced by Mayer [May35] in 1935 as a generalization of linear convexity. However, due to the naturality of this idea, its origins may go further back in time. We note that Mayer phrased the definition in the (much) wider context of Minkowski geometry, but this does not concern us in this thesis. It is a basic fact in convexity, that in the Euclidean space $\mathbb{R}^d$, a closed convex set, that is not equal to $\mathbb{R}^d$, is the intersection of closed half-spaces. Let $R > 0$. In the definition of a compact R-spindle convex set, radius R closed circular discs play a similar role to closed half-spaces (a precise definition will be stated later). One may say informally that the $R = \infty$ (or rather $R \to \infty$) limiting case corresponds to linear convexity.

We note that our use of the term spindle convexity is not exclusive to this concept. There are various terms in the literature used for R-spindle convex sets: Mayer used the (German) word "Überkonvexität" in [May35]. László Fejes Tóth in [FT82a], [FT82b] called such sets R-convex. Bezdek et al. [BLNP07] used the expression spindle convex. There are also papers where it is called ball-convex, λ-convex, etc. Since the notion of spindle convexity is very natural, it was reinvented several times in different contexts, which explains why so many different names are used for it. Its prevalence in the literature also points to its importance.

Fodor, Kevei and Vígh [FKV14] generalized the classical asymptotic formulas of Rényi and Sulanke [RS63, RS64] to random disc-polygons. In their probability model, the random disc-polygons are the intersection of all closed unit circles containing n i.i.d. uniform random points from a suitable convex disc K.

In this dissertation, we focus our attention on the one hand, the spindle convex version of Wendel's equality, in Chapter 2. On the other hand, we investigate the consequences of strengthening and weakening the smoothness conditions in the asymptotic formulas of Fodor, Kevei, and Vígh [FKV14] for the approximation of convex discs by random disc-polygons, see Chapters 3 and 4.

Wendel's equality gives the probability that the convex hull of $n > d$ i.i.d. random points in $\mathbb{R}^d$ with an origin-symmetric distribution does not contain the origin. Wendel's result has been extended by Wagner and Welzl [WW01] who proved that the symmetrical distributions are extremal in this sense. Wendel's result is an important tool in the Theory of Random Polytopes.

In Chapter 2, we consider the spindle convex analog of Wendel's problem in the plane and in higher dimensions, both for the uniform density in a ball and the Gaussian density in d-space.

The Gaussian case involves an interesting aspect of spindle convexity that is not often used, namely, we also allow random points to be farther away from each other than the diameter of the circular discs used in the definition of spindles resulting in their spindle convex hull covering the whole space.

The motivation behind the study of the spindle convex version of Wendel's equality is twofold. On the one hand, it is of special interest that the spindle formulation of the problem has positive probabilities already for two points in any dimension, while the original problem requires at least $d + 1$ points $\mathbb{R}^d$. The reason behind the difference in the number of points necessary to have non-zero probabilities is the type of convexity involved in each problem. The classical convex hull of two points is a line segment with zero area. Thus, the probability that the convex hull of two points taken at random contains the origin is zero. While the spindle convex hull of two points is a spindle (a set bounded by two circular arcs through the two points), which is a non-zero measure set (the probability that the two random points coincide is zero).

On the other hand, Wendel's equality is a useful tool to obtain results about random polytopes in the classical convex case. Given that many problems in classical convexity can be reformulated in terms of spindle convexity, having a Wendel-type equality in the spindle convex setting may prove very useful to solve problems in that realm.

In Chapter 2, we obtaine a Wendel-type equality for spindle convexity, see Theorem 1, which, similarly to the classical equality, can be evaluated for a fixed dimension and a fixed number of points. In Theorem 1, we also calculated the actual probabilities in the 2 and 3-dimensional case for two points, and in Theorem 2 for three random points in the plane. Nonetheless, evaluating our result for specific cases is more computationally demanding than the classical Wendel formula.

However, as a corollary of our results, one can observe that a theorem about the extremality of symmetric distributions, similar to one Wagner and Welzl proved for the classical convex case, is not attainable in spindle convexity. The reason is that we get different probabilities for our Wendel-type formula when evaluated for the uniform and normal distributions.

The difference in computability is explained by the tools used in each proof. Wendel's original proof relies on the distribution being symmetric with respect to the origin and Schläfli's formula for the number of cells into which h hyperplanes in general position divide R^d.

Wendel's proof is essentially a combinatorial argument that delivers a formula easy to compute. Unfortunately, the same combinatorial argument cannot be used in the spindle convex case. Instead, we must follow an analytical reasoning and evaluate the actual probabilities through integration. The main tool is a type of integral transformation often used in connection with problems on random polytopes: the linear Blaschke-Petkantschin formula.

One major difference compared to the classical convex case, which partly explains the need for an analytical argument, is that even if the random points can be separated from the origin by a hyperplane, they may not be separated from it by a supporting ball. Thus, one has to take into account their positions more precisely. We note that the use of the linear Blaschke-Petkantshin formula reduces the argument to lower dimensions in the case of two points.

In Chapters 3 and 4, we investigate the differentiability conditions of the asymptotic formulas of Fodor, Kevei and Vígh[FKV14] and the consequences of changing them. One motivation is to clarify the necessary and sufficient conditions for the formulas to hold and understand the relationship between smoothness and the behaviour of functionals such as the number of vertices and missed area.

We note that such investigations have been carried out in the classical case. For gradually weaker differentiability conditions, see, for example the results of Wieacker [Wie78], Bárány [Bár92], Schütt [Sch94] and Böröczky, Fodor and Hug [BFH10]. Under stronger differentiability conditions, Reitzner [Rei01, Rei04] obtained finite power series expansions (the length of the expansion depends on the degree of differentiability) for the expected number of vertices, missed volume, and also for other quantities, as the number of generating random points tends to infinity. The problem has not yet been explored to the same extent in the spindle convex case.

In Chapter 3, we obtain finite length asymptotic series expansions for the expected vertex number (and also the missed area) of random disc-polygons as n tends to infinity under the condition that the generating random point are from a convex domain whose boundary is $k+1$ times differentiable and has positive curvature everywhere, see Theorem 5 (vertex number) and Theorem 6 (missed area). The main tools are the argument following. We use the local representation of the boundary of the convex disc using a suitably differentiable function, see Lemma 7. We apply a statement on the inversion of certain functions by Gruber [Gru96], see Lemma 8. Finally, we use an asymptotic expansion of the incomplete beta function, see Lemma 9.

In Chapter 4, we examine the asymptotic behaviour of the expectations of the same quantities (number of vertices and missed area) under weaker smoothness assumptions. We prove that the asymptotic formulas of Fodor, Kevei and Vígh [FKV14] hold under the assumption that the convex domain slides freely in a circular disc, and a circle rolls freely in it. These conditions guarantee, on the one hand, that the disc is smooth (the boundary is C^1), on the other hand, that it is strictly convex. Moreover, Hug [Hug00] proved that the so-called spherical image map and the reverse spherical image map (which are both well-defined in this case and are inverses to each other) are both Lipschitz continuous, so differentiable almost everywhere. We also use the concept of generalized second order differentiability. Having a rolling circle and sliding freely in a circle also guarantee that the disc is spindle convex for all radii larger than or equal to the radius of the circle in which the disc slides freely. However, second order differentibility (and thus the existence of the usual curvature) does not necessarily hold. This is why we need a generalized notion of differentiability. We say that the boundary of the disc is differentiable twice in the generalized sense at a boundary point if the boundary can be approximated in a neighbourhood of this point by a parabola (like a second order Taylor polynomial). It is known (Alexandrov's theorem) that the boundary of a convex body is differentiable twice in the generalized sense almost everywhere with respect to the surface area measure. This defines a generalized curvature in a natural sense that coincides with the usual curvature at points where the boundary is differentiable twice, see precise definitions in Chapter 4. We prove, cf. Theorem 11, that the argument of Fodor, Kevei and Vígh can be carried out under these weaker assumptions with minor modifications, thus, extending the validity of their asymptotic formulas.

Chapter 2

Wendel's equality for intersections of balls

This chapter of the dissertation is based on the following published paper of the author:

[FMV23] F. Fodor, P. N. A. Montenegro, and V. Vígh, *On Wendel's equality for intersections of balls*, Aequationes Math. **97** (2023), no. 2, 439–451, DOI 10.1007/s00010-022-00912-3. MR4563622

2.1 Introduction and results

Wendel's equality [Wen62] is one of the classical results in geometric probability: it states that if $x_1, \dots, x_n$ are i.i.d. (independent and identically distributed) random points in $\mathbb{R}^d$ whose distribution is (centrally) symmetric with respect to the origin o, and the probability measure of hyperplanes are 0, then the probability that o is not contained in the convex hull $[x_1, \dots, x_n]$ is

$$\mathbb{P}(o \notin [x_1, \dots, x_n]) = \frac{1}{2^{n-1}} \sum_{i=0}^{d-1} \binom{n-1}{i}. \tag{2.1}$$

One can find a simple proof of (2.1) in Bárány [Bár07, pp. 94–95], which is independent of the distribution (under the above conditions) and essentially combinatorial in nature, using the Schläfli formula for the number of parts into which n independent hyperplanes divide the space.

It was proved by Wagner and Welzl [WW01], that o-symmetric distributions are extremal in this sense, meaning that if the distribution is absolutely continuous with respect to the Lebesgue measure, then the probability that the origin is not contained in the random polytope is at least the quantity in (2.1). Thus, (2.1) is essentially a lower bound. For more information on similar statements and further references, see also [SW08, Section 8.1.2].

Recently, Kabluchko and Zaporozhets [KZ20] investigated the related problem of finding the probability that the convex hull of n i.i.d. normally distributed random points in $\mathbb{R}^d$ contains a fixed points of space; they called these *absorption probabilities.* For more information we refer to the recent survey paper by Schneider [Sch17] and the book by Schneider and Weil [SW08].

In this chapter we study the following *spindle convex* variant of the above problems. Let $x, y \in \mathbb{R}^d$ be two points and $R > 0$. If $|x - y| \leq 2R$, then let the spindle $[x, y]_R$ determined by x and y be the intersection of all radius R closed balls that contain both x and y. If $|x - y| > 2R$, then we define $[x, y]_R = \mathbb{R}^d$. We say that a convex body $K \subset \mathbb{R}^d$ (compact convex set with non-empty interior) is spindle convex with radius R, or R-spindle convex if together with any two points $x, y \in K$, it contains the spindle $[x, y]_R$. It is known ([BLNP07]) that if a convex body $K \subset \mathbb{R}^d$ is spindle convex with radius R, then K is the intersection of all radius R closed balls that contain K. This latter property is called radius R ball-convexity.

Let $X \subset \mathbb{R}^d$. If $X \subset RB^d + v$ for some $v \in \mathbb{R}^d$, then the radius R spindle convex hull $[X]_R$ of K is defined as the intersection of all radius R closed balls containing X. If $X \not\subset RB^d + v$ for any $v \in \mathbb{R}^d$, then let $[X]_R = \mathbb{R}^d$. If $K \subset \mathbb{R}^d$ is spindle convex with radius R, and $X \subset K$, then $[X]_R \subset K$. For more information on spindle convexity, see, for example, the paper [BLNP07] by Bezdek et al. and the book [MMO19] by Martini, Montejano and Oliveros and the references therein.

First, we describe the o-symmetric R-spindle convex uniform model. Let $R > 0$, and let $K \subset \mathbb{R}^d$ be an o-symmetric convex body that is R-spindle convex. Let $x_1, \ldots, x_n$ be i.i.d. uniform random points from K. The uniform distribution in K is the unique probability distribution that is concentrated on K and whose density is equal to $1/A(K)$. We denote the radius R spindle convex hull of $x_1, \ldots, x_n$ by $K_n^R = [x_1, \ldots, x_n]_R$. By the

R-spindle convexity of K, the random ball-polytope K_n^R is contained in K. We ask the same question as in the classical convex case: what is the probability that $o \in K_n^R$ (or equivalenty, $o \notin K_n^R$)? We note that in this model we may always achieve by scaling (simultaneously K and radius R balls) that $R = 1$. Henceforth, in the following two theorems we assume that $R = 1$.

We study the special case when $K = rB^d$ with $0 < r \leq 1$. Then K is clearly spindle convex with radius $R = 1$. We wish to determine the probability

$$P(d, r, n) := \mathbb{P}(o \in [x_1, \ldots, x_n]_1).$$

In Section 2.2 we prove the following theorem:

Theorem 1 ([FMV23, Theorem 1.1])**.** *Let $K = rB^d$. Then*

$$P(d, r, 2) = \frac{\omega_{d-1}\omega_d}{(r^d\kappa_d)^2} \int_0^r \int_0^r \int_0^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2}\varphi \, d\varphi dr_2 dr_1,$$

where $\varphi(r_1, r_2) = \arcsin(r_1/2) + \arcsin(r_2/2)$*. In particular,*

$$P(2, 1, 2) = \frac{\sqrt{3}}{\pi} - \frac{1}{3} = 0.2179\ldots,$$
$$P(3, 1, 2) = \frac{1}{64}(23 + 12\sqrt{3}\pi - 8\pi^2) = 0.1459\ldots.$$

Furthermore, for the case of three points, we prove the following statement in Section 2.3.

Theorem 2 ([FMV23, Theorem 1.2])**.** *Let $K = B^2$. Then*

$$P(2, 1, 3) = \frac{-84\pi^2 - 477 + 360\sqrt{3}\pi}{144\pi^2} = 0.4594\ldots.$$

Finally, in Section 2.4, we study the Gaussian R-spindle convex model. Let $x_1, \ldots, x_n$ be i.i.d. random points from $\mathbb{R}^d$ distributed according to the standard normal distribution. The question is the same, what is the probability that $o \in K_n^R$? We note that in this second case, it may, and does, happen that $K_n^R = \mathbb{R}^d$. We give an integral formula for the probability that a Gaussian unit radius spindle contains the origin and evaluate it numerically in the plane, see (2.6).

2.2 Proof of Theorem 1

Note that it is the simplest case of the model when $n = 2$, and $K = rB^d$, where $0 < r \leq 1$ is a fixed number. This, of course, is of no interest in the classical version of Wendel's problem as $\mathbb{P}(o \in [x_1, x_2])$ since $[x_1, x_2]$ is a segment.

We note that the argument of the original proof of Wendel's equality based on Schäfli's formula cannot be used in the spindle convex setting. This is due to the fact that in order for o not to be in the (random) spindle $[x_1, x_2]_1$, the spindle has to be separated from o by a line. This is a geometric property that heavily depends on the actual positions of x_1 and x_2.

Let us examine what it means geometrically that $o \in [x_1, x_2]_1$. Let $M(x_1)$ denote the union of all open unit balls that contain o and x_1 on their boundary. Let $K(d, r, x_1)$ be the part of $rB^d \setminus M(x_1)$ that is in the closed half-space bounded by the hyperplane through o and orthogonal to x_1 which does not contain x_1. Furthermore, let $V(d, r, x_1) := V_d(K(r, x_1))$, that is the d-dimensional volume of $K(d, r, x_1)$. We depicted this region in Figure 2.1 when $d = 2$. We will only use $K(2, r, x_1)$ in our calculations, so, in order to simplify notation, we will denote it by $K(r, x_1) = K(2, r, x_1)$ and $V(r, x_1) = V(2, r, x_1)$.

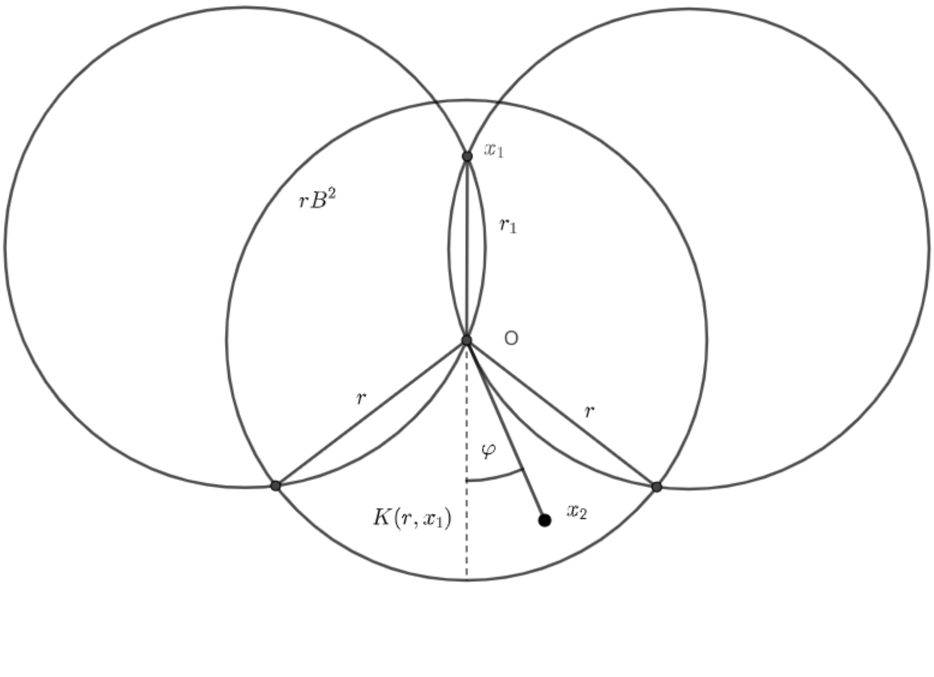

Figure 2.1: The region $K(r, x_1)$.

In order to evaluate $P(d,r,2)$, we use the linear Blaschke-Petkantschin formula. Let $G(d,2)$ denote the Grassmanian manifold of 2-dimensional linear subspaces of $\mathbb{R}^d$, and ν_2 be the unique rotation invariant Haar probability measure on $G(d,2)$. The 2-dimensional special case of the linear Blaschke-Petkantschin formula (see, for example, [SW08, Theorem 7.2.1 on p. 271]) says the following: If $f : (\mathbb{R}^d)^2 \to \mathbb{R}$ is a nonnative measurable function, then

$$\int_{(\mathbb{R}^d)^2} f\, d\lambda^2 = \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{L^2} f \nabla_2^{d-2}\, d\lambda_L^2\, \nu_2(dL), \tag{2.2}$$

where ∇_2 denotes the area of the parallelogram spanned by the vectors x_1, x_2 in L. The symbol λ denotes the Lebesgue measure in $\mathbb{R}^d$, and λ_L the (2-dimensional) Lebesgue measure in L.

Next, using polar coordinates for $x_1, x_2 \in L$, that is, $x_1 = r_1u_1$, $x_2 = r_2u_2$, where $u_1, u_2 \in S^1$, $r_1, r_2 \in \mathbb{R}_+$, we may write the right-hand-side of (2.2) as follows.

$$\begin{aligned}
&\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{L^2} f \nabla_2^{d-2}\, d\lambda_L^2\, \nu_2(dL) \\
&= \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{(S^1\times\mathbb{R})^2} f(r_1u_1, r_2u_2) \nabla_2^{d-2}\, r_1r_2dr_1du_1dr_2du_2\, \nu_2(dL) \\
&= \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{(S^1\times\mathbb{R})^2} f(r_1u_1, r_2u_2) r_1^{d-1} r_2^{d-1} \times \\
&\qquad\qquad \times |u_1 \times u_2|^{d-2} dr_1du_1dr_2du_2\, \nu_2(dL).
\end{aligned} \tag{2.3}$$

Now, from (2.3) and the rotational symmetry of the rB^d, we obtain that

$$\begin{aligned}
P(d,r,2) &= \frac{1}{(r^d\kappa_d)^2} \int_{rB^d} \int_{rB^d} \mathbf{1}(o \in [x_1, x_2]_1)\, dx_1dx_2 \\
&= \frac{1}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{S^1} \int_0^r \int_{S^1} \int_0^r \mathbf{1}(o \in [r_1u_1, r_2u_2]_1) r_1^{d-1} r_2^{d-1} \\
&\qquad \times |u_1 \times u_2|^{d-2} dr_1du_1dr_2du_2\, \nu_2(dL) \\
&= \frac{1}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{S^1} \int_0^r \int_{S^1} \int_0^r \mathbf{1}(o \in [r_1u_1, r_2u_2]_1) r_1^{d-1} r_2^{d-1} \\
&\qquad \times |u_1 \times u_2|^{d-2} dr_1du_1dr_2du_2 \\
&= \frac{1}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{S^1} \int_0^r \int_{S^1} \int_0^r \mathbf{1}(x_2 \in K(r, x_1)) r_1^{d-1} r_2^{d-1} \\
&\qquad \times |u_1 \times u_2|^{d-2} dr_2du_2dr_1du_1 \\
&= \frac{2\pi}{(r^d\kappa_d)^2} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_0^r \int_0^r \int_{-\varphi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} |u_1 \times u_2|^{d-2} du_2dr_2dr_1,
\end{aligned}$$

where

$$\varphi(r_1, r_2) = \arcsin(r_1/2) + \arcsin(r_2/2).$$

Let φ be the angle of u_2 and $-u_1$, as shown on Figure 2.1. Then

$$\begin{aligned}
P(d,r,2) &= \frac{2\pi}{(r^d\kappa_d)^2}\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2}\int_0^r\int_0^r\int_{-\varphi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1^{d-1}r_2^{d-1}|\sin\varphi|^{d-2}\,d\varphi dr_2dr_1 \\
&= \frac{4\pi}{(r^d\kappa_d)^2}\frac{\omega_{d-1}\omega_d}{\omega_1\omega_2}\int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_1^{d-1}r_2^{d-1}\sin^{d-2}\varphi\,d\varphi dr_2dr_1 \\
&= \frac{\omega_{d-1}\omega_d}{(r^d\kappa_d)^2}\int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_1^{d-1}r_2^{d-1}\sin^{d-2}\varphi\,d\varphi dr_2dr_1
\end{aligned}$$

The above integral can be evaluated by for any specific value of d using multiple integration by parts. In particular,

$$\begin{aligned}
P(2,r,2) &= \frac{4}{\pi r^4}\int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_2r_1d\varphi dr_2dr_1 \\
&= \frac{4}{\pi r^4}\int_0^r\int_0^r r_2r_1(\arcsin(r_1/2)+\arcsin(r_2/2))\,dr_2dr_1 \\
&= \frac{4}{\pi r^4}\left(\frac{r^2}{4}(r\sqrt{4-r^2}+2(r^2-2)\arcsin(r/2))\right) \\
&= \frac{1}{\pi r^2}\left(r\sqrt{4-r^2}+2(r^2-2)\arcsin(r/2)\right), \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
P(3,r,2) &= \frac{9}{2r^6}\int_0^r\int_0^r\int_0^{\varphi(r_1,r_2)} r_2^2r_1^2\sin\varphi\,d\varphi dr_2dr_1 \\
&= \frac{9}{2r^6}\left(\frac{r^2}{288}(-72+90r^2-4r^4+9r^6)\right. \\
&\qquad \left.+\frac{1}{4}\arcsin(r/2)(R\sqrt{4-r^2}(r^2-2)+4\arcsin(r/2))\right).
\end{aligned}$$

In particular,

$$\begin{aligned}
P(2,1,2) &= \frac{\sqrt{3}}{\pi}-\frac{1}{3} = 0.2179\ldots, \\
P(3,1,2) &= \frac{1}{64}(23+12\sqrt{3}\pi-8\pi^2) = 0.1459\ldots.
\end{aligned}$$

This finishes the proof of Theorem 1.

We conclude this section with the following statements.

Corollary 3 ([FMV23, Corollary 2.1]). *For any fixed $d \geq 2$, it holds that*

$$\lim_{r\to 0^+} P(d,r,2) = 0.$$

Furthermore, for any fixed $0 < r \le 1$, *it holds that*

$$\lim_{d\to\infty} P(d,r,2) = 0.$$

Proof. Note that, using $\arcsin x \le \pi x/2$ for $x \in [0,\pi/2]$ and $\sin x \le x$ for $x \in [0,\pi/2]$, we get that

$$\begin{aligned}
P(d,r,2) &\le \frac{C(d)}{r^{2d}} \int_0^r \int_0^r \int_0^{r_1+r_2} r_1^{d-1} r_2^{d-1} (r_1+r_2)^{d-2}\, d\varphi dr_2 dr_1 \\
&\le \frac{2^{d-1}C(d)}{r^{2d}} \int_0^r \int_{r_1}^r \int_0^{2r_2} r_2^{3d-4}\, d\varphi dr_2 dr_1 \\
&= \frac{2^{d}C(d)}{r^{2d}} \int_0^r \int_0^r r_2^{3d-3}\, dr_2 dr_1 \\
&= \frac{2^{d}C(d)}{r^{2d}} \frac{r^{3d-1}}{3d-2},
\end{aligned}$$

where the constant $C(d)$ depends only on the dimension d. From this it follows that

$$\lim_{r\to 0^+} P(d,r,2) = 0$$

for $d \ge 2$, as claimed.

In the proof of the second statement we use the fact that $\varphi(r_1,r_2) \le \pi/3$. Thus

$$\begin{aligned}
P(d,r,2) &\le \frac{\omega_{d-1}\omega_d}{r^{2d}\kappa_d^2} \int_0^r \int_0^r r_1^{d-1} r_2^{d-1} \left(\frac{\sqrt{3}}{2}\right)^{d-1} dr_2 dr_1 \\
&= \frac{\omega_{d-1}\omega_d}{d^2\kappa_d^2} \left(\frac{\sqrt{3}}{2}\right)^{d-1} = \frac{d-1}{d}\frac{\kappa_{d-1}}{\kappa_d} \left(\frac{\sqrt{3}}{2}\right)^{d-1}.
\end{aligned}$$

Since $\kappa_{d-1}/\kappa_d \sim c \cdot \sqrt{d}$ as $d \to \infty$, thus $P(d,r,2) \to 0$ as $d \to \infty$. (Here the symbol $\sim$ means the asymptotic equality of sequences, that is, $f(k) \sim g(k)$ if $\lim_{k\to\infty} f(n)/g(n) = 1$.) □

2.3 Proof of Theorem 2

The case when $n = 3$, can be treated, at least in the plane, as follows. We only consider when $r = 1$, that is, $K = B^2$. Let x_1, x_2, x_3 be i.i.d. uniform random points from B^2. Let

$$P(2,1,3) := \mathbb{P}(o \in [x_1,x_2,x_3]_1)$$

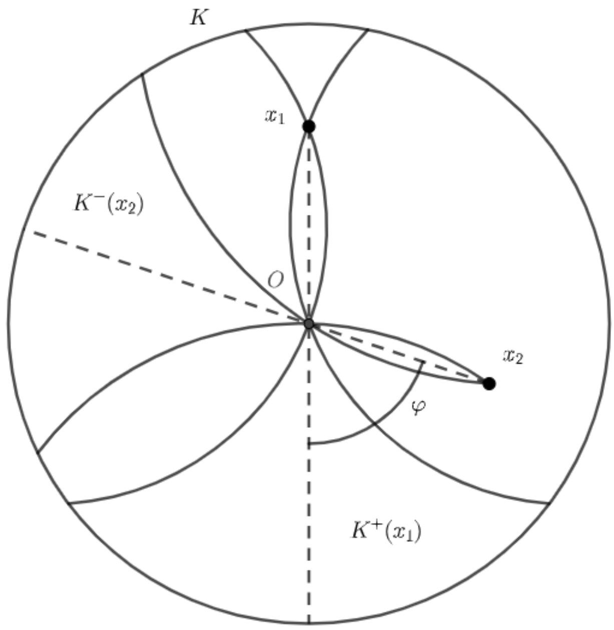

Figure 2.2: The case of three points

$$= \mathbb{P}(o \in [x_1, x_2]_1) + \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1)$$
$$= P(2, 1, 2) + \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1).$$

Let

$$\overline{P}(2, 1, 3) = \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1).$$

Due to the rotational symmetry of B^2, we may assume that $x_1 = (0, r_1)$. Let $x_2 = r_2 u_2$, where φ is the angle of u_2 and the negative half of the y-axis. Making use of the previously introduced notation, we write $K(x_1) = K(1, x_1)$ and, similarly, $K(x_2) = K(1, x_2)$. The ray ox_i divides $K(x_i)$ into two congruent parts. The part that is on the positive side of ox_i is denoted by $K^+(x_i)$, and the negative part is $K^-(x_i)$, as shown on Figure 2.2.

Let $V^+(x_i) = V_2(K^+(x_i))$ and $V^-(x_i) = V_2(K^-(x_i))$ for $i = 1, 2$. Then it holds that

$$V^+(x_i) = V^-(x_i) = \int_0^1 \int_0^{\varphi(r_i, r)} r\, d\varphi dr = \int_0^1 (\arcsin(r_i/2) + \arcsin(r/2)) r\, dr$$
$$= \frac{1}{12}\left(3\sqrt{3} - \pi + 6 \arcsin(r_i/2)\right).$$

We distinguish four cases according to the relative position of x_1 and x_2.

Case 1. $r_2 \leq r_1$ *and* $x_2 \notin [x_1, o]_1$.

In this case, $\varphi \in [\varphi(r_1, r_2), \pi - \arcsin(r_1/2) + \arcsin(r_2/2)]$. Then

$$
\begin{aligned}
P_1 &= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \notin [x_1, o]_1 \text{ and } r_1 \geq r_2) \\
&= \frac{2\pi}{\pi^3} \int_0^1 \int_0^{r_1} \int_{\varphi(r_1,r_2)}^{\pi - \arcsin(r_1/2) + \arcsin(r_2/2)} \left(V^+(x_1) + V^-(x_2) + \frac{\pi - \varphi}{2} \right) r_1 r_2 d\varphi dr_2 dr_1 \\
&= \frac{1}{\pi^2} \int_0^1 \int_0^{r_1} \int_{\varphi(r_1,r_2)}^{\pi - \arcsin(r_1/2) + \arcsin(r_2/2)} \Big(\sqrt{3} - \frac{\pi}{3} + \arcsin(r_1/2) \\
&\qquad + \arcsin(r_2/2) + \frac{\pi - \varphi}{2} \Big) r_1 r_2 d\varphi dr_2 dr_1 \\
&= -\frac{5}{72} - \frac{1}{\pi^2} + \frac{5}{4\sqrt{3}\pi}.
\end{aligned}
$$

Case 2. $r_2 \geq r_1$ and $x_1 \notin [x_2, o]_1$. By the symmetry of x_1 and x_2,

$$
\begin{aligned}
P_2 &= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_1 \notin [x_2, o]_1 \text{ and } r_1 \leq r_2) \\
&= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \notin [x_1, o]_1 \text{ and } r_1 \geq r_2) \\
&= -\frac{5}{72} - \frac{1}{\pi^2} + \frac{5}{4\sqrt{3}\pi}.
\end{aligned}
$$

Case 3. $x_2 \in [x_1, o]_1$.

In this case $r_1 \geq r_2$ and $\varphi \in [\pi - \arcsin(r_1/2) + \arcsin(r_2/2), \pi]$. Then $K(x_2) \subset K(x_1)$, thus

$$
\begin{aligned}
P_3 &= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \in [x_1, o]_1) \\
&= \frac{2\pi}{\pi^3} \int_0^1 \int_0^{r_1} \int_{\pi - \arcsin(r_1/2) + \arcsin(r_2/2)}^{\pi} V(x_1) r_1 r_2 d\varphi dr_2 dr_1 \\
&= \frac{1}{\pi^2} \int_0^1 \int_0^{r_1} \int_{\pi - \arcsin(r_1/2) + \arcsin(r_2/2)}^{\pi} \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} + \arcsin(r_1/2) \right) r_1 r_2 d\varphi dr_2 dr_1 \\
&= \frac{99 - 24\sqrt{3}\pi + 4\pi^2}{576\pi^2}.
\end{aligned}
$$

Case 4. $x_1 \in [x_2, o]_1$. Again, by the symmetry of x_1 and x_2,

$$
\begin{aligned}
P_4 &= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_1 \in [x_2, o]_1) \\
&= \mathbb{P}(o \notin [x_1, x_2]_1 \text{ and } o \in [x_1, x_2, x_3]_1 \text{ and } x_2 \in [x_1, o]_1) \\
&= \frac{99 - 24\sqrt{3}\pi + 4\pi^2}{576\pi^2}.
\end{aligned}
$$

Thus, taking into account the symmetry with respect to the line ox_1, we obtain that

$$
\overline{P}(2, 1, 3) = 2(P_1 + P_2 + P_3 + P_4) = \frac{-36\pi^2 - 477 + 216\sqrt{3}\pi}{144\pi^2}.
$$

Thus,

$$P(2,1,3) = P(2,1,2) + \overline{P}(2,1,3) = \frac{-84\pi^2 - 477 + 360\sqrt{3}\pi}{144\pi^2} = 0.4594\dots.$$

We note that the actual calculation can be carried out, at least numerically, for any $0 < r \le 1$. Furthermore, the cases of $n = 4, 5, \dots$ are essentially similar, although the case analysis grows significantly more complicated as n increases.

Finally, we note that according to Wendel's equality (2.1),

$$\mathbb{P}(0 \in [x_1, x_2, x_3]) = \frac{1}{4} < P(2,1,3).$$

The above inequality is expected as the spindle convex hull is strictly larger than the classical spindle convex hull and thus the corresponding probabilities must behave the same way.

2.4 The case of normally distributed random points

In this subsection we consider the model in which $R = 1$ and $x_1, \dots, x_n$ are i.i.d. random points in $\mathbb{R}^d$ that are distributed according to the standard normal distribution with density function

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2}},\ x \in \mathbb{R}^d.$$

Here we need to use the part of the definition of the spindle convex hull that normally does not come into play when the random points are chosen from a convex body that is spindle convex with radius less than or equal to 1. Namely, if $x, y \in \mathbb{R}^d$ are such that $|x - y| > 2$, then $[x, y]_1 := \mathbb{R}^d$.

We are interested in the following probability

$$P_N(d, 1, n) := \mathbb{P}(o \in [x_1, \dots, x_n]_1).$$

It is clear that

$$\mathbb{P}(o \in [x_1, \dots, x_n]) \le \mathbb{P}(o \in [x_1, \dots, x_n]_1)$$

as $[X] \subset [X]_1$ for any $X \subset \mathbb{R}^d$.

Let E be the event that $|x_1 - x_2| \le 2$. Then

$$P_N(d, 1, 2) = \mathbb{P}(o \in [x_1, x_2]_1 \text{ and } E) + \mathbb{P}(E^c),$$

where E^c is the complement of E, as E^c automatically implies that $o \in [x_1, x_2]_1$.

Let l denote the length of the random segment $[x_1, x_2]$. It is known (see [Mat99, p. 438] and the historical references therein) that the density of $s := l^2/4$ is

$$g(s) = \frac{s^{\frac{d}{2}-1}e^{-s}}{\Gamma(d/2)}, \ 0 < s < \infty. \tag{2.5}$$

Thus,

$$\mathbb{P}(E^c) = \int_1^\infty g(s)\, ds = \frac{\gamma(d/2, 1)}{\Gamma(d/2)},$$

where $\Gamma(\cdot)$ is Euler's gamma function, and $\gamma(d/2, x)$ denotes the lower incomplete gamma function.

Using the linear Blaschke–Petkantschin formula (2.3) and the rotational invariance of the standard normal distribution we obtain that

$$\begin{aligned}
&\mathbb{P}(o \in [x_1, x_2]_1 \text{ and } E) \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}(o \in [x_1, x_2]_1 \text{ and } E)\, e^{-\frac{|x_1|^2+|x_2|^2}{2}}\, dx_1 dx_2 \\
&= \frac{1}{(2\pi)^d} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{G(d,2)} \int_{L^2} \mathbf{1}(o \in [x_1, x_2]_1 \text{ and } E) \Delta^{d-2}(x_1, x_2)\, e^{-\frac{|x_1|^2+|x_2|^2}{2}} dx_1 dx_2\, \nu_2(dL) \\
&= \frac{1}{(2\pi)^d} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_{L^2} \mathbf{1}(o \in [x_1, x_2]_1 \text{ and } E) \Delta^{d-2}(x_1, x_2)\, e^{-\frac{|x_1|^2+|x_2|^2}{2}} dx_1 dx_2.
\end{aligned}$$

In order to evaluate the above integral, we use polar coordinates $x_1 = r_1 u_1$ and $x_2 = r_2 u_2$, $r_1, r_2 \geq 0$, $u_1, u_2 \in S^1$. Let φ be the angle of $-u_1$ and u_2, as before. For $2 - r_1 \leq r_2 \leq \sqrt{4 - r_1^2}$, let

$$\psi(r_1, r_2) = \pi - \arccos\left(\frac{r_1^2 + r_2^2 - 4}{2r_1 r_2}\right).$$

We distinguish two cases according to r_2. When $0 \leq r_2 \leq 2 - r_1$, then $-\varphi(r_1, r_2) \leq \varphi \leq \varphi(r_1, r_2)$, and when $2 - r_1 \leq r_2 \leq \sqrt{4 - r_1^2}$, then $-\varphi(r_1, r_2) \leq \varphi \leq -\psi(r_1, r_2)$ and $\psi(r_1, r_2) \leq \varphi(r_1, r_2)$, see Figure 2.3.

By the rotational symmetry of the normal distribution, integration with respect to u_1 is a just a multiplication by 2π. Then, we obtain that

$$\mathbb{P}(o \in [x_1, x_2]_1 \text{ and } E)$$

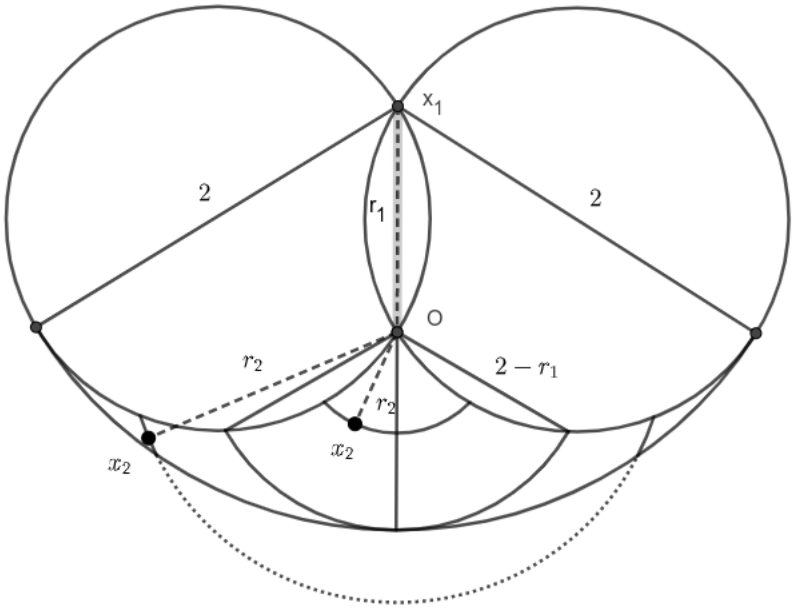

Figure 2.3: Integration bounds in φ according to r_2

$$= \frac{2}{(2\pi)^{d-1}} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_0^2 \int_0^{2-r_1} \int_0^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2}(\varphi)\, e^{-\frac{r_1^2+r_2^2}{2}} d\varphi dr_2 dr_1$$
$$+ \frac{2}{(2\pi)^{d-1}} \frac{\omega_{d-1}\omega_d}{\omega_1\omega_2} \int_0^2 \int_{2-r_1}^{\sqrt{4-r_1^2}} \int_{\psi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2}(\varphi)\, e^{-\frac{r_1^2+r_2^2}{2}} d\varphi dr_2 dr_1.$$

The above integrals can be evaluated, at least numerically, for any specific value of d. In particular, for $d = 2$, we obtain for the first integral

$$\frac{1}{\pi} \int_0^2 \int_0^{2-r_1} \int_0^{\varphi(r_1,r_2)} r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}} d\varphi dr_2 dr_1$$
$$= \frac{1}{\pi} \int_0^2 \int_0^{2-r_1} (\arcsin(r_1/2) + \arcsin(r_2/2)) r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}} dr_2 dr_1$$
$$\approx 0.079214.$$

The second integral is

$$\frac{1}{\pi} \int_0^2 \int_{2-r_1}^{\sqrt{4-r_1^1}} \int_{\psi(r_1,r_2)}^{\varphi(r_1,r_2)} r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}} d\varphi dr_2 dr_1$$
$$= \frac{1}{\pi} \int_0^2 \int_{2-r_1}^{\sqrt{4-r_1^2}} \left(\arcsin(r_1/2) + \arcsin(r_2/2) - \pi + \arccos\left(\frac{r_1^2 + r_2^2 - 4}{2r_1 r_2} \right) \right)$$
$$\times r_1 r_2\, e^{-\frac{r_1^2+r_2^2}{2}} dr_2 dr_1$$
$$\approx 0.01866.$$

For $d = 2$,

$$\mathbb{P}(E^c) = \frac{\gamma(1,1)}{\Gamma(1)} = \frac{1}{e} = 0.367879\ldots,$$

thus, in summary,

$$P_N(2,1,2) = 0.465753\ldots. \tag{2.6}$$

Chapter 3

Series expansions for random disc-polygons

This chapter of the dissertation is based on the following publication of the author:

[FMP24] F. Fodor and N. A. Montenegro Pinzón, *Series expansions for random disc-polygons in smooth plane convex bodies*, J. Appl. Probab. **61** (2024), no. 4, Published online May 16, 2024, DOI 10.1017/jpr.2024.27.

3.1 Introduction and results

Reconstructing a possibly unknown set, or some of its characteristic quantities, from a random sample of points is a classical and a much investigated problem that arises naturally in various fields, like stereology (see, for example, Baddeley and Jensen [BJ05]), computational geometry (see Goodman, O'Rourke and Tóth [GOT18], statistical quality control (see Devroye and Wise [DW80]), etc. Estimating the shape, volume, surface area, and other characteristic quantities of sets is of interest both in geometry and statistics, although the investigated aspects are in many cases different in the respective fields. For an overview of set estimation see, for example, Cuevas and Rodríguez-Casal [CRC03]. The set may be quite arbitrary but often various restrictions are imposed on it. One common such restriction that received much attention is when the set is required to be convex. In such a setting polytopes spanned by random samples of points from the set form a natural estimator. The theory of random polytopes is a rich and lively field with numerous applications. For a recent review and further references see, for example, Schneider [Sch17]. The convex hull is an optimal estimator if no other restrictions are imposed on the set other than convexity. However, in this chapter we study another estimator under further assumptions on K, namely, that the degree of smoothness of the boundary of K is prescribed to be C^{k+1} and it also assumed that the curvature is positive everywhere. Under these circumstances, using congruent circles to form the hull of the sample yields better performance than the classical convex hull.

Since the case when the number of random points is fixed is notoriously difficult, it has become common to investigate the asymptotic behaviour of functionals associated with random polytopes as the number of points in the sample tends to infinity. The investigations of the asymptotic behaviour of random polytopes started with the classical papers by Rényi and Sulanke [RS63,RS64] in the 1960s. They studied the following model in the plane. Let K be a convex body (a compact convex set with nonempty interior) in d dimensional Euclidean space $\mathbb{R}^d$ and let $x_1, \ldots, x_n$ be independent random points from K selected according to the uniform probability distribution.

The convex hull $K_n = [x_1, \ldots, x_n]$ of $x_1, \ldots, x_n$ is called a (uniform) random polytope in K. Rényi and Sulanke [RS63, RS64] proved asymptotic formulas for the expected number $f_0(K_n)$ of vertices of K_n and the expectation of the missed area $A(K \setminus K_n)$ under the assumption that the boundary ∂K of K is sufficiently smooth, and also in the case when K is itself a convex polygon. Wieacker [Wie78] extended this to the d-dimensional ball B^d, and Bárány [Bár92] for d-dimensional convex bodies with at least C^3_+ smooth boundary (three times continuously differentiable with everywhere positive Gauss-Kronecker curvature). Schütt [Sch94] removed all smoothness conditions, and Böröczky, Fodor and Hug [BFH10] extended the results for nonuniform distributions and weighted volume difference.

Let $V_i(\cdot)$, $i = 1, \ldots, d$ denote the i-th intrinsic volume of a convex body. In particular, $V_d(\cdot)$ is the volume (Lebesgue measure), $V_{d-1}(\cdot)$ is the surface volume, and $V_1(\cdot)$ is a constant multiple of the mean width. Reitzner [Rei04] established a power series expansion of the quantity $\mathbb{E}(V_i(K) - V_i(K_n))$ for all $i = 1, \ldots, d$ as $n \to \infty$ under stronger smoothness conditions on the boundary of K.

Theorem 4 ([Rei04]). *Let K be a convex body in $\mathbb{R}^d$ with $V_d(K) = 1$ whose boundary ∂K is C_+^{k+1} for some integer $k \geq 2$. Then*

$$\begin{aligned}&\mathbb{E}(V_i(K) - V_i(K_n)) \\ &\qquad = c_2^{(i,d)}(K)n^{-\frac{2}{d+1}} + c_3^{(i,d)}(K)n^{-\frac{3}{d+1}} + \ldots + c_k^{(i,d)}(K)n^{-\frac{k}{d+1}} + O(n^{-\frac{k+1}{d+1}}) \quad (3.1)\end{aligned}$$

as $n \to \infty$. Moreover, $c_{2m+1}^{(i,d)} = 0$ for $m \leq d/2$ if d is even, and $c_{2m+1}^{(i,d)} = 0$ for m if d is odd.

Under the same conditions as in Theorem 4, one can obtain from (3.1) a series expansion for the number of vertices $\mathbb{E}(f_0(K_n))$ via Efron's identity [BE65]

$$\mathbb{E}(f_0(K_n)) = d_2(K)n^{\frac{d-1}{d+1}} + d_3(K)n^{\frac{d-2}{d+1}} + \ldots + d_{k-2}(K)n^{\frac{d-k+1}{d+1}} + O(n^{\frac{d-k+2}{d+1}})$$

as $n \to \infty$.

Gruber [Gru96] proved the case of Theorem 4 when $i = 1$. Using properties of the convex floating body, Reitzner established the planar case for area ($d = 2$, $i = 2$) of Theorem 4 in [Rei01]. In particular, Reitzner proved that

$$d_4(K) = c_4^{(2,2)}(K) = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3}{2}}\int_{\partial K} k(x)\kappa^{\frac{1}{3}}(x)\,\mathrm{d}x,$$

where $\Gamma(\cdot)$ is Euler's gamma function, $k(x)$ is the affine curvature (for information about the affine curvature see, for example [Bla23, pp. 12–15].) and $\kappa(x)$ is the curvature of ∂K at x, and integration on the boundary ∂K of K is with respect to arc-length.

For more information about approximations of convex bodies by classical random polytopes we refer to the book by Schneider and Weil [RSWW08], and the survey articles by Bárány [Bár08], Reitzner [Rei10], and by Schneider [Sch17], and by Weil and Wieacker [WW93].

When estimating a planar convex body under curvature restrictions, naturally, it may be more advantageous to use suitably curved arcs to form the boundary of the approximating set that fit K better than line segments. One of the simplest such constructions uses radius R circular arcs and the resulting (convex) hull is called, among other names, the R-spindle convex hull, for precise definitions see below. The radius should be chosen in such a way that the (generalised) random polygon is still contained in K. This imposes the condition on R that it should be at least as large as the maximum radius of curvature of ∂K. However, similarly to the classical convex case, difficulties arise when R is equal to the maximal radius of curvature, so this case usually needs a separate treatment using different methods.

In this chapter, we study the R-spindle convex variant of the above probability model in the Euclidean plane $\mathbb{R}^2$. Let $R > 0$ be fixed, and let $x, y \in \mathbb{R}^2$ be such that their distance is at most $2R$. We call the intersection $[x, y]_R$ of all closed circular discs of radius R that contain both x and y the R-spindle of x and y. A set $X \subseteq \mathbb{R}^2$ is called *R-spindle convex* if from $x, y \in X$ it follows that $[x, y]_R \subseteq X$. Spindle convex sets are also convex in the usual linear sense. In this chapter we restrict our attention to compact spindle convex sets. One can show (cf. Corollary 3.4 on page 205 in [BLNP07]) that a convex disc (compact convex set in $\mathbb{R}^2$ with nonempty interior) is R-spindle convex if it is the intersection of (not necessarily finitely many) closed circular discs of radius R. The intersection of finitely many closed circular discs of radius R is called a convex R-disc-polygon. Let X be a compact set which is contained in a closed circular disc of radius R. The intersection of all R-spindle convex discs containing X is called the R-spindle convex hull of X, and it is denoted by $[X]_R$. If $X \subset K$ for an R-spindle convex disc K, then $[X]_R \subset K$. A prominent class of R-spindle convex sets that are directly relevant in this paper is provided by convex discs whose boundary is C^2_+ smooth with curvature $\kappa(x) \geq 1/R$ for all boundary points $x \in \partial K$ (see [Sch14, §2.5 and 3.2]). For more detailed information about spindle convexity we refer to Bezdek et al. [BLNP07] and Martini, Montejano and Oliveros [MMO19].

We note that there exist further generalisations of spindle convexity, most notably, the concept of L-convexity in which the translates of a fixed convex body L play the role of the radius R closed disc, for more information see, for example, Lángi, Naszódi and Talata [LNT13]. Another further generalisation is H-convexity introduced by Kabluchko, Marynych and Molchanov [KMM22], where the hull of a set is generated by intersections of transformed copies of a fixed convex set C by a set H of affine transformations. A similar concept (see, for example, Mani-Levitska [ML93]) to R-spindle convexity, called α-convexity, also exists, where the α-convex hull of a set is defined as the complement of the union of all radius r open balls disjoint from the set. The α-convex hull of a finite sample is different from its R-spindle convex hull as it is nonconvex while the R-convex hull is always convex. We note that the α-convex hull can be used to estimate not necessarily convex sets as well, see, for example, Paterio-Lopez and Rodríguez-Casal [PLRC08], Rodríguez-Casal [RC07] and Pateiro-López [PL08], where several such results are proved about random samples chosen from the set according to an absolute continuous probability distribution.

A convex R-disc-polygon is clearly R-spindle convex. We consider a single radius R disc and a single point also R-disc-polygons, albeit trivial ones. The non-smooth points of the boundary of a nontrivial convex R-disc-polygon are called vertices. The vertices divide the boundary into a union of radius R circular arcs of positive arc-length, we call edges. Thus, a nontrivial convex R-disc-polygon has an equal number of edges and vertices, just like a classical convex polygon, except the sides are radius R circular arcs. The radius R disc has one edge and no side, and a single point has one vertex and no side.

Our probability model is the following. Let K be convex disc with at least C^2_+ smooth boundary and let R be such that $\kappa(x) > 1/R$ for all $x \in \partial K$. Let $x_1, \ldots, x_n$ be independent random points in K chosen according to the uniform probability distribution. The R-spindle convex hull $K^R_n = [x_1, \ldots, x_n]_R$ is called a *uniform random R-disc-polygon* in K., which is a convex R-disc-polygon. It is clear that K^R_n has an equal number of vertices and sides with probability one, and its vertex set is formed by some of the random points $x_1, \ldots, x_n$. Let $f_0(K^R_n)$ denote the number of vertices of K^R_n. We note that in [PL08] the radius r_n of the discs used in the estimation of an α-convex set tends to zero as $n \to \infty$. In our model, we use suitable fixed radius discs in order to guarantee that the R-spindle convex hull of the random sample is contained in K. However, after the statements of our main results, we briefly discuss what happens to the quality of the approximation when the radius R tends to the limits of its possible range.

Fodor, Kevei and Vígh proved [FKV14, Thm 1.1 on p. 901] that under the above conditions the following hold.

$$\mathbb{E}(f_0(K^R_n)) = z_1(K)n^{\frac{1}{3}} + o\left(n^{\frac{1}{3}}\right), \tag{3.2}$$

$$\mathbb{E}(A(K \setminus K^R_n)) = A(K)z_1(K)n^{-\frac{2}{3}} + o\left(n^{-\frac{2}{3}}\right), \tag{3.3}$$

as $n \to \infty$, where

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x.$$

In the above formula $\Gamma(\cdot)$ is Euler's gamma function, $A(K)$ denotes the area of K, and integration on ∂K is with respect to arc-length.

We note that (3.2) and (4.2) are connected by an Efron-type [BE65] identity (see [FKV14, (5.10.) on p. 910]), which states that

$$\mathbb{E}(f_0(K_n^R)) = n\frac{\mathbb{E}(A(K \setminus K_{n-1}^R))}{A(K)}.$$

In this chapter we prove following theorems that provide a power series expansion of $\mathbb{E}(f_0(K_n^R))$ and $\mathbb{E}(A(K \setminus K_n^R))$ in the case when ∂K satisfies stronger differentiability conditions.

Theorem 5 ([FMP24, Theorem 2])**.** *Let $k \geq 2$ be an integer, and let K be a convex disc with C_+^{k+1} smooth boundary. Then for all $R > \max_{x \in \partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(f_0(K_n^R)) = z_1(K)n^{\frac{1}{3}} + \ldots + z_{k-1}(K)n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}})$$

as $n \to \infty$. All coefficients $z_1, \ldots, z_k$ can be determined explicitly. In particular,

$$\begin{aligned} z_1(K) &= \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{\frac{1}{3}} \mathrm{d}x, \\ z_2(K) &= 0, \\ z_3(K) &= -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3A(K)}{2}} \int_{\partial K} \frac{\kappa''(x)}{3(\kappa(x) - \frac{1}{R})^{\frac{4}{3}}} \\ &\qquad + \frac{2R^2\kappa^2(x) + 7R\kappa(x) - 1}{2R^2(\kappa(x) - \frac{1}{R})^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x) - \frac{1}{R})^{\frac{7}{3}}} \mathrm{d}x. \end{aligned}$$

By the spindle convex version of Efron's identity we obtain the following corollary.

Theorem 6 ([FMP24, Theorem 3])**.** *Let $k \geq 2$ be an integer, and let K be a convex disc with C_+^{k+1} smooth boundary. Then for all $R > \max_{x \in \partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(A(K \setminus K_n^R)) = z_1'(K)n^{-\frac{2}{3}} + \ldots + z_{k-1}'(K)n^{-\frac{k}{3}} + O(n^{-\frac{k+1}{3}})$$

as $n \to \infty$, where $z_i'(K) = A(K)z_i(K)$ for $i = 1, \ldots, k$.

We note that we only evaluate $z_i(K)$, $i = 1, 2, 3$ explicitly in this paper because the calculation, although possible, becomes more complicated as i increases, even when K is a closed disc. The coefficients $z_i(K)$ depend only on R, the area of K, and on the power series expansion of the local representation of the boundary of K, see (3.8), in particular, the derivatives of κ up to order $i-1$.

Although Theorems 5 and 6 are only valid for $R > R_M = \max_{x\in\partial K} 1/\kappa(x)$, it may also be interesting to look at the behaviour of the coefficients $z_i(K)$ at the limits of the range of R. When $R \to \infty$, the integral in $z_1(K)$ tends to the affine arc-length of ∂K, see [FKV14]. For $z_3(K)$, direct calculation yields that

$$\lim_{R\to\infty} \frac{\kappa''(x)}{3(\kappa(x)-\frac{1}{R})^{\frac{4}{3}}} + \frac{2R^2\kappa^2(x)+7R\kappa(x)-1}{2R^2(\kappa(x)-\frac{1}{R})^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x)-\frac{1}{R})^{\frac{7}{3}}} = k(x)\kappa^{\frac{1}{3}}(x),$$

where $k(x)$ is the affine curvature of ∂K at x, cf. also (3.1).

On the other hand, when $R \to R_M^+$, then

$$\lim_{R\to R_M^+} z_1(K) = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{R_M}\right)^{\frac{1}{3}} \mathrm{d}x, \tag{3.4}$$

where the integrand is bounded, nonnegative, and zero in exactly those points where $\kappa(x) = 1/R_M$. We conjecture that the right-hand-side of (3.4) is equal to $\lim_{n\to\infty} \mathbb{E}f_0(K_n^R))n^{-1/3}$ when $R = R_M$ and K is not a closed disc. However, this asymptotic expectation is not known.We also note, that $z_1(K)$ is a monotonically decreasing function of R, which shows that it is indeed more advantageous to use circular arcs to form the hull of the random sample of n point in order to approximate K better. Although the order of magnitude in n of the approximation is the same as in the linearly convex case, the main coefficient is smaller.

Furthermore, we note that in the particular case when $K = B^2$ and $R > 1$, then

$$z_1(B) = \sqrt[3]{\frac{2}{3\pi}}\Gamma\left(\frac{5}{3}\right)2\pi\left(1-\frac{1}{R}\right)^{\frac{1}{3}}, \quad z_2(B) = 0,$$
$$z_3(B) = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3\pi}{2}}2\pi\frac{2R^2+7R-1}{2R^2(1-\frac{1}{R})^{\frac{1}{3}}}.$$

If $R \to 1^+$, then $z_1(B) \to 0$, and $z_3(B) \to -\infty$, and both are monotonically increasing functions showing that the quality of approximation improves as R tends to 1. This behaviour comes as no surprise as the expected number of vertices behaves fundamentally differently from the previously discussed situation when $K \neq B$; the order of magnitude in n is different if $K = B$ as we will see below. Finally, we note that we also suspect that $z_3(K)$ behaves similarly as $z_3(B)$ when $R \to R_M^+$ but this is not clear from its current form.

It was proved in [FKV14] that

$$\mathbb{E}(f_0(B(R)_n^R)) = \frac{\pi^2}{2} + o(1),$$
$$\mathbb{E}(A(B(R) \setminus B(R)_n^R)) = \frac{R^2\pi^3}{2}\frac{1}{n} + o\left(\frac{1}{n}\right)$$

as $n \to \infty$. The unusual behaviour of $\mathbb{E}(f_0(B(R)_n^R))$, i.e. that it tends to a finite constant, was explained by Marynych and Molchanov [MM22]. They proved, in the much wider context of L-convexity (see also Fodor, Papvári, Vígh [FPV20]) that $\mathbb{E}(f_0(B(R)_n^R))$ tends to the expectation of the number of vertices of the polar of the zero cell of a Poisson line process whose intensity measure on $\mathbb{R}$ is the $A(B(R))^{-1} = 1/(R^2\pi)$ times the Lebesgue measure, and whose directional distribution is uniform on S^1, see [MM22, (6.1)]. In Section 4, we calculate the (the first three terms of) the power series expansion of $\mathbb{E}(f_0(B(R)_n^R))$ for the sake of completeness. This gives the speed of convergence of $\mathbb{E}(f_0(B(R)_n^R))$ to $\pi^2/2$. We note that here we only quoted the result of Marynych and Molchanov in the plane, however, they proved in $\mathbb{R}^d$.

The rest of the chapter is organized as follows. In Section 3.2, we briefly recall from [FKV14] the necessary background and describe how $\mathbb{E}(f_0(B(R)_n^R))$ can be calculated. In Section 3.3, we provide the power series expansions of the involved geometric quantities. In Section 3.4, we quote a power series expansion of the incomplete beta function from Gruber [Gru96]. We prove Theorem 5 in Section 3.5. Finally, in Section 3.6, we treat the case when $K = B(R)$.

3.2 Expectation of the number of vertices

Our arguments are based on the methods of Rényi and Sulanke [RS63] and Gruber [Gru96].We also note that, compared to those of [PL08], our methods essentially depend on the higher regularity and smoothness of the boundary of K and the explicit local power series expansion of ∂K. Notice note that it is enough to prove the theorem for

$R=1$, from that the statement for general R follows by a scaling argument. Therefore, we omit R from the notation in the proof.

Due to the C^{k+1}_+ condition, K is both smooth, i.e. has a unique supporting line at each boundary point, and strictly convex. Let $u_x \in S^1$ denote the unique outer unit normal vector to K at x, and for $u \in S^1$ let x_u be the (again) unique boundary point where the outer unit normal is equal to u.

We use B° to denote the interior of B. A subset D of K is a *disc-cap of K* if $D = K \setminus (B^\circ + p)$ for some point $p \in \mathbb{R}^2$. It was proved in [FKV14] that for a disc-cap $D = K \cap (B^\circ + p)$ there exists a unique point $x_0 \in \partial K \cap D$ and $t \geq 0$ such that $B + p = B + x_0 - (1+t)u_{x_0}$. We call x_0 the vertex and t the height of D.

We may assume that $o \in \operatorname{int} K$. Let $A = A(K) = V_2(K)$. Let $X_n = \{x_1, \dots, x_n\}$ be a sample of i.i.d. uniform random points from K. For $x_i, x_j \in X_n$, we denote by $x_i x_j$ the shorter unit circular arc connecting x_i and x_j with the property that x_i and x_j are in counterclockwise order on the arc. Let

$$\mathcal{E}(K_n) = \{x_i x_j : x_i, x_j \in X_n \text{ and } x_i x_j \text{ is an edge of } K_n\}$$

the set of directed edges of K_n. For $x_i, x_j \in X_n$, let C_{ij} be the disc-cap determined by the disc of $x_i x_j$, and $A_{ij} = A(C_{ij})$. Note that $x_i x_j \in \mathcal{E}(K_n)$ exactly when all the other $n-2$ random points of X_n are in $K \setminus C_{ij}$. Thus, due to the independence of the random points,

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \sum \frac{1}{A^n} \int_K \dots \int_K \mathbf{1}\{x_i x_j \in \mathcal{E}(K_n)\}\, \mathrm{d}x_1 \dots \mathrm{d}x_n \\ &= \binom{n}{2} \frac{1}{A^2} \int_K \int_K \left(1 - \frac{A_{12}}{A}\right)^{n-2} + \left(1 - \frac{A_{21}}{A}\right)^{n-2} \mathrm{d}x_1 \mathrm{d}x_2, \end{aligned} \tag{3.5}$$

where in the first line summation extends over all ordered pairs of distinct points from X_n. Now, we use the same re-parametrization for the pair (x_1, x_2) as in [FKV14]. Let

$$(x_1, x_2) = \Phi(u, t, u_1, u_2),$$

where $u, u_1, u_2 \in S^1$ and $0 \leq t \leq t_0(u)$ are chosen such that

$$C(u,t) = C_{12},$$

and

$$(x_1, x_2) = (x_u - (1+t)u + u_1, x_u - (1+t)u + u_2). \tag{3.6}$$

The vectors u_1 and u_2 are the unique outer unit normals of $\partial B + x_u - (1+t)u$ at x_1 and x_2, respectively. For fixed u and t, both u_1 and u_2 are contained in the same arc $L(u,t)$ of S^1 whose length is denoted by $\ell(u,t)$. The uniqueness of the vertex and height of disc-caps guarantees that the map Φ is well-defined, bijective, and differentiable on a suitable domain of (u,t,u_1,u_2). The Jacobian of Φ is

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right)|u_1 \times u_2|.$$

Let $A(u,t)$ denote the area of the disc-cap with vertex x_u and height t. For each $u \in S^1$, let $t_0(u)$ be maximal such that $A(u,t_0(u)) \geq 0$. Then, after the change of variables we get from (3.5) that

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_0(u)}\int_{L(u,t)}\int_{L(u,t)}\left(1 - \frac{A(u,t)}{A}\right)^{n-2} \\ &\qquad \times \left(1 + t - \frac{1}{\kappa(x_u)}\right)|u_1 \times u_2|\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t\mathrm{d}u \\ &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_0(u)}\left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\mathrm{d}t\mathrm{d}u,\end{aligned}$$

where

$$\begin{aligned}J(u,t) &= \left(1 + t - \frac{1}{\kappa(x_u)}\right)\int_{L(u,t)}\int_{L(u,t)}|u_1 \times u_2|\mathrm{d}u_1\mathrm{d}u_2 \\ &= 2\left(1 + t - \frac{1}{\kappa(x_u)}\right)(\ell(u,t) - \sin\ell(u,t)).\end{aligned}$$

We note that due to the C^2_+ property of ∂K, $J(u,t) \leq C$ for some $0 < C \leq 6(2\pi+1)$ that depends only on K.

Let $0 < \delta < A$ be an arbitrary but fixed small number. Let $0 < t_1$ be such that for arbitrary $t \in [t_1, t_0(u)]$ and $u \in S^1$ it holds that $A(u,t) \geq \delta$.

Then

$$\begin{aligned}&\int_{S^1}\int_{t_1}^{t_0(u)}\left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\mathrm{d}t\mathrm{d}u \\ &\ll \int_{S^1}\int_{t_1}^{t_0(u)}\left(1 - \frac{A(u,t)}{A}\right)^{n-2}\mathrm{d}t\mathrm{d}u \\ &\ll \int_{t_1}^{2}\left(1 - \frac{\delta}{A}\right)^{n-2}\mathrm{d}t \\ &\ll \left(1 - \frac{\delta}{A}\right)^{n-2}\end{aligned}$$

$$\ll e^{-\frac{\delta(n-2)}{A}},$$

thus with the choice of a suitably small δ,

$$\mathbb{E}(f_0(K_n^1)) = \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u + O(e^{-n}). \tag{3.7}$$

In the following sections we evaluate the integral (3.7) under different smoothness assumptions on ∂K.

3.3 Power series expansions

Let $k \geq 2$ be an integer and $K \subset \mathbb{R}^2$ a convex disc with a C_+^{k+1} boundary ($k+1$ times continuously differentiable with everywhere positive curvature). We will use the following statement which originates from Schneider [Sch81], later generalized by Gruber [Gru96]. We state it in the form used by Reitzner [Rei04], but only for $d = 2$.

Lemma 7. *Let K be a convex disc with C_+^{k+1} boundary for some integer $k \geq 2$. Then there exist constants $\alpha, \beta > 0$ depending only on K such that the following holds for every boundary point x of K. If $x = 0$ and the (unique) tangent line of K at x is $\mathbb{R}$, then there is an α neighbourhood of x in which the boundary of K can be represented by a convex function $f(\sigma)$ of differentiability class C^{k+1} in $\mathbb{R}$. Moreover, all derivatives of f up to order $k+1$ are uniformly bounded by β.*

Let $u \in S^1$ and let $x = x_u \in \partial K$. Assume that K is in the position described in Lemma 7. Let f be the function that represents the boundary of K in an α neighbourhood of x. Then f is of the form

$$f(\sigma) = b_2(u)\sigma^2 + \ldots + b_k(u)\sigma^k + O(\sigma^{k+1}),$$

where the coefficients $b_i = b_i(u)$, $i = 2, \ldots, k$ depend on u. In the foregoing we will suppress the dependence of coefficients on u (and thus on x) when we work with a fixed u. We will only indicate dependence when u is used in the argument.

We recall the following facts from the differential geometry of plane curves. Let $r(s)$ be the arc-length parametrization of ∂K with $r(0) = x$ in the neighbourhood of x such that the following hold. With the above assumptions on K, let the vector $r'(0)$, and the unit normal vector $r''(0)/\kappa(0) = -u$ form the basis of a Cartesian coordinate system, in which we denote the coordinate along the r'-axis by σ, and the r''-axis by η. Then

$$\sigma = \sigma(s) = s - \frac{\kappa^2(0)}{3!}s^3 - 3\kappa(0)\kappa'(0)\frac{s^4}{4!} + O(s^5),$$

$$\eta = \eta(s) = \kappa(0)\frac{s^2}{2} + \kappa'(0)\frac{s^3}{3!} + (\kappa''(0) - \kappa^3(0))\frac{s^4}{4!} + O(s^5). \tag{3.8}$$

From the equality $f(\sigma(s)) = \eta(s)$ we can identify the coefficients $b_2, \ldots, b_k$. In particular,

$$b_2 = \frac{\kappa(0)}{2}, \quad b_3 = \frac{\kappa'(0)}{6}, \quad b_4 = \frac{\kappa''(0) + 3\kappa^3(0)}{24}.$$

With a slight abuse of notation, in the above formulas we use κ to denote the curvature as a function of s, which is different from previous usage. Later, we will also use the same letter when the curvature is a function of the outer unit normal u. Moreover, when u (s or x) is fixed, we suppress the dependence of κ on u (s or x, respectively). It will always be clear from the context which function we consider.

We will also use the following statement due to Gruber [Gru96], see also Reitzner [Rei04] (we state it again only for $d = 2$, so this is a simpler version of the original theorem):

Lemma 8. *Let*

$$\eta = \eta(\sigma) = b_m\sigma^m + \ldots + b_k\sigma^k + O(\sigma^{k+1})$$

for $0 \leq \sigma \leq \alpha$, $2 \leq m \leq k$ be a strictly increasing function. Then there are coefficients $c_1, \ldots, c_{k-m+1}$ and a constant $\gamma > 0$ such that the inverse function $\sigma = \sigma(\eta)$ has the following representation

$$\sigma = \sigma(\eta) = c_1\eta^{\frac{1}{m}} + \ldots + c_{k-m+1}\eta^{\frac{k-m+1}{m}} + O(\eta^{\frac{k-m+2}{m}})$$

for $0 \leq \eta \leq \gamma$. The coefficients $c_1, \ldots, c_{k-m+1}$ can be determined explicitly in terms of $b_m, \ldots, b_k$. In particular,

i) $c_1 = \frac{1}{b_m^{\frac{1}{m}}}$,

ii) $c_2 = -\frac{b_{m+1}}{m b_m^{\frac{m+2}{m}}}$,

iii) $c_3 = -\frac{b_{m+2}}{m b_m^{\frac{m+3}{m}}} + \frac{(m+3)b_{m+1}^2}{2m^2 b_m^{\frac{2m+3}{m}}}$.

For $t \geq 0$, let the unit radius lower semicircle with centre $(0, 1+t)$ be represented by the function

$$g_t(\sigma) = t + 1 - \sqrt{1-\sigma^2} = t + 1 - \sum_{i=0}^{\infty} (-1)^i \binom{\frac{1}{2}}{i} \sigma^{2i}$$
$$= t + g_2\sigma^2 + \ldots + g_{2i}\sigma^{2i} + \ldots,$$

for $\sigma \in [-1, 1]$, where

$$g_2 = \frac{1}{2}, \quad g_3 = 0, \quad g_4 = \frac{1}{8}.$$

Let $\sigma_+ = \sigma_+(t) > 0$ and $\sigma_- = \sigma_-(t) < 0$ such that

$$f(\sigma_+) = g_t(\sigma_+), \quad \text{and} \quad f(\sigma_-) = g_t(\sigma_-).$$

For sufficiently small $\sigma > 0$, it holds that

$$t = t(\sigma) = f(\sigma) - 1 + \sqrt{1-\sigma^2} = u_2\sigma^2 + \ldots + u_k\sigma^k + O(\sigma^{k+1}),$$

where, in particular,

$$u_2 = b_2 - g_2, \quad u_3 = b_3, \quad u_4 = b_4 - g_4.$$

We note that, subsequently, we express coefficients in terms of the u_i's (as long as it does not become too complicated) as they carry all information about ∂K and the circle. We will only substitute their values when we determine our final answer.

Since $u_2 > 0$ by the conditions on ∂K, Lemma 8 yields

$$\sigma_+ = \sigma_+(t) = c_1 t^{\frac{1}{2}} + \ldots + c_{k-1} t^{\frac{k-1}{2}} + O(t^{\frac{k}{2}}), \tag{3.9}$$

where

$$c_1 = u_2^{-\frac{1}{2}}, \quad c_2 = -\frac{u_3}{2u_2^2}, \quad c_3 = \frac{5u_3^2 - 4u_2u_4}{8u_2^{\frac{7}{2}}}.$$

Similarly, we obtain that

$$\sigma_- = \sigma_-(t) = \tilde{c}_1 t^{\frac{1}{2}} + \ldots + \tilde{c}_{k-1} t^{\frac{k-1}{2}} + O(t^{\frac{k}{2}}), \tag{3.10}$$

where the coefficients $\tilde{c}_1, \ldots, \tilde{c}_{k-1}$ can be determined explicitly. In particular,

$$\tilde{c}_1 = -c_1, \quad \tilde{c}_2 = c_2, \quad \tilde{c}_3 = -c_3.$$

Thus, using (3.9) and (3.10), the area of the disc cap $C(u,t)$ is

$$\begin{aligned} A(u,t) &= \int_{\sigma_-}^{\sigma_+} g_t(\sigma) - f(\sigma)\,\mathrm{d}\sigma = \int_{\sigma_-}^{\sigma_+} t - u_2\sigma^2 - \ldots - u_k\sigma^k + O(\sigma^{k+1})\,\mathrm{d}\sigma \\ &= \left[t\sigma - \frac{u_2}{3}\sigma^3 - \ldots - \frac{u_k}{k+1}\sigma^{k+1} + O(\sigma^{k+2}) \right]_{\sigma_-}^{\sigma_+} \\ &= a_1 t^{\frac{3}{2}} + a_2 t^2 + \ldots + a_{k-1} t^{\frac{k+1}{2}} + O(t^{\frac{k+2}{2}}). \end{aligned} \tag{3.11}$$

where the coefficients $a_1, \ldots, a_{k-1}$ can be expressed explicitly. In particular,

$$a_1 = \frac{4}{3}u_2^{-\frac{1}{2}}, \quad a_2 = 0, \quad a_3 = \frac{5u_3^2 - 4u_2u_4}{10u_2^{\frac{7}{2}}}.$$

Now we turn to expressing the Jacobian $J(u,t)$ in the form of a series expansion in t. Using (3.9) and (3.10), we get

$$\begin{aligned} \ell(u,t) &= \int_{\sigma_-}^{\sigma^+} \sqrt{1 + (g_t'(\sigma))^2}\,\mathrm{d}\sigma = \int_{\sigma_-}^{\sigma^+} \sqrt{\frac{1}{1-\sigma^2}}\,\mathrm{d}\sigma = [\arcsin\sigma]_{\sigma_-}^{\sigma_+} \\ &= h_1 t^{\frac{1}{2}} + h_2 t + \ldots + h_{k-1} t^{\frac{k-1}{2}} + O(t^{\frac{k}{2}}), \end{aligned} \tag{3.12}$$

where the coefficients $h_1, \ldots, h_{k-1}$ can be expressed explicitly. In particular,

$$h_1 = 2u_2^{-\frac{1}{2}}, \quad h_2 = 0, \quad h_3 = \frac{15u_3^2 + 4u_2(u_2 - 3u_4)}{12u_2^{\frac{7}{2}}}.$$

We note that the coefficients c_1, c_2, c_3 (also $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$), a_1, a_2, a_3 and h_1, h_2, h_3 were calculated in [FKV14, pp. 911–912] with a different notation.

Now, using (3.12), we get that

$$\ell(u,t) - \sin\ell(u,t) = \sum_{i=0}^{\infty} (-1)^i \frac{\ell^{2i+1}(u,t)}{(2i+1)!} = l_1 t^{\frac{3}{2}} + \ldots + l_{k-1} t^{\frac{k+1}{2}} + O(t^{\frac{k+2}{2}}),$$

where the coefficients $l_1, \ldots, l_{k-1}$ can be calculated explicitly. In particular,

$$l_1 = \frac{4}{3}u_2^{-\frac{3}{2}}, \quad l_2 = 0, \quad l_3 = \frac{25u_3^2 + 4u_2(u_2 - 5u_4)}{10u_2^{\frac{9}{2}}}. \tag{3.13}$$

Then

$$J(u,t) = 2\left(1 + t - \frac{1}{\kappa(x_u)}\right)(\ell(u,t) - \sin\ell(u,t))$$

$$= j_1 t^{\frac{3}{2}} + \ldots + j_{k-1} t^{\frac{k+1}{2}} + O(t^{\frac{k+2}{2}}), \tag{3.14}$$

where, where the coefficients $j_1, \ldots, j_{k-1}$ can be calculated explicitly. In particular,

$$j_1 = \frac{8u_2^{-\frac{3}{2}}(\kappa - 1)}{3\kappa}, \quad j_2 = 0, \quad j_3 = \frac{8u_2^{-\frac{3}{2}}}{3} + \frac{25u_3^2 + 4u_2(u_2 - 5u_4)}{5u_2^{\frac{9}{2}}} \frac{(\kappa - 1)}{\kappa}.$$

For a fixed n, let $y = y(u, t)$ be defined by

$$\frac{y}{n-2} = \frac{A(u,t)}{A}.$$

Then, by (3.11) and using Lemma 8 for $\sqrt{t}$ and then squaring, we obtain that

$$t = p_1 \left(\frac{y}{n-2}\right)^{\frac{2}{3}} + \ldots + p_{k-1} \left(\frac{y}{n-2}\right)^{\frac{k}{3}} + O\left(\left(\frac{y}{n-2}\right)^{\frac{k+1}{3}}\right), \tag{3.15}$$

where the coefficients $p_1, \ldots, p_{k-1}$ can be calculated explicitly. In particular,

$$p_1 = \left(\frac{3A}{4}\right)^{\frac{2}{3}} u_2^{\frac{1}{3}}, \quad p_2 = 0, \quad p_3 = \frac{9A(-5u_3^2 + 4u_2u_4)}{320u_2^2}.$$

Then, substituting (3.15) into (3.14), we obtain

$$J\left(u, \frac{y}{n-2}\right) = q_1 \left(\frac{y}{n-2}\right) + \ldots + q_{k-1} \left(\frac{y}{n-2}\right)^{\frac{k+1}{3}} + O\left(\left(\frac{y}{n-2}\right)^{\frac{k+2}{3}}\right), \tag{3.16}$$

where the coefficients $q_1, \ldots, q_{k-1}$ can be calculated explicitly. In particular,

$$q_1 = j_1 p_1^{\frac{3}{2}}, \quad q_2 = 0, \quad q_3 = j_3 p_1^{\frac{5}{2}} + \frac{3 j_1 p_3 p_1^{\frac{1}{2}}}{2}.$$

In the coefficients q_1, q_3 we used j_1, j_3 and p_1, p_3 instead of the u_i's in order to simplify notation.

3.4 The incomplete beta function

In evaluating the integral (3.7), we use the following expansion of the incomplete beta-function from Gruber [Gru96].

Lemma 9 (Gruber [Gru96])**.** *Let $\beta \in \mathbb{R}$. There are coefficients $\gamma_1, \gamma_2, \ldots \in \mathbb{R}$ depending on β which can be determined explicitly such that for a fixed $l = 1, 2, \ldots$ and $0 < \alpha \leq 1$*

$$\int_0^{\alpha n} \left(1 - \frac{t}{n}\right)^n t^\beta \, dt = \Gamma(\beta + 1) + \frac{\gamma_1}{n} + \ldots + \frac{\gamma_l}{n^l} + O\left(\frac{1}{n^{l+1}}\right), \text{ as } n \to \infty.$$

In particular,

$$\gamma_1 = -\frac{\Gamma(\beta+3)}{2}, \quad \gamma_2 = -\frac{\Gamma(\beta+4)}{3} + -\frac{\Gamma(\beta+5)}{8}.$$

If α is chosen from a closed subinterval of $(0,1]$, then the constant in $O(\cdot)$ can be chosen independent of α.

In our calculations, we need the following corollary of Lemma 9.

Lemma 10 ([FMP24, Lemma 4])**.** *Under the same assumptions as in Lemma 9, it holds that*

$$\int_0^{\alpha(n-2)} \left(1 - \frac{t}{n-2}\right)^{n-2} t^\beta \, dt = \Gamma(\beta+1) + \frac{\gamma_1'}{n} + \ldots + \frac{\gamma_l'}{n^l} + O\left(\frac{1}{n^{l+1}}\right), \text{ as } n \to \infty.$$

In particular,

$$\gamma_1' = -\frac{\Gamma(\beta+3)}{2}, \quad \gamma_2' = -\frac{\Gamma(\beta+4)}{3} - 2\Gamma(\beta+3).$$

If α is chosen from a closed subinterval of $(0,1]$, then the constant in $O(\cdot)$ can be chosen independent of α.

Proof. Using (9) and

$$\frac{n}{n-2} = \frac{1}{1-\frac{2}{n}} = 1 + \frac{2}{n} + \frac{4}{n^2} + \ldots$$

we obtain

$$\begin{aligned}
&\int_0^{\alpha(n-2)} \left(1 - \frac{t}{n-2}\right)^{n-2} t^\beta \, dt \\
&= \Gamma(\beta+1) + \frac{\gamma_1}{n}\frac{n}{n-2} + \ldots + \frac{\gamma_l}{n^l}\frac{n^l}{(n-2)^l} + O\left(\frac{1}{n^{l+1}}\frac{n^{l+1}}{(n-2)^{l+1}}\right) \\
&= \Gamma(\beta+1) + \frac{\gamma_1'}{n} + \ldots + \frac{\gamma_l'}{n^l} + O\left(\frac{1}{n^{l+1}}\right),
\end{aligned}$$

from which we can get the coefficients $\gamma_1', \ldots, \gamma_l'$ by simple calculation. □

3.5 Proof of Theorem 5

Substituting (3.16) in the integral (3.7) and using (3.15), we obtain that

$$\begin{aligned}
\mathbb{E}(f_0(K_n^1)) &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u + O(e^{-n}) \\
&= \binom{n}{2}\frac{1}{A^2}\frac{1}{n-2}\int_{S^1}\int_0^{\tau(n-2)}\left(1 - \frac{y}{n-2}\right)^{n-2}
\end{aligned}$$

$$\times J\left(u, \frac{y}{n-2}\right) t'\left(\frac{y}{n-2}\right) \mathrm{d}y\mathrm{d}u + O(e^{-n}).$$

We evaluate the inner integral as follows. Collecting the terms according to the exponent of $y/(n-2)$ and also the error term yield

$$\begin{aligned}
&\binom{n}{2}\frac{1}{A^2}\frac{1}{n-2}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} J\left(u,\frac{y}{n-2}\right) t'\left(\frac{y}{n-2}\right) \mathrm{d}y \\
&= v_1\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{\frac{5}{3}}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{\frac{2}{3}}\,\mathrm{d}y + \ldots + \qquad (3.17)\\
&\quad + v_{k-1}\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{\frac{k+3}{3}}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{\frac{k}{3}}\,\mathrm{d}y \\
&\quad + O\left(\frac{1}{(n-2)^{\frac{k-2}{3}}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{\frac{k+1}{3}}\,\mathrm{d}y\right).
\end{aligned}$$

as $n\to\infty$. The coefficients $v_1,\ldots,v_{k-1}$ can be determined explicitly. In particular,

$$v_1 = \frac{2}{3}p_1q_1, \quad v_2 = 0, \quad v_3 = \frac{4}{3}q_1p_3 + \frac{2}{3}p_1q_3.$$

Here we use p_1, p_3 and q_1, q_3 to express v_1, v_3 for the sake of brevity. Of course, they can also be expressed explicitly in terms of the u_i's.

We evaluate the above integrals one-by-one using Lemma 10. In particular, the first integral is as follows:

$$\begin{aligned}
&v_1\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{\frac{5}{3}}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{\frac{2}{3}}\,\mathrm{d}y \\
&= \sqrt[3]{\frac{2}{3A}}\frac{(\kappa-1)^{\frac{1}{3}}}{\kappa}\frac{n(n-1)}{(n-2)^{\frac{5}{3}}}\left(\Gamma\left(\frac{5}{3}\right) - \frac{\Gamma\left(\frac{10}{3}\right)}{2}\frac{1}{n} + \ldots\right) \\
&= \sqrt[3]{\frac{2}{3A}}\frac{(\kappa-1)^{\frac{1}{3}}}{\kappa}\left(\Gamma\left(\frac{5}{3}\right) n^{\frac{1}{3}} + \left(\frac{7}{3}\Gamma\left(\frac{5}{3}\right) - \frac{\Gamma\left(\frac{10}{3}\right)}{2}\right)\frac{1}{n^{\frac{2}{3}}} + \ldots\right),
\end{aligned}$$

where in the last line we used the binomial series expansion

$$\frac{n(n-1)}{(n-2)^{5/3}} = n^{\frac{1}{3}} + \frac{7}{3}n^{-\frac{2}{3}} + \ldots.$$

The second (nonzero) integral is the following:

$$v_3\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{\frac{7}{3}}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{\frac{4}{3}}\,\mathrm{d}y$$

$$
\begin{aligned}
&= \frac{v_3}{2A^2}\frac{n(n-1)}{(n-2)^{\frac{7}{3}}}\left(\Gamma\left(\frac{7}{3}\right) - \frac{\Gamma\left(\frac{13}{3}\right)}{2}\frac{1}{n} + \dots\right) \\
&= \frac{v_3}{2A^2}\left(\Gamma\left(\frac{7}{3}\right)n^{-\frac{1}{3}} + \left(\frac{11\Gamma\left(\frac{7}{3}\right)}{3} - \frac{\Gamma\left(\frac{13}{3}\right)}{2}\right)n^{-\frac{4}{3}} + \dots\right),
\end{aligned}
$$

where we used the binomial series expansion

$$
\frac{n(n-1)}{(n-2)^{\frac{7}{3}}} = n^{-\frac{1}{3}} + \frac{11}{3}n^{-\frac{4}{3}} + \dots .
$$

Evaluating the $k-1$ integrals in (3.17) and collecting the terms, including the error term, we obtain that

$$
\binom{n}{2}\frac{1}{A^2}\int_0^{t_1}\left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\mathrm{d}t
= w_1 n^{\frac{1}{3}} + w_2 n^0 + \dots + w_{k-1} n^{-\frac{k-1}{3}} + O(n^{-\frac{k}{3}}),
$$

where, in principle, all coefficients $w_1, \dots, w_{k-1}$ can be calculated explicitly. In particular,

$$
\begin{aligned}
w_1(u) &= \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right)\frac{(\kappa(u)-1)^{\frac{1}{3}}}{\kappa(u)}, \\
w_2(u) &= 0, \\
w_3(u) &= -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3A}{2}}\left(\frac{\kappa''(u)}{3(\kappa(u)-1)^{\frac{4}{3}}\kappa(u)} + \frac{2\kappa^2(u) + 7\kappa(u) - 1}{2(\kappa(u)-1)^{\frac{1}{3}}\kappa(u)} - \frac{5(\kappa'(u))^2}{9(\kappa(u)-1)^{\frac{7}{3}}\kappa(u)}\right),
\end{aligned}
$$

where we recall that κ is a function of u.

Finally, integration with respect to u yields that

$$
\begin{aligned}
\mathbb{E}(f_0(K_n^1)) &= \int_{S^1} w_1(u)n^{\frac{1}{3}} + w_2(u)n^0 + \dots + w_{k-1}(u)n^{-\frac{k-1}{3}} + O(n^{-\frac{k}{3}})\,\mathrm{d}u \\
&= z_1(K)n^{\frac{1}{3}} + z_2(K)n^0 + \dots + z_{k-1}(K)n^{\frac{k-1}{3}} + O(n^{-\frac{k}{3}}),
\end{aligned}
$$

where, again, all coefficient can be found explicitly. In particular,

$$
\begin{aligned}
z_1(K) &= \int_{S^1} w_1(u)\,\mathrm{d}u = \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}(\kappa(x)-1)^{\frac{1}{3}}\,\mathrm{d}x, \\
z_2(K) &= 0, \\
z_3(K) &= \int_{S^1} w_3(u)\,du = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3A}{2}}\int_{\partial K}\frac{\kappa''(x)}{3(\kappa(x)-1)^{\frac{4}{3}}} \\
&\qquad + \frac{2\kappa(x)^2 + 7\kappa(x) - 1}{2(\kappa(x)-1)^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x)-1)^{\frac{7}{3}}}\,dx
\end{aligned}
$$

where we use that if ∂K is C^2_+ smooth and $f(u)$ is a measurable function on S^1, then $\int_{S^1} f(u)\mathrm{d}u = \int_{\partial K} f(u_x)\kappa(x)\mathrm{d}x$, (cf. formula (2.62) in [Sch14]). This finishes the proof of Theorem 5.

3.6 The case of the unit circle

For the sake of completeness, we consider the case when $K = B(R)$. Since $\mathbb{E}(f_0(B(R)^R_n))$ is independent of R, we may assume that $R = 1$. We will use the simpler notation $B^1_n = B(1)^1_n$. In [FKV14, p. 916] it was proved that

$$\mathbb{E}(f_0(B^1_n)) = \binom{n}{2} 4 \int_0^\pi \sin(\sigma)\left(1 - \frac{\sin(\sigma)+\sigma}{\pi}\right)^{n-1} \mathrm{d}\sigma.$$

Let

$$\frac{y}{n-1} = \frac{\sin(\sigma)+\sigma}{\pi}.$$

Since $\sin(\sigma)+\sigma$ is a strictly monotonically increasing analytic function on $[0,\pi]$, its inverse is also a strictly monotonically increasing analytic function by the Lagrange inversion theorem. Then σ has a power series expansion in terms of $y/(n-1)$ around $y = 0$ as follows

$$\sigma = c_1\left(\frac{y}{n-1}\right) + c_3\left(\frac{y}{n-1}\right)^3 + \ldots + c_{2k+1}\left(\frac{y}{n-1}\right)^{2k+1} + \ldots,$$

where all coefficients can be calculated explicitly. In particular,

$$c_1 = \frac{\pi}{2}, \quad c_3 = \frac{\pi^3}{96}, \quad c_5 = \frac{\pi^5}{1920}.$$

Thus,

$$\sin(\sigma) = e_1\left(\frac{y}{n-1}\right) + e_3\left(\frac{y}{n-1}\right)^3 + \ldots + e_{2k+1}\left(\frac{y}{n-1}\right)^{2k+1} + \ldots,$$

where the coefficients can be calculated explicitly. In particular,

$$e_1 = \frac{\pi}{2}, \quad e_3 = -\frac{\pi^3}{96}, \quad e_5 = -\frac{\pi^5}{1920}.$$

Therefore

$$\mathbb{E}(f_0(B^1_n)) = \binom{n}{2}\frac{4}{n-1}\int_0^{n-1}\left(1 - \frac{y}{n-1}\right)^{n-1} \sin\left(\sigma\left(\frac{y}{n-1}\right)\right)\sigma'\left(\frac{y}{n-1}\right)\mathrm{d}y$$

$$
\begin{aligned}
&= f_1\binom{n}{2}\frac{4}{(n-1)^2}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y\,dy \\
&+ f_3\binom{n}{2}\frac{4}{(n-1)^4}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^3\,dy \\
&+ \ldots + f_{2k+1}\binom{n}{2}\frac{4}{(n-1)^{2k+2}}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^{2k+1}\,dy + \ldots,
\end{aligned}
$$

where all coefficients $f_1, \ldots, f_{2k+1}, \ldots$ can be evaluated explicitly using Lemma 9 and the binomial series expansion of $n/(n-1)^{2k+1}$. In particular,

$$
f_1 = \frac{\pi^2}{4}, \quad f_3 = \frac{\pi^4}{96}, \quad f_5 = \frac{11\pi^6}{15360}.
$$

Thus, by Lemma 10 the first integral yields

$$
\begin{aligned}
&\binom{n}{2}\frac{\pi^2}{(n-1)^2}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y\,dy \\
&= \frac{\pi^2}{2}\frac{n}{n-1}\left(\Gamma(2) - \frac{\Gamma(4)}{2}\frac{1}{n-1} + \left(\frac{-\Gamma(5)}{3} + \frac{\Gamma(6)}{8}\right)\frac{1}{(n-1)^2} + \ldots\right) \\
&= \frac{\pi^2}{2}\left(1 - \frac{2}{n} + \frac{2}{n^2} + \ldots\right)
\end{aligned}
$$

The second integral yields

$$
\begin{aligned}
&f_3\binom{n}{2}\frac{4}{(n-1)^4}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^3\,dy \\
&= \frac{\pi^4}{48}\frac{n}{(n-1)^3}\left(\Gamma(4) - \frac{\Gamma(6)}{3}\frac{1}{n-1} + \left(-\frac{\Gamma(7)}{3} + \frac{\Gamma(8)}{8}\right)\frac{1}{(n-1)^2} + \ldots\right)
\end{aligned}
$$

Thus, for any $k \geq 0$,

$$
\mathbb{E}(f_0(B_n^1)) = w_0 n^0 + w_1 n^{-1} + w_2 n^{-2} + \ldots + w_k n^{-k} + \ldots, \tag{3.18}
$$

where all coefficient $w_0, \ldots, w_k$ can be calculated explicitly. In particular,

$$
w_0 = \frac{\pi^2}{2}, \quad w_1 = -\pi^2, \quad w_2 = \frac{\pi^4 + 8\pi^2}{8}, \quad w_3 = \frac{13\pi^2}{3} - \frac{11\pi^4}{24}. \tag{3.19}
$$

Chapter 4

The less smooth case

4.1 Introduction and results

In this chapter we discuss the case when the boundary of K is assumed to have weaker differentiability conditions. The motivation for this lies, at least partly, in the work of Schütt [Sch94], where he extended the validity of the previously proved asymptotic formula for the missing volume in the model of linear random polytopes to the case with no smoothness assumption at all. However, we note that Schütt's formula, although valid without smoothness conditions, gives 0 for many convex bodies, most notably for polytopes. In our case this does not happen because of the bounds on the curvatures.

Here we try to follow this path and extend the validity of formulas (3.2) and (3.3) under the weakest smoothness conditions our method currently allows. This is probably not the most general version, however, it shows that it is not the differentiability that is most important but the existence of the (positive) minimum and the maximum of the (generalized) curvature.

The asymptotic formulas (3.2) and (3.3) were proved in [FKV14]. We show that the original argument in [FKV14] can be carried out in this more general context with minor changes.

Our main conditions on the smoothness of ∂K will be the existence of a rolling circle and that K slides freely in a circle. More precisely, the definitions are the following.

A convex disc K has a *rolling circle* if there exists a real number $r_0 > 0$ with the property that any $x \in \partial K$ lies in some closed circular disc of radius r_0 contained in K, for a definition in a more general context see [Sch14, p. 164]. According to a theorem of Blaschke [Bla56], if the boundary of K is C^2_+ smooth, then it has a rolling circle; this is often called "Blaschke's rolling theorem". The existence of the rolling circle guarantees that the boundary of K is smooth in the sense it has a unique outer unit normal at each point, that is, no vertices are permitted. This also yields that ∂K is continuously differentiable, so it is C^1. Moreover, Hug proved in [Hug00] that the existence of a rolling circle is equivalent to the exterior unit normal being a Lipschitz function on ∂K. However, having a rolling circle does not guarantee that ∂K is differentiable twice.

We say that a convex disc K *slides freely in a circular disc* $R_0 B^2$ of radius R_0 if for each $x \in \partial K$ there exists a $p \in \mathbb{R}^2$ such that $x \in \partial(R_0 B^2) + p$ and $K \subset R_0 B^2 + p$. Again, for a definition in a more general context see [Sch14, p. 156]. The property that K slides freely in a circle yields that K is strictly convex, that is, it boundary does not contain segments. Hug [Hug00] proved that if K slides freely in a circle, then the reverse spherical image map (the map that assigns to a unit vector u the boundary point x_u where the outer unit normal of K is u) is a Lipschitz map from the unit circle to ∂K.

If K has both a rolling circle and slides freely in a circle, then it is smooth and strictly convex. Then both the spherical image map and it reverse are well-defined on the whole boundary ∂K and the whole of S^1, respectively, and they are inverses to one another, and both are Lipshitz continuous by Hug's results. Thus, in this case, by Rademacher's theorem both the spherical image map and its reverse are differentiable almost everywhere.

By the theorem of Bezdek et al [BLNP07], if K is an R-spindle convex disc, then it slides freely in a circle of radius R. The converse is also true, if the convex disc K slides freely in a circle of radius R, then it is the intersection of all supporting circles so it is R-spindle-convex. So R-spindle convexity, for convex discs, is equivalent to sliding in a circle of radius R. This is an important, although weak, smoothness property.

Since we do not require ∂K to be of class C^2, we need a notion of generalized second order differentiability and generalized curvature: We say that ∂K is twice differentiable in the generalized sense if it can be approximated by a quadratic function in the following sense: Let $x \in \partial K$. If K is positioned in such a way that $x = o$ and $\mathbb{R}$ is a support line of K, then in a neighborhood of the origin o, ∂K is the graph of a convex function f

defined on an open interval containing o satisfying

$$f(z) = \frac{\kappa(x)}{2} z^2 + o(z^2)$$

as $z \to 0$. We call $\kappa(x)$ the generalized curvature of ∂K at x. Note that if f is twice differentiable at x in the usual sense, then $\kappa(x)$ coincides with the usual curvature. We do not distinguish the generalized curvature from the usual curvature in notation, as the difference is always clear from the context. According to the classical result of Alexandrov (see P.M. Gruber [Gru07] or R. Schneider [Sch14]), ∂K is twice differentiable in the generalized sense at almost all points with respect to the arc-length. We call a boundary point x of K normal, if ∂K is differentiable twice in the generalized sense at x. We denote the set of normal boundary points by $\mathcal{M}(K)$. It is known that if $x \in \partial K$ is normal, then the spherical image map is differentiable (in the usual sense) at x, see [Sch14].

Our main results are described in the following theorem.

Theorem 11. *Let K be a convex disc that has a rolling circle and slides freely in a circle of R_0. Then for any $R > R_0$, it holds that*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^R)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x, \tag{4.1}$$

and

$$\lim_{n\to\infty} \mathbb{E}(A(K \setminus K_n^R)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}} \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x. \tag{4.2}$$

Let K satisfy the conditions of Theorem 11. Then ∂K is C^1 smooth and $\kappa(x) > 1/R_0 > 1$ for all normal boundary points $x \in \partial K$.

Since the conditions of Theorem 11 imply that ∂K is C^1, the statements of the following lemma hold at every normal boundary point.

Lemma 12. *Let K be a convex disc which has a rolling circle and slides freely in a circle of radius $R_0 < 1$. Let $R = 1$ and let $x \in \partial K$ normal boundary point with $\kappa(x) > 1$. With the same notation as in 3.2 it holds that*

$$\lim_{t\to 0^+} \ell(x_u, t) t^{-1/2} = 2\sqrt{\frac{2}{\kappa(x) - 1}}, \tag{4.3}$$

$$\lim_{t\to 0^+} A(u_x, t) \cdot t^{-3/2} = \frac{4}{3}\sqrt{\frac{2}{\kappa(x) - 1}}. \tag{4.4}$$

Lemma 12 was proved in [FKV14] for the case when ∂K is C^2_+. We observe that the proof is valid in our case without any changes.

We quote the following technical statement from [FKV14] which guarantees that we can bound the area of disc-caps from below uniformly in u for sufficiently small height t. Let the radius of the rolling circle of K be r_0.

Then the assumptions on K and (4.4) yield that there exists $0 < \hat{t} < r_0$ such that for any regular direction $u \in S^1$

$$A(u,t) \geq \frac{1}{2}\left(\frac{4}{3}\sqrt{\frac{2r_0}{1-r_0}}\right) t^{\frac{3}{2}} \quad \text{for } t \in [0, \hat{t}]. \tag{4.5}$$

4.2 Proof of Theorem 11

We only prove the asymptotic formula for the expectation of the number of vertices (4.1), as the one for the missed area (4.2) follows from it via Efron's identity in a standard way.

The idea of the proof is that the lack of second differentiability causes no problem because we can still integrate as the Jacobian is the same (almost everywhere) as in the usual case by Hug's results [Hug00]. Formally, we will integrate on a subset of ∂K where all necessary things (generalized curvature, Jacobian, etc.) exist. This is a measurable set and its complement is a set of measure 0.

We assume again that the radius of spindle convexity is 1 and obtain the general form of the theorem by scaling. We use the same notation as in Section 3.2 and recall that

$$\begin{aligned} \mathbb{E}(f_0(K_n^1)) &= \sum \frac{1}{A^n}\int_K \cdots \int_K \mathbf{1}\{x_i x_j \in \mathcal{E}(K_n)\}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= \binom{n}{2}\frac{1}{A^2}\int_K\int_K \left(1-\frac{A_{12}}{A}\right)^{n-2} + \left(1-\frac{A_{21}}{A}\right)^{n-2} \mathrm{d}x_1\mathrm{d}x_2, \end{aligned} \tag{4.6}$$

see (3.5). In the first line summation extends over all ordered pairs of distinct points from $X_n = \{x_1, \ldots, x_n\}$. We use the same reparametrization for the pair (x_1, x_2) as in Section 3.2 (and also in [FKV14]), see (3.6).

Since the conditions of Theorem11 guarantee that both the spherical image map and the reverse spherical image map are Lipschitz, they are differentiable almost everywhere by Rademacher's theorem, thus the Jacobian of the map Φ is

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right)|u_1 \times u_2|$$

almost everywhere. Therefore, by the reparametrization and the same argument as in Section 3.2, we obtain that

$$\mathbb{E}(f_0(K_n^1)) = \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u + O(n^{-2}).$$

where

$$J(u,t) = 2\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t)),$$

see (3.7).

Let $h(n) = (c\ln n/n)^{2/3}$, where c is a suitable positive constant that depends only on K and which we determine later. From the existence of the rolling circle it follows that there exists $n_0 \in \mathbb{N}$ and $\gamma_1 > 0$, depending only on K, such that if $n > n_0$, then $h(n) < t_1$, and $A(u,t) > \gamma_1 \cdot h(n)^{3/2}$ for all $h(n) \le t \le t_1$ and for all $u \in S^1$.

Since $R_0 < 1$, by the result of Hug [Hug00] $\kappa(x_u) \ge 1/R_0 > 1$ in each normal direction, which yields that $0 \le (1+t-1/\kappa(x_u)) < 3$ for any regular direction $u \in S^1$. Furthermore, $\ell(u,t) - \sin\ell(u,t) \le 2\pi + 1 < 8$ for all $0 \le t \le t_1$ and $u \in S^1$. Hence, for any fixed regular direction $u \in S^1$ and any $n > n_0$, it holds that

$$J(u,t) \le 48. \tag{4.7}$$

Thus, using (4.7), we obtain that for all $n > n_0$,

$$\begin{aligned}
&\binom{n}{2}\frac{1}{A^2}\int_{h(n)}^{t_1}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\\
&\le n^2\frac{24}{A^2}\int_{h(n)}^{t_1}\left(1-\frac{\gamma_1 h(n)^{\frac{3}{2}}}{A}\right)^{n-2}\mathrm{d}t\\
&< n^2\frac{24}{A^2}\int_0^{t_1}\left(1-\frac{\gamma_1 c(\ln n/n)}{A}\right)^{n-2}\mathrm{d}t\\
&\le n^2\frac{48}{A^2}n^{-\frac{c\gamma_1}{A}},
\end{aligned}$$

so if $c > 5A/3\gamma_1$, this quantity is $o(n^{1/3})$.

Thus,

$$\lim_{n\to\infty}\mathbb{E}(f_0(K_n^1))n^{-\frac{1}{3}} = \lim_{n\to\infty} n^{\frac{5}{3}}\frac{1}{2A^2}\int_{S^1}\int_0^{h(n)}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u, \tag{4.8}$$

Now, for $n > n_0$ we define the following function for (regular) normal directions $u \in S^1$

$$\theta_n(u) = n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\mathrm{d}t, \tag{4.9}$$

so

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^1)) \cdot n^{-\frac{1}{3}} = \lim_{n\to\infty} \frac{2}{A^2} \int_{S^1} \theta_n(u)\, \mathrm{d}u.$$

We recall formula (11) from [BFRV09] that states the following. For any $\beta \geq 0$, $\omega > 0$ and $\alpha > 0$ we have that

$$\int_0^{g(n)} t^{\beta} (1 - \omega t^{\alpha})^n \,\mathrm{d}t \sim \frac{1}{\alpha \omega^{\frac{\beta+1}{\alpha}}} \cdot \Gamma\left(\frac{\beta+1}{\alpha}\right) \cdot n^{-\frac{\beta+1}{\alpha}}, \tag{4.10}$$

as $n \to \infty$, assuming

$$\left(\frac{(\beta+\alpha+1)\ln n}{\alpha\omega n}\right)^{\frac{1}{\alpha}} < g(n) < \omega^{-\frac{1}{\alpha}},$$

for sufficiently large n. We note that Lemma 9 also implies (4.10).

The conditions on K and formula (4.3) imply that there exists $\gamma_3 > 0$ such that $\ell(u,t) - \sin \ell(u,t) < \gamma_2 t^{3/2}$ for all $0 < t < t_0$ and normal directions $u \in S^1$. We recall that $1 + t - 1/\kappa(x_u) < 3$ for all normal directions $u \in S^1$ and $0 \leq t \leq t_1$. Now (4.5) and (4.10) with $\alpha = \beta = 3/2$ and $\omega = (2/(3A))\sqrt{2\rho/(1-\rho)}$ yield that there exists $\gamma_3 > 0$, depending only on S, such that $\theta_n(u) < \gamma_3$ for all normal $u \in S^1$ and sufficiently large n. Thus, by Lebesgue's dominated convergence theorem we may exchange the limit and the integral and obtain that

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^1)) \cdot n^{-\frac{1}{3}} = \frac{2}{A^2} \int_{S^1} \lim_{n\to\infty} \theta_n(u)\, \mathrm{d}u. \tag{4.11}$$

Let $u \in S^1$ be a normal direction and $\varepsilon \in (0,1)$ be arbitrary but fixed. It follows from Lemma 12 that

$$\begin{aligned} \lim_{n\to\infty} \theta_n(u) = \frac{4\sqrt{2}}{3} & \left(\frac{1}{\kappa(x_u)-1}\right)^{\frac{3}{2}} \\ \times & \left[\frac{\kappa(x_u)-1}{\kappa(x_u)} \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}} t^{\frac{3}{2}}\right)^{n-2} t^{\frac{3}{2}} \mathrm{d}t \right. \\ & \left. + \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}} t^{\frac{3}{2}}\right)^{n-2} t^{\frac{5}{2}} \mathrm{d}t \right], \end{aligned} \tag{4.12}$$

see [FKV14, p. 910]. Applying (4.10) with $\alpha = 3/2$, $\beta = 5/2$ yields that the second term of (4.12) is 0. Now, (4.10) yields that

$$\lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\, t^{\frac{3}{2}}\right)^{n-2} t^{\frac{3}{2}} \mathrm{d}t = \frac{2}{3}\left(\frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\right)^{-\frac{5}{3}} \Gamma\left(\frac{5}{3}\right).$$

Thus, putting everything together we obtain

$$\lim_{n\to\infty} \theta_n(u) = \frac{8\sqrt{2}}{9}\left(\frac{1}{\kappa(x_u)-1}\right)^{\frac{3}{2}} \frac{\kappa(x_u)-1}{\kappa(x_u)} \left(\frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\right)^{-\frac{5}{3}} \Gamma\left(\frac{5}{3}\right).$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E} f_0(K_n^1) \cdot n^{-\frac{1}{3}} &= \frac{2}{A^2}\int_{S^1} \lim_{n\to\infty} \theta_n(u)\mathrm{d}u \\
&= \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right) \int_{S^1} \frac{1}{\kappa(x_u)} \left(\kappa(x_u)-1\right)^{\frac{1}{3}} \mathrm{d}u \\
&= \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right) \int_{\partial S} \left(\kappa(x)-1\right)^{\frac{1}{3}} \mathrm{d}x.
\end{aligned}$$

From this we get (4.1) by scaling.

Chapter 5

Concluding remarks and open problems

As already noted in Chapter 2, the methods used in Section 2.2 for the case of three points in the plane could, in theory, be applied for four or more points although the argument would involve more case analysis. This may only be worth writing down if one could use it for an arbitrary number of points. However, that seems difficult due to the increasing number of cases.

In connection with Wendel's problem, we mention the open question of the spindle convex variant of Sylvester's problem. It asks for the probability that four i.i.d. uniform random points from a spindle convex container K are (not) in spindle convex position, that is, they form (or do not form) the vertices of a disc-4-gon. It also makes sense to ask the same question for other probability distributions as well.

In Chapter 3, the advantage of using spindle convexity against classical convexity is most apparent when $K = rB^2$, where r is close to 1. In this case, the constants for spindle convex approximation are significantly smaller than for classical approximation.

We note that using essentially the same methods, but with significantly more technical difficulties, it is reasonable to say that we could extend results of this chapter to the L-convex case for the expected number of vertices and missed area.

Another important quantity in the plane is the perimeter of the random disc-polygons. In the paper [FKV14] by Fodor, Kevei and Vígh, there is an asymptotic formula about the expected perimeter difference of K and K_n^r which requires that ∂K is C_+^5 using a similar local expansion of the boundary as in Chapter 3. This suggests that one could perform a similar calculation for the perimeter in order to obtain a finite series expansion. However, this calculation seems to be much more involved than that of the missed area and vertex number.

Although we only deal with probabilities and expectations in this thesis, we note that asymptotic upper and lower bounds of matching orders of magnitude in n have been proved by Fodor and Vígh [FV18] and Fodor, Grünfelder and Vígh [FGV22] for the missed area under certain restrictions. However, no lower or upper bounds are known for the variance of the perimeter. For this question, methods of Reitzner [Rei03] could be used.

Also, the natural question about similar results in higher dimensions arises. However, very little information is available for this case. Recently, Marynych and Molchanov [MM22] proved an asymptotic formula for the expected number of k-dimensional faces of uniform random L-polytopes when the container from which the random points are selected is L itself. When L and the container K are different, essentially no formulas are known. Here, one of the difficulties lies in the fact that while d i.i.d. uniform random points selected from a convex body that is spindle convex with radius r span a hyperplane with probability 1 in $\mathbb{R}^d$, they may not lie on a sphere of radius r. Besides this geometric problem, more work is required to overcome other technical difficulties that arise when trying to adapt classical convex methods for the spindle convex case.

With regards to the results in Chapter 4, the natural question arises whether these asymptotic formulas are also valid without any smoothness conditions, similar to the classical convex case. In the paper by Böröczky, Fodor and Hug [BFH10] a completely different method was used to prove a similar asymptotic formula for the classical convex case with no smoothness conditions. However, it seems that that argument cannot be adapted easily to the spindle convex case. One reason for this is the essential use of affine transformations, which are not allowed in the spindle convex setting. It remains to be seen whether this can be circumvented somehow.

In a recent paper, Fodor, Kevei and Vígh [FKV23] proved similar asymptotic formulas for the case when K is a disc-polygon. Probably, their argument could be used to deal with a finite number of non-smooth points, but likely not when there are infinitely many of them.

Despite the above, we still think that the method of Böröczky, Fodor and Hug [BFH10] is the most hopeful that could deal with all kinds of extremes on the boundary.

We have already mentioned earlier that Fodor, Kevei and Vígh [FKV14] proved an asymptotic formula for the expectation of the perimeter difference of K and K_n^R. Their formula is valid if ∂K is C_+^5. It is an open question whether the differentiability conditions on ∂K could be relaxed while maintaining the validity of the formula. In this case, the difficulty lies in the fact that the proof uses several terms of local Taylor polynomial expansions of ∂K which would be unavailable under weaker differentiability conditions.

Chapter 6

Summary

In this dissertation we consider some spindle convex analogs of classical problems in the theory of random polytopes. Spindle convexity is when the (spindle convex) hull of a set S is produced by the intersection of all equal radius R closed balls containing S, for a more detailed description and precise definitions see Section 1.1. The closed balls play a similar role as closed half-spaces do in classical convexity.

Most of the mathematical content of the dissertation is based on the following two publications of the author:

[FMV23] F. Fodor, P. N. A. Montenegro, and V. Vígh, *On Wendel's equality for intersections of balls*, Aequationes Math. **97** (2023), no. 2, 439–451, DOI 10.1007/s00010-022-00912-3. MR4563622

[FMP24] F. Fodor and N. A. Montenegro Pinzón, *Series expansions for random disc-polygons in smooth plane convex bodies*, J. Appl. Probab. **61** (2024), no. 4, Published online May 16, 2024, DOI 10.1017/jpr.2024.27.

In Chapter 1, we give a general introduction to the subject. Section 1.1 contains some of the necessary definitions and notations. We leave the more specific terms and notations to the individual chapters. Section 1.2 provides a brief overview on the history of the relevant results of theory of random polytopes.

Chapter 2 contains the results of the paper [FMV23] about the spindle convex variant of Wendel's equality.

Wendel's equality [Wen62] is a classical result in geometric probability which states that if $x_1, \dots, x_n$ are i.i.d. random points in $\mathbb{R}^d$ whose distribution is (centrally) symmetric with respect to the origin o, and the probability measure of hyperplanes are 0, then the probability that o is not contained in the convex hull $[x_1, \dots, x_n]$ is

$$\mathbb{P}(o \notin [x_1, \dots, x_n]) = \frac{1}{2^{n-1}} \sum_{i=0}^{d-1} \binom{n-1}{i}. \tag{6.1}$$

In Chapter 2, we investigate the analogous question for spindle convexity. If K_n^R denotes the radius R spindle convex hull of n i.i.d. random points selected from the o-symmetric convex body $K \subset \mathbb{R}^d$ according to the uniform probability distribution, then what is the probability that $o \in K_n^R$? We note that in this model we may always achieve by scaling (simultaneously K and radius R circles) that $R = 1$. Henceforth, in the following two theorems we assume that $R = 1$.

First, we study the special case when $K = rB^d$ with $0 < r \leq 1$. Let

$$P(d, r, n) := \mathbb{P}(o \in [x_1, \dots, x_n]_1).$$

In Section 2.2 we prove the following theorem:

Theorem 1 ([FMV23, Theorem 1.1],)**.** *Let $K = rB^d$. Then*

$$P(d, r, 2) = \frac{\omega_{d-1}\omega_d}{(r^d \kappa_d)^2} \int_0^r \int_0^r \int_0^{\varphi(r_1, r_2)} r_1^{d-1} r_2^{d-1} \sin^{d-2} \varphi \, d\varphi dr_2 dr_1,$$

where $\varphi(r_1, r_2) = \arcsin(r_1/2) + \arcsin(r_2/2)$. In particular,

$$P(2, 1, 2) = \frac{\sqrt{3}}{\pi} - \frac{1}{3} = 0.2179\dots,$$
$$P(3, 1, 2) = \frac{1}{64}(23 + 12\sqrt{3}\pi - 8\pi^2) = 0.1459\dots.$$

Furthermore, for the case of three points, we prove the following statement in Section 2.3.

Theorem 2 ([FMV23, Theorem 1.2])**.** *Let $K = B^2$. Then*

$$P(2, 1, 3) = \frac{-84\pi^2 - 477 + 360\sqrt{3}\pi}{144\pi^2} = 0.4594\dots.$$

In Section 2.4, we study the Gaussian R-spindle convex model. Let $x_1, \ldots, x_n$ be i.i.d. random points from $\mathbb{R}^d$ distributed according to the standard normal distribution. The question is the same, what is the probability that $o \in K_n^R$? We note that in this second case, it may, and does, happen that $K_n^R = \mathbb{R}^d$. We give an integral formula for the probability that a Gaussian unit radius spindle contains the origin and evaluate it numerically in the plane, see (2.6).

Chapter 3 contains the results of the paper [FMP24]. In Chapter 3, we study uniform random disc-polygons in convex discs in the plane where the boundary of the convex disc K is assumed to have a higher degree of smoothness. Let $k \geq 2$ be an integer and $K \subset \mathbb{R}^2$ a convex disc that has a boundary that is C_+^{k+1}, meaning that it is $k+1$ times continuously differentiable and has strictly positive curvature $\kappa(x)$ for all boundary points $x \in \partial K$. Let K_n^R denote the R-spindle convex hull (an R-disc-polygon) of n i.i.d. uniform random points from K where it is assumed that $R > \max 1/\kappa(x)$ for $x \in \partial K$.

In the special case when $k = 1$, Fodor, Kevei and Vígh [FKV14] proved asymptotic formulas, among others, for the expected number of vertices $\mathbb{E} f_0(K_n^R)$ and so-called missed areas $A(K \setminus K_n^R)$. In Chapter 3 we further study these expectations and prove finite expansions for them in terms of powers of n as $n \to \infty$. In particular, the main result of this chapter is the following theorem.

Theorem 5 ([FMP24, Theorem 2])**.** *Let $k \geq 2$ be an integer, and let K be a convex disc with C_+^{k+1} smooth boundary. Then for all $R > \max_{x \in \partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(f_0(K_n^R)) = z_1(K) n^{\frac{1}{3}} + \ldots + z_k(K) n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}})$$

as $n \to \infty$. All coefficients $z_1, \ldots, z_k$ can be determined explicitly. In particular,

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}} \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{\frac{1}{3}} \mathrm{d}x,$$

$$z_2(K) = 0,$$

$$z_3(K) = -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3A(K)}{2}} \int_{\partial K} \frac{\kappa''(x)}{3(\kappa(x) - \frac{1}{R})^{\frac{4}{3}}} + \frac{2R^2\kappa^2(x) + 7R\kappa(x) - 1}{2R^2(\kappa(x) - \frac{1}{R})^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x) - \frac{1}{R})^{\frac{7}{3}}} \mathrm{d}x.$$

By the spindle convex version of Efron's identity we obtain, as a corollary, a similar expansion for the missed area $A(K \setminus K_n^R)$, see Theorem 6 ([FMP24, Theorem 3]).

In the case when $K = B(R)$, that is, K is a radius R closed circular disc, the expected number of vertices behaves fundamentally differently from the previously discussed situation. It was proved in [FKV14] that

$$\mathbb{E}(f_0(B(R)_n^R)) = \frac{\pi^2}{2} + o(1).$$

as $n \to \infty$.

In Section 3.6, applying the same method as before, we prove a finite expansion of the expected number of vertices $\mathbb{E}(f_0(B(R)_n^R))$ in terms of the powers of n, see formulas (3.18) and (3.19).

Finally, in Chapter 4, we extend the validity of the original asymptotic formulas proved by Fodor, Kevei and Vígh in [FKV14] for the expected number of vertices and missed area under weaker smoothness conditions. In [FKV14] it was assumed that ∂K is C^2_+ smooth and the radius of spindle convexity is strictly larger maximal radius of curvature of ∂K. In Chapter 4, we only assume that K has a rolling circle (there exists $r_0 > 0$ such that for every $x \in K$ there is a radius r_0 circular disc that is contained in K and which contains x on its boundary), and that K slides freely in a circle of radius R_0 (meaning that for each $x \in \partial K$ there is a circle of radius R_0 that contains K and x is on its boundary). These conditions naturally imply that K is R-spindle convex for all $R \geq R_0$. We prove the following statements:

Theorem 11 (p. 51)**.** *Let K be a convex disc that has a rolling circle and slides freely in a circle of R_0. Then for any $R > R_0$, it holds that*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^R)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x,$$

and

$$\lim_{n\to\infty} \mathbb{E}(A(K \setminus K_n^R)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}} \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x.$$

Chapter 7

Összefoglaló

Jelen dolgozat a geometriai valószínűség és sztochasztikus geometria témaköréhez tartozik. Különböző eloszlások szerint választott független véletlen pontok által meghatározott alakzatok geometriai tulajdonságait vizsgáljuk. A véletlen politópok irodalmában szokásos klasszikus konvexitás fogalom helyett az ún. orsókonvexitás fogalmát használjuk. Egy $\mathbb{R}^d$ euklideszi térbeli X kompakt halmazt $R > 0$ sugárral orsókonvexnek nevezünk, ha X benne van egy R sugarú zárt körlmezeben és megegyezik az őt tartalmazó összes R sugarú körlemez metszetével. Véges sok R sugarú zát körlemez metszetét körpoligonnak nevezzük.

A dolgozat gerincét két megjelent cikk adja: [FMV23, FMP24]. Továbbá a disszertáció tartalmaz nem publikált eredményeket is (4. fejezet). Az 1. fejezetben a legalapvetőbb jelölések és definíciók kerülnek bevezetésre, amit a témakör egy rövid töréneti összefoglalója és az elért eredmények ismertetése követ. A 2. fejezet a [FMV23] cikk tartalmán alapszik és a nevezetes Wendel-egyenlőség orsókonvex változatát vizsgálja. Itt arra vagyunk kíváncsiak, hogy egy origóra szimmetrikus eloszlás szerint választott véletlen pontok konvex burka mekkora valószínűséggel tartalmazza az origót. A klasszikus esetben ez a valószínűség csak a véletlen pontok számától függ, az eloszlástól nem. Az orsókonvex esetben az véletlen pontokat egyrészt egy origó középpontú gömbből az egyenletes eloszlás szerint, másrészt a standard normális eloszlás szerint választjuk. Meghatározzuk, hogy mekkora valószínűséggel van benne az origó az orsókonvex burkukban, ha a pontok száma kettő vagy három. Elviekben ez a módszer több pontra is alkalmazható lenne, de a sok eset miatt gyorsan elbonyolodik. A 2. fejezet fő eredményeit az 1 tétel, a 2 tétel, és a 2.4. alfejezetben a (2.6) formula tartalmazza.

A 3. fejezetben, ami az [FMP24] cikk tartalmán alapszik, Gruber [Gru96] és Reitzner [Rei01, Rei04] eredményeit terjesztjük ki orsókonvex lemezek véletlen körpoligonokkal való közelítésére. Konkrétan, a befoglaló K lemezről feltesszük, hogy határa $k+1$-szer folytonosan differenciálható ($k \geq 2$) és emellett a feltétel mellett véges sok tagból álló sorfejtést igazolunk a kimaradó terület, illetve a csúcsszám várható értékére, ha a pontok száma a végtelenbe tart és az orsókonvexitás sugara szigorúan nagyobb, mint a befoglaló lemez határának maximális görbületi sugara. Továbbá, ugyanazzal a módszerrel, megvizsgáljuk azt az esetet is, amikor a befoglaló K konvex lemez épp az R sugarú zárt körlmez, és erre is véges sorfejtést adunk, ha $n \to \infty$. A 3. fejezet fő eredményeit az 5. tétel, a 6. tétel, és a 3.6. alfejezetben a (3.18) és (3.19) formulák tartalmazzák.

A dolgozat 4. fejezetében Fodor, Kevei és Vígh [FKV14] véletlen körpoligonok kimaradó területére és csúcsszámára vonatkozó aszimptotikus formuláit terjesztjük ki konvex lemezek egy tágabb simasági osztályára. A [FKV14] cikkbeli eredmények feltételezték, hogy K határa C^2_+ és az orsókonvexitás sugara R szigorúan nagyobb, mint K határának maximális görbületi sugara. A 4. fejezetben K határáról csak azt tesszük fel, hogy rendelkezik gördülőkörrel, illetve szabadon siklik egy R_0 sugarú körben, illetve hogy $R > R_0$. A 3. fejezet fő eredményei a 11. tételben vannak összefoglalva. A 4. fejezet tartalma nem publikált.

Az 5. fejezet a dolgozatban leírt eredményekkel kapcsolatos néhány megjegyzést és nyitott problémát tartalmaz. A 6. fejezet a disszertáció angol nyelvű összefoglalója.

Chapter 8

The author's publications

I declare that as of the submission of this thesis, I have the following two publications, and that both of them are used in the dissertation.

[FMV23] F. Fodor, P. N. A. Montenegro, and V. Vígh, *On Wendel's equality for intersections of balls*, Aequationes Math. **97** (2023), no. 2, 439–451, DOI 10.1007/s00010-022-00912-3. MR4563622

[FMP24] F. Fodor and N. A. Montenegro Pinzón, *Series expansions for random disc-polygons in smooth plane convex bodies*, J. Appl. Probab. **61** (2024), no. 4, Published online May 16, 2024, DOI 10.1017/jpr.2024.27.

Chapter 9

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