POINTWISE CONVERGENCE OF FOURIER AND CONJUGATE SERIES OF PERIODIC FUNCTIONS IN TWO VARIABLES

PhD THESIS

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The development of the theory of Fourier series in mathematical analysis began in the 18th century. The first exactly proved result was published in DIRICHLET's (1805–1859) paper in 1829. That theorem concerns the convergence of Fourier series of piecewise monotonic functions. The result of this theorem was extended by JORDAN (1838–1922) to functions of bounded variation in 1881. In the literature it is known as Diriclet–Jordan test.

In the first part of our dissertation two classical and two later results of the theory of single Fourier series are introduced. The former ones are: the Dini test on the pointwise convergence of Fourier series (DINI (1845–1918), italian mathematician) and the Pringsheim test on the pointwise convergence of series conjugate to Fourier series (PRINGSHEIM (1850–1941), german mathematician). The latter theorems are: the quantitative version of the well-known Diriclet–Jordan test by BOJANIĆ [2] and its further developped version by TELYAKOVSKII [18]. In the further part of the dissertation these theorems are extended to functions in two variables and their applicability to functions in the multiplicative Lipscitz/Zygmund classes is examined.

1 Known results in one dimension

Dini test

Given a periodic (with period 2π) complex-valued function $f \in L^1(\mathbb{T})$, where $\mathbb{T} := [-\pi, \pi)$ is the one-dimensional torus. We consider the pointwise convergence of Fourier series

$$f(x) \sim \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikx}, \tag{1}$$

where the Fourier coefficients of function f are defined by

$$\hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) e^{-iku} \,\mathrm{d}u, \quad k \in \mathbb{Z}.$$

The simple properties of these follow by the Riemann–Lebesgue lemma (see, e.g., [19, Vol. I, p. 48]):

$$\hat{f}(k) \to 0 \quad \text{as} \quad |k| \to \infty.$$

The unsymmetric partial sums of the series in (1) are defined by

$$S_{m,n}(f;x) := \sum_{k=m}^{n} \hat{f}(k) e^{ikx}, \quad m,n \in \mathbb{Z}, \quad m \le n.$$

In the particular case when $m = -n$, the shorter notation $S_n(f;x)$ ($n \in \mathbb{N}$) is used; and they are called the symmetric partial sums.

The Dini test reads as follows.

Theorem 1.1. *Assume* $f \in L^1(\mathbb{T})$.

(i) *If for some* $x_0 \in \mathbb{T}$,

$$\frac{f(x_0-u)+f(x_0+u)-2f(x_0)}{u} \in L^1(\mathbb{T}), \tag{2}$$

then $S_n(f;x_0) \to f(x_0)$ *as* $n \to \infty$.

(ii) *If for some* $x_0 \in \mathbb{T}$,

$$\frac{f(x_0+u)-f(x_0)}{u} \in L^1(\mathbb{T}), \tag{3}$$

then $S_{m,n}(f;x_0) \to f(x_0)$ *as* $m \to -\infty$ *and* $n \to \infty$.

The proof of Statement (i) is well known (see, e.g., [19, Vol. I, p. 52] in the case when $x_0 := 0$). It hinges on the Riemann–Lebesgue lemma and the representation

$$S_n(f;x_0) - f(x_0) = \frac{1}{2\pi}\int_{\mathbb{T}} [f(x_0-u)+f(x_0+u)-2f(x_0)] D_n(u)\,\mathrm{d}u,$$

where $D_n(u)$ is the Dirichlet kernel. The proof of Statement (ii) is less known (see, e.g., [3]).

Condition (3) is clearly satisfied for every $x_0 \in \mathbb{T}$ if f is in the periodic Lipschitz class $\mathrm{Lip}(\alpha)$ for some $\alpha > 0$. Likewise, condition (2) is satisfied for every $x_0 \in \mathbb{T}$ if f is in periodic Zygmund class $\mathrm{Zyg}(\alpha)$ for some $\alpha > 0$.

Pringsheim test

The series conjugate to the Fourier series in (1), or briefly: the conjugate series, is defined by

$$\sum_{k\in\mathbb{Z}} (-i \operatorname{sign} k)\hat{f}(k) e^{ikx}, \tag{4}$$

whose unsymmetric and symmetric partial sums are denoted by $\widetilde{S}_{m,n}(f;x)$ and $\widetilde{S}_n(f;x)$, respectively.

We recall that the function conjugate to f, or briefly: the conjugate function $\widetilde{f}$, is defined as a Cauchy principal value integral:

$$\widetilde{f}(x) := (\mathrm{P.V.})\frac{1}{\pi}\int_0^{\pi} \frac{f(x-u)-f(x+u)}{2\tan\frac{1}{2}u}\,\mathrm{d}u = \lim_{\varepsilon\to 0+0}\left\{\frac{1}{\pi}\int_{\varepsilon}^{\pi}\right\}.$$

As is well-known $\widetilde{f}(x)$ exists at almost every $x \in \mathbb{T}$, whenever $f \in L^1(\mathbb{T})$, but generally $\widetilde{f} \notin L^1(\mathbb{T})$.

Statement (i) in the next theorem is known as the Pringsheim test (see, e.g., [19, Vol. I, p. 52]).

Theorem 1.2. *Assume $f \in L^1(\mathbb{T})$.*

(i) *If for some $x_0 \in \mathbb{T}$,*

$$\frac{f(x_0+u)-f(x_0-u)}{u} \in L^1(\mathbb{T}), \tag{5}$$

then $\widetilde{f}(x_0)$ exists in the sense of Lebesgue integral and $\widetilde{S}_n(f;x_0) \to \widetilde{f}(x_0)$ as $n \to \infty$.

(ii) *If for some $x_0 \in \mathbb{T}$,*

$$\frac{f(x_0+u)-f(x_0)}{u} \in L^1(\mathbb{T}), \tag{6}$$

then $\widetilde{S}_{m,n}(f;x_0) \to \widetilde{f}(x_0)$ as $m \to -\infty$ and $n \to \infty$.

The proof of Statement (i) hinges on the Riemann–Lebesgue lemma and the representation

$$\widetilde{S}_n(f;x) = \frac{1}{\pi}\int_{\mathbb{T}} f(x-u)\widetilde{D}_n(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{T}}[f(x-u)-f(x+u)]\widetilde{D}_n(u)\,\mathrm{d}u,$$

where $\widetilde{D}_n(u)$ is the conjugate Dirichlet kernel. A proof of the less known Statement (ii) can be found, e.g., in [3] by Chernoff.

Conditions (5) and (6) are clearly satisfied at every $x_0 \in \mathbb{T}$ if f is in the periodic Lipschitz class $\mathrm{Lip}(\alpha)$ for some $\alpha > 0$.

Theorems of Bojanić and Telyakovskii

According to the Dirichlet–Jordan theorem, the Fourier series of a periodic function f of bounded variation converges to $\frac{1}{2}[f(x-0)+f(x+0)]$ at each point x, that is,

$$\lim_{n\to\infty} S_n(f,x) = \frac{1}{2}[f(x-0)+f(x+0)].$$

For the rate of this convergence, Bojanić [2] gave the following estimate in the case when $f(x) = \frac{1}{2}[f(x-0)+f(x+0)]$.

Theorem 1.3. *If a periodic function f is of bounded variation on the interval $[-\pi,\pi]$, then the following estimate holds for every x and $n = 1, 2, \dots$:*

$$|S_n(f,x)-f(x)| \le \frac{3}{n}\sum_{k=1}^{n} V\left(\varphi_x, \left[0, \frac{\pi}{k}\right]\right), \tag{7}$$

where $\varphi_x(u) := f(x+u)+f(x-u)-2f(x)$, $u \in [0,\pi]$.

We note that the function $\varphi_x(t)$ is of bounded variation, too; and it is continuous at the point $t = 0$. Therefore the total variation function $V(\varphi_x,[0,t])$ is also continuous at $t = 0$; in particular, we have

$$V\left(\varphi_x, \left[0, \frac{\pi}{k}\right]\right) \to 0, \quad k \to \infty.$$

Hence it follows that the expression on the right-hand side of (7) converges to zero as $n \to \infty$, that is, Theorem 1.3 is a sharpening of the Dirichlet–Jordan theorem.

The statement of Theorem 1.3 was developped by Telyakovskii [18] as follows.

Theorem 1.4. *Let $m_1 = 1 < m_2 < \cdots < m_p < \ldots$ be a sequence of natural numbers such that the condition*

$$\sum_{p=p_0}^{\infty} \frac{1}{m_p} \le \frac{A}{m_{p_0}}, \quad p_0 = 1, 2, \ldots \tag{8}$$

is satisfied, where $A > 1$ is a constant. If a function f is of bounded variation, then for every μ and x the following estimate holds:

$$\begin{aligned} |f(x) - S_\mu(f,x)| &= \left| \sum_{|k|=\mu+1}^{\infty} \hat{f}(k) e^{ikx} \right| \le \\ &\le \left| \sum_{|k|=\mu+1}^{m_{p_0}-1} \hat{f}(k) e^{ikx} \right| + \sum_{p=p_0}^{\infty} \left| \sum_{k=m_p}^{m_{p+1}-1} \hat{f}(k) e^{ikx} \right| \le \\ &\le \frac{CA}{\mu+1} \sum_{k=1}^{\mu+1} V\left(\varphi_x, \left[0, \frac{\pi}{k}\right]\right), \end{aligned}$$

where $m_{p_0-1} \le \mu < m_{p_0}$ and A is the constant occurring in (8).

Following the scheme of Telyakovskii's proof and making use of Lemma 1.5 (see [12, Lemma 1-2]), one can achieve the following stronger estimate:

$$\sum_{p=p_0}^{\infty} \max_{m_p \le m \le M < m_{p+1}} \left| \sum_{k=m}^{M} \hat{f}(k) e^{ikx} \right| \le \frac{(\pi+4)A}{\pi m_{p_0}} \sum_{k=1}^{m_{p_0}} V\left(\varphi_x, \left[0, \frac{\pi}{k}\right]\right).$$

We extend this form of statement to functions in two variables.

Lemma 1.5. *If $m_1 = 1 < m_2 < \ldots < m_p < \ldots$ is a sequence of natural numbers such that the condition (8) is satisfied, then the following estimates hold:*

$$\sum_{p=p_0}^{\infty} \max_{m_p \le m \le M < m_{p+1}} \left| \sum_{k=m}^{M} \frac{\sin ku}{k} \right| \le \frac{\pi A}{m_{p_0} u}, \qquad 0 < u \le \pi,$$

$$\sum_{p=1}^{\infty} \max_{m_p \le m \le M < m_{p+1}} \left| \sum_{k=m}^{M} \frac{\sin ku}{k} \right| \le (\pi+2)A, \qquad u \in \mathbb{R}.$$

2 New results in two dimensions

Extension of the Dini test to double Fourier series

The double Fourier series of a complex-valued periodic (with period 2π) function $f \in L^1(\mathbb{T}^2)$ is defined by

$$f(x,y) \sim \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \hat{f}(k,l) e^{i(kx+ly)}, \tag{9}$$

wherethe Fourier coefficients $\hat{f}(k,l)$ az f are defined by

$$\hat{f}(k,l) := \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(u,v) e^{-i(ku+lv)} \,\mathrm{d}u \,\mathrm{d}v, \quad (k,l) \in \mathbb{Z}^2.$$

By the Riemann–Lebesgue lemma (see, e.g., [19, Vol. II, p. 301]), if $f \in L^1(\mathbb{T}^2)$, then

$$\hat{f}(k,l) \to 0 \quad \text{as} \quad \max\{|k|,|l|\} \to \infty.$$

This fact will be of vital importance in the proofs of our theorems.

The unsymmetric rectangular partial sums of the series in (9) are defined by

$$S_{m_1,n_1;m_2,n_2}(f;x,y) := \sum_{k=m_1}^{n_1} \sum_{l=m_2}^{n_2} \hat{f}(k,l) e^{i(kx+ly)}, \quad m_j, n_j \in \mathbb{Z}, \quad m_j \le n_j, \quad j = 1,2.$$

In the particular case when $m_j = -n_j$ ($n_j \in \mathbb{N}$), we use the shorter notation $S_{n_1,n_2}(f;x,y)$, and they are called the symmetric rectangular partial sums.

In our first theorem [10, Theorem 1], we give a sufficient condition for the convergence of the symmetric rectangular partial sums of the Fourier series in (9) at a given point $(x_0,y_0) \in \mathbb{T}^2$. This convergence also depends on the convergence of the single Fourier series of the so-called marginal functions $f(x,y_0)$, $x \in \mathbb{T}$, and $f(x_0,y)$, $y \in \mathbb{T}$, at $x := x_0$ and $y := y_0$, respectively. For these single Fourier series, we use the following notations:

$$f(x,y_0) \sim \sum_{k\in\mathbb{Z}} f(\cdot,y_0)^\wedge(k) e^{ikx}, \tag{10}$$

where

$$f(\cdot,y_0)^\wedge(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u,y_0) e^{-iku} \,\mathrm{d}u, \quad k \in \mathbb{Z};$$

and analogously

$$f(x_0,y) \sim \sum_{l\in\mathbb{Z}} f(x_0,\cdot)^\wedge(l) e^{ily}, \tag{11}$$

where

$$f(x_0,\cdot)^\wedge(l) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x_0,v) e^{-ilv} \,\mathrm{d}v, \quad l \in \mathbb{Z}.$$

Theorem 2.1. *Assume $f \in L^1(\mathbb{T}^2)$, $A, A_1, A_2 \in \mathbb{C}$, and for some $(x_0,y_0) \in \mathbb{T}^2$,*

$$u^{-1}v^{-1}\Delta_{2,2}(f;x_0,y_0;u,v;A_1+A_2-A) \in L^1(\mathbb{T}^2).$$

If the symmetric partial sums of the single Fourier series in (10) and (11) converge to A_1 and A_2 at $x := x_0$ and $y := y_0$, respectively, then

$$S_{n_1,n_2}(f;x_0,y_0) \to A \quad \textit{as} \quad n_j \to \infty,\ j = 1,2. \tag{12}$$

Conversely, if (12) is satisfied and if the symmetric partial sums of one of the Fourier series in (10) and (11) converge, then so do the symmetric partial sums of the other Fourier series.

Combining Statement (i) in Theorem 1.1 with Theorem 2.1 in the special case when $A = A_1 = A_2 := f(x_0, y_0)$ yields the following consequence [10, Corollary 1].

Corollary 2.1. *Assume* $f \in L^1(\mathbb{T}^2)$, $f(\cdot, y_0) \in L^1(\mathbb{T})$ *and* $f(x_0, \cdot) \in L^1(\mathbb{T})$ *for some* $(x_0, y_0) \in \mathbb{T}^2$. *If*

$$u^{-1}v^{-1}\Delta_{2,2}(f; x_0, y_0; u, v) \in L^1(\mathbb{T}^2), \tag{13}$$

$$u^{-1}[f(x_0 - u, y_0) + f(x_0 + u, y_0) - 2f(x_0, y_0)] \in L^1(\mathbb{T}),$$

and

$$v^{-1}[f(x_0, y_0 - v) + f(x_0, y_0 + v) - 2f(x_0, y_0)] \in L^1(\mathbb{T}),$$

then we have $S_{n_1,n_2}(f; x_0, y_0) \to f(x_0, y_0)$ *as* $n_j \to \infty$ $(j = 1, 2)$.

In second theorem [10, Theorem 2], we give a sufficient condition for the convergence of the unsymmetric rectangular partial sums of Fourier series in (9) at a given point.

Theorem 2.2. *Assume* $f \in L^1(\mathbb{T}^2)$ *and for some* $(x_0, y_0) \in \mathbb{T}^2$,

$$u^{-1}v^{-1}\Delta_{1,1}(f; x_0, y_0; u, v) \in L^1(\mathbb{T}^2). \tag{14}$$

If the unsymmetric partial sums of the single Fourier series in (10) and (11) converge to $f(x_0, y_0)$, *then*

$$S_{m_1,n_1;m_2,n_2}(f; x_0, y_0) \to f(x_0, y_0) \quad \text{as} \quad m_j \to -\infty \text{ and } n_j \to \infty, \ j = 1, 2. \tag{15}$$

Conversely, if (15) is satisfied and if the unsymmetric partial sums of one of the Fourier series in (10) and (11) converge $f(x_0, y_0)$, *then so do the unsymmetric partial sums of the other Fourier series.*

Combining Statement (ii) in Theorem 1.1 with Theorem 2.2 yields the following corollary [10, Corollary 2].

Corollary 2.2. *Assume* $f \in L^1(\mathbb{T}^2)$, $f(\cdot, y_0) \in L^1(\mathbb{T})$ *and* $f(x_0, \cdot) \in L^1(\mathbb{T})$ *for some* $(x_0, y_0) \in \mathbb{T}^2$. *If condition (14) and the following two more conditions are satisfied:*

$$u^{-1}[f(u, y_0) - f(x_0, y_0)] \in L^1(\mathbb{T}) \quad \text{and} \quad v^{-1}[f(x_0, v) - f(x_0, y_0)] \in L^1(\mathbb{T}),$$

then (15) is also satisfied.

It is obvious that if $f \in L^1(\mathbb{T}^2) \cap \mathrm{Zyg}(\alpha, \beta)$ for some $\alpha, \beta > 0$, then condition (13) is satisfied at every point (x_0, y_0). Likewise, if $f \in L^1(\mathbb{T}^2) \cap \mathrm{Lip}(\alpha, \beta)$ for some $\alpha, \beta > 0$, then condition (14) is satisfied at every point (x_0, y_0).

Extension of the Pringsheim test to double Fourier series

The series conjugate to the double Fourier series in (9) can be defined in several ways. The conjugate series with respect to the first variable is defined by

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}(-i\,\mathrm{sign}\,k)\hat{f}(k,l)e^{i(kx+ly)}, \tag{16}$$

the conjugate series with respect to the second variable is defined by

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}(-i\,\mathrm{sign}\,l)\hat{f}(k,l)e^{i(kx+ly)}, \tag{17}$$

and the conjugate series with respect to both variables is defined by

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}(-i\,\mathrm{sign}\,k)(-i\,\mathrm{sign}\,l)\hat{f}(k,l)e^{i(kx+ly)}. \tag{18}$$

The unsymmetric rectangular partial sums of series (16)–(18) are denoted by

$$\widetilde{S}^{(1,0)}_{m_1,n_1;m_2,n_2}(f;x,y), \quad \widetilde{S}^{(0,1)}_{m_1,n_1;m_2,n_2}(f;x,y), \quad \text{and} \quad \widetilde{S}^{(1,1)}_{m_1,n_1;m_2,n_2}(f;x,y),$$

respectively. The symmetric rectangular partial sums of the same series are denoted by $\widetilde{S}^{(1,0)}_{n_1,n_2}(f;x,y)$, $\widetilde{S}^{(0,1)}_{n_1,n_2}(f;x,y)$ and $\widetilde{S}^{(1,1)}_{n_1,n_2}(f;x,y)$, respectively.

In the investigation of convergence of double conjugate series, the conjugate series of the single Fourier series (10) and (11) play important roles. They are the following ones:

$$\sum_{k\in\mathbb{Z}}(-i\,\mathrm{sign}\,k)f(\cdot,y_0)^\wedge(k)e^{ikx}, \tag{19}$$

$$\sum_{l\in\mathbb{Z}}(-i\,\mathrm{sign}\,l)f(x_0,\cdot)^\wedge(l)e^{ily}. \tag{20}$$

In our next theorem [11, Theorem 1] we give a sufficient and necessary condition for the convergence of thy symmetric rectangular partial sums of conjugate series (16).

Theorem 2.3. *Assume $f\in L^1(\mathbb{T}^2)$. If for some $(x_0,y_0)\in\mathbb{T}^2$,*

$$u^{-1}v^{-1}\Delta_{1,2}(f;x_0,y_0;u,v)\in L^1(\mathbb{T}^2), \tag{21}$$

then the limit of $\widetilde{S}^{(1,0)}_{n_1,n_2}(f;x_0,y_0)$ as $n_j\to\infty$ $(j=1,2)$ exists if and only if the symmetric partial sums of the conjugate series (19) converge at $x:=x_0$, and in this case two limits coincide.

The symmetric counterpart of Theorem 2.3 reads as follows [11, Theorem 2].

Theorem 2.4. *Assume $f \in L^1(\mathbb{T}^2)$. If for some $(x_0, y_0) \in \mathbb{T}^2$,*

$$u^{-1}v^{-1}\Delta_{2,1}(f; x_0, y_0; u, v) \in L^1(\mathbb{T}^2), \tag{22}$$

then the limit of $\widetilde{S}^{(0,1)}_{n_1,n_2}(f; x_0, y_0)$ as $n_j \to \infty$ $(j = 1, 2)$ exists if and only if the symmetric partial sums of the conjugate series (20) converge at $y := y_0$, and in this case the two limits coincide.

Under a stronger condition, one can prove the convergence of the unsymmetric rectangular partial sums of the conjugate series (16)–(18) [11, Theorem 3].

Theorem 2.5. *Assume $f \in L^1(\mathbb{T}^2)$ and that for some $(x_0, y_0) \in \mathbb{T}^2$,*

$$u^{-1}v^{-1}\Delta_{1,1}(f; x_0, y_0; u, v) \in L^1(\mathbb{T}^2). \tag{23}$$

(i) *The limit of $\widetilde{S}^{(1,0)}_{m_1,n_1;m_2,n_2}(f; x_0, y_0)$ as $m_j \to -\infty$ and $n_j \to \infty$ $(j = 1, 2)$ exists if and only if the unsymmetric partial sums of the conjugate series (19) converge at $x := x_0$, and in this case the two limits coincide.*

(ii) *The limit of $\widetilde{S}^{(0,1)}_{m_1,n_1;m_2,n_2}(f; x_0, y_0)$ as $m_j \to -\infty$ and $n_j \to \infty$ $(j = 1, 2)$ exists if and only if the unsymmetric partial sums of the conjugate series (20) converge at $y := y_0$, and in this case the two limits coincide.*

(iii) *The limit of $\widetilde{S}^{(1,1)}_{m_1,n_1;m_2,n_2}(f; x_0, y_0)$ as $m_j \to -\infty$ and $n_j \to \infty$ $(j = 1, 2)$ exists.*

We note that condition (21) in Theorem 2.3 is certainly satisfied at every $(x_0, y_0) \in \mathbb{T}^2$ if $f \in \mathrm{LZ}(\alpha, \beta)$ for some $\alpha, \beta > 0$; condition (22) in Theorem 2.4 is satisfied at every $(x_0, y_0) \in \mathbb{T}^2$ if $f \in \mathrm{ZL}(\alpha, \beta)$ for some $\alpha, \beta > 0$; and condition (23) in Theorem 2.5 is satisfied if $f \in \mathrm{Lip}(\alpha, \beta)$ for some $\alpha, \beta > 0$.

Combining Theorems 2.3–2.5 with the Theorem 1.2 yields the following corollaries [11, Corollary 1-3].

Corollary 2.3. *Assume $f \in L^1(\mathbb{T}^2)$ and that condition (21) is satisfied for some $(x_0, y_0) \in \mathbb{T}^2$. If $f(\cdot, y_0) \in L^1(\mathbb{T})$ and*

$$u^{-1}[f(x_0 + u, y_0) - f(x_0 - u, y_0)] \in L^1(\mathbb{T}),$$

then $\widetilde{S}^{(1,0)}_{n_1,n_2}(f; x_0, y_0) \to f(\cdot, y_0)^{\sim}(x_0)$ as $n_j \to \infty$ $(j = 1, 2)$.

Corollary 2.4. *Assume $f \in L^1(\mathbb{T}^2)$ and that condition (22) is satisfied for some $(x_0, y_0) \in \mathbb{T}^2$. If $f(x_0, \cdot) \in L^1(\mathbb{T})$ and*

$$v^{-1}[f(x_0, y_0 + v) - f(x_0, y_0 - v)] \in L^1(\mathbb{T}),$$

then $\widetilde{S}^{(0,1)}_{n_1,n_2}(f; x_0, y_0) \to f(x_0, \cdot)^{\sim}(y_0)$ as $n_j \to \infty$ $(j = 1, 2)$.

Corollary 2.5. *Assume $f \in L^1(\mathbb{T}^2)$ and that condition (23) is satisfied for some $(x_0, y_0) \in \mathbb{T}^2$.*

(i) *If $f(\cdot, y_0) \in L^1(\mathbb{T})$ and*

$$u^{-1}[f(x_0 + u, y_0) - f(x_0, y_0)] \in L^1(\mathbb{T}),$$

akkor $\widetilde{S}^{(1,0)}_{m_1,n_1;m_2,n_2}(f; x_0, y_0) \to f(\cdot, y_0)^{\sim}(x_0)$ as $m_j \to -\infty$ and $n_j \to \infty$ $(j = 1, 2)$.

(ii) *If $f(x_0, \cdot) \in L^1(\mathbb{T})$ and*

$$v^{-1}[f(x_0, y_0 + v) - f(x_0, y_0)] \in L^1(\mathbb{T}),$$

then $\widetilde{S}^{(0,1)}_{m_1,n_1;m_2,n_2}(f; x_0, y_0) \to f(x_0, \cdot)^{\sim}(y_0)$ as $m_j \to -\infty$ and $n_j \to \infty$ $(j = 1, 2)$.

Extension of Telyakovskii's theorem to function in two variables

Telyakovskii's theorem is extended as follows [12, Theorem 3].

Theorem 2.6. *Let $m_1 = 1 < m_2 < \dots < m_p < \dots$ and $n_1 = 1 < n_2 < \dots < n_q < \dots$ be sequences of natural numbers such that the conditions*

$$\sum_{p=p_0}^{\infty} \frac{1}{m_p} \le \frac{A}{m_{p_0}}, \quad p_0 = 1, 2, \dots, \tag{24}$$

$$\sum_{q=q_0}^{\infty} \frac{1}{n_q} \le \frac{B}{n_{q_0}}, \quad q_0 = 1, 2, \dots, \tag{25}$$

are satisfied, where $A, B > 1$ are constants. If a periodic function f is of bounded variation over the rectangle $[-\pi, \pi] \times [-\pi, \pi]$ in the sense of Hardy and Krause (see [4]), then the following estimate holds for all natural numbers p_0, q_0 and all points (x, y):

$$\begin{aligned}\sum_{p=p_0}^{\infty} \sum_{q=q_0}^{\infty} \max_{m_p \le m \le M < m_{p+1}} \max_{n_q \le n \le N < n_{q+1}} \left| \sum_{|k|=m}^{M} \sum_{|l|=n}^{N} \hat{f}(k, l) e^{i(kx+ly)} \right| \le \\ \le \frac{(\pi + 4)^2 AB}{m_{p_0} n_{q_0}} \sum_{k=1}^{m_{p_0}} \sum_{l=1}^{n_{q_0}} V\left(\varphi_{xy}, \left[0, \frac{\pi}{k}\right] \times \left[0, \frac{\pi}{l}\right]\right),\end{aligned} \tag{26}$$

$$\begin{aligned}\sum_{p=p_0}^{\infty} \max_{m_p \le m \le M < m_{p+1}} \left| \sum_{|k|=m}^{M} \sum_{|l|=0}^{n_{q_0}-1} \hat{f}(k, l) e^{i(kx+ly)} \right| \le \\ \le \frac{(\pi + 4)A}{m_{p_0}} \sum_{k=1}^{m_{p_0}} V\left(\varphi_x(f(\cdot, y)), \left[0, \frac{\pi}{k}\right]\right) + \\ + \frac{(\pi + 4)^2 AB}{m_{p_0} n_{q_0}} \sum_{k=1}^{m_{p_0}} \sum_{l=1}^{n_{q_0}} V\left(\varphi_{xy}, \left[0, \frac{\pi}{k}\right] \times \left[0, \frac{\pi}{l}\right]\right)\end{aligned} \tag{27}$$

and

$$(28) \qquad \sum_{q=q_0}^{\infty} \max_{n_q \le n \le N < n_{q+1}} \left| \sum_{|k|=0}^{m_{p_0}-1} \sum_{|l|=n}^{N} \hat{f}(k,l) e^{i(kx+ly)} \right| \le$$

$$\le \frac{(\pi+4)B}{n_{q_0}} \sum_{l=1}^{n_{q_0}} V\left(\varphi_y(f(x,\cdot)), \left[0, \frac{\pi}{l}\right]\right) +$$

$$+\frac{(\pi+4)^2 AB}{m_{p_0} n_{q_0}} \sum_{k=1}^{m_{p_0}} \sum_{l=1}^{n_{q_0}} V\left(\varphi_{xy}, \left[0, \frac{\pi}{k}\right] \times \left[0, \frac{\pi}{l}\right]\right),$$

where A and B are the constants occurring in (24) and a (25), and

$$\varphi_{xy}(u,v) := f(x+u, y+v) + f(x-u, y+v) +$$
$$+f(x+u, y-v + f(x-u, y-v) - 4f(x,y), \quad (u,v) \in [0,\pi] \times [0,\pi].$$

An immediate consequence of Theorem 2.6 is the following [12, Corollary].

Corollary 2.6. *If a periodic (with period 2π) function $f(x,y)$ is of bounded variation over the rectangle $[-\pi,\pi] \times [-\pi,\pi]$ and*

$$(29) \quad f(x,y) = \frac{1}{4}[f(x-0,y-0) - f(x-0,y+0) - f(x+0,y-0) + f(x+0,y+0)],$$

then for all integers $m,n \ge 0$ we have

$$|S_{m,n}(f;x,y) - f(x,y)| \le \frac{C_1 A}{m+1} \sum_{k=1}^{m+1} V\left(\varphi_x(f(\cdot,y)), \left[0, \frac{\pi}{k}\right]\right) +$$

$$+\frac{C_2 B}{n+1} \sum_{l=1}^{n+1} V\left(\varphi_y(f(x,\cdot)), \left[0, \frac{\pi}{l}\right]\right) +$$

$$+\frac{C_3 AB}{(m+1)(n+1)} \sum_{k=1}^{m+1} \sum_{l=1}^{n+1} V\left(\varphi_{xy}, \left[0, \frac{\pi}{k}\right] \times \left[0, \frac{\pi}{l}\right]\right).$$

Clearly, the inequality in Theorem 2.6 is stronger than the latter one. We note that Móricz [13, Theorem 3] proved the inequality in Corollary 2.6 in a different way.

We also note that Corollary 2.6 is a two-dimensional extension of Theorem 1.3 by Bojanić. Furthermore, Theorem 2.7 below by Hardy [8], which is a two-dimensional extension of the classical Dirichlet–Jordan theorem (see, e.g., [19, Vol. I, p. 57]) can be early obtained from Corollary 2.6.

Theorem 2.7 (Hardy). *If a periodic (with period 2π) function f is of bounded variation over the rectangle $[-\pi,\pi] \times [-\pi,\pi]$ and satisfies condition (29), then its double Fourier series converges to $f(x,y)$ at each point (x,y).*

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