Weighted Tree Generating Regular Systems and Crisp-Determinization of Weighted Tree Automata

PhD Thesis

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Chapter 1

Introduction

In computer science, a tree is a widely used abstract data type. In particular, we can use trees to represent or manipulate hierarchical data. For instance, each of the following applications involves a tree-like abstract data type: the directory structure of each file system, the class-hierarchy in object-oriented programming without allowing multiple inheritance, abstract syntax trees for computer languages, parse trees in Natural Language Processing (NLP), Document Object Models ("DOM tree") of XML and HTML documents, *etc.* Interestingly, even JSON and YAML documents can be considered as trees, but they are typically represented in a different way.

In this PhD thesis we deal only with finite trees over ranked alphabets. A *ranked alphabet* Σ is a finite and nonempty set of symbols in which we associate with each symbol a unique rank, *i.e.*, a nonnegative integer. For each nonnegative integer k, we denote the set of all symbols in Σ of rank k by $\Sigma^{(k)}$. Then a *tree over* Σ is a finite, labeled, and ordered tree such that if a node of the tree has k children, then that node is labeled by an element of $\Sigma^{(k)}$. The set of all trees over Σ is denoted by T_Σ. Furthermore, each subset of T_Σ is called a *tree language over* Σ.

The classical model of finite-state tree automata (for short: fta) [62, 69, 71, 72] was invented to recognize a tree language over some ranked alphabet. An *fta* A *over a ranked alphabet* Σ consists of a finite and nonempty set Q (states), a family $\delta = (\delta_k \mid k \text{ is an integer})$ of relations $\delta_k \subseteq Q^k \times \Sigma^{(k)} \times Q$ (k-ary transitions), and a set $F \subseteq Q$ (root states). Then a *tree* ξ *over* Σ *is recognized by* A if we can associate to each node of ξ a state in the following way: (1) if a node is labeled by a symbol $\sigma \in \Sigma^{(k)}$ and the states associated to that node and its k children are q and $q_1, \ldots, q_k$, respectively, then $(q_1, \ldots, q_k, \sigma, q)$ is a k-ary transition in δ_k and (2) the state associated to the root of ξ is a root state. The tree language recognized by A is called a *recognizable tree language*. Moreover, two fta are said to *equivalent* if they recognize the same tree language. It is well known that with fta qualitative properties of recognizable tree languages can be described, such as emptiness, finiteness, *etc.*. For surveys on the theory of fta we refer to [22, 34, 43].

In parallel and later, further concepts were introduced and proved to be equivalent to fta such as tree generating regular systems (for short: tgrs) [18]; rational tree languages [34, 43, 73]; monadic second-order logic for trees [24, 73]; regular tree grammars [18, 43]; representable tree languages [43]. It is also known that a tree language is recognizable by an fta if and only if it is the image of a local tree language under a deterministic tree relabeling [34, 43, 71].

Later the idea came up to describe not only qualitative but also quantitative properties of recognizable tree languages, like degree of ambiguity or costs of acceptance. Clearly, each tree language can be considered as a mapping from the set of input trees to the Boolean semiring $\{0, 1\}$. Moreover, by replacing the Boolean semiring in such a mapping by any other semiring B, and allowing that the mapping associates arbitrary elements of B to the trees, a way was opened to describe also those quantitative properties. More precisely, each quantitative property can be interpreted as a mapping from the set of input trees to some semiring or more generally to the carrier set of some weight structure. Mappings describing quantitative properties of tree languages are called *weighted tree languages* (or *formal power series over trees*). To recognize such weighted tree languages, the model of weighted tree automata (for short: wta) was invented. The concept of wta is a natural extension of the concept of fta by adding weights to each transition and to each state; then the operations of the weight algebra allow to combine the transition weights while processing the input tree. The first such wta over a complete distributive lattice was introduced in [56] (also see [37]) under the name fuzzy tree automata. Over the years, several other weight algebras were used to enrich the expressive power of wta: *e.g.*, fields [10], commutative semirings [7], multioperator monoids [39, 40, 59, 60], strong bimonoids [1, 3, 66], and tree-valuation monoids [28]. In this thesis we will consider the model of wta over strong bimonoids. A strong bimonoid [21, 30, 32, 66] is basically a semiring in which the distributivity laws need not hold.

The theory of wta has a huge literature. Several questions have been studied throughout the years, *e.g.*, the pumping lemma for wta [13] and the determinization problem for wta [16, 19, 41]. Furthermore, similarly to the unweighted case, additional concepts were invented and shown to be equivalent to wta, see *e.g.*, weighted regular tree grammars [7] and the Kleene theorem for wta [7, 29]; monadic second-order logic and the Büchi-Elgot-Trakhtenbrot's theorem for recognizable weighted tree languages [31, 40] (*cf.* [25–27] for the string case); and weighted representable tree languages [51, 52]. It is also known that each each weighted tree language recognized by a wta is the image of a local weighted tree language under a deterministic tree relabeling [38]. For a survey on the theory of wta we refer to [35, 41, 42].

In this thesis we will consider the following two topics. The first one is the equivalence of wta and weighted tree generating regular systems (for short: wtgrs) over semirings. In [4] the concept of wtgrs over a strong bimonoid was introduced as a natural extension of the concept of tree generating regular system (for short: tgrs) [18] to the weighted case. The semantics of wtgrs was not defined as a straightforward generalization of the original semantics of tgrs. In fact, an alternative, but essentially equivalent semantics was introduced for tgrs, of which the generalization to the weighted case opens a way to prove the equivalence of tgrs and Boolean wtgrs, and the desired equivalence of wta and wtgrs (like the equivalence of fta and tgrs in [18]). We recall the main results of [4] in Chapter 4 (*cf.* Theorems 4.2.8 and 4.3.4 and 4.4.5).

The second topic is the crisp-determinization problem. The *determinization problem* shows up if we wish to specify a problem (*e.g.*, a tree language) in a nondeterministic way and to calculate its solution (*e.g.*, membership) in a deterministic way. More precisely, the determinization problem asks the following: for a given nondeterministic device $\mathcal{A}$ of a given type (or class), does there exist a bottom-up deterministic device $\mathcal{A}'$ of the same type which is equivalent to $\mathcal{A}'$?

It is well known that the determinization problem is solved positively for the class of all fta (*cf.*, *e.g.*, [73, Thm. 1], [34, Thm. 3.8] , and [43, Thm. 2.2.6]), *i.e.*, for each fta A, there is an equivalent bottom-up deterministic fta A'. The construction of A' from A is called powerset construction. However, the situation changes drastically if we consider the class of all wta. More precisely, there exists a wta to which there does not exist an equivalent bottom-up deterministic wta [7, 10, 36, 59]. On the other side, there are subclasses of he class of all wta for which the determinization problem can be solved positively [16, Cor. 4.9 and Thm. 4.24], [41, Thm. 3.17], and [19, Thm. 5.2].

A special case of determinization of wta is when we require that the resulting deterministic wta is crisp-deterministic. We call a wta over a strong bimonoid *crisp-deterministic* if it is total and bottom-up deterministic, and each of its transitions carries either the additive or the multiplicative unit of the underlying strong bimonoid; weights different from these units may only appear at the root of the given input tree. Then the *crisp-determinization problem* (of wta over strong bimonoids) deals with the following question: for a given wta $\mathcal{A}$, does there exist a crisp-deterministic wta $\mathcal{A}'$ such that $\mathcal{A}'$ is equivalent to $\mathcal{A}$? If the answer to this question is "yes", *i.e.*, such a wta $\mathcal{A}'$ exists, then we say that the wta $\mathcal{A}$ is *crisp-determinizable*.

Without doubt, the notion of crisp-deterministic wta is quite restrictive. However, in spite of this fact, it is worth to investigate crisp-deterministic wta as they have a strong relationship with fuzzy questions (*cf.* [42, Ch. 19]). In Chapter 5, we give a sufficient condition for wta to be crisp-determinizable (*cf.* Theorems 5.2.8 and 5.2.12), and also we prove both undecidability and decidability results regarding the crisp-determinization problem (*cf.* Theorems 5.3.7 and 5.3.14 and 5.4.15). To prove the decidability results, we need new pumping lemmas for wta. We show them in Section 3.2 (*cf.* Theorems 3.2.3 and 3.2.4).

We close the introduction with the following notes to the reader. In this thesis we will give several constructions, each of them takes some object(s) as input and delivers some other object(s) as output. We will use the expression "can construct" in the following sense. There is an algorithm, *i.e.*, a finite number of exact, finite instructions, such that for each input given effectively[1], the algorithm terminates after a finite number of steps and produces a correct output.

Finally, in order to avoid many repetitions of similar conditions like "Let Σ be a ranked alphabet." or "Let B be a strong bimonoid", *etc.*, we will use general conventions to fix such conditions (*cf.* p. 15 and 21, *etc.*) Moreover, in order to emphasize the main results of the thesis, we put them into a gray colored box. For such statements, we do not use the mentioned conventions but show all the necessary conditions inside the box.

[1]It is difficult to give an exact definition of the expression "given effectively" as its definition crucially depends on the class of input objects and on the algorithm for which we define it. Hence, in this thesis the expression "given effectively" means that each restriction in the smallest set of reasonable restrictions, which are crucial for the termination of the algorithm on each input, is satisfied.

Chapter 2

Preliminaries

In this chapter we recall fundamental notions and notations. In most of the cases, we follow the formalism used in the corresponding parts of [42]. We organize this chapter as follows. In Section 2.1 we introduce general notations and recall basic concepts. In Section 2.2 we recall some fundamental notions and notations from the theory of formal tree languages, and then, in Section 2.3 from the theory of finite-state tree automata and recognizable tree languages. Finally, in Section 2.4, we recall basic notions and notations from the theory of universal algebra, and consider particular algebras, which satisfy certain algebraic laws.

2.1 Basic concepts

General notations. We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ of natural numbers, by $\mathbb{N}_+$ the set $\mathbb{N} \setminus \{0\}$, and by $\mathbb{Z}$ the set of integers. For every $m, n \in \mathbb{N}$, we define $[m, n] = \{i \in \mathbb{N} \mid m \leq i \leq n\}$. Moreover, we abbreviate $[1, n]$ by $[n]$. Note that $[0] = \emptyset$.

For every $a, b \in \mathbb{N}$, we denote by $\max(a, b)$ and $\min(a, b)$ the maximum and the minimum of a and b with respect to $\leq$, respectively. In the usual way, we extend $\max$ and $\min$ to each finite subset of $\mathbb{N}$. For each $N \subseteq \mathbb{N}$, we denote by $\max(N)$ (respectively, $\min(N)$) the maximum (the minimum, respectively) of N.

Sometimes we use the set $\mathbb{N} \cup \{\infty\}$. We abbreviate that set by $\mathbb{N}_\infty$, and, in the natural way, we extend the operations $+$ and $\min$ to $\mathbb{N}_\infty$, *i.e.*, we set $a + \infty = \infty + a = \infty$ and $\min(a, \infty) = \min(\infty, a) = a$ for each $a \in \mathbb{N}_\infty$. In a similar way we proceed with an extension of $\mathbb{N}$ by $-\infty$ and the operations $+$ and $\max$.

Let A be a set. We denote by $|A|$ the *cardinality of* A, by $\mathcal{P}(A)$ the *set of all subsets of* A, and by $\mathcal{P}_{\text{fin}}(A)$ the *set of all finite subsets of* A. Evidently, if A is finite, then we have $\mathcal{P}(A) = \mathcal{P}_{\text{fin}}(A)$.

Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be sets. The *Cartesian product of* $A_1, \ldots, A_n$, denoted by $A_1 \times \ldots \times A_n$, is the set $\{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for each } i \in [n]\}$. Moreover, the *$n$-fold Cartesian product of* A is the Cartesian product of $A \times \ldots \times A$, where A appears exactly

n times. As usual, we abbreviate $A \times \ldots \times A$ by A^n. In particular, $A^0 = \{()\}$. Let V be a subset of $A_1 \times \ldots \times A_n$. Sometimes we call V a *relation (on $A_1, \ldots, A_n$)*. Moreover, for each $i \in [n]$, we call the mapping $\mathrm{pr}_i : V \to A_i$ defined, for each $(a_1, \ldots, a_n) \in V$, by $\mathrm{pr}_i(a_1, \ldots, a_n) = a_i$, the *$i$th projection mapping of V into A_i*.

Strings. A *string over A* is a finite sequence $w = a_1 \cdots a_n$ with $n \in \mathbb{N}$ and $a_i \in A$ for each $i \in [n]$. In this case we call n the *length of the string* w, and denote it also by $\mathrm{len}(w)$. For every $k, \ell \in [n]$ with $k < \ell$, we define $w(k) = a_k$ and $w(k \ldots \ell) = a_k \cdots a_\ell$. The *empty string*, denoted by ε, is the string of length 0. For each $n \in \mathbb{N}$, we denote by A^n the *set of strings over A of length n*. Moreover, we denote by A^* the *set of all strings over A*, *i.e.*, we have $A^* = \bigcup_{n \in \mathbb{N}} A^n$. Furthermore, for every strings v and w in A^*, we denote by vw the *concatenation of v and w* and by $\mathrm{prefix}(v)$ the *set* $\{v' \in A^* \mid (\exists u \in A^*) : v = v'u\}$ *of prefixes of* v.

Observe that, for each $n \in \mathbb{N}$, the notation A^n is overloaded in the following sense: it denotes both (a) the n-fold Cartesian product of A and (b) the set of strings over A of length n. Of course, formally these sets are different, but since there exists a bijection between them, we find it acceptable to use the same notation.

An *alphabet* is a finite and nonempty set. Let A be an alphabet. Then each subset $L \subseteq A^*$ is called a *(formal) language over A*. For every languages L_1 and L_2 over A, the *concatenation of L_1 and L_2*, denoted by $L_1 \cdot L_2$, is the language

$$L_1 \cdot L_2 = \{w_1 w_2 \mid w_1 \in L_1, w_2 \in L_2\}$$

over A.

Binary relations. Let A and B be sets. A *binary relation (on A and B)* is a subset of $A \times B$. Let R be a binary relation on A and B. For each pair (a, b) in $A \times B$, we sometimes write aRb to indicate that $(a, b) \in R$. Moreover, for each $a \in A$, we define $R(a) = \{b \in B \mid aRb\}$, and furthermore, for each $A' \subseteq A$, we set $R(A') = \bigcup_{a \in A'} R(a)$. The *inverse of R*, denoted by R^{-1}, is the binary relation $\{(b, a) \mid aRb\}$ on $B \times A$. If $A = B$, then we call R a *binary relation (on A)*.

Let R be a binary relation on A. We say that R is

- *reflexive* if aRa for every $a \in A$,
- *symmetric* if aRb implies that bRa for every $a, b \in A$,
- *antisymmetric* if aRb and bRa imply that $a = b$ for every $a, b \in A$, and
- *transitive* if aRb and bRc imply that aRc for every $a, b, c \in A$.

We call R an *equivalence relation (on A)* if it is reflexive, symmetric, and transitive. If R is an equivalence relation, then, for each $a \in A$, the *equivalence class of a with respect to R*, denoted by $[a]_R$, is the set $[a]_R = \{b \in A \mid aRb\}$, and furthermore, the *factor set of A modulo R*, denoted by A/R, is the set $\{[a]_R \mid a \in A\}$.

A binary relation on A is said to be a *partial ordering (on A)* if it is reflexive, antisymmetric, and transitive. Let $\preceq$ be a partial ordering on A. For every $(a, b) \in A^2$, we denote by $a \prec b$ that $a \preceq b$ and $a \neq b$. We call $(A, \preceq)$ a *partially ordered set (with respect to $\preceq$)* (for short: *poset*). Let $A' \subseteq A$ and $p \in A$. We say that p is an *upper bound of A'* (a *lower bound of A'*) if $a \preceq p$ (respectively, $p \preceq a$) for each $a \in A'$. We call p the *supremum of A' (with respect to $\preceq$)*, denoted by $\sup_{\preceq}(A')$, if p is an upper bound of A' and $p \preceq b$ for every upper bound b of A'. We say that p is the *infimum of A' (with respect to $\preceq$)*, denoted by $\inf_{\preceq}(A')$, if p is a lower bound of A' and $b \preceq p$ for each lower bound b of A'.

A partial ordering $\preceq$ on A is called a *linear ordering (on A)* if, for each $(a, b) \in A^2$, we have $a \preceq b$ or $b \preceq a$. Let $\preceq$ be a linear ordering on A. For every subset $A' \subseteq A$ and element $a \in A'$, we say that a *is minimal in* A' if $a = \inf_{\preceq}(A')$. Clearly, if A' is finite and nonempty, then there exists a unique minimal element in A', which we denote by $\min_{\preceq}(A')$.

Mappings. Let $B \neq \emptyset$ and $f \subseteq A \times B$. We say that f is a *mapping (from A to B)*, denoted by $f : A \to B$, if, for each $a \in A$, there exists a unique $b \in B$ such that afb. In this case we write $f(a) = b$ as usual. Let $f : A \to B$ be a mapping. For every $a \in A$ and $b \in B$, if we have $f(a) = b$, then sometimes we denote this fact also by $a \overset{f}{\mapsto} b$ or just by $a \mapsto b$ if f is clear from the context. We say that f is

- *injective* if $a \neq b$ implies that $f(a) \neq f(b)$ for every $a, b \in A$,
- *surjective* if, for each $b \in B$, there exists $a \in A$ such that $f(a) = b$,
- *bijective* if f is injective and surjective.

Note that if f is bijective, then $|A| = |B|$.

The *image of f*, denoted by $\mathrm{im}(f)$, is the set $\mathrm{im}(f) = \{f(a) \mid a \in A\}$. For each subset $A' \subseteq A$, the *restriction of f to A'*, denoted by $f|_{A'}$, is the mapping $f|_{A'} : A' \to B$ defined, for each $a \in A'$, by $f|_{A'}(a) = f(a)$. We denote the *set of all mappings from A to B* by B^A. For every mappings $f_1, f_2 \in B^A$, we write $f_1 = f_2$ if, for each $a \in A$, we have $f_1(a) = f_2(a)$.

Let C be a nonempty set, and $g : B \to C$. The *composition of f and g*, denoted by $g \circ f$, is the mapping $(g \circ f) : A \to C$ defined, for each $a \in A$, by $(g \circ f)(a) = g(f(a))$.

Let A be nonempty. For every $k \in \mathbb{N}$ and mapping $h : A^k \to A$, we say that h is a *k-ary operation on A*. For each $k \in \mathbb{N}$, we denote by $\mathrm{Ops}^{(k)}(A)$ the *set of all k-ary operations on A*, and furthermore, by $\mathrm{Ops}(A)$ the set $\bigcup_{k \in \mathbb{N}} \mathrm{Ops}^{(k)}(A)$. The *identity mapping on A*, denoted by id_A, is the mapping $\mathrm{id}_A \in \mathrm{Ops}^{(1)}$ defined, for each $a \in A$, by $\mathrm{id}_A(a) = a$.

For every subsets $A' \subseteq A$ and $O \subseteq \mathrm{Ops}(A)$, we say that A' is *closed under the operations in O* if, for every $k \in \mathbb{N}$, k-ary operation $h \in O$, and $(a_1, \ldots, a_k) \in (A')^k$, we have $h(a_1, \ldots, a_k) \in A'$. We denote by $\langle A' \rangle_O$ the smallest subset of A, of which A' is subset and which is closed under the operations in O.

Let I be a set. An *I-indexed family over A* (or just: a family over A) is a mapping $f : I \to A$. Let f be an I-indexed family over A. Sometimes we denote f also by $(a_i \mid i \in I)$ where $a_i = f(i)$ for each $i \in I$. We say that f is *finite* if I is finite. Moreover, we also say that I is the index set of f. Let $f = (A_i \mid i \in I)$ be an I-indexed family over $\mathcal{P}(A)$. We say that f is a *a partitioning of A (with respect to I)* if we have $\bigcup_{i \in I} A_i = A$, and $A_i \cap A_j = \emptyset$ for every $i, j \in I$ with $i \neq j$.

2.2 Trees and contexts

Here we recall some fundamental notions and notations from the theory of formal tree languages [22, 34, 43].

Ranked sets. A *ranked set* is a pair $(\Sigma, \mathrm{rk}_\Sigma)$ in which Σ is a finite (and possibly empty) set and $\mathrm{rk}_\Sigma : \Sigma \to \mathbb{N}$ is a mapping, called *rank mapping*. Let $(\Sigma, \mathrm{rk}_\Sigma)$ be a ranked set. For each $k \in \mathbb{N}$, the *set of all k-ary symbols in Σ*, denoted by $\Sigma^{(k)}$, is the set $\Sigma^{(k)} = \{\sigma \in \Sigma \mid \mathrm{rk}_\Sigma(\sigma) = k\}$. Sometimes we write $\sigma^{(k)}$ to indicate that $\sigma \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$. We define $\mathrm{maxrk}(\Sigma) = \max\{k \in \mathbb{N} \mid \Sigma^{(k)} \neq \emptyset\}$. Each ranked set $(\Sigma, \mathrm{rk}_\Sigma)$, in which Σ is an alphabet, is called a *ranked alphabet*. Furthermore, a ranked alphabet $(\Sigma, \mathrm{rk}_\Sigma)$ is called a *string ranked alphabet* if $\Sigma = (\Sigma^{(1)} \cup \Sigma^{(0)})$, $|\Sigma^{(1)}| \geq 1$, and $|\Sigma^{(0)}| = 1$.

In the rest of this PhD thesis, Σ will abbreviate an arbitrary ranked set $(\Sigma, \mathrm{rk}_\Sigma)$ such that $\Sigma^{(0)} \neq \emptyset$ if not specified otherwise.

Trees and tree languages. Let H be a set disjoint from Σ. The *set of Σ-trees over H*, denoted by $\mathrm{T}_\Sigma(H)$, is the smallest set T such that

(i) H is a subset of T and

(ii) if $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in T^k$, then $\sigma(\xi_1, \ldots, \xi_k) \in T$.

If H is clear from the context, then we refer to each element of $\mathrm{T}_\Sigma(H)$ as a *Σ-tree* (or just: *tree*). As usual, for each $\alpha \in \Sigma^{(0)}$, we sometimes abbreviate the tree $\alpha()$ by α. Moreover, we write T_Σ for $\mathrm{T}_\Sigma(\emptyset)$. We call each subset L of T_Σ a *Σ-tree language* (or just: *tree language*).

The *set of positions* (or: Gorn addresses [46]) of trees is defined by the mapping $\mathrm{pos} : \mathrm{T}_\Sigma(H) \to \mathcal{P}_{\mathrm{fin}}(\mathbb{N}_+^*)$, where, for each $\xi \in \mathrm{T}_\Sigma(H)$, we define $\mathrm{pos}(\xi)$ as follows:

(i) if ξ is in $\Sigma^{(0)} \cup H$, then we let $\mathrm{pos}(\xi) = \{\varepsilon\}$ and

(ii) if $\xi = \sigma(\xi_1, \ldots, \xi_k)$ for some $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in \mathrm{T}_\Sigma(H)$, then we let $\mathrm{pos}(\xi) = \{\varepsilon\} \cup \{iv \mid i \in [k], v \in \mathrm{pos}(\xi_i)\}$.

The *height* and the *size* of a tree $\xi \in \mathrm{T}_\Sigma(H)$ are $\mathrm{height}(\xi) = \max(\{\mathrm{len}(v) \mid v \in \mathrm{pos}(\xi)\})$ and $\mathrm{size}(\xi) = |\mathrm{pos}(\xi)|$, respectively.

Let ξ and ζ be in $\mathrm{T}_\Sigma(H)$, and $v \in \mathrm{pos}(\xi)$. The *label of* ξ *at* v, denoted by $\xi(v)$, the *subtree of* ξ *at* v, denoted by $\xi|_v$, and the *replacement of the subtree of* ξ *at* v *by* ζ, denoted by $\xi[\zeta]_v$, are defined as follows:

(i) if ξ is in $\Sigma^{(0)} \cup H$, then $v = \varepsilon$ and we let $\xi(\varepsilon) = \xi|_\varepsilon = \xi$, and $\xi[\zeta]_\varepsilon = \zeta$ and

(ii) if $\xi = \sigma(\xi_1, \ldots, \xi_k)$ for some $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in \mathrm{T}_\Sigma(H)$, then
- for $v = \varepsilon$, we let $\xi(\varepsilon) = \sigma$, $\xi|_\varepsilon = \xi$, and $\xi[\zeta]_\varepsilon = \zeta$ and
- for $v = iv'$ with $i \in [k]$ and $v' \in \mathrm{pos}(\xi_i)$, we let $\xi(v) = \xi_i(v')$, $\xi|_v = \xi_i|_{v'}$, and $\xi[\zeta]_v = \sigma(\xi_1, \ldots, \xi_{i-1}, \xi_i[\zeta]_{v'}, \xi_{i+1}, \ldots, \xi_k)$.

Let σ be in $\Sigma \cup H$. We denote the set $\{v \in \mathrm{pos}(\xi) \mid \xi(v) = \sigma\}$ by $\mathrm{pos}_\sigma(\xi)$. Moreover, we say that σ *occurs in* ξ if $\mathrm{pos}_\sigma(\xi) \neq \emptyset$.

Contexts. Let $\square$ be a symbol such that $\square \notin \Sigma$. For each $\xi \in \mathrm{T}_\Sigma(\{\square\})$ and each $v \in \mathrm{pos}(\xi)$, we abbreviate the tree $\xi[\square]_v$ by $\xi|^v$. The *set of all* Σ*-contexts*, denoted by C_Σ, is the set $\mathrm{C}_\Sigma = \{\xi \in \mathrm{T}_\Sigma(\{\square\}) \mid \mathrm{pos}_\square(\xi) = 1\}$. Hence, each Σ-context is a Σ-tree over the set $\{\square\}$ in which $\square$ appears precisely once, as a leaf. We mention that the set C_Σ of all Σ-contexts can be defined also inductively as follows:

(i) $\square$ is in C_Σ,

(ii) $\sigma(\xi_1, \ldots, \xi_{i-1}, c, \xi_{i+1}, \ldots, \xi_k)$ is in C_Σ whenever $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, $i \in [k]$, $(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^{k-1}$, and $c \in \mathrm{C}_\Sigma$, and

(iii) every Σ-context can be obtained by applying the rules (i) and (ii) a finite number of times.

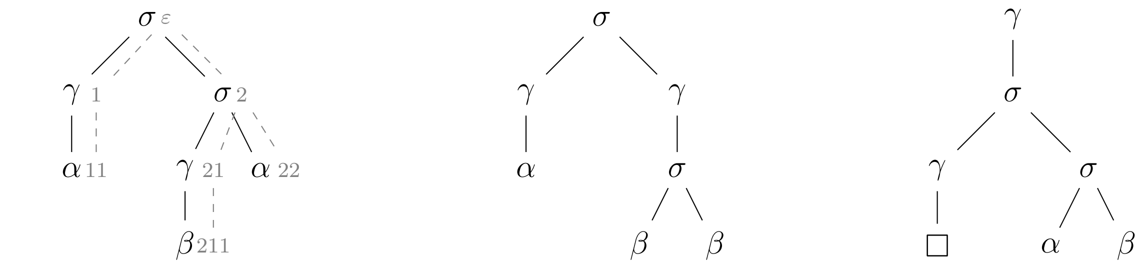
Let $c \in \mathrm{C}_\Sigma$ with $\{v\} = \mathrm{pos}_\square(c)$, and ζ be in $\mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$. We abbreviate $c[\zeta]_v$ by $c[\zeta]$. Thus, we obtain $c[\zeta]$ from the Σ-context c by replacing the leaf $\square$ by ζ. Clearly, if ζ is a Σ-context, then so is $c[\zeta]$. Moreover, for each $n \in \mathbb{N}$, we define the n*-th power of* c, denoted by c^n, by induction as follows: $c^0 = \square$ and $c^{n+1} = c[c^n]$.

Example 2.2.1. Let $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$. Figure 2.1 shows Σ-trees over $\{\square\}$ as follows. Figure 2.1(a) illustrates the Σ-tree $\xi = \sigma(\gamma(\alpha), \sigma(\gamma(\beta), \alpha))$ together with its positions in gray color and dashed lines. Observe that, *e.g.*, we have $\xi(2) = \sigma$, $\xi|_2 = \sigma(\gamma(\beta), \alpha)$, and $\mathrm{pos}_\sigma(\xi) = \{\varepsilon, 2\}$, *i.e.*, σ occurs in ξ. Figure 2.1(b) depicts the replacement of the subtree of ξ at position 2 by the Σ-tree $\gamma(\sigma(\beta, \beta))$. Finally, Figure 2.1(c) shows the Σ-context $\gamma(\sigma(\gamma(\square), \sigma(\alpha, \beta)))$. $\triangle$

2.3 Finite-state tree automata

In this section we recall basic notions and notations from the theory of finite-state tree automata and recognizable tree languages from [22, 34, 43].

Finite-state tree automata. A *finite-state tree automaton over* Σ (for short: Σ-fta, or just fta) [34, 43] is a triple $A = (Q, \delta, F)$ where



(a) *a Σ-tree together with its positions* **(b)** *replacement of the subtree $\xi|_2$ by the Σ-tree $\gamma(\sigma(\beta, \beta))$* **(c)** *a Σ-context*

Figure 2.1. *Illustrations of the Σ-trees over $\{\square\}$ given in Example 2.2.1*

- Q is a finite nonempty set (*states*) such that $Q \cap \Sigma = \emptyset$,
- $\delta = (\delta_k \mid k \in \mathbb{N})$ is a family of relations $\delta_k \subseteq Q^k \times \Sigma^{(k)} \times Q$ (*transition relations*), where we consider Q^k as a set of strings over Q of length k, and
- $F \subseteq Q$ (*set of root states*).

Let $A = (Q, \delta, F)$ be a Σ-fta. We call A *total* (respectively, *bottom-up deterministic* or for short: *bu deterministic*) if, for every $k \in \mathbb{N}$, $w \in Q^k$, and $\sigma \in \Sigma^{(k)}$, there exists at least (respectively, at most) one $q \in Q$ such that $(w, \sigma, q) \in \delta_k$.

Semantics. We mention that, for fta, two semantics can be defined: the initial algebra semantics [34, 42, 43] and the run semantics [22, 42]. We recall that the two kinds of semantics coincide, *cf.*, *e.g.*, [42, Lm. 2.13.1]. In this thesis we use only the run semantics.

For this, let $\xi \in \mathrm{T}_\Sigma$. A *run of A on ξ* is a mapping $\rho : \mathrm{pos}(\xi) \to Q$. Let ρ be a run of A on ξ, and $q \in Q$. We say that ρ is

- a *q-run* if $\rho(\varepsilon) = q$,
- *valid* if, for every $v \in \mathrm{pos}(\xi)$, it holds that $\big(\rho(v1) \cdots \rho(vk), \xi(v), \rho(v)\big) \in \delta_k$ where $\xi(v) \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$, and
- *accepting* if ρ is valid and $\rho(\varepsilon) \in F$.

We denote the *set of all q-runs* (*all valid q-runs*, *all accepting q-runs*) *of A on ξ* by $\mathrm{Run}_A(q, \xi)$ (respectively, $\mathrm{Run}^{\mathrm{v}}_A(q, \xi)$ and $\mathrm{Run}^{\mathrm{a}}_A(q, \xi)$). Moreover, we set

$$\mathrm{Run}_A(\xi) = \bigcup_{q \in Q} \mathrm{Run}_A(q, \xi) \quad , \quad \mathrm{Run}^{\mathrm{v}}_A(\xi) = \bigcup_{q \in Q} \mathrm{Run}^{\mathrm{v}}_A(q, \xi) \quad , \quad \text{and}$$

$$\mathrm{Run}^{\mathrm{a}}_A(\xi) = \bigcup_{q \in Q} \mathrm{Run}^{\mathrm{a}}_A(q, \xi) \ .$$

Then the *semantics of* A, denoted by $\mathrm{L}(A)$, is the Σ-tree language defined by

$$\mathrm{L}(A) = \{\xi \in \mathrm{T}_\Sigma \mid \mathrm{Run}^{\mathrm{a}}_A(\xi) \neq \emptyset\} \ .$$

We say that two Σ-fta A and A' are *equivalent* if $\mathrm{L}(A) = \mathrm{L}(A')$. Furthermore, a Σ-tree language L is *recognizable* if there exists a Σ-fta A such that $\mathrm{L}(A) = L$. For the theory of recognizable Σ-tree languages we refer to [22, 34, 43]. Next we recall two well known results on Σ-fta.

Lemma 2.3.1. *cf.* [43, Thm. 2.4.2] For every recognizable Σ-tree languages L_1 and L_2, also the Σ-tree language $L_1 \cup L_2$ is recognizable. □

Lemma 2.3.2. *cf.* [43, Thm. 2.2.6] For each Σ-fta A, we can construct a total and bu deterministic Σ-fta A' such that A and A' are equivalent. □

Now we recall that each Σ-fta can be depicted as a particular Σ-hypergraph.

Fta-hypergraphs. Formally, a Σ-*hypergraph* (*cf.* [9, 23, 49]) is a pair $g = (Q, E)$, where Q is a finite set (*nodes*) and $E \subseteq \bigcup_{k \in \mathbb{N}} Q^k \times \Sigma^{(k)} \times Q$ is a finite set (*hyperedges*). For two Σ-hypergraphs $g = (Q, E)$ and $g' = (Q', E')$, we write $g \subseteq g'$ (respectively, $g = g'$) if we have $Q \subseteq Q'$ and $E \subseteq E'$ (respectively, $Q = Q'$ and $E = E'$).

We can illustrate a Σ-hypergraph as a picture in the following way *cf.* [42]. Let $g = (Q, E)$ be a Σ-hypergraph. We represent each node $q \in Q$ as a circle with q in its center. Furthermore, we depict each hyperedge $(q_1 \cdots q_k, \sigma, q) \in E$ as a box with σ in its center; this box has exactly one outgoing arrow that leads to the representation of the node q, and it has k incoming arrows which come from the representations of the nodes $q_1, \ldots, q_k$, respectively. The string $q_1 \cdots q_k$ determines the order among $q_1, \ldots, q_k$ as follows: starting from the unique outgoing arrow and moving counter-clockwise around the box, the i-th incoming arrow comes from the representation of the i-th component of the string $q_1 \cdots q_k$.

Finally, for each Σ-fta $A = (Q, \delta, F)$, the *fta-hypergraph of* A, denoted by g_A, is the Σ-hypergraph $g_A = (Q, \bigcup_{k \in \mathbb{N}} \delta_k)$.

Example 2.3.3. Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-fta

$$A = (\{q_l, q_r, q_f, q_t\}, \delta, \{q_f\})$$

with transition relations $\delta_2 = \{(q_l q_r, \sigma, q_f), (q_r q_r, \sigma, q_t), (q_f q_r, \sigma, q_f), (q_t q_r, \sigma, q_t)\}$, $\delta_0 = \{(\varepsilon, \alpha, q_l), (\varepsilon, \alpha, q_r)\}$, and $\delta_k = \emptyset$ for every $k \in (\mathbb{N} \setminus \{0, 2\})$. Clearly, since both $(\varepsilon, \alpha, q_l)$ and $(\varepsilon, \alpha, q_r)$ are in δ_0, the Σ-fta A is not bu deterministic. Moreover, since, for the string $q_l q_f$ in Q^2 and $\sigma \in \Sigma^{(2)}$, there does not exist $q \in \{q_l, q_r, q_f, q_t\}$ such that $(q_l q_f, \sigma, q) \in \delta_2$, the fta A is not total either. Figure 2.2 depicts the fta-hypergraph of A.

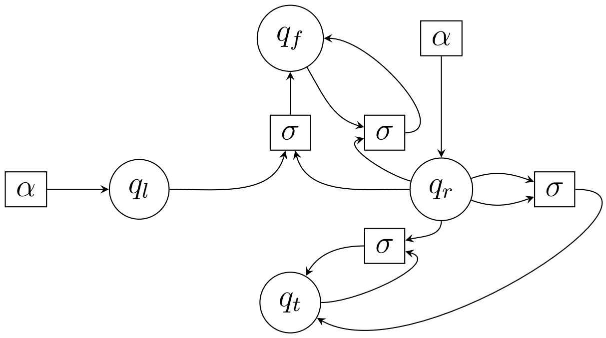
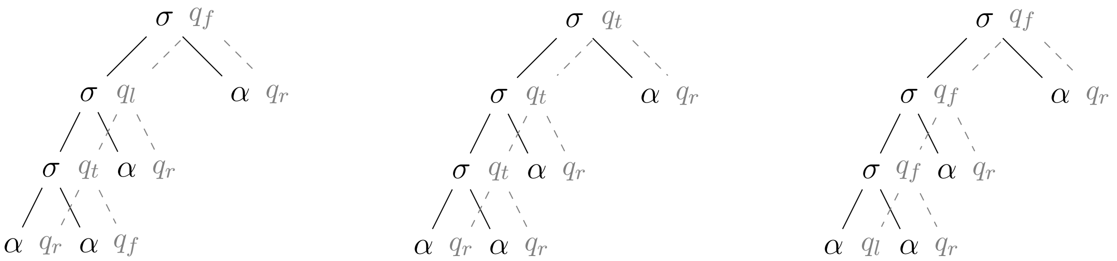


Figure 2.2. *The fta-hypergraph g_A of the Σ-fta A defined in Example 2.3.3*



(a) *a not valid q_f-run* **(b)** *a valid but not accepting q_t-run* **(c)** *an accepting q_f-run*

Figure 2.3. *Runs of the Σ-fta A defined in Example 2.3.3 on the Σ-tree ξ_3*

Next we represent runs of A as pictures as follows. Let $c = \sigma(\square, \alpha)$. Obviously, c is a Σ-context. For each $n \in \mathbb{N}_+$, we define the Σ-tree ξ_n by $\xi_n = c^n[\alpha]$. Figure 2.3 illustrates the Σ-tree ξ_3 with three runs of the Σ-fta A in gray color and dashed lines in the following way. Figure 2.3(a) shows a q_f-run ρ of A. Note that, since $(\rho(11)\rho(12), \sigma, \rho(1)) = (q_t q_r, \sigma, q_l)$ is not in δ_2, this run ρ is not valid, and thus, it is not accepting either. Figure 2.3(b) illustrates a valid q_t-run; however, since q_t is not a root state, this is not accepting either. Finally, Figure 2.3(c) depicts an accepting q_f-run.

In fact, for each $\xi \in \mathrm{T}_\Sigma$, we have $\xi \in \{\xi_n \mid n \in \mathbb{N}_+\}$ if and only if there exists a unique accepting run of A on ξ. Thus, $\mathrm{L}(A) = \{\xi_n \mid n \in \mathbb{N}_+\}$. $\triangle$

2.4 Weight structures

Here we recall some fundamental notions and notations from the theory of universal algebra [20, 47], and consider particular algebras which satisfy certain algebraic laws.

Universal algebra. A Σ-*algebra* is a pair $\mathsf{A} = (A, \theta)$ which consists of a nonempty set A (*carrier set*) and a Σ-indexed family θ over $\mathrm{Ops}(A)$ (Σ-*interpretation* or *interpretation of* Σ) such that, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have $\theta(\sigma) \in \mathrm{Ops}^{(k)}(A)$. We denote by $\theta(\Sigma)$ the set $\{\theta(\sigma) \mid \sigma \in \Sigma\}$ of operations.

Let $\mathsf{A} = (A, \theta)$ be a Σ-algebra. A *subalgebra of* A is a Σ-algebra (A', θ') such that $A' \subseteq A$, the set A' is closed under the operations in $\theta(\Sigma)$, and, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have $\theta'(\sigma) = \theta(\sigma)|_{(A')^k}$. For each $A' \subseteq A$, the *subalgebra of* A *generated by* A' is the subalgebra $(\langle A' \rangle_{\theta(\Sigma)}, \theta)$ of A. The *smallest subalgebra of* A is the subalgebra of A generated by $\emptyset$.

We say that A is *finite* if the set A is finite, and it is *locally finite*, if, for each finite subset $A' \subseteq A$, the set $\langle A' \rangle_{\theta(\Sigma)}$ is finite. Moreover, we call A *computable* if A is a recursively enumerable set with tests for equality and, for each $\sigma \in \Sigma$, the operation $\theta(\sigma)$ is computable (*e.g.* by a Turing machine).

Let $\sim$ be an equivalence relation on A. We call $\sim$ a *congruence relation on* A if, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(a_1, \ldots, a_k), (b_1, \ldots, b_k) \in A^k$, the relation $a_i \sim b_i$ for each $i \in [k]$ implies that we have $\theta(\sigma)(a_1, \ldots, a_k) \sim \theta(\sigma)(b_1, \ldots, b_k)$.

Let $\sim$ be a congruence relation on A. The *quotient algebra of* A *modulo* $\sim$ is the Σ-algebra $\mathsf{A}/\sim = (A/\sim, \theta/\sim)$, where $A/\sim$ is the factor set of A modulo $\sim$ and $\theta/\sim$ is defined, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(a_1, \ldots, a_k) \in A^k$, by $(\theta/\sim)(\sigma)([a_1]_\sim, \ldots, [a_k]_\sim) = [\theta(\sigma)(a_1, \ldots, a_k)]_\sim$. For the well-definedness of $\theta/\sim$ we refer to [47, p. 36].

Let $\mathsf{A}_1 = (A_1, \theta_1)$ and $\mathsf{A}_2 = (A_2, \theta_2)$ be Σ-algebras. Furthermore, let $h : A_1 \to A_2$ be a mapping. We say that h is a Σ-*algebra homomorphism (from* A_1 *to* A_2*)* if, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(a_1, \ldots, a_k) \in (A_1)^k$, we have

$$h(\theta_1(\sigma)(a_1, \ldots, a_k)) = \theta_2(\sigma)(h(a_1), \ldots, h(a_k)) \ .$$

If h is bijective, then h is a Σ-*algebra isomorphism*. If there exists such an isomorphism h, then we say that A_1 and A_2 are *isomorphic*, and we denote this fact by $\mathsf{A}_1 \cong \mathsf{A}_2$.

Let $\mathsf{A} = (A, \theta)$ be a Σ-algebra, and $\sim$ be a congruence on A. Then the mapping $h : A \to A/\sim$ defined, for each $a \in A$, by $h(a) = [a]_\sim$ is a Σ-algebra homomorphism from A to $\mathsf{A}/\sim$.

Next we give two examples of Σ-algebras.

Example 2.4.1. One of the well known Σ-algebras is the Σ-*term algebra* $\mathsf{Term}_\Sigma = (\mathrm{T}_\Sigma, \theta_\Sigma)$, where $\theta_\Sigma(\sigma)(\xi_1, \ldots, \xi_k) = \sigma(\xi_1, \ldots, \xi_k)$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$. △

In the rest of this thesis, if $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ for some $n \in \mathbb{N}_+$ and it is clear from the context, then we sometimes denote a Σ-algebra (A, θ) by $(A, \theta(\sigma_1), \ldots, \theta(\sigma_n))$ and refer to it as an algebra.

Example 2.4.2. We consider the algebra $\mathsf{Nt} = (\mathbb{N}, +, \cdot, 0)$, where $+$ and $\cdot$ denote the usual addition and multiplication over $\mathbb{N}$, respectively. Furthermore, we consider the algebra $\mathsf{0}_{+,\cdot} = (\{0\}, +, \cdot, 0)$, where $+$ and $\cdot$ are the usual addition and multiplication over $\mathbb{N}$ restricted to the set $\{0\}$. Clearly, $\mathsf{0}_{+,\cdot}$ is a finite subalgebra of Nt generated by the set $\{0\}$. Interestingly, $\mathsf{0}_{+,\cdot}$ coincides with the smallest subalgebra of Nt. However, the subalgebra of Nt generated by the set $\{0, 1\}$ equals Nt.

Let $k \in \mathbb{N}_+$, and $\equiv_k$ be a binary relation on $\mathbb{N}$ defined, for every $m, n \in \mathbb{N}$, by

$$m \equiv_k n \text{ iff both } m \text{ and } n \text{ give the same remainder when they are divided by } k \ .$$

Trivially, $\equiv_k$ is a congruence relation on Nt. Now we consider the quotient algebra

$$\mathsf{Nt}/{\equiv_k} \ = \ (\mathbb{N}/{\equiv_k}, +_k, \cdot_k, [0]_{\equiv_k})$$

of Nt modulo $\equiv_k$, where $\mathbb{N}/{\equiv_k} \ = \ \{[0]_{\equiv_k}, [1]_{\equiv_k}, \ldots, [k-1]_{\equiv_k}\}$, and $+_k$ and $\cdot_k$ are the usual addition and multiplication modulo k, respectively. Note that the mapping $h : \mathbb{N} \to \mathbb{N}/{\equiv_k}$ defined, for each $n \in \mathbb{N}$, by $h(n) = [n]_{\equiv_k}$, is a homomorphism from Nt to $\mathsf{Nt}/{\equiv_k}$. $\triangle$

Properties of binary operations. Let B be a nonempty set. Each element in $\mathrm{Ops}^{(2)}(B)$ is called a *binary operation*. Let $\odot$ be a binary operation on B. We say that $\odot$ is

- *associative* if $(a \odot b) \odot c = a \odot (b \odot c)$ for every $a, b, c, \in B$,
- *commutative* if $a \odot b = b \odot a$ for every $a, b \in B$, and
- *idempotent* if $a \odot a = a$ for each $a \in B$.

An element $e \in B$ is an *identity element (of $\odot$)* if $e \odot a = a \odot e = a$ for every $a \in B$. If $e \in B$ is an identity element, then it is unique.

Let $\oplus$ and $\otimes$ be two binary operations on B, and let $a, b, c \in B$. We say that $\otimes$ is

- *right distributive (with respect to $\oplus$)* if $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$,
- *left distributive (with respect to $\oplus$)* if $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, and
- *distributive (with respect to $\oplus$)* if it is both right distributive and left distributive (with respect to $\oplus$).

Furthermore, $\oplus$ and $\otimes$ satisfy the *absorption axioms* if $a \otimes (a \oplus b) = a$ and $a \oplus (a \otimes b) = a$.

Example 2.4.3. We consider the Σ-algebra Nt defined in Example 2.4.2. Obviously, both $+$ and $\cdot$ are associative and commutative operations. The identity elements of $+$ and $\cdot$ are 0 and 1, respectively. Moreover, $\cdot$ is distributive with respect to $+$. $\triangle$

Semigroups and monoids. A *semigroup* is an algebra $(B, \odot)$ such that $\odot$ is an associative binary operation on B. Moreover, a *monoid* is an algebra $(B, \odot, e)$ such that $(B, \odot)$ is a semigroup and the nullary operation e is an identity of $\odot$. A semigroup $(B, \odot)$ is *commutative* if $\odot$ is commutative. Similarly, we define commutative monoids.

Let $(B, \odot, e)$ be a monoid. We extend $\odot$ to finitely many arguments. Let I be a finite set with $I = \{i_1, \dots, i_k\}$ for some $k \in \mathbb{N}$. If (a) $I \subseteq \mathbb{N}$ and $i_1 < \dots < i_k$ or (b) $\odot$ is commutative, then we define the operation $\bigodot_I : B^I \to B$ such that, for each I-indexed family $(b_i \mid i \in I)$ of elements in B, we have

$$\bigodot_I (b_i \mid i \in I) = \begin{cases} b_{i_1} \odot \dots \odot b_{i_k} & \text{if } I \neq \emptyset \\ e & \text{otherwise} \end{cases} .$$

We abbreviate $\bigodot_I (b_i \mid i \in I)$ by $\bigodot(b_i \mid i \in I)$ or just by $\bigodot_{i \in I} b_i$. Moreover, if $I = [k]$ for some $k \in \mathbb{N}$, then sometimes we denote $\bigodot_{i \in [k]} b_i$ also by $\bigodot_{i=1}^{k} b_i$. Note that, in particular, we have $\bigodot_{i=1}^{k} b_i = b_1 \odot \dots \odot b_k$ for each $k \in \mathbb{N}_+$, and $\bigodot_{i \in \emptyset} b_i = e$.

Let $(B, \odot, e)$ be a commutative monoid. We say that $(B, \odot, e)$ is *complete* if, for each index set I, there exists a mapping $\sum_I^{\odot} : B^I \to B$ such that for each I-indexed family $(b_i \mid i \in I)$ over B, the following statements hold true (*cf.* [33, p. 124]):

- if $I = \{j\}$, then $\sum_{i \in \{j\}}^{\odot} b_i = b_j$,
- if $I = \{j, j'\}$, then $\sum_{i \in \{j, j'\}}^{\odot} b_i = b_j \odot b_{j'}$, and
- for each set J and each partitioning $(I_j \mid j \in J)$ of I, we have

$$\sum_{i \in I}^{\odot} b_i = \sum_{j \in J}^{\odot} \Big(\sum_{i \in I_j}^{\odot} b_i \Big) ,$$

where $\sum_{i \in I}^{\odot} b_i$ is an abbreviation for $\sum_I^{\odot} (b_i \mid i \in I)$. Let $(b_i \mid i \in I)$ be a finite I-indexed family of elements of B. It is easy to see that, if $(B, \odot, e)$ is complete, then we have $\sum_{i \in I}^{\odot} b_i = \bigodot_{i \in I} b_i$. Sometimes we also write $\sum_{i \in I}^{\odot} b_i$ for $\bigodot_{i \in I} b_i$ even if $(B, \odot, e)$ is not complete.

Next we give an example of a complete monoid.

Example 2.4.4. The commutative monoid $(\mathbb{N}_\infty, +, 0)$ is complete with the mapping

$$\sum_I^{+} : (\mathbb{N}_\infty)^I \to \mathbb{N}_\infty \quad \text{with}$$

$$(n_i \mid i \in I) \mapsto \begin{cases} \sum_{i \in J} n_j & \text{if } \{n_i \mid i \in I\} \subseteq \mathbb{N} \text{ and } J = \{i \in I \mid n_i \neq 0\} \text{ is finite} \\ \infty & \text{otherwise} \end{cases} .$$

△

Strong bimonoids. A *strong bimonoid* [21, 30, 32, 66] is an algebra

$$\mathsf{B} = (B, \oplus, \otimes, \mathbb{0}, \mathbb{1}) \ ,$$

where $(B, \oplus, \mathbb{0})$ is a commutative monoid, $(B, \otimes, \mathbb{1})$ is a monoid, $\mathbb{0} \neq \mathbb{1}$, and $\mathbb{0}$ is an *annihilator for* $\otimes$, *i.e.*, $b \otimes \mathbb{0} = \mathbb{0} \otimes b = \mathbb{0}$ holds true for every $b \in B$. The operations $\oplus$ and $\otimes$ are called *addition* and *multiplication*, respectively. Let $\mathsf{B} = (B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid. We say that B is

- *commutative* if $\otimes$ is commutative,
- *idempotent* if $\oplus$ is idempotent,
- *right distributive* if $\otimes$ is right distributive (with respect to $\oplus$),
- *left distributive* if $\otimes$ is left distributive (with respect to $\oplus$),
- *distributive* if it is right distributive and left distributive,
- *zero-sum free* if $a \oplus b = \mathbb{0}$ implies $a = b = \mathbb{0}$ for every $a, b \in B$,
- *zero-divisor free* if $a \otimes b = \mathbb{0}$ implies $a = \mathbb{0}$ or $b = \mathbb{0}$ for every $a, b \in B$,
- *positive* if it is zero-sum free and zero-divisor free,
- *complete* if $(B, \oplus, \mathbb{0})$ is complete,
- *additively locally finite* if $(B, \oplus, \mathbb{0})$ is locally finite,
- *multiplicatively locally finite* if $(B, \otimes, \mathbb{1})$ is locally finite, and
- *bi-locally finite* if it is additively locally finite and multiplicatively locally finite.

For every $n \in \mathbb{N}$ and $b \in B$, we define the elements nb and b^n in B by induction as follows: $0b = \mathbb{0}$ and $(n+1)b = b \oplus nb$, and $b^0 = \mathbb{1}$ and $b^{n+1} = b \otimes b^n$, respectively.

Example 2.4.5. *cf.* [30, Ex. 1] and [42, Ex. 2.6.10] Here we recall some examples of strong bimonoids.

1. The algebra $\mathsf{PlusMin} = (\mathbb{N}_\infty, +, \min, 0, \infty)$ is a commutative strong bimonoid. Moreover, it is also complete (*cf.* Example 2.4.4). However, it is not bi-locally finite. Furthermore, it is not distributive, because there exist $a, b, c \in \mathbb{N}_\infty$ such that $\min(a, b+c) \neq \min(a, b) + \min(a, c)$ (*e.g.*, take $a = b = c \neq 0$).
2. Let $(C, +, 0)$ be a commutative monoid, and $B = \{f \in \mathrm{Ops}^{(1)}(C) \mid f(0) = 0\}$. We extend $+$ to B by a pointwise addition on elements of B, *i.e.*, for every $f, g \in B$ and $c \in C$, we define $(f+g)(c) = f(c) + g(c)$. Moreover, we define the operation $\diamond$ on B such that, for every $f, g \in B$ and $c \in C$, we have $(f \diamond g)(c) = g(f(c))$. Finally, we denote by $\tilde{0}$ the mapping $\tilde{0} : C \to C$ such that $\tilde{0}(c) = 0$ for each $c \in C$. Then the algebra $(B, +, \diamond, \tilde{0}, \mathrm{id}_C)$ is a strong bimonoid. We mention that this algebra is called a *near semiring (over C)* [54, 58]. Ob-

serve that the condition $f(0) = 0$ guarantees that $(f \diamond \tilde{0}) = \tilde{0}$. Also, note that, except for trivial cases, the operation $\diamond$ is left distributive over $+$, but not right distributive.

3. We consider the algebra $(\Gamma^* \cup \{\infty\}, \wedge, \cdot, \infty, \varepsilon)$, where
   - $\wedge$ is the longest common prefix operation,
   - $\cdot$ is the usual concatenation of strings, and
   - ∞ is a new element such that $s \wedge \infty = \infty \wedge s = s$ and $s \cdot \infty = \infty \cdot s = \infty$ for each s in $\Gamma^* \cup \{\infty\}$.

   Obviously, it is a left distributive but not right distributive strong bimonoid (consider, *e.g.*, if $\Gamma = \{a, b, c\}$, then $abc = (a \wedge ab) \cdot bc \neq (a \cdot bc) \wedge (ab \cdot bc) = ab$). We note that this strong bimonoid occurs in investigations for natural language processing, see [65].
4. We recall the strong bimonoid $\mathsf{Stb} = (\mathbb{N}, \oplus, \odot, 0, 1)$ from [30, Ex. 25], where the two commutative operations $\oplus$ and $\odot$ on $\mathbb{N}$ are defined as follows. For every $a \in \mathbb{N}$, let $0 \oplus a = a$, $0 \odot a = 0$, and $1 \odot a = a$. Moreover, let

$$a \oplus b = \begin{cases} b & \text{if } b \text{ is even} \\ b+1 & \text{if } b \text{ is odd} \end{cases},$$

   for every $a, b \in \mathbb{N}_+$ with $a \leq b$, and let

$$a \odot b = \begin{cases} b+1 & \text{if } b \text{ is even} \\ b & \text{if } b \text{ is odd} \end{cases},$$

   for every $a, b \in (\mathbb{N}_+ \setminus \{1\})$ with $a \leq b$, where $+$ denotes the usual addition on $\mathbb{N}$. Clearly, it is not distributive (*e.g.* $2 \odot (2 \oplus 3) = 5 \neq 4 = (2 \odot 2) \oplus (2 \odot 3)$). Moreover, it is bi-locally finite but not locally finite, see [42, Ex. 2.6.10].

For further examples of strong bimonoids we refer to [30, Ex. 1], [21, Ex. 2.2], [32, Ex. 2.1], and [42, Ex. 2.6.10]. △

Semirings. A *semiring* [45, 50] is a distributive strong bimonoid. A semiring $\mathsf{S} = (S, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is *complete (as a semiring)* if it is complete as a strong bimonoid and the following equalities hold for every index set I, I-indexed family $(s_i \mid i \in I)$, and $s \in S$ (*cf.*, *e.g.*, [33, p. 125] and [36]):

$$\sum_{i\in I}^{\oplus} s \otimes s_i = s \otimes \Big(\sum_{i\in I}^{\oplus} s_i\Big) \quad \text{and} \quad \sum_{i\in I}^{\oplus} s_i \otimes s = \Big(\sum_{i\in I}^{\oplus} s_i\Big) \otimes s \ .$$

In the rest of this thesis, if we say that a semiring S is complete, then we mean by that S is complete as a semiring.

Example 2.4.6. *cf.* [42, Ex. 2.6.9]. Here we show some examples of semirings.

1. The *Boolean semiring* $\mathsf{Boole} = (\mathbb{B}, \vee, \wedge, 0, 1)$, where $\mathbb{B} = \{0, 1\}$ (the truth values) and $\vee$ and $\wedge$ denote disjunction and conjunction, respectively. Furthermore, it is complete with the mapping

$$\sum_I^{\vee} : \mathbb{B}^I \to \mathbb{B} \quad \text{with} \quad (b_i \mid i \in I) \mapsto \begin{cases} 1 & \text{if there exists } i \in I \text{ such that } b_i = 1 \\ 0 & \text{otherwise} \end{cases} .$$

2. The algebra $\mathsf{Nt} = (\mathbb{N}, +, \cdot, 0)$ given in Example 2.4.2 extended with the identity element 1 of $\cdot$ is, in fact, a semiring. Hence, in the rest of this thesis we write $\mathsf{Nat} = (\mathbb{N}, +, \cdot, 0, 1)$, and refer to that algebra as the *semiring of natural numbers*.
3. The *semiring* $\mathsf{Int} = (\mathbb{Z}, +, \cdot, 0, 1)$ *of integers*.
4. The semiring $\mathsf{MaxPlus} = (\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$.
5. The semiring $\mathsf{MinPlus} = (\mathbb{N}_{\infty}, \min, +, \infty, 0)$. It is complete with the mapping

$$\sum_I^{\min} : (\mathbb{N}_\infty)^I \to \mathbb{N}_\infty \quad \text{with} \quad (n_i \mid i \in I) \mapsto \inf(n_i \mid i \in I) \ .$$

6. The *semiring* $\mathsf{Lang}_\Gamma = (\mathcal{P}(\Gamma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ *of formal languages* where $\cdot$ denotes the concatenation of languages. It is complete with the mapping

$$\sum_I^{\cup} : (\mathcal{P}(\Gamma^*))^I \to \mathcal{P}(\Gamma^*) \quad \text{with} \quad (L_i \mid i \in I) \mapsto \bigcup_{i \in I} L_i \ .$$

Each semiring except 6 is commutative. Moreover, the semirings 1–2 and 4–6 are positive. The Boolean semiring is finite, and hence, it is bi-locally finite. Furthermore, also the semirings 4–6 are additively locally finite. For further examples of semirings we refer to [42, Ex. 2.6.9]. $\triangle$

Lattices. Here we recall some basic notions from the theory of lattices [12, 48] and [20, Ch. 1]. A *lattice* is an algebra $\mathsf{L} = (L, \vee, \wedge)$ in which $\vee$ (the *join*) and $\wedge$ (the *meet*) are binary operations, $(L, \vee)$ and $(L, \wedge)$ are commutative semigroups, the operations $\vee$ and $\wedge$ are idempotent and satisfy the absorption axioms.

Let $\mathsf{L} = (L, \vee, \wedge)$ be a lattice. We say that L is *bounded* if there exist elements $\mathbb{0}$ and $\mathbb{1}$ in L such that $\mathbb{0} \vee a = a$ and $\mathbb{1} \wedge a = a$ for every $a \in L$. We denote a bounded lattice also by $\mathsf{L} = (L, \vee, \wedge, \mathbb{0}, \mathbb{1})$. Recall that each bounded lattice is a bi-locally finite and commutative strong bimonoid [30, Ex. 1]. Hence, a bounded lattice $(L, \vee, \wedge, \mathbb{0}, \mathbb{1})$ is said to be *complete as a strong bimonoid* if $(L, \vee, \mathbb{0})$ is complete. If this is the case, then we have $\sum_{i \in \emptyset}^{\vee} a_i = \mathbb{0}$.

Here we show two examples of lattices.

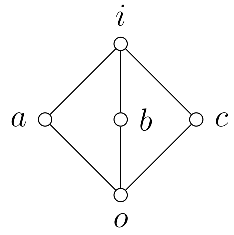


Figure 2.4. *Visualization of the bounded lattice* M_3 *given in Example 2.4.7(2)*

Example 2.4.7. *cf.* [42, Ex. 2.6.15].

1. Let A be a set. Then $\mathsf{PS}_A = (\mathcal{P}(A), \cup, \cap, \emptyset, A)$ is a bounded lattice. Moreover, since the monoid $(\mathcal{P}(A), \cup, \emptyset)$ is complete with the mapping

$$\sum_I^{\cup} : (\mathcal{P}(A))^I \to \mathcal{P}(A) \quad \text{with} \quad (A_i \mid i \in I) \mapsto \bigcup_{i \in I} A_i \ ,$$

the lattice PS_A is also complete.

2. [48, Fig. 2] and [20, Fig. 5] Let $M_3 = \{o, a, b, c, i\}$ be a set with five elements. Moreover, we consider the binary relation $\preceq$ on M_3 such that

$$o \prec a \prec i \quad \text{and} \quad o \prec b \prec i \quad \text{and} \quad o \prec c \prec i$$

and $x \not\prec y$ for any other combination $x, y \in M_3$. Obviously, $\preceq$ is a partial ordering, *i.e.*, $(M_3, \preceq)$ is a poset. Then we consider the binary operations $\vee$ and $\wedge$ on M_3 defined, for every $x, y \in M_3$, by

$$x \vee y = \sup_{\preceq}(\{x, y\}) \quad \text{and} \quad x \wedge y = \inf_{\preceq}(\{x, y\}) \ .$$

Clearly, the algebra $\mathsf{M}_3 = (M_3, \vee, \wedge, o, i)$ is a bounded lattice, which is not distributive. The lattice M_3 is visualized on Figure 2.4.

For further examples of lattices we refer to [42, Ex. 2.6.15]. △

In the rest of this thesis, $\mathsf{B} = (B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ *is an arbitrary strong bimonoid if not specified otherwise.*

Chapter 3

Weighted tree automata and pumping lemmas

This chapter is organized as follows. In Section 3.1, we recall fundamental notions, notations, and results of weighted tree languages and weighted tree automata from [41, 42]. Moreover, in Section 3.2 we present our pumping lemmas for runs of weighted tree automata.

3.1 The model

In order to recall the fundamental notions and notations, we adopt the formalism used in the corresponding part of [42].

Weighted sets and weighted tree languages. Let A be a set, and $f : A \to B$ be a mapping. We also say that f is a B-*weighted set* (or just: *weighted set*). The *support of* f *with respect to* B, denoted by $\mathrm{supp}_{\mathsf{B}}(f)$, is defined by $\mathrm{supp}_{\mathsf{B}}(f) = \{a \in A \mid f(a) \neq \mathbb{0}\}$. A (Σ, B)-*weighted tree language* (or just: *weighted tree language*) is a B-weighted set $\psi : \mathrm{T}_\Sigma \to B$.

Next we give some examples of weighted tree languages. Each of them is connected to the set of positions. Firstly, we give the weighted tree language $\#_{\max}$. For this, we assume that Σ contains two binary symbols σ and ω. Then the weighted tree language $\#_{\max}$ assigns to each tree $\xi \in \mathrm{T}_\Sigma$ the number of occurrences of that symbol out of σ and ω, which occurs the most times in ξ.

Example 3.1.1. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. Then we consider the mapping $\#_{\max} : \mathrm{T}_\Sigma \to \mathbb{N}_{-\infty}$ defined, for each $\xi \in \mathrm{T}_\Sigma$, by

$$\#_{\max}(\xi) = \max(|\operatorname{pos}_\sigma(\xi)|, |\operatorname{pos}_\omega(\xi)|) \ .$$

Obviously, $\#_{\max}$ is a $(\Sigma, \mathsf{MaxPlus})$-weighted tree language, where $\mathsf{MaxPlus}$ is the semiring defined in Example 2.4.6(4). $\triangle$

Our second example is a variant of the above one. We assume that Σ contains a binary symbol σ. Then it counts, for each $\xi \in \mathrm{T}_\Sigma$, how many times σ occurs in ξ. For that, this time we consider a different ranked alphabet, and use a different strong bimonoid as weight structure.

Example 3.1.2. [4, Ex. 2] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Then we consider the mapping $\#_\sigma : \mathrm{T}_\Sigma \to \mathbb{N}_\infty$ defined, for each $\xi \in \mathrm{T}_\Sigma$, by

$$\#_\sigma(\xi) = |\operatorname{pos}_\sigma(\xi)| \enspace .$$

Evidently, $\#_\sigma$ can be considered as a $(\Sigma, \mathsf{MinPlus})$-weighted tree language, where $\mathsf{MinPlus}$ is the semiring defined in Example 2.4.6(5). $\triangle$

As the last example we define a weighted tree language split, which partitions the set T_Σ into four subsets as follows. We assume that Σ contains a binary symbol σ and a unary symbol γ. Then split splits T_Σ into four partitions: (1) the set of trees in which none of σ and γ occurs, (2) and (3) the set of trees in which only σ and only γ occurs, respectively, and (4) the set of trees in which both σ and γ occur. As weight structure, we consider the bounded lattice M_3.

Example 3.1.3. Let $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$. We consider the bounded lattice M_3 given in Example 2.4.7(2). Next we define the mapping $\mathrm{split} : \mathrm{T}_\Sigma \to M_3$, for each $\xi \in \mathrm{T}_\Sigma$, as follows:

$$\mathrm{split}(\xi) = \begin{cases} i & \text{if } \xi = \alpha \enspace , \\ a & \text{if } \xi \in (\mathrm{T}_{\{\sigma,\alpha\}} \setminus \{\alpha\}) \enspace , \\ b & \text{if } \xi \in (\mathrm{T}_{\{\gamma,\alpha\}} \setminus \{\alpha\}) \enspace , \\ o & \text{otherwise} \enspace . \end{cases}$$

Clearly, split is a (Σ, M_3)-weighted tree language. $\triangle$

Weighted tree automata. A *weighted tree automaton over* Σ *and* B (for short: (Σ, B)-wta, or just: wta) [41, 42] is a triple $\mathcal{A} = (Q, \delta, F)$, where

- Q is a finite and nonempty set (*states*) such that $Q \cap \Sigma = \emptyset$,
- $\delta = (\delta_k \mid k \in \mathbb{N})$ is a family of mappings $\delta_k : Q^k \times \Sigma^{(k)} \times Q \to B$ (*transition mappings*)[1], where we consider Q^k as a set of strings over Q of length k, and
- $F : Q \to B$ is a mapping (*root weight mapping*).

Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Sometimes, for each $q \in Q$, we abbreviate $F(q)$ by F_q. We say that $\mathcal{A}$ is *total* (respectively, *bottom-up deterministic*, or for short:

[1]For each $k \in \mathbb{N}_+$ with $k > \operatorname{maxrk}(\Sigma)$, since $\Sigma^{(k)} = \emptyset$, we have $\delta_k : \emptyset \to B$.

bu deterministic) if, for every $k \in \mathbb{N}$, $w \in Q^k$, and $\sigma \in \Sigma^{(k)}$, there exists at least (respectively, at most) one $q \in Q$ such that $\delta_k(w, \sigma, q) \neq \mathbb{0}$.

Semantics. We mention that, for $\mathcal{A}$, the following two semantics can be defined: the initial algebra semantics and the run semantics [3, 41, 42, 66]. In general, the two kinds of semantics may differ [30]; however, if B is a semiring or $\mathcal{A}$ is bu deterministic, then they coincide [14, Lm. 4.1.13] and [66, Thm. 4.1] and [3, Thm. 3.10]. In this PhD thesis we deal only with the run semantics.

In order to define the run semantics, the concept of run of $\mathcal{A}$ on a tree $\xi \in \mathrm{T}_\Sigma$ is crucial. However, to prove our pumping lemmas (*cf.* Theorems 3.2.3 and 3.2.4), we need a more general definition of the run. For this, let $\zeta \in \mathrm{T}_\Sigma(\{\Box\})$. A *run of $\mathcal{A}$ on ζ* is a mapping $\rho : \mathrm{pos}(\zeta) \to Q$. Let ρ be a run of $\mathcal{A}$ on ζ, and $q \in Q$. We say that ρ is

- a *q-run* if $\rho(\varepsilon) = q$,
- *valid* if $\delta_k\big(\rho(v1) \cdots \rho(vk), \zeta(v), \rho(v)\big) \neq \mathbb{0}$ for every $v \in \mathrm{pos}(\zeta)$ with $\zeta(v) \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$, and
- *accepting* if it is valid and $F_{\rho(\varepsilon)} \neq \mathbb{0}$.

We denote the *set of all q-runs* (*all valid q-runs*, *all accepting q-runs*) *of $\mathcal{A}$ on ζ* by $\mathrm{Run}_\mathcal{A}(q, \zeta)$ (respectively, $\mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q, \zeta)$ and $\mathrm{Run}^{\mathrm{a}}_\mathcal{A}(q, \zeta)$). Furthermore, we define the following sets

$$\mathrm{Run}_\mathcal{A}(\zeta) = \bigcup_{q \in Q} \mathrm{Run}_\mathcal{A}(q, \zeta) \quad , \quad \mathrm{Run}^{\mathrm{v}}_\mathcal{A}(\zeta) = \bigcup_{q \in Q} \mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q, \zeta) \quad , \quad \text{and}$$
$$\mathrm{Run}^{\mathrm{a}}_\mathcal{A}(\zeta) = \bigcup_{q \in Q} \mathrm{Run}^{\mathrm{a}}_\mathcal{A}(q, \zeta) \ .$$

Let $v \in \mathrm{pos}(\zeta)$. We define the mapping $\rho|_v : \mathrm{pos}(\zeta|_v) \to Q$ such that, for each $v' \in \mathrm{pos}(\zeta|_v)$, we have $\rho|_v(v') = \rho(vv')$. Obviously, $\rho|_v \in \mathrm{Run}_\mathcal{A}(\zeta|_v)$, and thus, we call it the *run induced by ρ at position v*.

The *weight of the run ρ of $\mathcal{A}$ for ζ*, denoted by $\mathrm{wt}_\mathcal{A}(\zeta, \rho)$, is the element in B defined by induction on the structure of ζ as follows:

(i) if $\zeta = \Box$, then $\mathrm{wt}_\mathcal{A}(\Box, \rho) = \mathbb{1}$ and
(ii) if $\zeta = \sigma(\zeta_1, \ldots, \zeta_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\zeta_1, \ldots, \zeta_k) \in (\mathrm{T}_\Sigma(\{\Box\}))^k$, then

$$\mathrm{wt}_\mathcal{A}(\zeta, \rho) = \Big(\bigotimes_{i=1}^{k} \mathrm{wt}_\mathcal{A}(\zeta_i, \rho|_i) \Big) \otimes \delta_k\big(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)\big) \ . \tag{3.1}$$

If confusion is ruled out, then sometimes we drop the index $\mathcal{A}$ from $\mathrm{wt}_\mathcal{A}(\zeta, \rho)$ and write just $\mathrm{wt}(\zeta, \rho)$ for the weight of ρ.

The *(run) semantics of $\mathcal{A}$*, denoted by $[\![\mathcal{A}]\!]$, is the (Σ, B)-weighted tree language

$[\![\mathcal{A}]\!] : \mathrm{T}_\Sigma \to B$ defined, for each $\xi \in \mathrm{T}_\Sigma$, by

$$[\![\mathcal{A}]\!](\xi) = \bigoplus_{\rho \in \mathrm{Run}_\mathcal{A}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \ .$$

Note that, for every $\xi \in \mathrm{T}_\Sigma$ and ρ in $\mathrm{Run}_\mathcal{A}(\xi) \setminus \mathrm{Run}^\mathrm{a}_\mathcal{A}(\xi)$, either there exists $v \in \mathrm{pos}(\xi)$ with $\xi(v) \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$ such that $\delta_k\big(\rho(v1)\cdots\rho(vk), \zeta(v), \rho(v)\big) = \mathbb{0}$, *i.e.*, $\rho \notin \mathrm{Run}^\mathrm{v}_\mathcal{A}(\xi)$, or we have $F_{\rho(\varepsilon)} = \mathbb{0}$, and thus, $\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} = \mathbb{0}$. For this, we have

$$\bigoplus_{\rho \in \mathrm{Run}_\mathcal{A}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} = \bigoplus_{\rho \in \mathrm{Run}^\mathrm{v}_\mathcal{A}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} = \bigoplus_{\rho \in \mathrm{Run}^\mathrm{a}_\mathcal{A}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \ .$$

Moreover, if $\mathcal{A}$ is total and bu deterministic, then, for each $\xi \in \mathrm{T}_\Sigma$, there is a unique valid run ρ_ξ of $\mathcal{A}$ on ξ, *i.e.*, we have $\{\rho_\xi\} = \mathrm{Run}^\mathrm{v}_\mathcal{A}(\xi)$, with $[\![\mathcal{A}]\!](\xi) = \mathrm{wt}(\xi, \rho_\xi) \otimes F_{\rho_\xi(\varepsilon)}$ *cf.* [42, Lm. 4.2.1(3b)]. We will use the above equalities without any reference.

For two (Σ, B)-wta $\mathcal{A}$ and $\mathcal{A}'$, we say that $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent* if $[\![\mathcal{A}]\!] = [\![\mathcal{A}']\!]$. Furthermore, for each (Σ, B)-weighted tree language $\psi : \mathrm{T}_\Sigma \to B$, we say that r is *(run) recognizable* if there exists a (Σ, B)-wta $\mathcal{A}$ such that $r = [\![\mathcal{A}]\!]$.

Representation of wta by fta-hypergraphs. Recall that in Section 2.3 we have defined Σ-hypergraphs, and shown that each Σ-fta A can be represented by a particular Σ-hypergraph, which we call the fta-hypergraph of A. Now we show that also each (Σ, B)-wta can be represented by an fta-hypergraph but only with extra annotations *cf.* [42]. For this, let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. We first consider the Σ-hypergraph

$$g_\mathcal{A} = (Q, \bigcup_{k \in \mathbb{N}} \mathrm{supp}_\mathsf{B}(\delta_k)) \ .$$

Then we add to $g_\mathcal{A}$ the weights of transitions and the root weights of $\mathcal{A}$ as follows. For each $q \in Q$ with $F_q \neq \mathbb{0}$, we add F_q to the node which represents q. Otherwise, if we have $F_q = \mathbb{0}$, then we do not illustrate F_q in the picture. Moreover, for each transition of $\mathcal{A}$ with non-$\mathbb{0}$-weight, *i.e.*, element in $\bigcup_{k \in \mathbb{N}} \mathrm{supp}(\delta_k)$, we add its weight to its representing hyperedge. We call $g_\mathcal{A}$ the *fta-hypergraph of $\mathcal{A}$*.

Here we give some examples of wta and their fta-hypergraphs. Firstly, we show that the $(\Sigma, \mathsf{MaxPlus})$-weighted tree language $\#_\mathrm{max}$ defined in Example 3.1.1 is recognizable.

Example 3.1.4. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. Then we construct the $(\Sigma, \mathsf{MaxPlus})$-wta

$$\mathcal{A}_\mathrm{max} = (\{q_\sigma, q_\omega, q_s\}, \delta, F)$$

with $\delta_2(q_\sigma q_\sigma, \sigma, q_\sigma) = \delta_2(q_\omega q_\omega, \omega, q_\omega) = \delta_2(q_s q_s, \sigma, q_s) = \delta_2(q_s q_s, \omega, q_s) = 1$, and $\delta_0(\varepsilon, \alpha, q_\sigma) = \delta_0(\varepsilon, \alpha, q_\omega) = \delta_0(\varepsilon, \alpha, q_s) = \delta_2(q_\sigma q_\sigma, \omega, q_\sigma) = \delta_2(q_\omega q_\omega, \sigma, q_\omega) = 0$, and

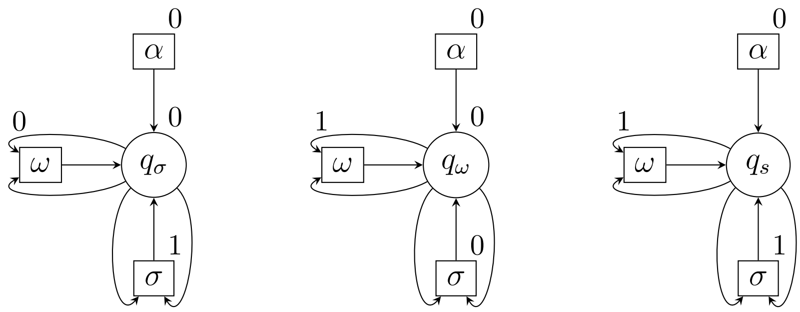


Figure 3.1. *The fta-hypergraph of the* $(\Sigma, \mathsf{MaxPlus})$*-wta* $\mathcal{A}_{\max}$ *defined in Example 3.1.4*

every other transition has weight $-\infty$, and $F_{q_\sigma} = F_{q_\omega} = 0$ and $F_{q_s} = -\infty$. Figure 3.1 shows the fta-hypergraph of $\mathcal{A}_{\max}$. Observe that, since both $\delta_0(\varepsilon, \alpha, q_\sigma)$ and $\delta_0(\varepsilon, \alpha, q_\omega)$ have weight 0, the wta $\mathcal{A}_{\max}$ is not bu deterministic. Moreover, since $\delta_2(q_\sigma q_\omega, \sigma, q) = -\infty$ for each $q \in \{q_\sigma, q_\omega, q_s\}$, the wta $\mathcal{A}_{\max}$ is not total either.

Next we illustrate some runs of $\mathcal{A}_{\max}$ in the following way. For this, let

$$\xi = \sigma(\sigma(\alpha, \sigma(\alpha, \alpha)), \sigma(\alpha, \alpha)) \enspace .$$

Figure 3.2 depicts the Σ-tree ξ with three runs of the wta $\mathcal{A}_{\max}$ in gray color and dashed lines as follows. Figure 3.2(a) shows a not valid q_σ-run (consider, *e.g.*, $\delta_2(q_\sigma q_\omega, \sigma, q_s) = -\infty$). Figure 3.2(b) illustrates a valid but not accepting q_s-run (recall that $F_{q_s} = -\infty$). Moreover, Figure 3.2(c) depicts an accepting q_σ-run.

Finally, we examine the semantics of $\mathcal{A}_{\max}$. Let $\xi \in \mathrm{T}_\Sigma$. Clearly, on ξ there are exactly three valid runs, which are as follows. For each $\mathrm{tag} \in \{\sigma, \omega, s\}$, we denote by ρ_{tag} the run of $\mathcal{A}_{\max}$ on ξ such that $\rho_{\mathrm{tag}}(v) = q_{\mathrm{tag}}$ for each $v \in \mathrm{pos}(\xi)$. Then we have $\mathrm{wt}(\xi, \rho_\sigma) = |\operatorname{pos}_\sigma(\xi)|$, $\mathrm{wt}(\xi, \rho_\omega) = |\operatorname{pos}_\omega(\xi)|$, and $\mathrm{wt}(\xi, \rho_s) = |\operatorname{pos}_\sigma(\xi)| + |\operatorname{pos}_\omega(\xi)|$.

Note that, since $F_{q_s} = -\infty$, out of the three valid runs, only ρ_σ and ρ_ω are accepting. Hence, we have

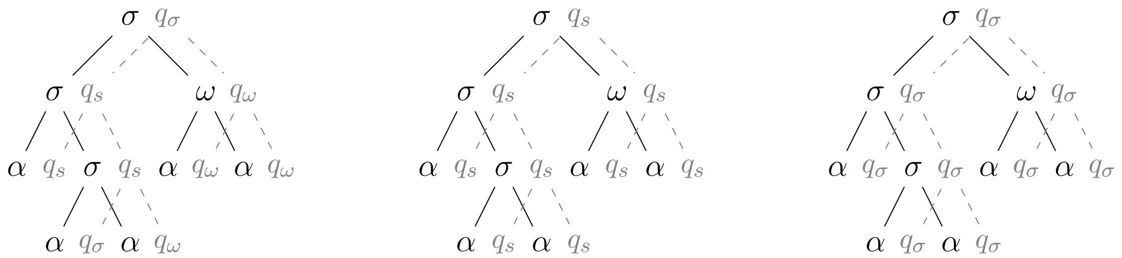
$$\begin{aligned} [\![\mathcal{A}_{\max}]\!](\xi) &= \max(\mathrm{wt}(\xi, \rho_\sigma) + F_{q_\sigma}, \mathrm{wt}(\xi, \rho_\omega) + F_{q_\omega}) \\ &= \max(|\operatorname{pos}_\sigma(\xi)|, |\operatorname{pos}_\omega(\xi)|) = \#_{\max}(\xi) \enspace , \end{aligned}$$

i.e., $\#_{\max}$ is recognizable. △

In the following example, we prove that the $(\Sigma, \mathsf{MinPlus})$-weighted tree language $\#_\sigma$ defined in Example 3.1.2 is recognizable.

Example 3.1.5. Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Then we construct the $(\Sigma, \mathsf{MinPlus})$-wta

$$\mathcal{A}_\sigma = (\{q\}, \delta, F) \enspace ,$$



(a) *a not valid q_σ-run* **(b)** *a valid but not accepting q_s-run* **(c)** *an accepting q_σ-run*

Figure 3.2. *Runs of the* $(\Sigma, \mathsf{MaxPlus})$*-wta* $\mathcal{A}_{\max}$ *defined in Example 3.1.4*

where $\delta_0(\varepsilon, \alpha, q) = 0$, $\delta_2(qq, \sigma, q) = 1$, and $F_q = 0$. (Note that in the fta-hypergraph shown in Figure 3.1 if we consider the state q_σ without the transition $\delta_2(q_\sigma q_\sigma, \omega, q_\sigma)$ and identify q_σ with q, then we obtain the fta-hypergraph of $\mathcal{A}_\sigma$.) Obviously, $\mathcal{A}_\sigma$ is both total and bu deterministic. Then, for each $\xi \in \mathrm{T}_\Sigma$, there is exactly one run of $\mathcal{A}_\sigma$ on ξ, which denote by ρ_ξ. Moreover, for each $\xi \in \mathrm{T}_\Sigma$, that run ρ_ξ is accepting, and hence, $[\![\mathcal{A}_\sigma]\!](\xi) = \mathrm{wt}(\xi, \rho_\xi) + F_q = |\operatorname{pos}_\sigma(\xi)| = \#_\sigma(\xi)$, *i.e.*, the $(\Sigma, \mathsf{MinPlus})$-weighted tree language $\#_\sigma$ is recognizable. $\triangle$

Finally, we show that also the (Σ, M_3)-weighted tree language split defined in Example 3.1.3 is recognizable.

Example 3.1.6. Let $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$. We consider the bounded lattice M_3 shown in Example 2.4.7(2). We construct the (Σ, M_3)-wta

$$\mathcal{A}_{\mathrm{split}} = (\{q\}, \delta, F)$$

such that $\delta_0(\varepsilon, \alpha, q) = i$, $\delta_1(q, \gamma, q) = b$, $\delta_2(qq, \sigma, q) = a$, and $F_q = i$. Figure 3.3 depicts the fta-hypergraph of $\mathcal{A}_{\mathrm{split}}$. Note that $\mathcal{A}_{\mathrm{split}}$ is both total and bu-deterministic as well.

Evidently, for each $\xi \in \mathrm{T}_\Sigma$, there is a unique run of $\mathcal{A}_{\mathrm{split}}$ on ξ, which we denote by ρ_ξ. Next we prove by induction on the structure of ξ the following statement:

$$\text{for each } \xi \in \mathrm{T}_\Sigma, \text{ we have } \mathrm{wt}(\xi, \rho_\xi) = \mathrm{split}(\xi) \ . \tag{3.2}$$

Induction base: For $\xi = \alpha$, we can calculate as follows:

$$\mathrm{wt}(\alpha, \rho_\alpha) = \delta_0(\varepsilon, \alpha, q) = i = \mathrm{split}(\alpha) \ .$$

Induction step: We proceed by case analysis. Firstly, assume that $\xi = \gamma(\xi_1)$ for some $\xi_1 \in \mathrm{T}_\Sigma$. Recall that we have $\mathrm{wt}(\xi, \rho_\xi) = \mathrm{wt}(\xi_1, \rho_{\xi_1}) \wedge \delta_1(q, \gamma, q)$, where $\delta_1(q, \gamma, q) = b$. Moreover, by I.H., we have $\mathrm{wt}(\xi_1, \rho_{\xi_1}) = \mathrm{split}(\xi_1)$, and thus,

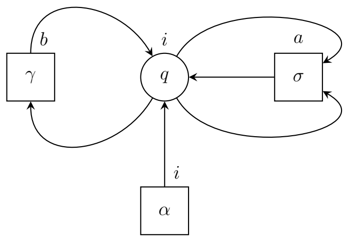


Figure 3.3. *The fta-hypergraph of the* (Σ, M_3)*-wta* $\mathcal{A}_{\mathrm{split}}$ *given in Example 3.1.6*

$\mathrm{wt}(\xi, \rho_\xi) = b$ if $\mathrm{pos}_\sigma(\xi_1) = \emptyset$, and $\mathrm{wt}(\xi, \rho_\xi) = o$ otherwise. Consequently, in this case $\mathrm{wt}(\xi, \rho_\xi) = \mathrm{split}(\xi)$ holds true.

Alternatively, now assume that $\xi = \sigma(\xi_1, \xi_2)$ for some $\xi_1, \xi_2 \in \mathrm{T}_\Sigma$. Clearly, we have $\mathrm{wt}(\xi, \rho_\xi) = \mathrm{wt}(\xi_1, \rho_{\xi_1}) \wedge \mathrm{wt}(\xi_2, \rho_{\xi_2}) \wedge \delta_2(qq, \sigma, q)$, where $\delta_2(qq, \sigma, q) = a$. Furthermore, by I.H., we have $\mathrm{wt}(\xi_i, \rho_{\xi_i}) = \mathrm{split}(\xi_i)$ for each $i \in \{1, 2\}$, and hence, $\mathrm{wt}(\xi, \rho_\xi) = a$ if $\mathrm{pos}_\gamma(\xi_i) = \emptyset$ for each $i \in \{1, 2\}$, and $\mathrm{wt}(\xi, \rho_\xi) = o$ otherwise. This completes the proof of (3.2). Then, for each $\xi \in \mathrm{T}_\Sigma$, we have

$$[\![\mathcal{A}_{\mathrm{split}}]\!](\xi) = \mathrm{wt}(\xi, \rho_\xi) \wedge F_q = \mathrm{wt}(\xi, \rho_\xi) = \mathrm{split}(\xi) \ ,$$

where the second equality is due to the fact that $F_q = i$, and the last equality follows from (3.2). △

Now we recall a well known result from the theory of wta.

Lemma 3.1.7. [41, Thm. 3.9] and [42, Lm. 10.9.2] Let $\mathsf{B}_1 = (B_1, \oplus_1, \otimes_1, \mathbb{0}_1, \mathbb{1}_1)$ and $\mathsf{B}_2 = (B_2, \oplus_2, \otimes_2, \mathbb{0}_2, \mathbb{1}_2)$ be strong bimonoids, and $h : B_1 \to B_2$ be a strong bimonoid homomorphism. Then, for each (Σ, B_1)-wta $\mathcal{A}$, we can construct a (Σ, B_2)-wta $h(\mathcal{A})$ such that $[\![h(\mathcal{A})]\!] = h \circ [\![\mathcal{A}]\!]$. □

3.2 Pumping lemmas

Here we prove pumping lemmas for runs of wta. We will use them in Section 5.4. With a pumping lemma one can achieve structural implications on small or particular large trees (*cf.* [43, Lm. 2.10.1] and [13, Lm. 5.5]). Since such pumping lemmas already exist for wta (*cf.* [13, Sect. 5]), the question may arise why we present another pumping lemmas. To answer that question we note that Borchardt's setting in [13] deals with bu deterministic wta over semirings and employs initial algebra semantics, whereas in our setting we deal with (arbitrary) wta over strong bimonoids and employ run semantics. Nevertheless, if we consider the class of all bu deterministic wta over semirings, then the two settings coincide. In order to prove our pumping lemmas, we first recall some fundamental notions and notations from [2].

Loops of wta. Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Furthermore, let $c \in \mathrm{C}_\Sigma$ with $\{v\} = \mathrm{pos}_\square(c)$ and $\theta \in \mathrm{Run}_\mathcal{A}(q, c)$ with $\theta(v) = p$ for some $p \in Q$. We call θ a *(q,p)-run of $\mathcal{A}$ on c*. For each $p \in Q$, we denote the *set of all (q,p)-runs* (*all valid (q,p)-runs*) *of $\mathcal{A}$ on c* by $\mathrm{Run}_\mathcal{A}(q, c, p)$ (respectively, $\mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q, c, p)$). Clearly, we have $\mathrm{Run}_\mathcal{A}(q, c) = \bigcup_{p \in Q} \mathrm{Run}_\mathcal{A}(q, c, p)$. Moreover, each run $\theta \in \mathrm{Run}_\mathcal{A}(q, c, q)$ is called a *loop*.

Combinations of runs of wta. Let $c \in \mathrm{C}_\Sigma$ with $\{v\} = \mathrm{pos}_\square(c)$, $\zeta \in \mathrm{T}_\Sigma(\{\square\})$, $q', q \in Q$, $\theta \in \mathrm{Run}_\mathcal{A}(q', c, q)$, and $\rho \in \mathrm{Run}_\mathcal{A}(q, \zeta)$. The *combination of θ and ρ (at v)*, denoted by $\theta[\rho]$, is the q'-run $\theta[\rho] : \mathrm{pos}(c[\zeta]) \to Q$ of $\mathcal{A}$ on $c[\zeta]$ defined, for each $u \in \mathrm{pos}(c[\zeta])$, as follows: if $u = vw$ for some $w \in \mathrm{pos}(\zeta)$, then we define $\theta[\rho](u) = \rho(w)$, otherwise we define $\theta[\rho](u) = \theta(u)$.

Left- and right subproducts. Let $c \in \mathrm{C}_\Sigma$ with $\{v\} = \mathrm{pos}_\square(c)$, and $\theta \in \mathrm{Run}_\mathcal{A}(c)$. We define two mappings $l_{c,\theta} : \mathrm{prefix}(v) \to B$ and $r_{c,\theta} : \mathrm{prefix}(v) \to B$ inductively on the length of their arguments (*cf.* [13, p. 526] for bu deterministic wta). Intuitively, we can split the product (3.1) yielding the element $\mathrm{wt}(c, \theta)$ in B into a left subproduct $l_{c,\theta}(\varepsilon)$ and a right subproduct $r_{c,\theta}(\varepsilon)$, where the border is given by the factor $\mathbb{1}$ coming from the weight of $\square$. Figure 3.4 shows the illustration of mappings $l_{c,\theta}$ and $r_{c,\theta}$.

Formally, let $w \in \mathrm{prefix}(v)$. Then, assuming that $c(w) = \sigma$, $\mathrm{rk}_\Sigma(\sigma) = k$ for some $k \in \mathbb{N}$, and $b = \delta_k(\theta(w1) \cdots \theta(wk), \sigma, \theta(w))$, we let

$$l_{c,\theta}(w) = \begin{cases} \mathbb{1} & \text{if } w = v \\ \bigotimes_{j=1}^{i-1} \mathrm{wt}(c|_{wj}, \theta|_{wj}) \otimes l_{c,\theta}(wi) & \text{if } wi \in \mathrm{prefix}(v) \text{ for some } i \in \mathbb{N}_+ \end{cases}$$

$$r_{c,\theta}(w) = \begin{cases} \mathbb{1} & \text{if } w = v \\ r_{c,\theta}(wi) \otimes \bigotimes_{j=i+1}^{k} \mathrm{wt}(c|_{wj}, \theta|_{wj}) \otimes b & \text{if } wi \in \mathrm{prefix}(v) \text{ for some } i \in \mathbb{N}_+ \end{cases} .$$

In the sequel, for every (Σ, B)-wta $\mathcal{A}$, Σ-context c, and run θ of $\mathcal{A}$ on c, we will abbreviate $l_{c,\theta}(\varepsilon)$ and $r_{c,\theta}(\varepsilon)$ by $l_{c,\theta}$ and $r_{c,\theta}$, respectively.

Lemma 3.2.1. [1, Obs. 6] and [2, Obs. 5.1] Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Then, for every Σ-context c and run θ of $\mathcal{A}$ on c, we have $\mathrm{wt}(c, \theta) = l_{c,\theta} \otimes r_{c,\theta}$.

Proof. We prove our statement by induction on the structure of c.

Induction base: Then we have $c = \square$ with $\mathrm{pos}(\square) = \{\varepsilon\}$, and thus, $l_{c,\theta} = r_{c,\theta} = \mathbb{1}$. Moreover, since $\mathrm{wt}(\square, \theta) = \mathbb{1}$, our statement evidently holds true.

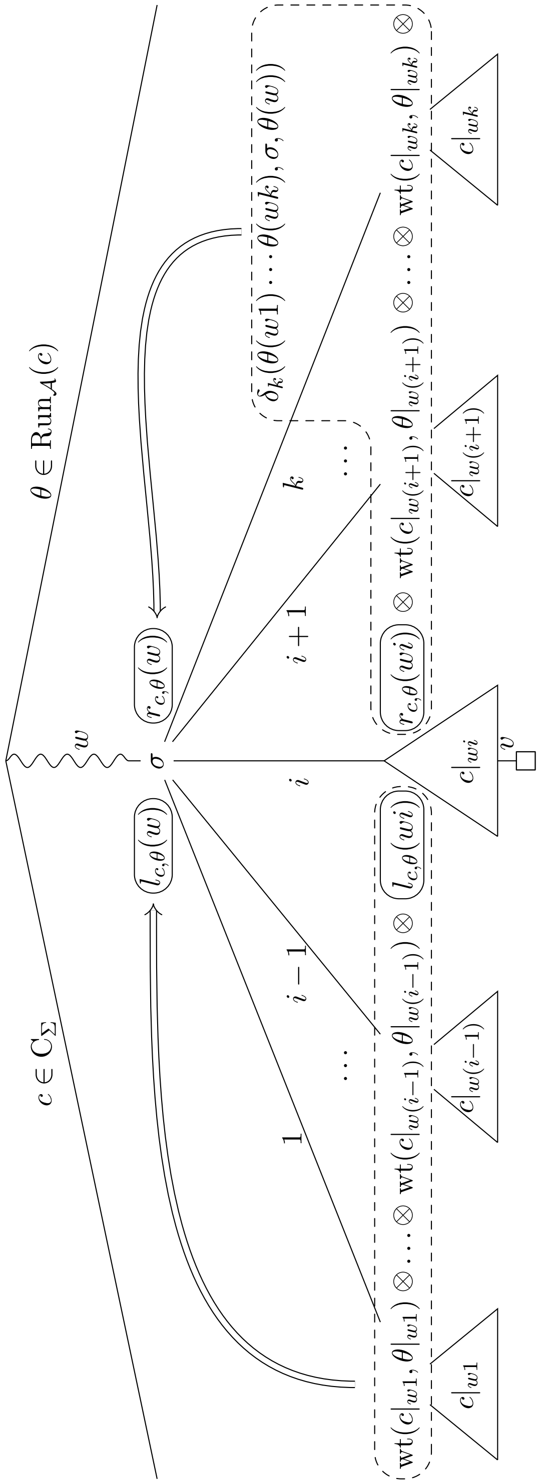


Figure 3.4. *Illustration of mappings $l_{c,\theta}$ and $r_{c,\theta}$ (cf. [2, Fig. 3])*

Induction step: There exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, $c' \in \mathrm{C}_\Sigma$, $i \in [k]$, and $(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^{(k-1)}$ such that $c = \sigma(\xi_1, \ldots, \xi_{i-1}, c', \xi_{i+1}, \ldots, \xi_k)$. Furthermore, we have

$$\mathrm{wt}(c', \theta|_i) = l_{c',\theta|_i} \otimes r_{c',\theta|_i} = l_{c,\theta}(i) \otimes r_{c,\theta}(i) \enspace ,$$

where the first equality holds true by I.H., and the second one is due to the facts that $l_{c',\theta|_i} = l_{c,\theta}(i)$ and $r_{c',\theta|_i} = r_{c,\theta}(i)$. Then, by assuming that

$$a = \bigotimes_{j=1}^{i-1} \mathrm{wt}(\xi_j, \theta|_j) \quad \text{and} \quad b = \bigotimes_{j=i+1}^{k} \mathrm{wt}(\xi_j, \theta|_j) \otimes \delta_k(\theta(1) \cdots \theta(k), \sigma, \theta(\varepsilon)) \enspace ,$$

we can calculate as follows:

$$\mathrm{wt}(c, \theta) = a \otimes \mathrm{wt}(c', \theta|_i) \otimes b = a \otimes l_{c,\theta}(i) \otimes r_{c,\theta}(i) \otimes b = l_{c,\theta} \otimes r_{c,\theta} \enspace ,$$

where the first equality is due to (3.1), the second equality follows from I.H., and the last equality holds true by the definitions of $l_{c,\theta}$ and $r_{c,\theta}$. This completes our proof. □

Lemma 3.2.2. [1, Lm. 7] and [2, Lm. 5.2] (also *cf.* [13, Lm. 5.1]) Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Then, for every Σ-context c, Σ-tree ξ, states q' and q in Q, (q', q)-run θ of $\mathcal{A}$ on c, and q-run ρ of $\mathcal{A}$ on ξ, we have $\mathrm{wt}(c[\xi], \theta[\rho]) = l_{c,\theta} \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c,\theta}$.

Proof. Similarly, we prove our statement by induction on the structure of c.

Induction base: Clearly, in this case we have $c = \Box$ and $q' = q$. Thus, we have $\Box[\xi] = \xi$ and $\theta[\rho] = \rho$ with $\mathrm{wt}(\Box[\xi], \theta[\rho]) = \mathrm{wt}(\xi, \rho)$. Consequently, we can calculate as follows:

$$\mathrm{wt}(\Box[\xi], \theta[\rho]) = \mathrm{wt}(\xi, \rho) = \mathbb{1} \otimes \mathrm{wt}(\xi, \rho) \otimes \mathbb{1} = l_{\Box,\theta} \otimes \mathrm{wt}(\xi, \rho) \otimes r_{\Box,\theta} \enspace ,$$

where the last equality is due to the definitions of $l_{\Box,\theta}$ and $r_{\Box,\theta}$.

Induction step: There exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, $c' \in \mathrm{C}_\Sigma$, $i \in [k]$, and $(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^{(k-1)}$ such that $c = \sigma(\xi_1, \ldots, \xi_{i-1}, c', \xi_{i+1}, \ldots, \xi_k)$. Clearly, we have $c[\xi] = \sigma(\xi_1, \ldots, \xi_{i-1}, c'[\xi], \xi_{i+1}, \ldots, \xi_k)$, and $(\theta|_i)[\rho]$ is a $\theta(i)$-run of $\mathcal{A}$ on $c'[\xi]$. Moreover, we have

$$\mathrm{wt}(c'[\xi], \theta|_i[\rho]) = l_{c',\theta|_i} \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c',\theta|_i} = l_{c,\theta}(i) \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c,\theta}(i) \enspace ,$$

where the first equality is due to I.H., and the second one follows from the facts that $l_{c',\theta|_i} = l_{c,\theta}(i)$ and $r_{c',\theta|_i} = r_{c,\theta}(i)$. Then, by assuming that

$$a = \bigotimes_{j=1}^{i-1} \mathrm{wt}(\xi_j, \theta|_j) \quad \text{and} \quad b = \bigotimes_{j=i+1}^{k} \mathrm{wt}(\xi_j, \theta|_j) \otimes \delta_k(\theta(1) \cdots \theta(k), \sigma, \theta(\varepsilon)) \enspace ,$$

we can calculate as follows:

$$\begin{aligned}\mathrm{wt}(c[\xi], \theta[\rho]) &= a \otimes \mathrm{wt}(c'[\xi], (\theta|_i)[\rho]) \otimes b = a \otimes l_{c,\theta}(i) \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c,\theta}(i) \otimes b \\ &= l_{c,\theta} \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c,\theta} \enspace ,\end{aligned}$$

where the first equality is due to (3.1), and the last one follows from the definition of $l_{c,\theta}$ and $r_{c,\theta}$. This finishes our proof. □

Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Then, for every Σ-context c, state q in Q, and loop θ in $\mathrm{Run}_{\mathcal{A}}(q, c, q)$, and nonnegative integer $n \in \mathbb{N}$, the *nth power of* θ, denoted by θ^n, is the (q, q)-run on c^n defined by induction as follows: (i) $\theta^0 = (\varepsilon \mapsto q)$ (recall that we have $c^0 = \square$) and (ii) $\theta^{n+1} = \theta[\theta^n]$.

Theorem 3.2.3. *[1, Thm. 8] and [2, Lm. 5.3] (also* cf. *[13, Lm. 5.3]) Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$, and B be a strong bimonoid. Moreover, let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Then, for every Σ-contexts c' and c, Σ-tree ξ, states q' and q in Q, (q', q)-run θ' of $\mathcal{A}$ on c', (q, q)-run θ of $\mathcal{A}$ on c, and q-run ρ of $\mathcal{A}$ on ξ, and for each $n \in \mathbb{N}$, we have*

$$\mathrm{wt}(c'[c^n[\xi]], \theta'[\theta^n[\rho]]) = l_{c',\theta'} \otimes (l_{c,\theta})^n \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c',\theta'} \enspace .$$

Proof. We first prove by induction the following statement:

$$\mathrm{wt}(c^n[\xi], \theta^n[\rho]) = (l_{c,\theta})^n \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \enspace . \tag{3.3}$$

Induction base: If $n = 0$, then we have $c^0 = \square$ and $\theta^0 = (\varepsilon \mapsto q)$, and thus, $\square[\xi] = \xi$ and $(\varepsilon \mapsto q)[\rho] = \rho$. Hence, we can calculate as follows:

$$\mathrm{wt}(\square[\xi], (\varepsilon \mapsto q)[\rho]) = \mathrm{wt}(\xi, \rho) = \mathbb{1} \otimes \mathrm{wt}(\xi, \rho) \otimes \mathbb{1} = (l_{c,\theta})^0 \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^0 \enspace ,$$

where the last equality is due to the facts that $(l_{c,\theta})^0 = \mathbb{1}$ and $(r_{c,\theta})^0 = \mathbb{1}$.

Induction step: Assume that (3.3) holds true for n. Now we consider Σ-context c^{n+1}, and the (q, q)-run θ^{n+1} of $\mathcal{A}$ on c^{n+1}. Then we have $c^{n+1} = c[c^n]$ and $\theta^{n+1} = \theta[\theta^n]$, and hence, $c^{n+1}[\xi] = c[c^n[\xi]]$ and $\theta^{n+1}[\rho] = \theta[\theta^n[\rho]]$. Thus, we can calculate as follows:

$$\begin{aligned}\mathrm{wt}(c[c^n[\xi]], \theta[\theta^n[\rho]]) &= l_{c,\theta} \otimes \mathrm{wt}(c^n[\xi], \theta^n[\rho]) \otimes r_{c,\theta} \\ &= l_{c,\theta} \otimes (l_{c,\theta})^n \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c,\theta} \\ &= (l_{c,\theta})^{n+1} \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^{n+1} \enspace ,\end{aligned}$$

where the first equality follows from Lemma 3.2.2, the second one holds true by I.H., and the last one is due to the facts that we have $(l_{c,\theta})^{n+1} = (l_{c,\theta} \otimes (l_{c,\theta})^n)$ and

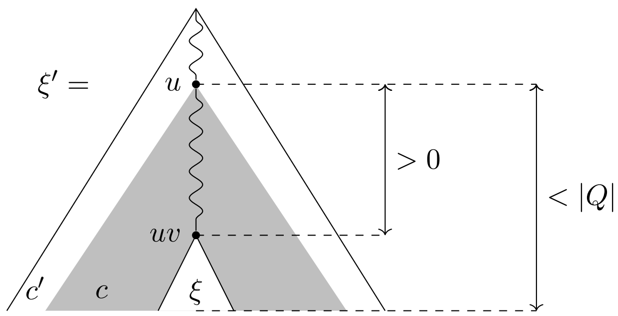


Figure 3.5. *Illustration of the decomposition of the tree ξ' in the proof of Theorem 3.2.4 along the positions u and uv (*cf.* [2, Fig. 2])*

$(r_{c,\theta})^{n+1} = ((r_{c,\theta})^n \otimes r_{c,\theta})$. This completes the proof of (3.3). Now we prove the statement of the lemma as follows:

$$\begin{aligned}\mathrm{wt}(c'[c^n[\xi]], \theta'[\theta^n[\rho]]) &= l_{c',\theta'} \otimes \mathrm{wt}(c^n[\xi], \theta^n[\rho]) \otimes r_{c',\theta'} \\ &= l_{c',\theta'} \otimes (l_{c,\theta})^n \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c',\theta'} \quad ,\end{aligned}$$

where the first equality is due to Lemma 3.2.2, and the second one follows from (3.3). This completes our proof. □

Theorem 3.2.4. *[1, Thm. 9] and [2, Thm. 5.4] (also cf. [13, Lm. 5.5]) Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$, and B be a strong bimonoid. Moreover, let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. For every Σ-tree ξ', state q' in Q, and q'-run ρ' of $\mathcal{A}$ on ξ', if $\mathrm{height}(\xi') \geq |Q|$, then there exist Σ-contexts c' and c, Σ-tree ξ, state q in Q, (q', q)-run θ' of $\mathcal{A}$ on c', (q, q)-run θ of $\mathcal{A}$ on c, and q-run ρ of $\mathcal{A}$ on ξ such that the following conditions hold true: $\xi' = c'[c[\xi]]$, $\rho' = \theta'[\theta[\rho]]$, $\mathrm{height}(c) > 0$, $\mathrm{height}(c[\xi]) < |Q|$, and, for each $n \in \mathbb{N}$, we have*

$$\mathrm{wt}(c'[c^n[\xi]], \theta'[\theta^n[\rho]]) = l_{c',\theta'} \otimes (l_{c,\theta})^n \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c',\theta'} \quad .$$

Proof. Assume that $\mathrm{height}(\xi') \geq |Q|$. Then there exist $u, v \in \mathbb{N}_+^*$ such that $uv \in \mathrm{pos}(\xi')$, $|v| > 0$, $\mathrm{height}(\xi'|_u) < |Q|$, and $\rho'(u) = \rho'(uv)$. Hence, we let $c' = (\xi'|^u)$, $c = ((\xi'|_u)|^v)$, and $\xi = \xi'|_{uv}$. Obviously, we have $\xi' = c'[c[\xi]]$. Figure 3.5 shows the decomposition of ξ' along the positions u and uv.

Moreover, we let $\theta' = (\rho'|_{\mathrm{pos}(c')})$, $\theta = ((\rho'|_u)|_{\mathrm{pos}(c)})$, and $\rho = (\rho'|_{uv})$. Then our statement follows from Theorem 3.2.3. This concludes our proof. □

Chapter conclusion. The author of this PhD thesis declares that his contribution to Theorems 3.2.3 and 3.2.4 is significant, and also that Theorems 3.2.3 and 3.2.4 are published in [1, 2].

Chapter 4

Weighted tree generating regular systems

In this chapter we will discuss the results on weighted tree generating regular systems presented in [4]. In Section 4.1 we explain why we introduce an alternative semantics, called reduction semantics, for tree generating regular systems, and why we define the concept of weighted tree generating regular system by the generalization of the reduction semantics to the weighted case.

In Section 4.2 we recall the concept of tree generating regular system with its derivation semantics and related results from [18]. Furthermore, we introduce our alternative semantics, show the equivalence of the two semantics, and prove normal form lemmas with our alternative semantics.

In Section 4.3 we introduce the concept of weighted tree generating regular system over a strong bimonoid, define its reduction semantics, and show the equivalence of tree generating regular systems and weighted tree generating regular systems over the Boolean semiring.

Finally, in Section 4.4 we prove the equivalence of wta and weighted tree generating regular systems.

4.1 The problem

In [4] the concept of weighted tree generating regular system (for short: wtgrs) was introduced and a further characterization of recognizable weighted tree languages was given. As weight structures of wtgrs, strong bimonoids [21, 30, 66] were used. The aim of that paper was to show that wtgrs and wta are equivalent (correspondingly to the fact that tgrs of [18] and fta are equivalent).

More precisely, (Σ, B)-wtgrs were defined such that the following two require-

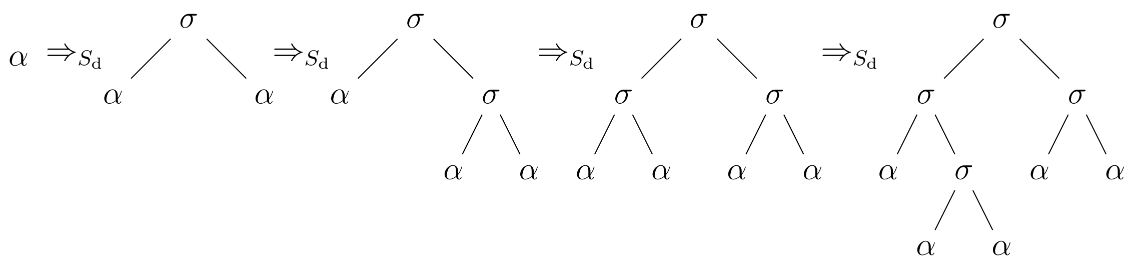


Figure 4.1. *An α-computation of P for the tree $\sigma(\sigma(\alpha, \sigma(\alpha, \alpha)), \sigma(\alpha, \alpha))$ under $\Rightarrow_{S_\mathrm{d}}$. Observe that we may replace the symbols α in an arbitrary order. (*cf. *[4, Fig. 1])*

ments were fulfilled:

(a) Each (Σ, Boole)-wtgrs $\mathcal{S}$ is "equivalent" to a Σ-tgrs S, and vice versa, where Boole is the Boolean semiring given in Example 2.4.6(1).

(b) Under some mild conditions, each (Σ, B)-wtgrs $\mathcal{S}$ is equivalent to a (Σ, B)-wta $\mathcal{A}$, and vice versa. (4.1)

In this chapter we recall the results of that paper. To fully understand those results, here we briefly recall the concept of tgrs and its derivation semantics introduced by Brainerd [18]. Moreover, we show that the seemingly natural generalization of the derivation semantics to the weighted case does not work, *i.e.*, it does not fulfill Requirement (4.1)(b). Finally, we explain the two characteristics of our alternative semantics, called reduction semantics, given in Subsection 4.2.1 for tgrs. In fact, the reduction semantics is essentially the same as the derivation semantics (*cf.* Theorem 4.2.8).

A Σ-*tgrs* (or just *tgrs*) S [18] consists of a ground term rewriting system [8, 22] P over some ranked alphabet Δ and a finite subset Z of designated trees over Δ. The ranked alphabet Δ is partitioned into as follows: the ranked alphabet Σ of *terminals* and the ranked set N of *nonterminals*. Moreover, we call elements of P *productions* and elements of Z *axioms*. The ground term rewrite relation $\Rightarrow_S$ induced by S is defined in the standard way (*cf.* [8, Def. 3.1.8]). Furthermore, the *derivation semantics of S* (for short: *d-semantics of S*) is the set of trees ξ over Σ such that there exist an axiom $\zeta \in Z$ and a ζ-computation of P for ξ under $\Rightarrow_S$, *i.e.*, $\zeta \Rightarrow_S^* \xi$. In order to illustrate the d-semantics of tgrs, Figure 4.1 shows an example of the tgrs S_d where

- Σ consists of the terminals $\sigma^{(2)}$ and $\alpha^{(0)}$; the ranked set N of nonterminals is empty,
- Z is a singleton set consisting of the axiom α, and
- P contains only the production $\alpha \to \sigma(\alpha, \alpha)$.

If in certain steps of a ζ-computation d of P for ξ under $\Rightarrow_S$ we could replace at incomparable positions[1], then there may exist several other ζ-computations of P for ξ under $\Rightarrow_S$. For instance, we can obtain another α-computation of P for the tree in Figure 4.1 if in the second step we replace the leftmost α in $\sigma(\alpha, \alpha)$.

Moreover, we define a (Σ, B)-wtgrs to be a Σ-tgrs in which to each production and to each axiom a weight in B is associated, *i.e.*, a (Σ, B)-wtgrs $\mathcal{S}$ consists of a Σ-tgrs $S = (N, Z, P)$, a mapping $wt : P \to B$ (production weight mapping), and a mapping $X : Z \to B$ (axiom weight mapping). The natural generalization of the d-semantics of tgrs to the weighted case, *i.e.*, the d-semantics of $\mathcal{S}$, would be as follows. For a tree ξ over Σ and an axiom $\zeta \in Z$, and for a ζ-computation d of P for ξ under $\Rightarrow_S$, to calculate the weight of d, we would multiply the weights of the productions in a fixed order determined by d by applying the multiplication operation $\otimes$ of B. Then we calculate the d-semantics of $\mathcal{S}$ for a tree ξ over Σ as follows: by using the addition operation $\oplus$ of B we sum up all weights of ζ-computations of P for ξ under $\Rightarrow_S$ multiplied by the axiom weight $X(\zeta)$. However, this is not suitable to fulfill Requirement (4.1)(b) for the following reason. When we associate a (Σ, B)-wtgrs $\mathcal{S}$ to a (Σ, B)-wta $\mathcal{A}$, more than one computation of P may correspond to a single run of $\mathcal{A}$. Furthermore, since the addition operation $\oplus$ of B is not necessarily idempotent, this may yield that the d-semantics of $\mathcal{S}$ and the semantics of $\mathcal{A}$ differ.

In order to avoid this phenomenon, we advocate an alternative semantics, called *reduction semantics* (for short: *r-semantics*), given in Subsection 4.2.1 for tgrs. The d-semantics and the r-semantics of tgrs are essentially equivalent (*cf.* Theorem 4.2.8). Moreover, we introduce the concept of wtgrs with the natural generalization of the r-semantics of tgrs to the weighted case (*cf.* Subsection 4.3.1). The r-semantics of a tgrs S has two characteristics:

(i) it is based on a restriction of the term rewriting relation, denoted by $\Rightarrow_{S,\mathrm{dp}}$, in which replacements can be performed only at the minimal position (with respect to the depth-first post-ordering of positions) at which a replacement is possible and

(ii) the r-semantics of S is the set of trees ζ over Σ such that there exist an axiom ξ and a ζ-computation of P for ξ under $\Rightarrow_{S,\mathrm{dp}}$.

Figure 4.2 shows an example of a computation by the rewrite relation $\Rightarrow_{S_\mathrm{r},\mathrm{dp}}$ of the Σ-tgrs S_r, where $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, $N = \emptyset$, $Z = \{\alpha\}$, and $P' = \{\sigma(\alpha, \alpha) \to \alpha\}$.

In conclusion, we introduce the r-semantics for the following reasons. Firstly, for each tgrs S there exists a tgrs S' such that the d-semantics of S is equal to the r-semantics of S' (see Figures 4.1 and 4.2). Vice versa, for each tgrs S there exists a tgrs S' such that the r-semantics of S is equal to the d-semantics of S'. This equivalence of the d-semantics and the r-semantics of tgrs is described in Subsection 4.2.2. Secondly, the concept of wtgrs introduced with the natural generalization of the r-semantics of tgrs to the weighted case fulfills Requirements (4.1)(a) and (b)

[1] We call two positions of a tree *incomparable* if none of them is a prefix of the other one.

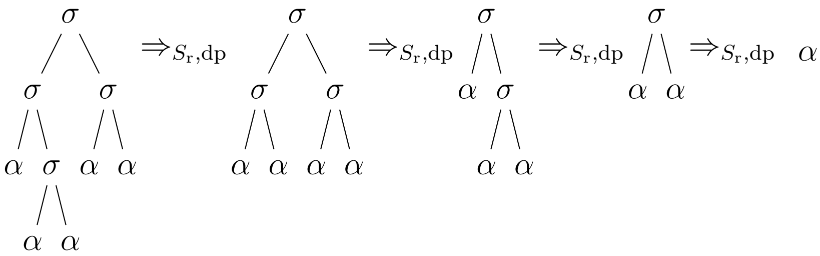


Figure 4.2. *A $\sigma(\sigma(\alpha,\sigma(\alpha,\alpha)),\sigma(\alpha,\alpha))$-computation of P' for α under $\Rightarrow_{S_{\mathrm{r}},\mathrm{dp}}$ (*cf.* [4, Fig. 2])*

(*cf.* Subsection 4.3.2 and Section 4.4, respectively).

4.2 Tree generating regular systems

This section consists of three parts. In Subsection 4.2.1 we recall the concept of ground term rewriting system from [8] and the concept of tree generating regular system with its derivation semantics from [18]. Moreover, we introduce our alternative semantics, called reduction semantics, for tree generating regular systems. In Subsection 4.2.2 we show the equivalence of the derivation semantics and the reduction semantics. Finally, in Subsection 4.2.3 we prove normal form lemmas for tree generating regular systems with reduction semantics. These normal form lemmas will be used to prove corresponding normal form lemmas for weighted tree generating regular systems over strong bimonoids with reduction semantics (*cf.* Subsection 4.3.3).

4.2.1 The model

Ground term rewriting systems. As already mentioned in Section 4.1, a tree generating regular system is a particular ground term rewriting system. For this, let $(\Delta, \mathrm{rk}_\Delta)$ be a ranked alphabet. Formally, a *ground term rewriting system over* Δ (for short: Δ-gtrs, or just: gtrs) [8, 22] is a finite set P of $(\mathrm{T}_\Delta)^2$.

Let P be a Δ-gtrs, and $p = (\eta, \kappa) \in P$. We call p a *production*, and, as usual, we denote it also by $\eta \to \kappa$. The *left-hand side of* p, denoted by $\mathrm{lhs}(p)$, and the *right-hand side of* p, denoted by $\mathrm{rhs}(p)$, are η and κ, respectively.

We define the *rewrite relation of* p, denoted by $\xrightarrow{p}$, as the binary relation on T_Δ such that

$$\begin{aligned}&\text{for every } \zeta, \xi \in \mathrm{T}_\Delta : \\ &\zeta \xrightarrow{p} \xi \quad \text{if there exists } v \in \mathrm{pos}(\zeta) \text{ such that } \zeta|_v = \eta \text{ and } \xi = \zeta[\kappa]_v \ .\end{aligned}$$

Sometimes we also write $\zeta \overset{v,p}{\to} \xi$ to make explicit the position v at which the production p is applied.

Later we will employ two subsets of $\overset{p}{\to}$: one will be used in the definition of derivation semantics (using the full set $\overset{p}{\to}$) and one will be used in the definition of reduction semantics (a strict subset of $\overset{p}{\to}$). Since both definitions use the same concept of computation, in order to avoid repetitions, we introduce a parameter. Formally, let $\rightsquigarrow = (\overset{p}{\rightsquigarrow} \mid p \in P)$ be a family of binary relations $\overset{p}{\rightsquigarrow}$ on T_Δ such that $\overset{p}{\rightsquigarrow} \subseteq \overset{p}{\to}$. Moreover, for each $k \in \mathbb{N}$, we consider P^k as a set of strings over P of length k. Now let $\zeta, \xi \in \mathrm{T}_\Delta$. A *$\zeta$-computation of P for ξ under $\rightsquigarrow$* is a string $d = p_1 \cdots p_k$ in P^k with $k \in \mathbb{N}$ and $p_i \in P$ for each $i \in [k]$ such that there exist $\zeta_0 \in \mathrm{T}_\Delta$ and $(\zeta_1, \ldots, \zeta_k) \in (\mathrm{T}_\Delta)^k$ with

- $\zeta = \zeta_0$,
- $\zeta_{i-1} \overset{p_i}{\rightsquigarrow} \zeta_i$ for each $i \in [k]$, and
- $\zeta_k = \xi$.

We denote a ζ-computation d of P for ξ under $\rightsquigarrow$ also by $\zeta \overset{d}{\rightsquigarrow} \xi$ to make explicit the computation.

Tree generating regular systems. For every ranked set (N, rk_N) with $\Sigma \cap N = \emptyset$, we define the ranked alphabet $(\Sigma \cup N, \mathrm{rk}_{\Sigma \cup N})$ such that, for each σ in $\Sigma \cup N$, we let $\mathrm{rk}_{\Sigma \cup N}(\sigma) = \mathrm{rk}_\Sigma(\sigma)$ if $\sigma \in \Sigma$ and $\mathrm{rk}_{\Sigma \cup N}(\sigma) = \mathrm{rk}_N(\sigma)$ otherwise. A *tree generating regular system over* Σ (for short: Σ-tgrs, or just: tgrs) [18] is a triple $S = (N, Z, P)$, where

- N is a ranked set (of *nonterminals*) such that $\Sigma \cap N = \emptyset$,
- $Z \subseteq \mathrm{T}_{\Sigma \cup N}$ is a finite set (of *axioms*), and
- P is a $(\Sigma \cup N)$-gtrs, *i.e.*, a finite set (of *productions*).

Now we recall the derivation semantics and define the reduction semantics. For this, let $S = (N, Z, P)$ be a Σ-tgrs.

Derivation semantics. We define $\Rightarrow_S = (\overset{p}{\Rightarrow}_S \mid p \in P)$ to be a family of binary relations $\overset{p}{\Rightarrow}_S$ on $\mathrm{T}_{\Sigma \cup N}$ such that $\overset{p}{\Rightarrow}_S = \overset{p}{\to}$. Let $\zeta, \xi \in \mathrm{T}_{\Sigma \cup N}$. A *$\zeta$-derivation of S for* ξ is a ζ-computation of P for ξ under $\Rightarrow_S$. We denote by $\mathrm{Der}_S(\zeta, \xi)$ the *set of all ζ-derivations of S for ξ*, and furthermore, by $\mathrm{Der}_S(\xi)$ the set $\bigcup_{\zeta \in Z} \mathrm{Der}_S(\zeta, \xi)$. The *derivation semantics of* S (for short: d-semantics of S), denoted by $\mathrm{L}_\mathrm{d}(S)$, is the Σ-tree language

$$\mathrm{L}_\mathrm{d}(S) = \{\xi \in \mathrm{T}_\Sigma \mid \mathrm{Der}_S(\xi) \neq \emptyset\} \ .$$

Two Σ-tgrs S and S' are *d-equivalent* if $\mathrm{L}_\mathrm{d}(S) = \mathrm{L}_\mathrm{d}(S')$. Moreover, for each Σ-tree language L, we say that L is *d-generated* if there exists a Σ-tgrs S such that $L = \mathrm{L}_\mathrm{d}(S)$.

Reduction semantics. For the reduction semantics, we define a particular subfamily of the family $\Rightarrow_S = (\stackrel{p}{\Rightarrow}_S \mid p \in P)$. We obtain it by requiring that productions may be applied at positions which are minimal, with respect to the depth-first post-order, among those positions at which a production can be applied. Formally, let $\zeta \in \mathrm{T}_\Sigma$. The *depth-first post-ordering on* $\mathrm{pos}(\zeta)$, denoted by $\preceq_{\mathrm{dp}}$, is the linear ordering defined, for every $w, v \in \mathrm{pos}(\zeta)$, by

$$w \preceq_{\mathrm{dp}} v \text{ iff } (v \in \mathrm{prefix}(w)) \vee (\exists u \in \mathrm{prefix}(w) \cap \mathrm{prefix}(v))(\exists i, j \in \mathbb{N}_+) : \\ (ui \in \mathrm{prefix}(w)) \wedge (uj \in \mathrm{prefix}(v)) \wedge (i < j) \enspace .$$

We let $w \prec_{\mathrm{dp}} v$ if $(w \preceq_{\mathrm{dp}} v)$ and $(w \neq v)$. Then, in particular, $\min_{\prec_{\mathrm{dp}}}(\mathrm{pos}(\zeta))$ is the leftmost leaf of ζ.

In the rest of this chapter, we will abbreviate $\preceq_{\mathrm{dp}}$ *and* $\prec_{\mathrm{dp}}$ *by* $\preceq$ *and* $\prec$*, respectively.*

Let $\mathrm{lhs}(P) = \{\mathrm{lhs}(p) \mid p \in P\}$. We define the family $\Rightarrow_{S,\mathrm{dp}} = (\stackrel{p}{\Rightarrow}_{S,\mathrm{dp}} \mid p \in P)$, where $\stackrel{p}{\Rightarrow}_{S,\mathrm{dp}}$ is the binary relation on $\mathrm{T}_{\Sigma \cup N}$ such that

$$\text{for every } \zeta, \xi \in \mathrm{T}_{\Sigma \cup N} \quad : \\ \zeta \stackrel{p}{\Rightarrow}_{S,\mathrm{dp}} \xi \quad \text{if } \zeta \stackrel{w,p}{\Rightarrow} \xi \text{ and } w = \min_{\prec}(\{v \in \mathrm{pos}(\zeta) \mid \zeta|_v \in \mathrm{lhs}(P)\}) \enspace .$$

Let $\zeta, \xi \in \mathrm{T}_{\Sigma \cup N}$. A *$\zeta$-reduction of S to ξ* is a ζ-computation of P for ξ under $\Rightarrow_{S,\mathrm{dp}}$. We denote by $\mathrm{Red}_S(\zeta, \xi)$ the *set of all ζ-reductions of S to ξ*, and furthermore, by $\mathrm{Red}_S(\zeta)$ the set $\bigcup_{\xi \in Z} \mathrm{Red}_S(\zeta, \xi)$. The *reduction semantics of S* (for short: r-semantics of S), denoted by $\mathrm{L_r}(S)$, is the Σ-tree language

$$\mathrm{L_r}(S) = \{\zeta \in \mathrm{T}_\Sigma \mid \mathrm{Red}_S(\zeta) \neq \emptyset\} \enspace .$$

Two Σ-tgrs S and S' are *r-equivalent* if $\mathrm{L_r}(S) = \mathrm{L_r}(S')$. Furthermore, for each Σ-tree language L, we say that L is *r-generated* if there exists a Σ-tgrs S such that $L = \mathrm{L_r}(S)$.

Observe that, for every Σ-tgrs S, a ζ-derivation of S for ξ starts with an axiom ζ and ends in a Σ-tree ξ; however, a ζ-reduction of S to ξ starts with a Σ-tree ζ and ends in an axiom ξ.

Here we give some examples of tgrs. Our first example shows that the tree language $\mathrm{L}(A)$ recognized by the fta A defined in Example 2.3.3 is d-generated.

Example 4.2.1. [4, Ex. 6] (also *cf.* [18, Ex. 3.4]) Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs

$$S = (\{C^{(0)}\}, \{C\}, \{\ C \to \sigma(C, \alpha) \ ,\ C \to \sigma(\alpha, \alpha)\ \}) \enspace .$$

Next we examine the d-semantics of S. For this, let $c = \sigma(\Box, \alpha)$, and for each $n \in \mathbb{N}_+$, we define the Σ-tree ξ_n by $\xi_n = c^n[\alpha]$. Figure 4.3 shows the C-derivation of S for ξ_3.

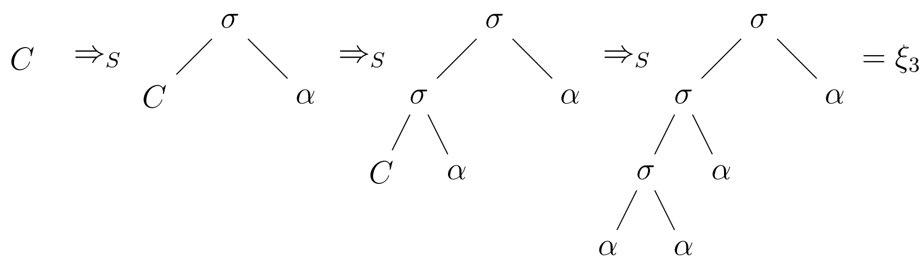


Figure 4.3. *The C-derivation of S defined in Example 4.2.1 for the tree ξ_3 (*cf.* [4, Fig. 3])*

Clearly, we have $\mathrm{L_d}(S) = \{\xi_n \mid n \in \mathbb{N}_+\} = \mathrm{L}(A)$, *i.e.*, the Σ-tree language $\mathrm{L}(A)$ is d-generated. $\triangle$

The next example shows that the d-semantics and the r-semantics do not always coincide.

Example 4.2.2. [4, Ex. 7] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs

$$S_\mathrm{d} = (\emptyset, \{\alpha\}, \{\ \alpha \to \sigma(\alpha, \alpha)\ \})\ .$$

Let $\xi = \sigma(\sigma(\alpha, \sigma(\alpha, \alpha)), \sigma(\alpha, \alpha))$. Figure 4.1 shows the α-derivation of S_d for ξ. Evidently, we have $\mathrm{L_d}(S_\mathrm{d}) = \mathrm{T}_\Sigma$. However, in case of the r-semantics, the axiom α must occur at the end of a reduction, but this cannot be achieved with the production $\alpha \to \sigma(\alpha, \alpha)$. Thus, we have $\mathrm{L_r}(S_\mathrm{d}) = \{\alpha\}$, *i.e.*, $\mathrm{L_d}(S_\mathrm{d}) \neq \mathrm{L_r}(S_\mathrm{d})$.

Although, the two semantics of S_d differ, interestingly, we can give another tgrs S_r such that the r-semantics of S_r and the d-semantics of S_d are the same. For this, it is sufficient to exchange the left-hand side and the right-hand side of the unique production $\alpha \to \sigma(\alpha, \alpha)$. Thus, we consider the Σ-tgrs

$$S_\mathrm{r} = (\emptyset, \{\alpha\}, \{\ \sigma(\alpha, \alpha) \to \alpha\ \})\ .$$

Figure 4.2 illustrates the ξ-reduction of S_r to α. Obviously, we have $\mathrm{L_r}(S_\mathrm{r}) = \mathrm{T}_\Sigma = \mathrm{L_d}(S_\mathrm{d})$, *i.e.*, the Σ-tree language is not just d-generated but also r-generated. $\triangle$

Later we will see that the phenomenon described in Example 4.2.2 holds true in general as well (*cf.* Lemma 4.2.7). Now from [18] we recall a result on the equivalence of fta and tgrs with d-semantics.

Theorem 4.2.3. *[18, Thm. 4.9] For each $L \subseteq \mathrm{T}_\Sigma$, the Σ-tree language L is recognizable if and only if it is d-generated.*

Next, also from [18], we recall normal forms of tgrs and corresponding normal form lemmas for tgrs with d-semantics. Let $S = (N, Z, P)$ be a Σ-tgrs, and let $p = (\eta \to \kappa)$ be in P. We say that p is

- *expansive* if $\eta = A$ and $\kappa = \sigma(A_1, \ldots, A_k)$,
- *contracting* if $\eta = \sigma(A_1, \ldots, A_k)$ and $\kappa = A$, and
- *a chain production* if $\eta = A$ and $\kappa = A'$

for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $(A_1, \ldots, A_k) \in (N^{(0)})^k$, and $A, A' \in N^{(0)}$. For every $A \in N$ and $p \in P$, we say that A *occurs in the left-hand side (right-hand side) of* p if A occurs in $\mathrm{lhs}(p)$ ($\mathrm{rhs}(p)$, respectively). Moreover, we say that S

- has a *single nonterminal axiom* if $Z = \{A\}$ for some $A \in N^{(0)}$, and there do not exist productions p_1 and p_2 in P such that A occurs in the left-hand side of p_1 and in the right-hand side of p_2,
- is *simple* if each production in P has the form either $A \to \sigma(A_1, \ldots, A_k)$, or $\sigma(A_1, \ldots, A_k) \to A$, or $A \to A'$ with $A, A' \in N^{(0)}$, $k \in \mathbb{N}$, $\sigma \in (\Sigma \cup N)^{(k)}$, and $(A_, \ldots, A_k) \in (N^{(0)})^k$, and
- is *expansive* if each production in P is expansive.

Note that each expansive tgrs is obviously a simple tgrs.

> *For each Σ-tgrs $S = (N, Z, P)$, if S has a single nonterminal axiom, then we emphasize this fact by writing Z_0 instead of Z,* i.e.*, $S = (N, Z_0, P)$, and identify Z_0 with its unique element.*

The corresponding normal form lemmas for tgrs with d-semantics are the following.

Lemma 4.2.4. [18, Lm. 3.10] For each Σ-tgrs S, we can construct a Σ-tgrs S' such that S' has a single nonterminal axiom and it is d-equivalent to S. $\square$

Lemma 4.2.5. [18, Lm. 3.12] For each Σ-tgrs S, which has a single nonterminal axiom, we can construct a Σ-tgrs S' such that also S' has a single nonterminal axiom, and furthermore, it is simple and d-equivalent to S. $\square$

Lemma 4.2.6. [18, Lm. 3.15] For each simple Σ-tgrs S, which has a single nonterminal axiom, we can construct a Σ-tgrs S' such that also S' has a single nonterminal axiom, and furthermore, it is expansive and d-equivalent to S. $\square$

4.2.2 Equivalence of the d-semantics and the r-semantics

We devote this subsection to show the equivalence of the d-semantics and the r-semantics of tgrs. For this, the following notations and definition are necessary.

Let $S = (N, Z, P)$ be a Σ-tgrs. For each production $p = (\eta \to \kappa)$ in P, we denote by $\mathrm{rel}(p)$ the production $\kappa \to \eta$. Moreover, for each subset P' of P, we let $\mathrm{rel}(P') = \{\mathrm{rel}(p) \mid p \in P'\}$.

For two Σ-tgrs $S = (N, Z, P)$ and $S' = (N, Z, \mathrm{rel}(P))$, we say that S and S' are *related*. For instance, the two Σ-tgrs S_{d} and S_{r} defined in Example 4.2.2 are related. Clearly, for each Σ-tgrs S, there exists exactly one Σ-tgrs S' such that S and S' are related. We denote this S' also by $\mathrm{rel}(S)$. Moreover, for each Σ-tgrs S, we have $S = \mathrm{rel}(\mathrm{rel}(S))$. Later we will use this fact without any reference.

Lemma 4.2.7. [4, Lm. 14] For each Σ-tgrs S, the following statements hold true: 1. $\mathrm{L}_{\mathrm{d}}(S) = \mathrm{L}_{\mathrm{r}}(\mathrm{rel}(S))$ and 2. $\mathrm{L}_{\mathrm{r}}(S) = \mathrm{L}_{\mathrm{d}}(\mathrm{rel}(S))$.

Proof. We first prove Statement 1. By Lemmas 4.2.4, 4.2.5, and 4.2.6, we may assume that S has a single nonterminal axiom and it is expansive. Let $S = (N, Z_0, P)$ and $\mathrm{rel}(S) = (N, Z_0, \mathrm{rel}(P))$. Then we prove, by induction on the structure of ζ, the following statement:

$$\begin{aligned}&\text{for every } \zeta \in \mathrm{T}_\Sigma \text{ and } A \in N^{(0)}:\\ &\text{we have } \mathrm{Der}_S(A, \zeta) \neq \emptyset \quad \text{iff} \quad \mathrm{Red}_{\mathrm{rel}(S)}(\zeta, A) \neq \emptyset\ .\end{aligned} \tag{4.2}$$

Induction base: Then we have $\zeta = \alpha$ for some $\alpha \in \Sigma^{(0)}$. Moreover, since S is expansive, and S and $\mathrm{rel}(S)$ are related, we have

$$\mathrm{Der}_S(A, \alpha) \neq \emptyset \quad \text{iff} \quad (A \to \alpha) \in P \quad \text{iff} \quad (\alpha \to A) \in \mathrm{rel}(P) \quad \text{iff} \quad \mathrm{Red}_{\mathrm{rel}(S)}(\alpha, A) \neq \emptyset\ .$$

Induction step: Then there exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\zeta_1, \ldots, \zeta_k \in \mathrm{T}_\Sigma$ such that $\zeta = \sigma(\zeta_1, \ldots, \zeta_k)$. Since S is expansive, and S and $\mathrm{rel}(S)$ are related, for every $A_1, \ldots, A_k \in N^{(0)}$, we have

$$(A \to \sigma(A_1, \ldots, A_k)) \in P \quad \text{iff} \quad (\sigma(A_1, \ldots, A_k) \to A) \in \mathrm{rel}(P)\ .$$

Moreover, by I.H., for each $i \in [k]$ and each $A_i \in N^{(0)}$, we have $\mathrm{Der}_S(A_i, \zeta_i) \neq \emptyset$ iff $\mathrm{Red}_{\mathrm{rel}(S)}(\zeta_i, A_i) \neq \emptyset$. Thus, we have

$$\begin{aligned}
&\mathrm{Der}_S(A, \zeta) \neq \emptyset\\
\text{iff}\ \ &\text{there exist } A_1, \ldots, A_k \in N^{(0)} \text{ such that } p = (A \to \sigma(A_1, \ldots, A_k)) \in P\\
&\qquad \text{and } \mathrm{Der}_S(A_i, \zeta_i) \neq \emptyset \text{ for each } i \in [k] \text{ with}\\
&\qquad A \stackrel{p}{\Rightarrow}_S \sigma(A_1, \ldots, A_k) \stackrel{d_1}{\Rightarrow}_S \cdots \stackrel{d_k}{\Rightarrow}_S \sigma(\zeta_1, \ldots, \zeta_k) = \zeta\ ,\\
&\qquad \text{where } d_i \in \mathrm{Der}_S(A_i, \zeta_i) \text{ for each } i \in [k]\\
\text{iff}\ \ &\text{there exist } A_1, \ldots, A_k \in N^{(0)} \text{ such that } \mathrm{rel}(p) = (\sigma(A_1, \ldots, A_k) \to A) \in \mathrm{rel}(P)\\
&\qquad \text{and } \mathrm{Red}_{\mathrm{rel}(S)}(\zeta_i, A_i) \neq \emptyset \text{ for each } i \in [k] \text{ with}\\
&\qquad \zeta = \sigma(\zeta_1, \ldots, \zeta_k) \stackrel{r_1}{\Rightarrow}_{\mathrm{rel}(S),\mathrm{dp}} \cdots \stackrel{r_k}{\Rightarrow}_{\mathrm{rel}(S),\mathrm{dp}} \sigma(A_1, \ldots, A_k) \stackrel{\mathrm{rel}(p)}{\Rightarrow}_{\mathrm{rel}(S),\mathrm{dp}} A\ ,\\
&\qquad \text{where } r_i \in \mathrm{Red}_{\mathrm{rel}(S)(\zeta_i, A_i)} \text{ for each } i \in [k]
\end{aligned}$$

iff $\quad \mathrm{Red}_{\mathrm{rel}(S)}(\zeta, A) \neq \emptyset$.

This completes the proof of (4.2). Finally, for each $\zeta \in \mathrm{T}_\Sigma$, we have

$$\zeta \in \mathrm{L_d}(S) \quad \text{iff} \quad \mathrm{Der}_S(Z_0, \zeta) \neq \emptyset \quad \text{iff}^{(*)} \quad \mathrm{Red}_{\mathrm{rel}(S)}(\zeta, Z_0) \neq \emptyset \quad \text{iff} \quad \zeta \in \mathrm{L_r}(\mathrm{rel}(S)) \ ,$$

where at $(*)$ we apply (4.2). This finishes the proof of Statement 1. To prove Statement 2, it is sufficient to see that we have

$$\mathrm{L_r}(S) = \mathrm{L_r}(\mathrm{rel}(\mathrm{rel}(S))) = \mathrm{L_d}(\mathrm{rel}(S)) \ ,$$

where the second equality is due to Statement 1. □

The main result of this subsection is as follows, *i.e.*, the d-semantics and the r-semantics of tgrs are equivalent.

Theorem 4.2.8. *[4, Thm. 15] Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$. Then, for each $L \subseteq \mathrm{T}_\Sigma$, the Σ-tree language L is d-generated if and only if it is r-generated.*

4.2.3 Normal forms of tgrs with r-semantics

In this subsection, we first define a new normal form for tgrs, called contracting tgrs. Then we prove normal form lemmas for tgrs with r-semantics. One may notice that these new normal form lemmas can be proven in a very similar way as Brainerd proves the corresponding normal form lemmas for tgrs with d-semantics in [18]. Hence, in order to avoid repetitions, we build up our proofs on the corresponding ones in [18] and exploit the equivalence of the d-semantics and the r-semantics of tgrs described in Subsection 4.2.2.

Let $S = (N, Z, P)$ be a Σ-tgrs. We say that S is *contracting* if each production in P is contracting. Clearly, each contracting tgrs is a simple tgrs. Moreover, for each Σ-tgrs S, if S is contracting, then $\mathrm{rel}(S)$ is expansive, and vice versa.

Construction 4.2.9. [18, in the proof of Lm. 3.10] Let $S_\mathrm{d} = (N, Z, P_\mathrm{d})$ be a Σ-tgrs. We construct the Σ-tgrs $S'_\mathrm{d} = (N', Z_0, P'_\mathrm{d})$ such that

- Z_0 consists of a single new nonterminal of rank 0, *i.e.*, $N \cap Z_0 = \emptyset$,
- we identify Z_0 with its one and only element, and
- $N' = N \cup Z_0$, and $P'_\mathrm{d} = P_\mathrm{d} \cup \{Z_0 \to \xi \mid \xi \in Z\}$.

△

Lemma 4.2.10. [4, Lm. 17] (also *cf.* [18, Lm. 3.10]) For each Σ-tgrs S_r, we can construct a Σ-tgrs S'_r such that S'_r has a single nonterminal axiom and it is r-equivalent to S_r.

$$\zeta = \begin{array}{c}\sigma\\ / \quad \backslash\\ \alpha \quad \begin{array}{c}\sigma\\ / \ \backslash\\ \alpha \ \ \alpha\end{array}\end{array} \overset{2,p_2}{\Rightarrow}_{S'_{\mathrm{r}},\mathrm{dp}} \begin{array}{c}\sigma\\ / \ \backslash\\ \alpha \ \ \alpha\end{array} \overset{\varepsilon,p_2}{\Rightarrow}_{S'_{\mathrm{r}},\mathrm{dp}} \alpha \overset{\varepsilon,p_1}{\Rightarrow}_{S'_{\mathrm{r}},\mathrm{dp}} Z_0$$

Figure 4.4. *A ζ-reduction of the tgrs S'_{r} constructed in Example 4.2.11 to Z_0*

Proof. If S_{r} already has a single nonterminal axiom, then we let $S'_{\mathrm{r}} = S_{\mathrm{r}}$ and we are done, otherwise we proceed as follows.

Let $S_{\mathrm{d}} = \mathrm{rel}(S_{\mathrm{r}})$. By applying Lemma 4.2.4 to S_{d}, we can construct a Σ-tgrs S'_{d} such that S'_{d} has a single nonterminal axiom and it is d-equivalent to S_{d}. (In fact, we obtain S'_{d} by applying Construction 4.2.9 to S_{d}.) Let $S'_{\mathrm{r}} = \mathrm{rel}(S'_{\mathrm{d}})$. Obviously, since S'_{d} and S'_{r} are related, also S'_{r} has a single nonterminal axiom. Then we have

$$\mathrm{L}_{\mathrm{r}}(S_{\mathrm{r}}) = \mathrm{L}_{\mathrm{d}}(S_{\mathrm{d}}) = \mathrm{L}_{\mathrm{d}}(S'_{\mathrm{d}}) = \mathrm{L}_{\mathrm{r}}(S'_{\mathrm{r}}) \ ,$$

where the equalities follow from Lemma 4.2.7(2), Lemma 4.2.4, and Lemma 4.2.7(1), respectively. $\square$

In the following example we show an application of Lemma 4.2.10.

Example 4.2.11. [4, Ex. 18] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs S_{r} constructed in Example 4.2.2. Evidently, S_{r} does not have a single nonterminal axiom. However, by following the proof of Lemma 4.2.10, we can construct the Σ-tgrs

$$S'_{\mathrm{r}} = (\{Z_0^{(0)}\}, Z_0, \{\ \underbrace{\alpha \to Z_0}_{p_1}\ ,\ \underbrace{\sigma(\alpha,\alpha) \to \alpha}_{p_2}\ \}) \ .$$

Obviously, S'_{r} has a single nonterminal axiom. Furthermore, by Lemma 4.2.10, it is r-equivalent to S_{r}. Note that, for each Σ-tree ζ, there is exactly one reduction in $\mathrm{Red}_{S'_{\mathrm{r}}}(\zeta) = \mathrm{Red}_{S'_{\mathrm{r}}}(\zeta, Z_0)$. For the Σ-tree $\zeta = \sigma(\alpha, \sigma(\alpha,\alpha))$, Figure 4.4 shows that ζ-reduction of S'_{r} to Z_0. $\triangle$

Construction 4.2.12. [18, in the proof of Lm. 3.12] Let $S_{\mathrm{d}} = (N, Z_0, \{p_1, \ldots, p_n\})$ be a Σ-tgrs with $n \in \mathbb{N}$ such that S_{d} has a single nonterminal axiom. For each $i \in [n]$, we let $p_i = (\eta_i \to \kappa_i)$. We can construct the Σ-tgrs $S'_{\mathrm{d}} = (N', Z_0, P'_{\mathrm{d}})$ such that N' contains of the following nonterminals:

- for every $i \in [n]$ and $v \in \mathrm{pos}(\eta_i)$, let $E_{i,v}$ be a new symbol such that $E_{i,v} \notin N$ and $E_{i,v} \in N'$ with $\mathrm{rk}_{N'}(E_{i,v}) = 0$,
- for every $i \in [n]$ and $u \in \mathrm{pos}(\kappa_i)$, let $F_{i,u}$ be a new symbol such that $F_{i,u} \notin N$ and $F_{i,u} \in N'$ with $\mathrm{rk}_{N'}(F_{i,u}) = 0$,

- for every $A \in N$, we let $A \in N'$ with $\mathrm{rk}_{N'}(A) = \mathrm{rk}_N(A)$, and
- there is no other nonterminal in N';

and the set P'_d consists of the following productions:

- for every $i \in [n]$ and $v \in \mathrm{pos}(\eta_i)$ with $\sigma = \eta_i(v)$ and $k = \mathrm{rk}_{\Sigma \cup N}(\sigma)$:

$$p_{i,-1,v} = (\sigma(E_{i,v1}, \dots, E_{i,vk}) \to E_{i,v}) \text{ is in } P'_\mathrm{d} \ . \tag{4.3a}$$

- for every $i \in [n]$:

$$p_{i,0,\varepsilon} = (E_{i,\varepsilon} \to F_{i,\varepsilon}) \text{ is in } P'_\mathrm{d} \ . \tag{4.3b}$$

- for every $i \in [n]$ and $u \in \mathrm{pos}(\kappa_i)$ with $\omega = \kappa_i(u)$ and $\ell = \mathrm{rk}_{\Sigma \cup N}(\omega)$:

$$p_{i,1,u} = (F_{i,u} \to \omega(F_{i,u1}, \dots, F_{i,u\ell})) \text{ is in } P'_\mathrm{d} \ . \tag{4.3c}$$

Moreover, for each $i \in [n]$, we define

$$(P'_\mathrm{d})_i = \{p_{i,-1,v} \mid v \in \mathrm{pos}(\eta_i)\} \cup \{p_{i,0,\varepsilon}\} \cup \{p_{i,1,u} \mid u \in \mathrm{pos}(\kappa_i)\} \ .$$

Observe that the family $((P'_\mathrm{d})_i \mid i \in [n])$ is a partitioning of P'_d. △

Lemma 4.2.13. [4, Lm. 19] (also *cf.* [18, Lm. 3.12]) For each Σ-tgrs S_r, we can construct a Σ-tgrs S'_r such that it is simple and r-equivalent to S_r. Moreover, if S_r has a single nonterminal axiom, then S'_r has that as well.

Proof. If S_r already has a single nonterminal axiom, then we continue; otherwise, by Lemma 4.2.10, we may assume that S_r has a single nonterminal axiom. If S_r is already simple, then we let $S'_\mathrm{r} = S_\mathrm{r}$, and we are done. Otherwise, we proceed as follows.

Let $S_\mathrm{d} = \mathrm{rel}(S_\mathrm{r})$. By applying Lemma 4.2.5 to S_d, we can construct a Σ-tgrs S'_d such that also S'_d has a single nonterminal axiom, and furthermore, it is simple and d-equivalent to S_d. (As a matter of fact, we obtain S'_d by applying Construction 4.2.12 to S_d.) Now we let $S'_\mathrm{r} = \mathrm{rel}(S'_\mathrm{d})$. Clearly, since S'_d and S'_r are related, also S'_r has a single nonterminal axiom and it is simple. Hence, we have

$$\mathrm{L_r}(S) = \mathrm{L_d}(S_\mathrm{d}) = \mathrm{L_d}(S'_\mathrm{d}) = \mathrm{L_r}(S'_\mathrm{r}) \ ,$$

where the equalities follow from Lemma 4.2.7(2), Lemma 4.2.5, and Lemma 4.2.7(1), respectively. □

The following example shows an application of Lemma 4.2.13.

$$\zeta = \sigma(\alpha, \sigma(\alpha, \alpha)) \overset{1,p'_{2,-1,1}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, \sigma(\alpha, \alpha)) \overset{21,p'_{2,-1,1}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, \sigma(F_{2,1}, \alpha)) \overset{22,p'_{2,-1,2}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, \sigma(F_{2,1}, F_{2,2})) \overset{2,p'_{2,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}}$$

$$\sigma(F_{2,1}, F_{2,\varepsilon}) \overset{2,p'_{2,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, E_{2,\varepsilon}) \overset{2,p'_{2,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, \alpha) \overset{2,p'_{2,-1,2}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, F_{2,2}) \overset{\varepsilon,p'_{2,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}}$$

$$F_{2,\varepsilon} \overset{\varepsilon,p'_{2,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} E_{2,\varepsilon} \overset{\varepsilon,p'_{2,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} \alpha \overset{\varepsilon,p'_{1,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} F_{1,\varepsilon} \overset{\varepsilon,p'_{1,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} E_{1,\varepsilon} \overset{\varepsilon,p'_{1,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{r}},\mathrm{dp}} Z_0$$

Figure 4.5. *A ζ-reduction of the tgrs S''_{r} defined in Example 4.2.14 to Z_0*

Example 4.2.14. [4, Ex. 20] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs S'_{r} given in Example 4.2.11. Recall that S'_{r} has a single nonterminal axiom. Moreover, observe that S'_{r} is not simple as both $\mathrm{lhs}(p_2)$ and $\mathrm{rhs}(p_2)$ are in T_Σ. Nevertheless, by following the proof of Lemma 4.2.13, we can construct the Σ-tgrs $S''_{\mathrm{r}} = (N'', Z_0, P''_{\mathrm{r}})$, where N'' consists of the following nonterminals:

- from p_1 we obtain $E^{(0)}_{1,\varepsilon}$ and $F^{(0)}_{1,\varepsilon}$,
- from p_2 we obtain $E^{(0)}_{2,\varepsilon}$, $F^{(0)}_{2,\varepsilon}$, $F^{(0)}_{2,1}$, and $F^{(0)}_{2,2}$, and
- inherited from S'_{r} we have $Z^{(0)}_0$,

and (P''_{r}) is partitioned by the family $((P''_{\mathrm{r}})_i \mid i \in \{1, 2\})$ with

$$(P''_{\mathrm{r}})_1 = \{ \underbrace{\alpha \to F_{1,\varepsilon}}_{p'_{1,-1,\varepsilon}} , \underbrace{F_{1,\varepsilon} \to E_{1,\varepsilon}}_{p'_{1,0,\varepsilon}} , \underbrace{E_{1,\varepsilon} \to Z_0}_{p'_{1,1,\varepsilon}} \} \text{ and}$$

$$(P''_{\mathrm{r}})_2 = \{ \underbrace{\alpha \to F_{2,1}}_{p'_{2,-1,1}} , \underbrace{\alpha \to F_{2,2}}_{p'_{2,-1,2}} , \underbrace{\sigma(F_{2,1}, F_{2,2}) \to F_{2,\varepsilon}}_{p'_{2,-1,\varepsilon}} , \underbrace{F_{2,\varepsilon} \to E_{2,\varepsilon}}_{p'_{2,0,\varepsilon}} , \underbrace{E_{2,\varepsilon} \to \alpha}_{p'_{2,1,\varepsilon}} \} .$$

Clearly, S''_{r} has a single nonterminal axiom and it is simple. Moreover, it is r-equivalent to S'_{r} by Lemma 4.2.13. Observe that, for each Σ-tree ζ, there is exactly one reduction in $\mathrm{Red}_{S''_{\mathrm{r}}}(\zeta) = \mathrm{Red}_{S''_{\mathrm{r}}}(\zeta, Z_0)$. For the Σ-tree $\zeta = \sigma(\alpha, \sigma(\alpha, \alpha))$, Figure 4.5 depicts that ζ-reduction of S''_{r} to Z'. $\triangle$

Lemma 4.2.15. [18, Lm. 3.13] Let $S_{\mathrm{d}} = (N, Z_0, P_{\mathrm{d}})$ be a Σ-tgrs such that S_{d} has a single nonterminal axiom and it is simple. Then it is decidable, for arbitrary $A \in N^{(0)}$, $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(A_1, \dots, A_k) \in (N^{(0)})^k$, whether the set $\mathrm{Der}_{S_{\mathrm{d}}}(A, \sigma(A_1, \dots, A_k))$ is not empty. $\square$

Construction 4.2.16. [18, in the proof of Lm. 3.15] Let $S_{\mathrm{d}} = (N, Z_0, P_{\mathrm{d}})$ be a Σ-tgrs such that S_{d} has a single nonterminal axiom and it is simple. We can construct a Σ-tgrs $S'_{\mathrm{d}} = (N, Z_0, P'_{\mathrm{d}})$ such that

$$\begin{aligned}&\text{for every } A \in N^{(0)},\ k \in \mathbb{N},\ \sigma \in \Sigma^{(k)}, \text{ and } (A_1, \ldots, A_k) \in (N^{(0)})^k\\ &\text{if the set } \mathrm{Der}_{S_{\mathrm{d}}}(A, \sigma(A_1, \ldots, A_k)) \text{ is not empty,}\\ &\text{then we put the production } A \to \sigma(A_1, \ldots, A_k) \text{ in } P'_{\mathrm{d}}\ .\end{aligned} \tag{4.4}$$

We note that, by Lemma 4.2.15, the condition of (4.4) is decidable. △

Lemma 4.2.17. [4, Lm. 21] (also *cf.* [18, Lm. 3.15]) For each Σ-tgrs S_{r}, we can construct a Σ-tgrs S'_{r} such that S'_{r} is contracting and r-equivalent to S_{r}. Moreover, if S_{r} has a single nonterminal axiom, then S'_{r} is so.

Proof. If S_{r} already has a single nonterminal axiom, then we continue; otherwise, by Lemma 4.2.10, we may assume that S_{r} has a single nonterminal axiom. Moreover, if S_{r} is already simple, then we continue; otherwise, by Lemma 4.2.13, we may assume that S_{r} is simple. If S_{r} is already contracting, then we let $S'_{\mathrm{r}} = S_{\mathrm{r}}$, and we are done. Otherwise, we proceed as follows.

Let $S_{\mathrm{d}} = \mathrm{rel}(S_{\mathrm{r}})$. By applying Lemma 4.2.6 to S_{d}, we can construct a Σ-tgrs S'_{d} such that also S'_{d} has a single nonterminal axiom, and furthermore, it is expansive and d-equivalent to S_{d}. (In fact, we obtain S'_{d} by applying Construction 4.2.16 to S_{d}.) Next we let $S'_{\mathrm{r}} = \mathrm{rel}(S'_{\mathrm{d}})$. Evidently, since S'_{d} has a single nonterminal axiom and it is expansive, and since S'_{d} and S'_{r} are related, the tgrs S'_{r} has a single nonterminal axiom as well and it is contracting. Thus, we have

$$\mathrm{L_r}(S_{\mathrm{r}}) = \mathrm{L_d}(S_{\mathrm{d}}) = \mathrm{L_d}(S'_{\mathrm{d}}) = \mathrm{L_r}(S'_{\mathrm{r}})\ ,$$

where the equalities follow from Lemma 4.2.7(2), Lemma 4.2.6, and Lemma 4.2.7(1), respectively. □

In the following example we show an application of Lemma 4.2.17.

Example 4.2.18. [4, Ex. 22] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs S''_{r} shown in Example 4.2.14. Recall that S''_{r} has a single nonterminal axiom and it is simple. However, S''_{r} is not contracting as the production $p'_{1,0,\varepsilon}$ is not contracting.

We follow the proof of Lemma 4.2.17, and thus, we first consider the Σ-tgrs $S''_{\mathrm{d}} = \mathrm{rel}(S''_{\mathrm{r}})$. In order to construct the expansive Σ-tgrs S'''_{d} d-equivalent to S''_{d}, now we consider particular derivations of S''_{d}. For this, we let $p_{i,-j,\varepsilon} = \mathrm{rel}(p'_{i,j,\varepsilon})$ for every $i \in \{1,2\}$ and $j \in \{-1,0,1\}$, and $p_{2,1,i} = \mathrm{rel}(p'_{2,-1,i})$ for each $i \in \{1,2\}$. Then these particular derivations of S''_{d} as follows:

$$Z_0 \overset{p_{1,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} E_{1,\varepsilon} \overset{p_{1,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} F_{1,\varepsilon} \overset{p_{1,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} \alpha \overset{p_{2,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} E_{2,\varepsilon} \overset{p_{2,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} F_{2,\varepsilon} \overset{p_{2,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} \sigma(F_{2,1}, F_{2,2})\ ,$$

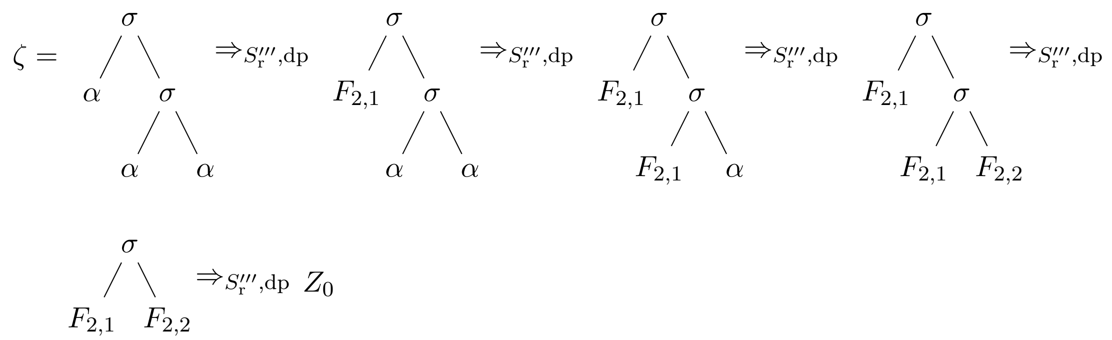


Figure 4.6. *A ζ-reduction of the tgrs S constructed in Example 4.2.18 to Z_0*

$$F_{2,1} \overset{p_{2,1,1}}{\Rightarrow}_{S''_{\mathrm{d}}} \alpha \overset{p_{2,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} E_{2,\varepsilon} \overset{p_{2,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} F_{2,\varepsilon} \overset{p_{2,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} \sigma(F_{2,1}, F_{2,2}) \ , \text{ and}$$
$$F_{2,2} \overset{p_{2,1,2}}{\Rightarrow}_{S''_{\mathrm{d}}} \alpha \overset{p_{2,-1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} E_{2,\varepsilon} \overset{p_{2,0,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} F_{2,\varepsilon} \overset{p_{2,1,\varepsilon}}{\Rightarrow}_{S''_{\mathrm{d}}} \sigma(F_{2,1}, F_{2,2}) \ .$$

Hence, the set $\mathrm{Der}_{S''_{\mathrm{d}}}(A, \sigma(A_1, A_2))$ is not empty for every $A \in N''$, $A_1 \in \{F_{2,1}, E_{2,\varepsilon}, F_{2,\varepsilon}\}$, and $A_2 \in \{F_{2,2}, E_{2,\varepsilon}, F_{2,\varepsilon}\}$; and also the set $\mathrm{Der}_{S''_{\mathrm{d}}}(A, \alpha)$ is not empty for each A in $N'' \setminus \{E_{2,\varepsilon}, F_{2,\varepsilon}\}$. Then we can construct the Σ-tgrs $S'''_{\mathrm{r}} = (N'', Z_0, P'''_{\mathrm{r}})$, where

$$P'''_{\mathrm{r}} = \Bigg(\Bigg(\bigcup_{\substack{A_1 \in \{F_{2,1}, E_{2,\varepsilon}, F_{2,\varepsilon}\} \\ A_2 \in \{F_{2,2}, E_{2,\varepsilon}, F_{2,\varepsilon}\} \\ A \in N''}} \{\sigma(A_1, A_2) \to A\} \Bigg) \cup \Bigg(\bigcup_{A \in N'' \setminus \{E_{2,\varepsilon}, F_{2,\varepsilon}\}} \{\alpha \to A\} \Bigg)\Bigg) \ .$$

Evidently, S'''_{r} has a single nonterminal axiom and it is contracting. Moreover, by Lemma 4.2.17, it is r-equivalent to S''_{r}. Note that, for each Σ-tree ζ, there are more than one ζ-reduction of S'''_{r} to Z_0. *E.g.*, for the Σ-tree $\zeta = \sigma(\alpha, \sigma(\alpha, \alpha))$, Figure 4.6 illustrates a ζ-reduction of S'''_{r} to Z_0. $\triangle$

4.3 Weighted tree generating regular systems

This section is built up as follows. In Subsection 4.3.1 we introduce the concept of weighted tree generating regular system with reduction semantics. In Subsection 4.3.2 we prove the equivalence of tgrs and weighted tree generating regular system over the Boolean semiring. Finally, in Subsection 4.3.3 we define normal forms for weighted tree generating regular systems and prove corresponding normal form lemmas. These normal form lemmas will be used to prove the equivalence of wta and weighted tree generating regular systems over strong bimonoids (*cf.* Section 4.4).

4.3.1 The model

A *weighted tree generating regular system over* Σ *and* B (for short: (Σ, B)-wtgrs, or just: wtgrs) [4] is a triple $\mathcal{S} = (S, wt, X)$, where

- $S = (N, Z, P)$ is a Σ-tgrs,
- $wt : P \to B$ is the *production weight mapping*, and
- $X : Z \to B$ is the *axiom weight mapping*.

We call S the *tgrs underlying* $\mathcal{S}$. Moreover, sometimes, for each $\xi \in Z$, we abbreviate $X(\xi)$ by X_ξ.

> *Observe that, since each* (Σ, B)*-wtgrs* $\mathcal{S}$ *is, basically, an extension of some* Σ*-tgrs, the concepts and abbreviations defined for tree generating regular systems are also available for weighted tree generating regular systems. In particular, we may write* $\Rightarrow_{\mathcal{S},\mathrm{dp}}$ *instead of* $\Rightarrow_{S,\mathrm{dp}}$.

Let $\mathcal{S} = (S, wt, X)$ be a (Σ, B)-wtgrs with $S = (N, Z, P)$. Moreover, let $\zeta, \xi \in \mathrm{T}_{\Sigma \cup N}$. Then we define $\mathrm{Red}_{\mathcal{S}}(\zeta, \xi) = \mathrm{Red}_S(\zeta, \xi)$. From now on, for every $r \in \mathrm{Red}_{\mathcal{S}}(\zeta, \xi)$, we denote ξ also by $\overset{\Rightarrow}{r}$. Let $r = (p_1 \cdots p_k) \in \mathrm{Red}_{\mathcal{S}}(\zeta, \xi)$ with $k \in \mathbb{N}$ and $p_i \in P$ for each $i \in [k]$. We say that r is

- *valid* if $wt(p_i) \neq \mathbb{0}$ for each $i \in [k]$ and
- *successful* if it is valid and $\xi \in \mathrm{supp}(X)$.

We denote the *set of all valid* (*successful*) ζ-reductions of $\mathcal{S}$ to ξ by $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, \xi)$ (respectively, $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, \xi)$). Furthermore, we define the sets

$$\mathrm{Red}_{\mathcal{S}}(\zeta) = \bigcup_{\xi \in Z} \mathrm{Red}_{\mathcal{S}}(\zeta, \xi) \quad \text{and} \quad \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta) = \bigcup_{\xi \in Z} \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, \xi) \quad \text{and}$$
$$\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \bigcup_{\xi \in Z} \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, \xi) \ .$$

The *weight of* r, denoted by $\mathrm{wt}_{\mathcal{S}}(r)$, is the element in B defined by

$$\mathrm{wt}_{\mathcal{S}}(r) = \bigotimes_{i=1}^{k} wt(p_i) \ .$$

In particular, if $\zeta = \xi$, then $k = 0$, and thus, $\mathrm{wt}_{\mathcal{S}}(r) = \mathbb{1}$.

We say that $\mathcal{S}$ is *finite-reductional* if, for each $\zeta \in \mathrm{T}_\Sigma$, the set $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$ is finite. If $\mathcal{S}$ is finite-reductional or B is complete, then the *(reduction) semantics of* $\mathcal{S}$, denoted by $[\![\mathcal{S}]\!]$, is the (Σ, B)-weighted tree language $[\![\mathcal{S}]\!] : \mathrm{T}_\Sigma \to B$ defined, for each $\zeta \in \mathrm{T}_\Sigma$, by

$$[\![\mathcal{S}]\!](\zeta) = \sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{\overset{\Rightarrow}{r}} \ .$$

Note that, for every $\zeta \in \mathrm{T}_\Sigma$ and $r = (p_1 \cdots p_n) \in \mathrm{Red}_{\mathcal{S}}(\zeta)$ with $n \in \mathbb{N}$ and $p_i \in P$ for each $i \in [n]$, if $r \notin \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$, then either there is an $i \in [n]$ such that $wt(p_i) = \mathbb{0}$, or we have $\overset{\Rightarrow}{r} \notin \mathrm{supp}_{\mathsf{B}}(X)$, *i.e.*, $X_{\overset{\Rightarrow}{r}} = \mathbb{0}$. Consequently, we have

$$\sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{\overset{\Rightarrow}{r}} = \bigoplus_{\xi \in \mathrm{supp}_{\mathsf{B}}(X)} \sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta,\xi)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{\xi} \ .$$

Later, we will use this fact without any reference.

For two (Σ, B)-wtgrs $\mathcal{S}$ and $\mathcal{S}'$, we say that $\mathcal{S}$ and $\mathcal{S}'$ are *r-equivalent* if $[\![\mathcal{S}]\!] = [\![\mathcal{S}']\!]$. Moreover, for each (Σ, B)-weighted tree language ψ, we say that ψ is *r-generated* if there exists a (Σ, B)-wtgrs $\mathcal{S}$ such that $[\![\mathcal{S}]\!] = \psi$.

The following example shows that the weighted tree language $\#_{\max}$ defined in Example 3.1.1 is r-generated.

Example 4.3.1. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. We first construct the Σ-tgrs

$$S = (\{A_\sigma^{(0)}, A_\omega^{(0)}, A_s^{(0)}\}, \{A_\sigma, A_\omega\}, P)$$

where P consists of the following productions:

$$\begin{aligned} P \ = \ \{ \ & \underbrace{\alpha \to A_\sigma}_{p_1} \ , \ \underbrace{\sigma(A_\sigma, A_\sigma) \to A_\sigma}_{p_2} \ , \ \underbrace{\omega(A_\sigma, A_\sigma) \to A_\sigma}_{p_3} \ , \\ & \underbrace{\alpha \to A_\omega}_{p_4} \ , \ \underbrace{\sigma(A_\omega, A_\omega) \to A_\omega}_{p_5} \ , \ \underbrace{\omega(A_\omega, A_\omega) \to A_\omega}_{p_6} \ , \\ & \underbrace{\alpha \to A_s}_{p_7} \ , \ \underbrace{\sigma(A_s, A_s) \to A_s}_{p_8} \ , \ \underbrace{\omega(A_s, A_s) \to A_s}_{p_9} \ , \ \underbrace{\omega(A_\sigma, A_\omega) \to A_\sigma}_{p_{10}} \ \} \ . \end{aligned}$$

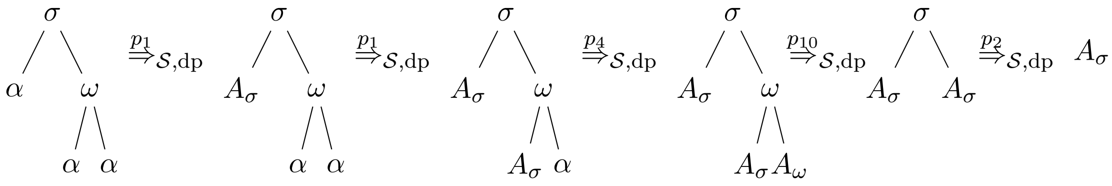
Then we construct the $(\Sigma, \mathsf{MaxPlus})$-wtgrs $\mathcal{S} = (S, wt, X)$ such that $wt(p_i) = 0$ for each $i \in \{1, 3, 4, 5, 7\}$, $wt(p_i) = 1$ for each $i \in \{2, 6, 8, 9\}$, $wt(p_{10}) = -\infty$, and $X(A_\sigma) = X(A_\omega) = 0$.

Let $\zeta = \sigma(\alpha, \omega(\alpha, \alpha))$. Next we consider some reductions in $\mathrm{Red}_{\mathcal{S}}(\zeta)$ as follows (*cf.* Figure 4.7). Figure 4.7(a) shows a not valid reduction in $\mathrm{Red}_{\mathcal{S}}(\zeta, A_\sigma)$. Figure 4.7(b) illustrates a valid but not successful reduction in $\mathrm{Red}_{\mathcal{S}}(\zeta, A_s)$. Finally, Figure 4.7(c) depicts a successful reduction in $\mathrm{Red}_{\mathcal{S}}(\zeta, A_\sigma)$.

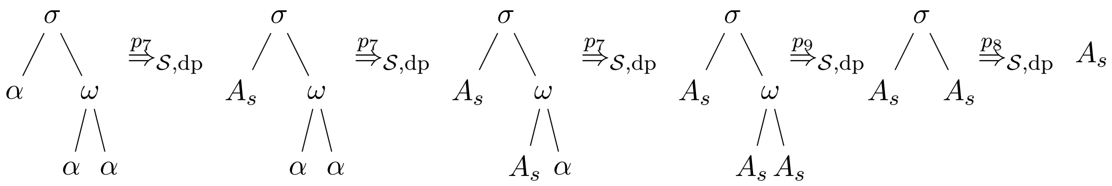
Now we examine the semantics of $\mathcal{S}$. For this, let $\zeta \in \mathrm{T}_\Sigma$. Obviously, there are exactly two reductions in $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$: one ends in A_σ and the other one in A_ω, *i.e.*, $\{r_\sigma\} = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, A_\sigma)$ and $\{r_\omega\} = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, A_\omega)$. Hence, $\mathcal{S}$ is finite-reductional. Observe that $\mathrm{wt}_{\mathcal{S}}(r_\sigma) = |\,\mathrm{pos}_\sigma(\zeta)|$ and $\mathrm{wt}_{\mathcal{S}}(r_\omega) = |\,\mathrm{pos}_\omega(\zeta)|$. Moreover, we have

$$\begin{aligned} [\![\mathcal{S}]\!](\zeta) &= \max(\mathrm{wt}_{\mathcal{S}}(r_\sigma) + X(A_\sigma), \mathrm{wt}_{\mathcal{S}}(r_\omega) + X(A_\omega)) \\ &= \max(|\,\mathrm{pos}_\sigma(\zeta)|, |\,\mathrm{pos}_\omega(\zeta)|) = \#_{\max}(\zeta) \ , \end{aligned}$$

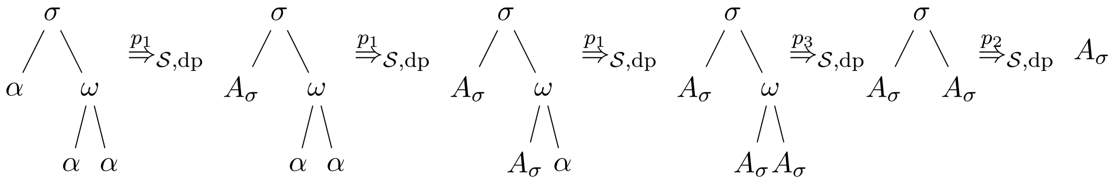
i.e., $\#_{\max}$ is r-generated. $\triangle$



(a) *a not valid reduction*



(b) *a valid but not successful reduction*



(c) *a successful reduction*

Figure 4.7. *Reductions of the* $(\Sigma, \mathsf{MaxPlus})$*-wtgrs* $\mathcal{S}$ *constructed in Example 4.3.1*

The following example proves that the weighted tree language $\#_\sigma$ defined in Example 3.1.2 is also r-generated.

Example 4.3.2. [4, Ex. 23] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs S_r constructed in Example 4.2.2. Then we construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}_\mathrm{r} = (S_\mathrm{r}, wt, X)$ where $wt(\sigma(\alpha, \alpha) \to \alpha) = 1$ and $X_\alpha = 0$. Recall that in case of $\Rightarrow_{\mathcal{S}_\mathrm{r},\mathrm{dp}}$ replacements can be performed only at the minimal position. Hence, for each Σ-tree ζ, there is exactly one reduction in $\mathrm{Red}^\mathrm{s}_{\mathcal{S}_\mathrm{r}}(\zeta)$, *i.e.*, $\mathcal{S}_\mathrm{r}$ is finite-reductional. Moreover, we have $[\![\mathcal{S}_\mathrm{r}]\!] = \#_\sigma$. $\triangle$

4.3.2 Equivalence of tgrs and wtgrs over the Boolean semiring

Next we show that Σ-tgrs and (Σ, Boole)-wtgrs are essentially the same, *i.e.*, Requirement (4.1)(a) is fulfilled. To prove the equivalence of Σ-tgrs and (Σ, Boole)-wtgrs, the following concept is necessary. For each (Σ, B)-wtgrs $\mathcal{S} = (S, wt, X)$ with $S = (N, Z, P)$, the *support tgrs of* $\mathcal{S}$, denoted by $\mathrm{supp}_\mathsf{B}(\mathcal{S})$, is the Σ-tgrs

$$\mathrm{supp}_\mathsf{B}(\mathcal{S}) = (N, \mathrm{supp}_\mathsf{B}(X), \mathrm{supp}_\mathsf{B}(wt)) \ .$$

Note that, for every $\zeta \in \mathrm{T}_\Sigma$ and $\xi \in \mathrm{supp}_{\mathsf{B}}(X)$, we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta,\xi) = \mathrm{Red}_{\mathrm{supp}_{\mathsf{B}}(\mathcal{S})}(\zeta,\xi)$, and thus, $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \mathrm{Red}_{\mathrm{supp}_{\mathsf{B}}(\mathcal{S})}(\zeta)$. We will use this fact without any reference.

Lemma 4.3.3. [4, Lm. 24] For a (Σ, Boole)-wtgrs $\mathcal{S}$, we have

$$\mathrm{supp}_{\mathsf{Boole}}(\llbracket \mathcal{S} \rrbracket) = \mathrm{L}_{\mathrm{r}}(\mathrm{supp}_{\mathsf{Boole}}(\mathcal{S})) \enspace .$$

Proof. Let $\mathcal{S} = (S, wt, X)$ with $S = (N, Z, P)$. Moreover, in the rest of this proof we abbreviate $\mathrm{supp}_{\mathsf{Boole}}$ by supp. We first prove the following statement:

$$\begin{aligned}&\text{for every } \zeta \in \mathrm{T}_\Sigma,\ \xi \in \mathrm{supp}(X), \text{ and } r \in \mathrm{Red}_{\mathcal{S}}(\zeta,\xi)\text{:}\\ &\text{we have } \mathrm{wt}_{\mathcal{S}}(r) \neq 0 \quad \text{iff} \quad r \in \mathrm{Red}_{\mathrm{supp}(\mathcal{S})}(\zeta,\xi) \enspace .\end{aligned} \tag{4.5}$$

Let $r = p_1 \cdots p_k$ with $k \in \mathbb{N}$ and $p_i \in P$ for each $i \in [k]$. Then we have

$$\begin{aligned}\mathrm{wt}_{\mathcal{S}}(r) \neq 0 \quad &\text{iff} \quad \text{we have } wt(p_i) \neq 0 \text{ for each } i \in [k]\\ &\text{iff} \quad \text{we have } p_i \in \mathrm{supp}(wt) \text{ for each } i \in [k]\\ &\text{iff} \quad \text{we have } r \in \mathrm{Red}_{\mathrm{supp}(\mathcal{S})}(\zeta,\xi) \enspace ,\end{aligned}$$

where the second equivalence follows from the fact that the semiring Boole is zero-divisor free. This completes the proof of (4.5). Now recall that the semiring B is complete (*cf.* Example 2.4.6(1)). Then, for each $\zeta \in \mathrm{T}_\Sigma$, we have

$$\begin{aligned}\zeta \in \mathrm{supp}(\llbracket \mathcal{S} \rrbracket) \quad &\text{iff} \quad \Big(\sum^{\vee}_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}}(r) \wedge X_{\vec{r}} \Big) \neq 0\\ &\text{iff}^{(*)} \quad (\exists \xi \in \mathrm{supp}(X))(\exists r \in \mathrm{Red}_{\mathcal{S}}(\zeta,\xi)) : \mathrm{wt}_{\mathcal{S}}(r) \neq 0\\ &\text{iff} \quad (\exists \xi \in \mathrm{supp}(X)) : \mathrm{Red}_{\mathrm{supp}(\mathcal{S})}(\zeta,\xi) \neq \emptyset \quad \text{iff} \quad \zeta \in \mathrm{L}_{\mathrm{r}}(\mathrm{supp}(\mathcal{S})) \enspace ,\end{aligned}$$

where at $(*)$ we use the fact that the semiring Boole is positive; and the last but one equivalence is due to (4.5). □

Theorem 4.3.4. *[4, Thm. 25] Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$. Moreover, let L be a Σ-tree language. Then the following statements are equivalent.*

1. *We can construct a Σ-tgrs such that $\mathrm{L}_{\mathrm{r}}(S) = L$.*
2. *We can construct a (Σ, Boole)-wtgrs such that $\mathrm{supp}_{\mathsf{Boole}}(\llbracket \mathcal{S} \rrbracket) = L$.*

Proof. In the rest of this proof, we abbreviate $\mathrm{supp}_{\mathsf{Boole}}$ by supp.

(1 ⇒ 2). Let $S = (N, Z, P)$. We can construct the (Σ, Boole)-wtgrs $\mathcal{S} = (S, wt, X)$ such that $\mathrm{supp}(X) = Z$ and $\mathrm{supp}(wt) = P$. Clearly, we have $S = \mathrm{supp}(\mathcal{S})$. Moreover, by Lemma 4.3.3, we have $\mathrm{L}_{\mathrm{r}}(S) = \mathrm{L}_{\mathrm{r}}(\mathrm{supp}(\mathcal{S})) = \mathrm{supp}(\llbracket \mathcal{S} \rrbracket)$.

$(2 \Rightarrow 1)$. Let $\mathcal{S} = (S, wt, X)$ with $S = (N, Z, P)$. Evidently, we can construct the Σ-tgrs $\mathrm{supp}(\mathcal{S})$ as described above. Then, by Lemma 4.3.3, we have $\mathrm{L_r}(\mathrm{supp}(\mathcal{S})) = \mathrm{supp}([\![\mathcal{S}]\!])$. $\square$

4.3.3 Normal forms of wtgrs

In the rest of this section we define three normal forms of wtgrs and prove corresponding normal form lemmas. For each (Σ, B)-wtgrs $\mathcal{S} = (S, wt, X)$, we say that $\mathcal{S}$ has a *single nonterminal axiom* (is *simple* or is *contracting*) if S has a single nonterminal axiom (respectively, is simple or is contracting).

Lemma 4.3.5. [4, Lm. 26] For each (Σ, B)-wtgrs $\mathcal{S}$ such that $\mathcal{S}$ is finite-reductional or B is complete, we can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ has a single nonterminal axiom and it is r-equivalent to $\mathcal{S}$. Moreover, if $\mathcal{S}$ is finite-reductional, then $\mathcal{S}'$ is so.

Proof. If $\mathcal{S}$ already has a single nonterminal axiom, then we let $\mathcal{S}' = \mathcal{S}$, and we are done. Otherwise, we proceed as follows.

Let $\mathcal{S} = (S_\mathrm{r}, wt, X)$ with $S_\mathrm{r} = (N, Z, P_\mathrm{r})$. By applying Lemma 4.2.10 to S_r, we can construct a Σ-tgrs S'_r such that S'_r has a single nonterminal axiom and it is r-equivalent to S_r. By following the denotations of Construction 4.2.9, we let $S'_\mathrm{r} = (N', Z_0, P'_\mathrm{r})$, where $P'_\mathrm{r} = \mathrm{rel}(P'_\mathrm{d})$. Then, we can construct the (Σ, B)-wtgrs $\mathcal{S}' = (S'_\mathrm{r}, wt', X')$ such that $X'(Z_0) = \mathbb{1}$, and wt' is defined as follows:
- $wt'(\mathrm{rel}(Z_0 \to \xi)) = X_\xi$ for each $\xi \in Z$ (*cf.* Construction 4.2.9) and
- $wt'(p) = wt(p)$ for each $p \in P_\mathrm{r}$.

Clearly, $\mathcal{S}'_\mathrm{r}$ has a single nonterminal axiom. Next, for each $\zeta \in \mathrm{T}_\Sigma$, we define the mapping $\varphi_\zeta : \mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta) \to \mathrm{Red}^\mathrm{s}_{\mathcal{S}'}(\zeta)$, for each $r \in \mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta)$, by $\varphi_\zeta(r) = r(\overrightarrow{\overrightarrow{r}} \to Z_0)$, where $(\overrightarrow{\overrightarrow{r}} \to Z_0) = \mathrm{rel}(Z_0 \to \overrightarrow{\overrightarrow{r}})$. Then we prove the following statement:

$$\text{for every } \zeta \in \mathrm{T}_\Sigma \text{ and } r \in \mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta)\text{:} \quad \text{we have } \mathrm{wt}_\mathcal{S}(r) \otimes X_{\overrightarrow{\overrightarrow{r}}} = \mathrm{wt}_{\mathcal{S}'}(\varphi_\zeta(r)) \ . \tag{4.6}$$

Since $\varphi_\zeta(r) = r(\overrightarrow{\overrightarrow{r}} \to Z_0)$, we can compute as follows:

$$\mathrm{wt}_\mathcal{S}(r) \otimes X_{\overrightarrow{\overrightarrow{r}}} = \mathrm{wt}_\mathcal{S}(r) \otimes wt'(\overrightarrow{\overrightarrow{r}} \to Z_0) = \mathrm{wt}_{\mathcal{S}'}(r(\overrightarrow{\overrightarrow{r}} \to Z_0)) = \mathrm{wt}_{\mathcal{S}'}(\varphi_\zeta(r)) \ ,$$

and thus, we finish the proof of (4.6). Now we prove the following statement:

$$\begin{array}{l}\text{for each } \zeta \in \mathrm{T}_\Sigma\text{, the mapping } \varphi_\zeta : \mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta) \to \mathrm{Red}^\mathrm{s}_{\mathcal{S}'}(\zeta) \text{ is bijective,} \\ \textit{i.e.}\text{, it is (a) injective and (b) surjective.}\end{array} \tag{4.7}$$

For this, we first show that Statement (a) holds true. Evidently, if $|\mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta)| \leq 1$, then φ_ζ is injective. Hence, we may assume that $|\mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta)| > 1$. Let $r_1, r_2 \in \mathrm{Red}^\mathrm{s}_\mathcal{S}(\zeta)$

such that $r_1 \neq r_2$. Then we have $\zeta \notin Z$, and thus, $\mathrm{len}(r_i) \geq 1$ for each $i \in \{1, 2\}$. Since $r_1 \neq r_2$, there exist an integer i in $[\mathrm{len}(r_1)] \cap [\mathrm{len}(r_2)]$ such that $r_1(1 \dots i-1) = r_2(1 \dots i-1)$ and $r_1(i) \neq r_2(i)$.

Let $i \in \{1, 2\}$. Moreover, let $p_i = (\overrightarrow{r_i} \to Z_0)$. Obviously, we have $p_i \in P'_{\mathrm{r}}$. Since $r_i \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$, we have $X_{\overrightarrow{r_i}} \neq \mathbb{0}$, and thus, $wt'(p_i) \neq \mathbb{0}$. Then we have $\varphi_\zeta(r_i) = rp_i$ such that $\zeta \overset{r_i}{\Rightarrow}_{\mathcal{S}',\mathrm{dp}} \overrightarrow{r_i} \overset{p_i}{\Rightarrow}_{\mathcal{S}',\mathrm{dp}} Z_0$. Then, since $r_1 \neq r_2$, we have $\varphi_\zeta(r_1) \neq \varphi_\zeta(r_2)$, *i.e.*, φ_ζ is injective. This concludes the proof of Statement (a), *i.e.*, φ_ζ is injective.

Now we prove Statement (b). Since $\mathcal{S}'$ has a single nonterminal axiom, we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0)$. Obviously, if $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0) = \emptyset$, then we also have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \emptyset$. Thus, we may assume that $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0) \neq \emptyset$. Let $r' = (p'_1 \cdots p'_k) \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0)$ with $k \in \mathbb{N}_+$ and $p'_i \in P'_{\mathrm{r}}$ for each $i \in [k]$. It follows from our construction that we have $\mathrm{lhs}(p'_k) \in Z$ with $X(\mathrm{lhs}(p'_k)) = wt'(p'_k) \neq \mathbb{0}$, *i.e.*, $\mathrm{lhs}(p'_k) \in \mathrm{supp}_{\mathsf{B}}(X)$. Moreover, since $\mathcal{S}'$ has a single nonterminal axiom, for each $i \in [k-1]$, the nonterminal Z_0 does not occur in the left-hand side of $r'(i)$. Hence, $r = p'_1 \cdots p'_{k-1}$ is in $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$. Note that we have $\varphi(r) = r'$. This finishes the proof of Statement (b), *i.e.*, φ_ζ is surjective. Furthermore, it concludes the proof of (4.7). Finally, for each $\zeta \in \mathrm{T}_\Sigma$, we have

$$\begin{aligned}
[\![\mathcal{S}]\!](\zeta) &= \sum_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)}^{\oplus} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{\overrightarrow{r}} = \sum_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)}^{\oplus} \mathrm{wt}_{\mathcal{S}'}(\varphi_\zeta(r)) \otimes (X')_{Z_0} \\
&= \sum_{r' \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)}^{\oplus} \mathrm{wt}_{\mathcal{S}'}(r') \otimes (X')_{Z_0} = [\![\mathcal{S}']\!](\zeta) \enspace ,
\end{aligned}$$

where the second equality holds true by (4.6); and the last but one equality is due to (4.7) and the fact that family

$$(\{r(\xi \to Z_0) \mid r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, \xi)\} \mid \xi \in Z)$$

is a partitioning of $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0)$. Also, observe that, by (4.7), for each $\zeta \in \mathrm{T}_\Sigma$, we have $|\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)| = |\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)|$. Thus, if $\mathcal{S}$ is finite-reductional, then $\mathcal{S}'$ is so. This completes our proof. $\square$

In the following example we show an application of Lemma 4.3.5.

Example 4.3.6. [4, Ex. 27] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}_{\mathrm{r}}$ constructed in Example 4.3.2. Clearly, $\mathcal{S}$ does not have a single nonterminal axiom. Thus we follow the proof of Lemma 4.3.5. Firstly, by applying Lemma 4.2.10 to S_{r}, we construct the Σ-tgrs S'_{r} such that S'_{r} has a single nonterminal axiom and it is r-equivalent to S_{r} (*cf.* Example 4.2.11). Then we construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}'_{\mathrm{r}} = (S'_{\mathrm{r}}, wt', X')$ such that $wt'(p_1) = W_\alpha = 0$, $wt'(p_2) = wt(p_2) = 1$, and $X'(Z_0) = 0$. Evidently, $\mathcal{S}'_{\mathrm{r}}$ has a single nonterminal axiom. Moreover, by Lemma 4.3.5, $\mathcal{S}'_{\mathrm{r}}$ is r-equivalent to $\mathcal{S}_{\mathrm{r}}$. Note that, since $\mathcal{S}_{\mathrm{r}}$ is finite-reductional, $\mathcal{S}'_{\mathrm{r}}$ is so. Recall that, since $\mathcal{S}'_{\mathrm{r}}$ is an extension of S'_{r}, Figure 4.4 shows a successful $\sigma(\alpha, \sigma(\alpha, \alpha))$-reduction of $\mathcal{S}'$ to Z_0. $\triangle$

Lemma 4.3.7. [4, Lm. 28] Let $\mathcal{S}$ be a (Σ, B)-wtgrs such that $\mathcal{S}$ is finite-reductional or B is complete. We can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ is simple and r-equivalent to $\mathcal{S}$. Moreover, if $\mathcal{S}$ is finite-reductional, then $\mathcal{S}'$ is so. Similarly, if $\mathcal{S}$ has a single nonterminal axiom, $\mathcal{S}'$ has so.

Proof. If $\mathcal{S}$ already has a single nonterminal axiom, then we proceed; otherwise, by Lemma 4.3.5, we may assume that $\mathcal{S}$ has a single nonterminal axiom. If $\mathcal{S}$ is already simple, then we let $\mathcal{S}' = \mathcal{S}$, and we are done. Otherwise, we continue as follows.

Let $\mathcal{S} = (S_{\mathrm{r}}, wt, X)$ with $S_{\mathrm{r}} = (N, Z_0, P_{\mathrm{r}})$ and $P_{\mathrm{r}} = \{(\kappa_1 \to \eta_1), \ldots, (\kappa_n \to \eta_n)\}$ for some $n \in \mathbb{N}$. Firstly, by applying Lemma 4.2.13 to S_{r}, we can construct a Σ-tgrs S'_{r} such that also S'_{r} has a single nonterminal axiom, and furthermore, it is simple and r-equivalent to S_{r}. As already mentioned in the proof of Lemma 4.2.13, in order to construct S'_{r}, we apply Construction 4.2.12 to $\mathrm{rel}(S_{\mathrm{r}})$. Hence, by following the denotations of Construction 4.2.12 and that in the proof of Lemma 4.2.13, we have $S'_{\mathrm{r}} = (N', Z_0, P'_{\mathrm{r}})$, where $P'_{\mathrm{r}} = \mathrm{rel}(P'_{\mathrm{d}})$. Secondly, we can construct the (Σ, B)-wtgrs $\mathcal{S}' = (S'_{\mathrm{r}}, wt', X)$, and we define wt' as follows:

- for every $i \in [n]$ and $v \in \mathrm{pos}(\eta_i)$, the production $p_{i,-1,v}$ defined in (4.3a) in Construction 4.2.12 is in P'_{d}, and hence, the production $p'_{i,1,v} = \mathrm{rel}(p_{i,-1,v})$ is in P'_{r} and we set $wt'(p'_{i,1,v}) = \mathbb{1}$,
- for each $i \in [n]$, the production $p_{i,0,\varepsilon}$ defined in (4.3b) in Construction 4.2.12 is in P'_{d}, and thus, the production $p'_{i,0,\varepsilon} = \mathrm{rel}(p_{i,0,\varepsilon})$ is in P'_{r} and we set $wt'(p'_{i,0,\varepsilon}) = wt(\kappa_i \to \eta_i)$, and
- for every $i \in [n]$ and $u \in \mathrm{pos}(\kappa_i)$, the production $p_{i,1,u}$ defined in (4.3c) in Construction 4.2.12 is in P'_{d}, and so, the production $p'_{i,-1,u} = \mathrm{rel}(p_{i,1,u})$ is in P'_{r} and we set $wt'(p'_{i,-1,u}) = \mathbb{1}$.

Moreover, for every $i \in [n]$, we define

$$(P'_{\mathrm{r}})_i = \{p'_{i,-1,u} \mid u \in \mathrm{pos}(\kappa_i)\} \cup \{p'_{i,0,\varepsilon}\} \cup \{p'_{i,1,v} \mid v \in \mathrm{pos}(\eta_i)\} \ .$$

Note that the family $((P'_{\mathrm{r}})_i \mid i \in [n])$ is a partitioning of P'_{r}. Furthermore, $\mathcal{S}'$ has a single nonterminal axiom and it is simple.

Next we establish a relationship between successful reductions of $\mathcal{S}$ and that of $\mathcal{S}'$. For every $w \in \mathbb{N}_+^*$ and $i \in [n]$, and each $u \in \mathrm{pos}(\kappa_i)$, we define a string over $\mathbb{N}_+^* \times (P'_{\mathrm{r}})_i$, denoted by $\mathbb{L}(wu, \kappa_i|_u)$, inductively as follows: assuming that $\kappa_i(u) \in (\Sigma \cup N)^{(\ell)}$ for some $\ell \in \mathbb{N}$, we let

$$\mathbb{L}(wu, \kappa_i|_u) = \mathbb{L}(wu1, \kappa_i|_{u1}) \cdots \mathbb{L}(wu\ell, \kappa_i|_{u\ell})(wu, p'_{i,-1,u}) \ .$$

Moreover, for every $w \in \mathbb{N}_+^*$ and $i \in [n]$, and each $v \in \mathrm{pos}(\eta_i)$, we define a string over $\mathbb{N}_+^* \times (P'_{\mathrm{r}})_i$, denoted by $\mathbb{R}(wv, \eta_i|_v)$, inductively as follows: assuming that $\eta_i(v) \in (\Sigma \cup N)^{(k)}$ for some $k \in \mathbb{N}$, we let

$$\mathbb{R}(wv, \eta_i|_v) = (wv, p'_{i,1,v})\, \mathbb{R}(wv1, \eta_i|_{v1}) \cdots \mathbb{R}(wvk, \eta_i|_{vk}) \ .$$

Let $\zeta \in \mathrm{T}_\Sigma$. We define a mapping $\varphi_\zeta : \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) \to \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)$ as follows. Recall that we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, Z_0)$ and $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0)$. Assume that $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, Z_0)$ is not empty. Then let $r = ((\kappa_{j_1} \to \eta_{j_1}) \cdots (\kappa_{j_m} \to \eta_{j_m})) \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, Z_0)$ with $m \in \mathbb{N}$ and $j_i \in [n]$ for each $i \in [m]$ such that

$$\zeta \overset{w_1, (\kappa_{j_1} \to \eta_{j_1})}{\Rightarrow_{\mathcal{S},\mathrm{dp}}} \cdots \overset{w_m, (\kappa_{j_m} \to \eta_{j_m})}{\Rightarrow_{\mathcal{S},\mathrm{dp}}} Z_0 \ .$$

Now we consider the string

$$s_r = \mathbb{L}(w_1, \kappa_{j_1})(w_1, p'_{j_1,0,\varepsilon})\, \mathbb{R}(w_1, \eta_{j_1}) \cdots \mathbb{L}(w_m, \kappa_{j_m})(w_m, p'_{j_m,0,\varepsilon})\, \mathbb{R}(w_m, \eta_{j_m}) \tag{4.8}$$

over $\mathbb{N}_+^* \times P'_{\mathrm{r}}$. Let $N = \mathrm{len}(s_r)$. Observe that, since r is successful, for each $i \in [N]$, we have $wt'(\mathrm{pr}_2(s_r(i))) \neq \mathbb{0}$. Moreover, we have $X(Z_0) \neq \mathbb{0}$. Nevertheless, it is easily possible that $\mathrm{pr}_2(s_r(1)) \cdots \mathrm{pr}_2(s_r(N))$ is not a ζ-reduction of S' to Z_0, because by decomposing the productions in P_{r} (*cf.* Construction 4.2.12), we may apply a production $\mathrm{pr}_2(s_r(i))$ for some $i \in [N]$ at a position $\mathrm{pr}_1(s_r(i))$, which is not minimal, with respect to the depth-first post-order, among those positions at which a production can be applied. That phenomenon is quite possible. Thus, we pause the proof of Lemma 4.3.7, and we give an example of this phenomenon here. However, for the sake of simplicity, we consider only the support tgrs of wtgrs.

Example 4.3.8. We consider the Σ-tgrs S'_{r} constructed in Example 4.2.11 and the Σ-tgrs S''_{r} given in Example 4.2.14. Note that each of them can be considered as a support tgrs of some wtgrs. Moreover, S'_{r} is not simple, but S''_{r} is so. Recall that S''_{r} is r-equivalent to S'_{r} (*cf.* Example 4.2.14). Now we consider the successful $\sigma(\alpha, \sigma(\alpha, \alpha))$-reduction r of S'_{r} to Z_0 shown in Figure 4.4, and construct the string s_r as described above. Then we have

$$\begin{aligned} s_r =& (21, p'_{2,-1,1})(22, p'_{2,-1,2})(2, p'_{2,-1,\varepsilon})(2, p'_{2,0,\varepsilon})(2, p'_{2,1,\varepsilon})\underline{(1, p'_{2,-1,1})}(2, p'_{2,-1,2})(\varepsilon, p'_{2,-1,\varepsilon}) \\ & (\varepsilon, p'_{2,0,\varepsilon})(\varepsilon, p'_{2,1,\varepsilon})(\varepsilon, p'_{1,-1,\varepsilon})(\varepsilon, p'_{1,0,\varepsilon})(\varepsilon, p'_{1,1,\varepsilon}) \ . \end{aligned}$$

Note that $\mathrm{pr}_2(s_r(1)) \cdots \mathrm{pr}_2(s_r(13))$ is not a $\sigma(\alpha, \sigma(\alpha, \alpha))$-reduction of S''_{r} to Z_0, because the position $\mathrm{pr}_1(s_r(1)) = 21$ is not minimal, with respect to the depth-first post-order, among those positions of $\sigma(\alpha, \sigma(\alpha, \alpha))$ at which a production can be applied. In this case that minimal position is $\mathrm{pr}_1(s_r(6)) = 1$. $\triangle$

After the brief pause, we proceed with the proof of Lemma 4.3.7. Recall that we can apply a production only at a position, which is minimal, with respect to the depth-first post-order, among those positions at which a production can be applied. Moreover, the decomposition of the productions in P_{r} (*cf.* Construction 4.2.12) also determine which production can be applied next. Thus, there exists a unique permutation s'_r of $s_r(1), \ldots, s_r(N)$ such that $\mathrm{pr}_2(s'_r(1)) \cdots \mathrm{pr}_2(s'_r(N))$ is a successful ζ-reduction of S' to Z_0, and every other permutation of $s_r(1), \ldots, s_r(N)$ is not even a ζ-reduction of S' to Z_0. Note that, in some cases we have $s'_r = s_r$; however, if $s'_r \neq s_r$, then we can obtain s'_r by moving some productions $p'_{i,-1,u}$ with $i \in [n]$ and $u \in \mathrm{pos}(\kappa_i)$ with $wt'(p'_{i,-1,u}) = \mathbb{1}$ forward. Then we let $\varphi_\zeta(r) = \mathrm{pr}_2(s'_r(1)) \cdots \mathrm{pr}_2(s'_r(N))$. Now we pause the proof of Lemma 4.3.7 again, and give an example where $s_r \neq s'_r$.

Example 4.3.9. Here we continue Example 4.3.8. Thus, we consider the string s_r constructed in Example 4.3.8. Then we consider the unique permutation s'_r of $s_r(1), \ldots, s_r(13)$ which is as follows:

$$\begin{aligned} s'_r = &\underline{(1, p'_{2,-1,1})}(21, p'_{2,-1,1})(22, p'_{2,-1,2})(2, p'_{2,-1,\varepsilon})(2, p'_{2,0,\varepsilon})(2, p'_{2,1,\varepsilon})(2, p'_{2,-1,2})(\varepsilon, p'_{2,-1,\varepsilon}) \\ &(\varepsilon, p'_{2,0,\varepsilon})(\varepsilon, p'_{2,1,\varepsilon})(\varepsilon, p'_{1,-1,\varepsilon})(\varepsilon, p'_{1,0,\varepsilon})(\varepsilon, p'_{1,1,\varepsilon}) \ . \end{aligned}$$

Observe that $\mathrm{pr}_2(s'_r(1)) \cdots \mathrm{pr}_2(s'_r(13))$ is a successful $\sigma(\alpha, \sigma(\alpha, \alpha))$-reduction of S''_{r} to Z_0 (*cf.* Figure 4.5). As a matter of fact, the only difference between s_r and s'_r is that $(1, p'_{2,-1,1})$ occurs at the beginning of the string s'_r. △

After our small example, we continue the proof of Lemma 4.3.7. Next we show properties of the mapping φ_ζ. Thus we first prove the following statement:

$$\text{for every } \zeta \in \mathrm{T}_\Sigma \text{ and } r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)\text{: we have } \mathrm{wt}_{\mathcal{S}}(r) = \mathrm{wt}_{\mathcal{S}'}(\varphi_\zeta(r)) \ . \tag{4.9}$$

Clearly, we have $wt(r(i)) \neq \mathbb{0}$ for each $i \in [\mathrm{len}(r)]$. Consequently, it follows from our construction that $wt'(\varphi_\zeta(r)(i)) \neq \mathbb{0}$ for each $i \in [\mathrm{len}(\varphi_\zeta(r))]$. Moreover, observe that, for every $i \in [\mathrm{len}(\varphi_\zeta(r))]$, if $\varphi_\zeta(r)(i) \notin \{p'_{j,0,\varepsilon} \mid j \in [n]\}$, then we have $wt'(\varphi_\zeta(r)(i)) = \mathbb{1}$, *i.e.*, we can leave the weight of the production $\varphi_\zeta(r)(i)$ out of our calculation. Assume that $r = (\kappa_{j_1} \to \eta_{j_1}) \cdots (\kappa_{j_m} \to \eta_{j_m})$ with $m \in \mathbb{N}$ and $j_i \in [n]$ for each $i \in [m]$. Furthermore, assume that $\varphi_\zeta(r)(k_1) = p'_{j_{i_1},0,\varepsilon}$ and $\varphi_\zeta(r)(k_2) = p'_{j_{i_2},0,\varepsilon}$ for some $k_1, k_2 \in [\mathrm{len}(\varphi_\zeta(r))]$ and $i_1, i_2 \in [m]$. By the definition of φ_ζ, we have $i_1 < i_2$ iff $k_1 < k_2$. Thus, we have

$$\mathrm{wt}_{\mathcal{S}}(r) = \bigotimes_{i=1}^{m} wt(\kappa_{j_i} \to \eta_{j_i}) = \bigotimes_{i=1}^{m} wt'(p'_{j_i,0,\varepsilon}) = \mathrm{wt}_{\mathcal{S}'}(\varphi_\zeta(r)) \ .$$

This completes the proof of (4.9). Now we prove the following statement:

$$\text{for each } \zeta \in \mathrm{T}_\Sigma, \text{ the mapping } \varphi_\zeta : \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) \to \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta) \text{ is bijective, } \textit{i.e.}, \text{ it is (a) injective and (b) surjective.} \tag{4.10}$$

We first prove Statement (a). Obviously, if $|\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)| \leq 1$, then φ_ζ is injective. Hence, we may assume that $|\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)| > 1$. Let $r_1, r_2 \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$ such that $r_1 \neq r_2$. Obviously, we have $s_{r_1} \neq s_{r_2}$, and thus, $s'_{r_1} \neq s'_{r_2}$. This completes the proof of Statement (a), *i.e.*, φ_ζ is injective.

Now we show that Statement (b) holds true. Evidently, if $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta) = \emptyset$, then we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \emptyset$ as well. Thus, we may assume that $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta) \neq \emptyset$. Let $r' \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)$ with $N = \mathrm{len}(r')$. Assume that, for each $i \in [N]$, we apply the production $r'(i)$ at w'_i for some $w'_i \in \mathbb{N}^*_+$. By our construction, there exists a unique permutation s_r of $(w'_1, r'(1)), \ldots, (w'_N, r'(N))$ of the form described in (4.8). Then the reduction $r = (\kappa_{j_1} \to \eta_{j_1}) \cdots (\kappa_{j_m} \to \eta_{j_m})$ is in $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$. Moreover, we have $\varphi_\zeta(r) = r'$. This concludes the proof of Statement (b), *i.e.*, φ_ζ is surjective. Moreover, this completes the proof of (4.10). Finally, since, for each $\zeta \in \mathrm{T}_\Sigma$, we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, Z_0)$ and $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0)$, we can calculate as follows:

$$\begin{aligned}
[\![\mathcal{S}]\!](\zeta) &= \bigoplus_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{Z_0} = \bigoplus_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}'}(\varphi_\zeta(r)) \otimes X_{Z_0} \\
&= \bigoplus_{r' \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)} \mathrm{wt}_{\mathcal{S}'}(r') \otimes X_{Z_0} = [\![\mathcal{S}']\!](\zeta) \ ,
\end{aligned}$$

where the second equality is due to (4.9); and the third one follows from the fact φ_ζ is a bijection between the sets $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$ and $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)$ by (4.10). Moreover, since, for each $\zeta \in \mathrm{T}_\Sigma$, the mapping φ_ζ is a bijection between the sets $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$ and $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)$, if $\mathcal{S}$ is finite-reductional, then $\mathcal{S}'$ is so. □

The following example demonstrates an application of Lemma 4.3.7.

Example 4.3.10. [4, Ex. 29] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}'_{\mathrm{r}}$ given in Example 4.3.6. Evidently, $\mathcal{S}'_{\mathrm{r}}$ has a single nonterminal axiom (*cf.* the production p_2). Thus, we follow the proof of Lemma 4.3.7. Firstly, by applying Lemma 4.2.13 to S'_{r}, we construct the Σ-tgrs S''_{r} such that S''_{r} has a single nonterminal axiom, it is simple and r-equivalent to S'_{r} (*cf.* Example 4.2.14). Secondly, we can construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}''_{\mathrm{r}} = (S''_{\mathrm{r}}, wt'', X')$ such that $wt''(p'_{2,\varepsilon,0}) = wt'(p_2) = 1$, and every other production has weight 0. Clearly, $\mathcal{S}''_{\mathrm{r}}$ has a single nonterminal axiom and it is simple. Moreover, by Lemma 4.3.7, it is r-equivalent to $\mathcal{S}'_{\mathrm{r}}$, and furthermore, since $\mathcal{S}'_{\mathrm{r}}$ is finite-reductional, $\mathcal{S}''_{\mathrm{r}}$ is so. Note that, since $\mathcal{S}''_{\mathrm{r}}$ is an extension of the tgrs S''_{r}, Figure 4.5 depicts a successful $\sigma(\alpha, \sigma(\alpha, \alpha))$-reduction of $\mathcal{S}''_{\mathrm{r}}$ to Z_0. △

In order to be able to construct a semantically equivalent contracting wtgrs (*cf.* Lemma 4.3.16(2)) for a wtgrs $\mathcal{S}$, it is crucial that each nonterminal of $\mathcal{S}$ of rank zero is r-useful, *i.e.*, it occurs in at least one successful reduction of $\mathcal{S}$. Fortunately, for a wtgrs $\mathcal{S}$, we can construct another wtgrs $\mathcal{S}'$, of which each nonterminal of rank zero is r-useful (*cf.* Lemma 4.3.13). To construct $\mathcal{S}'$, the following notions, notations, and result (*cf.* Lemma 4.3.12) are necessary.

Let $\mathcal{S} = (S, wt, X)$ be a (Σ, B)-wtgrs with $S = (N, Z_0, P)$ such that $\mathcal{S}$ has a single nonterminal axiom and it is simple. For each $A \in N$, we say that A is *r-useful (in $\mathcal{S}$)* if there exist $\zeta \in \mathrm{T}_\Sigma$, $\xi \in \mathrm{T}_{\Sigma\cup N}$, $r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, \xi)$, and $r' \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi, Z_0)$ such that A occurs in ξ and $(rr') \in \mathrm{Red}^{\mathrm{a}}_{\mathcal{S}}(\zeta, Z_0)$. Moreover, we say that $\mathcal{S}$ is *r-reduced* if each $A \in N^{(0)}$ is r-useful.

Example 4.3.11. Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}''_{\mathrm{r}}$ constructed in Example 4.3.10. Observe that, each nonterminal in $\mathcal{S}''_{\mathrm{r}}$ of rank zero is r-useful, and thus, $\mathcal{S}''_{\mathrm{r}}$ is r-reduced. △

For each ranked set $(N', \mathrm{rk}_{N'})$, we say that $(N', \mathrm{rk}_{N'})$ is a *ranked subset of* (N, rk_N) if $N' \subseteq N$ and $\mathrm{rk}_{N'}(A) = \mathrm{rk}_N(A)$ for each $A \in N'$.

A *regular tree grammar over* Σ (for short: Σ-rtg, or just: rtg) [22, 34, 43] is a Σ-tgrs $G = (N, Z, P)$ such that $N = N^{(0)}$, $Z \subseteq N$, $|Z| = 1$, and each production in P has the form $A \to \kappa$, where $A \in N$ and $\kappa \in \mathrm{T}_{\Sigma\cup N}$. Let $G = (N, Z, P)$ be a Σ-rtg. For each $A \in N$, we say that A is *useful (in G)* if there exist $\zeta \in \mathrm{T}_{\Sigma\cup N}$ and $\xi \in \mathrm{T}_\Sigma$ such that A occurs in ζ, $\mathrm{Der}_G(Z_0, \zeta) \neq \emptyset$, and $\mathrm{Der}_G(\zeta, \xi) \neq \emptyset$. Furthermore, we say that G is *reduced* if each $A \in N$ is useful.

Lemma 4.3.12. *cf.* [22, Prop. 2.1.3] For each Σ-rtg G, we can construct a Σ-rtg G' such that G' is reduced and it is d-equivalent to G. □

Lemma 4.3.13. Let $\mathcal{S}$ be a (Σ, B)-wtgrs such that $\mathcal{S}$ is finite-reductional or B is complete. We can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ is r-reduced and r-equivalent to $\mathcal{S}$. Moreover, if $\mathcal{S}$ has one of the following properties, then so does $\mathcal{S}'$: having single nonterminal axiom, simple, and finite-reductional.

Proof. If $\mathcal{S}$ already has a single nonterminal axiom, then we continue; otherwise, by Lemma 4.3.5, we may assume that. Then, if $\mathcal{S}$ is already simple, then we proceed; otherwise, by Lemma 4.3.7, we may assume that. If $\mathcal{S}$ is already r-reduced, then we set $\mathcal{S}' = \mathcal{S}$ and we are done.

Otherwise, we proceed as follows. Let $\mathcal{S} = (S, wt, X)$ with $S = (N, Z_0, P)$. Recall that we have $\mathrm{supp}_{\mathsf{B}}(\mathcal{S}) = (N, \mathrm{supp}_{\mathsf{B}}(X), \mathrm{supp}_{\mathsf{B}}(wt))$. Clearly, if either $\mathrm{supp}_{\mathsf{B}}(X) = \emptyset$ or $\mathrm{supp}_{\mathsf{B}}(wt) = \emptyset$, then we have $\mathrm{supp}_{\mathsf{B}}([\![\mathcal{S}]\!]) = \emptyset$. If this is the case, then we let $S' = (\{Z_0^{(0)}\}, Z_0, \emptyset)$ and $\mathcal{S}' = (S', wt', X')$ with $wt' : \emptyset \to B$ and $X'(Z_0) = \mathbb{1}$. Obviously, $\mathcal{S}'$ has a single nonterminal axiom, and it is simple, finite-reductional, r-reduced and r-equivalent to $\mathcal{S}$; and we are done.

Consequently, in the rest of this proof we may assume that neither $\mathrm{supp}_{\mathsf{B}}(X) = \emptyset$ nor $\mathrm{supp}_{\mathsf{B}}(wt) = \emptyset$. By following the proof of Lemma 4.2.17, we can construct the Σ-tgrs $\widehat{S} = (N, Z_0, \mathrm{rel}(P'_{\mathrm{d}}))$ such that $\widehat{S}$ is contracting (*cf.* Construction 4.2.16) and it is r-equivalent to $\mathrm{supp}_{\mathsf{B}}(\mathcal{S})$.

We consider the ranked subset (N_0, rk_{N_0}) of (N, rk_N) such that $N_0 = N^{(0)}$, and then, we can construct the Σ-rtg $G = (N_0, Z_0, P'_{\mathrm{d}})$. Next we prove, by induction on the structure of ζ, the following statement:

$$\begin{aligned} &\text{for every } \zeta \in \mathrm{T}_\Sigma \text{ and } A \in N_0\text{:} \\ &\text{we have } \mathrm{Der}_G(A,\zeta) \neq \emptyset \quad \text{iff} \quad \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, A) \neq \emptyset \ . \end{aligned} \tag{4.11}$$

Induction base: There exists $\alpha \in \Sigma^{(0)}$ such that $\zeta = \alpha$. Then we have

$$\begin{aligned} \mathrm{Der}_G(A,\alpha) \neq \emptyset \quad &\text{iff} \quad (A \to \alpha) \in P'_{\mathrm{d}} \\ &\text{iff} \quad (\alpha \to A) \in \mathrm{rel}(P'_{\mathrm{d}}) \quad \text{iff} \quad \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\alpha, A) \neq \emptyset \ , \end{aligned}$$

where the last equivalence follows from our construction and from the proof of Lemma 4.2.17.

Induction step: Then there exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\zeta_1, \ldots, \zeta_k \in \mathrm{T}_\Sigma$ such that $\zeta = \sigma(\zeta_1, \ldots, \zeta_k)$. Evidently, for every $A_1, \ldots, A_k \in N_0$, we have

$$\begin{aligned} (A \to \sigma(A_1, \ldots, A_k)) \in P'_{\mathrm{d}} \quad &\text{iff} \quad (\sigma(A_1, \ldots, A_k) \to A) \in \mathrm{rel}(P'_{\mathrm{d}}) \\ &\text{iff} \quad \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\sigma(A_1, \ldots, A_k), A) \neq \emptyset \ , \end{aligned}$$

where the second equivalence follows from our construction and from the proof of Lemma 4.2.17. Moreover, by I.H., for each $i \in [k]$, and each $A_i \in N_0$, we have $\mathrm{Der}_G(A_i, \zeta_i) \neq \emptyset$ iff $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i) \neq \emptyset$. Thus, we have

$$\begin{aligned} &\mathrm{Der}_G(A,\zeta) \neq \emptyset \\ \text{iff}\ \ &\text{there exist } A_1, \ldots, A_k \in N_0 \text{ such that } p = (A \to \sigma(A_1, \ldots, A_k)) \in P'_{\mathrm{d}} \\ &\quad \text{and } \mathrm{Der}_G(A_i, \zeta_i) \neq \emptyset \text{ for each } i \in [k] \text{ with} \\ &\quad A \stackrel{p}{\Rightarrow}_G \sigma(A_1, \ldots, A_k) \stackrel{d_1}{\Rightarrow}_G \cdots \stackrel{d_k}{\Rightarrow}_G \sigma(\zeta_1, \ldots, \zeta_k) = \zeta \ , \\ &\quad \text{where } d_i \in \mathrm{Der}_G(A_i, \zeta_i) \text{ for each } i \in [k] \\ \text{iff}\ \ &\text{there exist } A_1, \ldots, A_k \in N_0 \text{ such that } r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\sigma(A_1, \ldots, A_k), A) \\ &\quad \text{and } \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i) \neq \emptyset \text{ for each } i \in [k] \text{ with} \\ &\quad \zeta = \sigma(\zeta_1, \ldots, \zeta_k) \stackrel{r_1}{\Rightarrow}_{\mathcal{S},\mathrm{dp}} \cdots \stackrel{r_k}{\Rightarrow}_{\mathcal{S},\mathrm{dp}} \sigma(A_1, \ldots, A_k) \stackrel{r}{\Rightarrow}_{\mathcal{S},\mathrm{dp}} A \ , \\ &\quad \text{where } r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i) \text{ for each } i \in [k] \\ \text{iff}\ \ &\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, A) \neq \emptyset \ . \end{aligned}$$

This completes the proof of (4.11). Moreover, by (4.11) and its proof, for $\zeta \in \mathrm{T}_\Sigma$, we have $\mathrm{Der}_G(Z_0, \zeta) \neq \emptyset$ iff $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, Z_0) \neq \emptyset$, *i.e.*, G is reduced iff $\mathcal{S}$ is r-reduced.

By Lemma 4.3.12, we can construct a Σ-rtg G' such that G' is reduced and d-equivalent to G. As a matter fact, it follows from the proof of [22, Prop. 2.1.3] (*cf.* Lemma 4.3.12) that we have $G' = (N', Z_0, P')$ such that $N' \subseteq N_0$ and $P' \subseteq P'_{\mathrm{d}}$. Observe that we have $N' \subseteq N^{(0)}$. Let P_0 be a subset of $\mathrm{supp}_{\mathsf{B}}(wt)$ such that

- for every $k \in \mathbb{N}$, $\sigma \in (\Sigma \cup N)^{(k)}$, $(A_1, \ldots, A_k) \in (N_0)^k$, and $A \in N_0$, if the production $p = (\sigma(A_1, \ldots, A_k) \to A)$ is in $\mathrm{supp}_{\mathsf{B}}(wt)$, $\sigma \in (\Sigma \cup (N \setminus N_0) \cup N')$, $(A_1, \ldots, A_k) \in (N')^k$, and $A \in N'$, then we put p in P_0,
- for every $A, B \in N_0$, if the production $p = (A \to B)$ is in $\mathrm{supp}_{\mathsf{B}}(wt)$, and both A and B are in N', then we put p in P_0, and
- for every $A \in N_0$, $k \in \mathbb{N}$, $\sigma \in (\Sigma \cup N)^{(k)}$, and $(A_1, \ldots, A_k) \in (N_0)^k$, if the production $p = (A \to \sigma(A_1, \ldots, A_k))$ is in $\mathrm{supp}_{\mathsf{B}}(wt)$, $A \in N'$, $\sigma \in (\Sigma \cup (N \setminus N_0) \cup N')$, and $(A_1, \ldots, A_k) \in (N')^k$, then we put p in P_0.

We define $\widehat{N} = \{A \in N \mid (\exists p \in P_0) : A \text{ occurs in } \mathrm{lhs}(p) \text{ or in } \mathrm{rhs}(p)\}$, *i.e.*, $\widehat{N}$ consists of all the nonterminals occurring in the left-hand side or in the right-hand side of the productions in P_0. Then we consider the ranked subset $(\widehat{N}, \mathrm{rk}_{\widehat{N}})$ of (N, rk_N). Now we can construct the Σ-tgrs $S_0 = (\widehat{N}, Z_0, P_0)$. Moreover, we can construct the (Σ, B)-wtgrs $\mathcal{S}' = (S_0, wt_0, X)$, where $wt_0(p) = wt(p)$ for each $p \in P_0$. Note that, since G' is reduced, the wtgrs $\mathcal{S}'$ is r-reduced.

It follows from our construction that, for each $\zeta \in \mathrm{T}_\Sigma$, we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)$. Furthermore, for every $\zeta \in \mathrm{T}_\Sigma$ and $r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$, we have $\mathrm{wt}_{\mathcal{S}}(r) = \mathrm{wt}_{\mathcal{S}'}(r)$. Thus, for each $\zeta \in \mathrm{T}_\Sigma$, we can compute as follows:

$$[\![\mathcal{S}]\!](\zeta) = \sum\nolimits^{\oplus}_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{Z_0} = \sum\nolimits^{\oplus}_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)} \mathrm{wt}_{\mathcal{S}'}(r) \otimes X_{Z_0} = [\![\mathcal{S}']\!](\zeta) \ .$$

This concludes our proof. □

Our example shows an application of Lemma 4.3.13.

Example 4.3.14. Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the Σ-tgrs

$$S = (\{A^{(3)}, B^{(0)}, C^{(0)}\}, \{C\}, \{\ A(B,B,B) \to B\ ,\ \ \alpha \to B\ ,\ \ \sigma(B,B) \to B\ \})\ .$$

Then we construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S} = (S, wt, X)$ such that $wt(\alpha \to B) = 0$, $wt(\sigma(B,B) \to B) = 2$, $wt(A(B,B,B) \to B) = 3$, and $X(C) = 1$. Evidently, $\mathcal{S}$ has a single nonterminal axiom, and it is simple. Moreover, since, for each $\zeta \in \mathrm{T}_\Sigma$, the set $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \emptyset$, the wtgrs $\mathcal{S}$ is finite-reductional.

Evidently, the nonterminal B is not r-useful, *i.e.*, $\mathcal{S}$ is not r-reduced. Therefore, by following the proof of Lemma 4.3.13, we can construct a $(\Sigma, \mathsf{MinPlus})$-wtgrs

$\mathcal{S}' = ((\{C^{(0)}\}, \{C\}, \emptyset), wt', X)$ with $wt' : \emptyset \to B$. Clearly, $\mathcal{S}'$ r-reduced, and by Lemma 4.3.13, it is r-equivalent to $\mathcal{S}$. △

Let $\mathcal{S} = (S, wt, X)$ be a (Σ, B)-wtgrs with $S = (N, Z, P)$. We say that $\mathcal{S}$ *has nullary nonterminal axioms* if $Z \subseteq N^{(0)}$. Obviously, if $\mathcal{S}$ has a single nonterminal axiom, then $\mathcal{S}$ has nullary nonterminal axioms.

Lemma 4.3.15. *cf.* [4, Lm. 16] Let $\mathcal{S}$ be a (Σ, B)-wtgrs such that $\mathcal{S}$ has nullary nonterminal axioms and it is contracting. Then $\mathcal{S}$ is finite-reductional.

Proof. Let $\mathcal{S} = (S, wt, X)$ with $S = (N, Z, P)$. We first prove, by induction on the structure of ζ, the following statement:

$$\text{for every } \zeta \in \mathrm{T}_\Sigma \text{ and } A \in N^{(0)}, \text{ the set } \mathrm{Red}_{\mathcal{S}}(\zeta, A) \text{ is finite} . \tag{4.12}$$

<u>Induction base:</u> Then we have $\zeta = \alpha$ for some $\alpha \in \Sigma^{(0)}$. Furthermore, since $\mathcal{S}$ is contracting, $\mathrm{Red}_{\mathcal{S}}(\alpha, A) \subseteq \{(\alpha \to A)\}$, *i.e.*, the set $\mathrm{Red}_{\mathcal{S}}(\alpha, A)$ is finite as required.

<u>Induction step:</u> Then there exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\zeta_1, \ldots, \zeta_k \in \mathrm{T}_\Sigma$ such that $\zeta = \sigma(\zeta_1, \ldots, \zeta_k)$. By I.H., for each $i \in [k]$ and each $A_i \in N^{(0)}$, the set $\mathrm{Red}_{\mathcal{S}}(\zeta_i, A_i)$ is finite. Thus, we have

$$|\mathrm{Red}_{\mathcal{S}}(\zeta, A)| \leq \sum_{A_1, \ldots, A_k \in N^{(0)}} \Big(|\mathrm{Red}_{\mathcal{S}}(\zeta_1, A_1)| \cdot \ldots \cdot |\mathrm{Red}_{\mathcal{S}}(\zeta_k, A_k)|\Big) .$$

This completes the proof of (4.12). Let $\zeta \in \mathrm{T}_\Sigma$. Then, since $\mathcal{S}$ has nullary nonterminal axioms, we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) \subseteq \mathrm{Red}_{\mathcal{S}}(\zeta) = \bigcup_{A \in Z} \mathrm{Red}_{\mathcal{S}}(\zeta, A)$. Moreover, by (4.12), for each $A \in Z$, the set $\mathrm{Red}_{\mathcal{S}}(\zeta, A)$ is finite. Consequently, $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$ is finite, *i.e.*, $\mathcal{S}$ is finite-reductional as desired. □

Lemma 4.3.16. [4, Lm. 30] Let B be a semiring and $\mathcal{S}$ be a (Σ, B)-wtgrs such that $\mathcal{S}$ is finite-reductional or B is complete.

1. There exists a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ has a single nonterminal axiom, and it is contracting and r-equivalent to $\mathcal{S}$.
2. If, in addition, B is computable and $\mathcal{S}$ is finite-reductional, then we can construct $\mathcal{S}'$.

Proof. We first prove Statement 1. If $\mathcal{S}$ already has a single nonterminal axiom, then we proceed; otherwise, by Lemma 4.3.5, we may assume that. Similarly, if $\mathcal{S}$ is already simple, then we continue; otherwise, by Lemma 4.3.7, we may assume that. If $\mathcal{S}$ is already contracting, then we set $\mathcal{S}' = \mathcal{S}$, and we are done. Otherwise, we proceed as follows.

By Lemma 4.3.13, we may assume that $\mathcal{S}$ is r-reduced. Let $\mathcal{S} = (S_{\mathrm{r}}, wt, X)$ with $S_{\mathrm{r}} = (N, Z_0, P)$. If we have (a) $\mathrm{supp}_{\mathsf{B}}(wt) = \emptyset$ or (b) $\mathrm{supp}_{\mathsf{B}}(X) = \emptyset$, then, by the proof of Lemma 4.3.13, we have $P = \emptyset$, *i.e.*, $\mathcal{S}$ is contracting, and thus, we set $\mathcal{S}' = \mathcal{S}$, and we are done. Hence, we may assume that neither Assumption (a) nor Assumption (b) holds true. Firstly, by applying Lemma 4.2.17 to S_{r}, we can construct the Σ-tgrs S'_{r} such that also S'_{r} has a single nonterminal axiom, and furthermore, it is contracting and r-equivalent to S_{r}. As already mentioned in the proof of Lemma 4.2.17, to obtain S'_{r} we apply Construction 4.2.16 to S_{r}. Thus, by following the denotations of Construction 4.2.16 and the denotations of Lemma 4.2.17, we have $S'_{\mathrm{r}} = (N, Z_0, \mathrm{rel}(P'_{\mathrm{d}}))$. Next we can construct the (Σ, B)-wtgrs $\mathcal{S}' = (S'_{\mathrm{r}}, wt', X)$ such that wt' is defined as follows:

$$\begin{aligned}
&\text{for every } k \in \mathbb{N},\ \sigma \in \Sigma^{(k)},\ (A_1, \ldots, A_k) \in (N^{(0)})^k, \text{ and } A \in N^{(0)}: \\
&\text{if the production } p = (\sigma(A_1, \ldots, A_k) \to A) \text{ is in } \mathrm{rel}(P'_{\mathrm{d}})\ , \\
&\text{then we set } wt'(p) = \sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\sigma(A_1,\ldots,A_k),A)} \mathrm{wt}_{\mathcal{S}}(r)\ .
\end{aligned} \tag{4.13}$$

Obviously, since S'_{r} has a single nonterminal axiom and it is contracting, also $\mathcal{S}'$ has these properties. Moreover, since $\mathcal{S}'$ has a single nonterminal axiom, *i.e.*, it has nullary nonterminal axioms, by Lemma 4.3.15 , $\mathcal{S}'$ is finite-reductional regardless of whether $\mathcal{S}$ is finite-reductional or not. Next we prove, by induction on the structure of ζ, the following statement:

$$\begin{aligned}
&\text{for every } \zeta \in \mathrm{T}_\Sigma \text{ and } A \in N^{(0)}: \\
&\text{we have } \sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta,A)} \mathrm{wt}_{\mathcal{S}}(r) = \sum^{\oplus}_{r' \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta,A)} \mathrm{wt}_{\mathcal{S}'}(r')\ .
\end{aligned} \tag{4.14}$$

Induction base: Clearly, there exists $\alpha \in \Sigma^{(0)}$ such that $\zeta = \alpha$. Then, by the proof of Lemma 4.2.17, we have $\mathrm{Red}_{\mathcal{S}}(\alpha, A) \neq \emptyset$ iff $(\alpha \to A) \in \mathrm{rel}(P'_{\mathrm{d}})$. If $\mathrm{Red}_{\mathcal{S}}(\alpha, A) = \emptyset$ and $(\alpha \to A) \notin \mathrm{rel}(P'_{\mathrm{d}})$, then, respectively, we have

$$\Big(\sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\alpha,A)} \mathrm{wt}_{\mathcal{S}}(r) \Big) = \mathbb{0} \quad \text{and} \quad \Big(\sum^{\oplus}_{r' \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\alpha,A)} \mathrm{wt}_{\mathcal{S}'}(r') \Big) = \mathbb{0}\ ,$$

where the latter follows from the fact that $\mathcal{S}'$ is contracting, *i.e.*, $\mathrm{Red}_{\mathcal{S}'}(\alpha, A) = \emptyset$. Consequently, our statement holds true. Now assume that we have $\mathrm{Red}_{\mathcal{S}}(\alpha, A) \neq \emptyset$ and $(\alpha \to A) \in \mathrm{rel}(P'_{\mathrm{d}})$. Then we have

$$\sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\alpha,A)} \mathrm{wt}_{\mathcal{S}}(r) = wt'(\alpha \to A) = \sum^{\oplus}_{r' \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\alpha,A)} \mathrm{wt}_{\mathcal{S}'}(r')\ ,$$

where the first equality is due to (4.13), and the second one follows from the fact that $\mathcal{S}'$ is contracting.

Induction step: Then there exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\zeta_1, \ldots, \zeta_k \in \mathrm{T}_\Sigma$ such that

$\zeta = \sigma(\zeta_1, \ldots, \zeta_k)$. By the proof of Lemma 4.2.17, for every $A_1, \ldots, A_k \in N^{(0)}$, we have $\mathrm{Red}_{\mathcal{S}}(\sigma(A_1, \ldots, A_k), A) \neq \emptyset$ iff $(\sigma(A_1, \ldots, A_k) \to A) \in \mathrm{rel}(P'_{\mathrm{d}})$. Moreover, by I.H., for each $i \in [k]$, and each $A_i \in N^{(0)}$, we have $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i) \neq \emptyset$ iff $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta_i, A_i) \neq \emptyset$, and

$$\sum^{\oplus}_{r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i)} \mathrm{wt}_{\mathcal{S}}(r_i) = \sum^{\oplus}_{r'_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta_i, A_i)} \mathrm{wt}_{\mathcal{S}'}(r'_i) \ .$$

Thus, we can compute as follows:

$$\begin{aligned}
&\sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, A)} \mathrm{wt}_{\mathcal{S}}(r) \\
&= \sum^{\oplus}_{\substack{(A_1,\ldots,A_k) \in (N^{(0)})^k \\ \forall i \in [k]: r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i) \\ r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\sigma(A_1,\ldots,A_k), A)}} \Big(\Big(\bigotimes_{i=1}^{k} \mathrm{wt}_{\mathcal{S}}(r_i) \Big) \otimes \mathrm{wt}_{\mathcal{S}}(r) \Big) \\
&= \bigoplus_{(A_1,\ldots,A_k) \in (N^{(0)})^k} \Big(\bigotimes_{i=1}^{k} \Big(\sum^{\oplus}_{r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta_i, A_i)} \mathrm{wt}_{\mathcal{S}}(r_i) \Big) \otimes \sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\sigma(A_1,\ldots,A_k), A)} \mathrm{wt}_{\mathcal{S}}(r) \Big) \\
&= \bigoplus_{(A_1,\ldots,A_k) \in (N^{(0)})^k} \Big(\bigotimes_{i=1}^{k} \Big(\sum^{\oplus}_{r'_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta_i, A_i)} \mathrm{wt}_{\mathcal{S}'}(r'_i) \Big) \otimes wt'(\sigma(A_1, \ldots, A_k) \to A) \Big) \\
&= \sum^{\oplus}_{\substack{(A_1,\ldots,A_k) \in (N^{(0)})^k \\ \forall i \in [k]: r'_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta_i, A_i)}} \Big(\bigotimes_{i=1}^{k} \Big(\mathrm{wt}_{\mathcal{S}'}(r'_i) \Big) \otimes wt'(\sigma(A_1, \ldots, A_k) \to A) \Big) \\
&= \sum^{\oplus}_{r' \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta, A)} \mathrm{wt}_{\mathcal{S}'}(r') \ ,
\end{aligned}$$

where the second and the last but one equalities follow from the fact that B is a semiring; and third equality is due to I.H. and (4.13). This completes the proof of (4.14). Finally, for each $\zeta \in \mathrm{T}_{\Sigma}$, we can compute as follows:

$$[\![\mathcal{S}]\!](\zeta) = \sum^{\oplus}_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{Z_0} = \sum^{\oplus}_{r' \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)} \mathrm{wt}_{\mathcal{S}'}(r') \otimes X_{Z_0} = [\![\mathcal{S}']\!](\zeta) \ ,$$

where the second equality is due to (4.14) and the fact that B is a semiring. This concludes the proof of Statement 1. Now we show that also Statement 2 holds true. We first prove, by contradiction, the following statement:

$$\begin{aligned}&\text{for every } k \in \mathbb{N},\ \sigma \in \Sigma^{(k)},\ (A_1, \ldots, A_k) \in (N^{(0)})^k, \text{ and } A \in N^{(0)} \ , \\ &\text{the set } \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\sigma(A_1, \ldots, A_k), A) \text{ is finite} \ .\end{aligned} \tag{4.15}$$

Note that, by Construction 4.2.16, we have

$$\mathrm{Red}_{\mathcal{S}}(\sigma(A_1,\dots,A_k),A) \neq \emptyset \quad \text{iff} \quad (\sigma(A_1,\dots,A_k) \to A) \in \mathrm{rel}(P'_{\mathrm{d}}) \ .$$

Moreover, recall that we have $\mathrm{supp}_{\mathsf{B}}(wt) = \emptyset$ and $\mathrm{supp}_{\mathsf{B}}(X) \neq \emptyset$. Moreover, let $\zeta \in \mathrm{T}_\Sigma$. Then we have

$$\begin{aligned}
&\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta, Z_0) \neq \emptyset \\
&\text{iff there exists } r' = (p_1 \cdots p_n) \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}'}(\zeta, Z_0) \text{ with } n \in \mathbb{N}_+ \text{ and } p_i \in \mathrm{supp}_{\mathsf{B}}(wt') \\
&\qquad\qquad \text{for each } i \in [n] \\
&\text{iff there exists } r = (r_1 \cdots r_n) \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\zeta, Z_0) \text{ with } n \in \mathbb{N}_+ \text{ and} \\
&\qquad r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\mathrm{lhs}(p_i), \mathrm{rhs}(p_i)) \text{ for some } p_i \in \mathrm{supp}_{\mathsf{B}}(wt') \text{ for each } i \in [n] \\
&\text{iff } \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta, Z_0) \neq \emptyset \ .
\end{aligned}$$

Hence, if there exists $i \in [n]$ such that the set $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\mathrm{lhs}(p_i), \mathrm{rhs}(p_i))$ is not finite, then $\mathcal{S}$ is not finite-reductional. It is a contradiction. This completes the proof of (4.15). Moreover, since B is computable, for each production $\sigma(A_1,\dots,A_k) \to A$, we can compute $wt'(\sigma(A_1,\dots,A_k) \to A)$ in (4.13). Thus, we can construct the wtgrs $\mathcal{S}'$. $\square$

The following example shows an application of Lemma 4.3.16.

Example 4.3.17. [4, Ex. 31] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Note that the semiring $\mathsf{MinPlus}$ given in Example 2.4.6(5) is computable and complete. We consider the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}''_{\mathrm{r}}$ constructed in Example 4.3.10. Obviously, $\mathcal{S}''_{\mathrm{r}}$ has a single nonterminal axiom and it is simple; however, it is not contracting. For this, we follow the proof of Lemma 4.3.16. Observe that $\mathcal{S}''_{\mathrm{r}}$ is r-reduced (*cf.* Example 4.3.11). Firstly, by Lemma 4.2.17, we can construct a Σ-tgrs S'''_{r} such that also S'''_{r} has a single nonterminal axiom, and furthermore, it is contracting and r-equivalent to S''_{r} (*cf.* Example 4.2.18). Next, to compute the weights of productions of S'''_{r} in $\mathcal{S}'''_{\mathrm{r}}$, we consider the following reduction of $\mathcal{S}''_{\mathrm{r}}$, where we use a left brace to show three different ways to finish that reduction:

$$\begin{aligned}
\sigma(F_{2,\varepsilon}, F_{2,\varepsilon}) &\overset{p'_{2,0,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \sigma(E_{2,\varepsilon}, F_{2,\varepsilon}) \overset{p'_{2,1,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \sigma(\alpha, F_{2,\varepsilon}) \overset{p'_{2,-1,1}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, F_{2,\varepsilon}) \\
&\overset{p'_{2,0,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, E_{2,\varepsilon}) \overset{p'_{2,1,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, \alpha) \overset{p'_{2,-1,2}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \sigma(F_{2,1}, F_{2,2}) \\
&\overset{p'_{2,-1,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} F_{2,\varepsilon} \overset{p'_{2,0,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} E_{2,\varepsilon} \\
&\overset{p'_{2,1,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} \alpha \left\{ \begin{array}{l}
\overset{p'_{2,-1,1}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} F_{2,1} \\
\overset{p'_{2,-1,2}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} F_{2,2} \\
\overset{p'_{1,-1,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} F_{1,\varepsilon} \overset{p'_{1,0,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} E_{1,\varepsilon} \overset{p'_{1,1,\varepsilon}}{\Rightarrow}_{\mathcal{S}''_{\mathrm{r}},\mathrm{dp}} Z_0 \ .
\end{array}\right.
\end{aligned}$$

Recall that we have $wt'(p'_{2,0,\varepsilon}) = 1$ and every other production of S''_{r} has weight 0 in $\mathcal{S}''_{\mathrm{r}}$. Then we can construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}'''_{\mathrm{r}} = (S'''_{\mathrm{r}}, wt'', X')$ such that

- for each production $p = (\sigma(A_1, A_2) \to A)$ with $A_1 \in \{F_{2,1}, E_{2,\varepsilon}\}$, $A_2 \in \{F_{2,2}, E_{2,\varepsilon}\}$, we set $wt''(p) = 1$ if $A \in (N'' \setminus \{F_{2,\varepsilon}\})$, and $wt''(p) = 0$ otherwise;
- for each production $p = (\sigma(A_1, A_2) \to A)$ such that the pair (A_1, A_2) is in the set $\{(F_{2,\varepsilon}, F_{2,2}), (F_{2,1}, F_{2,\varepsilon}), (E_{2,\varepsilon}, F_{2,\varepsilon}), (F_{2,\varepsilon}, E_{2,\varepsilon})\}$, we set $wt''(p) = 2$ if $A \in (N'' \setminus \{F_{2,\varepsilon}\})$, and $wt''(p) = 1$ otherwise; and
- for each production $p = (\sigma(F_{2,\varepsilon}, F_{2,\varepsilon}) \to A)$, we set $wt''(p) = 3$ if $A \in (N'' \setminus \{F_{2,\varepsilon}\})$, and $wt''(p) = 2$ otherwise; and
- for each production $p = (\alpha \to A)$ with $A \in (N'' \setminus \{E_{2,\varepsilon}, F_{2,\varepsilon}\})$, we set $wt''(p) = 0$.

Evidently, $\mathcal{S}'''_{\mathrm{r}}$ has a single nonterminal axiom and it is contracting. Recall that, by Lemma 4.3.15, it is finite-reductional. Finally, by Lemma 4.3.16, it is r-equivalent to $\mathcal{S}''_{\mathrm{r}}$. △

4.4 Equivalence of wta and wtgrs

In this section we first define another normal form for wtgrs and prove a corresponding form lemma (*cf.* Lemma 4.4.1). This normal form is crucial to prove the equivalence of wta and wtgrs. Moreover, we show that, if a wta and a wtgrs are related (*cf.* Definition 4.4.3), then their semantics coincide (*cf.* Lemma 4.4.4). Finally, we show that a weighted tree language is recognizable iff it is r-generated (*cf.* Theorem 4.4.5).

For each (Σ, B)-wtgrs $\mathcal{S} = (S, wt, X)$ with $S = (N, Z, P)$, we say that $\mathcal{S}$ is *(contracting) production complete* [4] if, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $(A_1, \dots, A_k) \in (N^{(0)})^k$, and $A \in N^{(0)}$, the production $\sigma(A_1, \dots, A_k) \to A$ is in P.

Lemma 4.4.1. *cf.* [4, Lm. 32] Let $\mathcal{S}$ be a (Σ, B)-wtgrs such that $\mathcal{S}$ has nullary nonterminal axioms and it is contracting. We can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that also $\mathcal{S}'$ has nullary nonterminal axioms and it is contracting, and furthermore, it is production complete and r-equivalent to $\mathcal{S}$.

Proof. If $\mathcal{S}$ is already production complete, then we let $\mathcal{S}' = \mathcal{S}$, and we are done. Otherwise, we proceed as follows. Let $\mathcal{S} = (S, wt, X)$ with $S = (N, Z, P)$. Then we construct the (Σ, B)-wtgrs $\mathcal{S}' = (S', wt', X)$ with $S' = (N, Z, P')$ such that, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $(A_1, \dots, A_k) \in (N^{(0)})^k$, and $A \in N^{(0)}$, we put the production $p = (\sigma(A_1, \dots, A_k) \to A)$ in P', and set $wt'(p) = wt(p)$ if $p \in P$, and $wt'(p) = \mathbb{0}$ otherwise. Evidently, $\mathcal{S}'$ has nullary nonterminal axioms and it is contracting. Now let $\zeta \in \mathrm{T}_\Sigma$. Finally, since each production in $P' \setminus P$ has weight $\mathbb{0}$, we have $\mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta) = \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}'}(\zeta)$, and furthermore, $\mathrm{wt}_{\mathcal{S}}(r) = \mathrm{wt}_{\mathcal{S}'}(r)$ for each $r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\zeta)$. Thus, we have $[\![\mathcal{S}]\!](\zeta) = [\![\mathcal{S}']\!](\zeta)$. □

Next we show an application of Lemma 4.4.1.

Example 4.4.2. Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}_{\mathrm{r}}'''$ constructed in Example 4.3.17. Recall that $\mathcal{S}_{\mathrm{r}}'''$ has nullary nonterminal axioms and it is contracting. Observe that $\mathcal{S}_{\mathrm{r}}'''$ is not production complete as $\sigma(Z_0, Z_0) \to Z_0$ is not a production of the underlying tgrs of $\mathcal{S}_{\mathrm{r}}'''$. However, by Lemma 4.4.1, we can construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}_{\mathrm{r}}''''$ such that $\mathcal{S}_{\mathrm{r}}''''$ has nullary nonterminal axioms and it is contracting, and furthermore, it is production complete and r-equivalent to $\mathcal{S}_{\mathrm{r}}'''$. $\triangle$

Now we explain when we say that a wta and a wtgrs are related.

Definition 4.4.3. Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. Moreover, let $\mathcal{S} = (S, wt, X)$ be a (Σ, B)-wtgrs with $S = (N, Z, P)$ such that $\mathcal{S}$ has nullary nonterminal axioms and it is contracting and production complete. We say that $\mathcal{A}$ and $\mathcal{S}$ are *related* if the following conditions are satisfied:

- $Q = N = N^{(0)}$,
- $Z = \mathrm{supp}_{\mathsf{B}}(F)$,
- $P = \{\sigma(w) \to q \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in Q^k, q \in Q\}$,
- $wt(\sigma(w) \to q) = \delta_k(w, \sigma, q)$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $w \in Q^k$, and $q \in Q$, and
- $X = F|_Z$.

$\triangle$

Clearly, for each (Σ, B)-wta $\mathcal{A}$, there is exactly one (Σ, B)-wtgrs $\mathcal{S}$ such that $\mathcal{A}$ and $\mathcal{S}$ are related. We denote this $\mathcal{S}$ also by $\mathrm{rel}(\mathcal{A})$. Also, vice versa, for each contracting and production complete (Σ, B)-wtgrs $\mathcal{S}$ such that $\mathcal{S}$ has nullary nonterminal axioms, there is exactly one (Σ, B)-wta $\mathcal{A}$ such that $\mathcal{A}$ and $\mathcal{S}$ are related. We denote this $\mathcal{A}$ also by $\mathrm{rel}(\mathcal{S})$.

Lemma 4.4.4. [4, Lm. 33] Let $\mathcal{A}$ be a (Σ, B)-wta and $\mathcal{S}$ be a (Σ, B)-wtgrs such that $\mathcal{A}$ and $\mathcal{S}$ are related. Then we have $[\![\mathcal{A}]\!] = [\![\mathcal{S}]\!]$.

Proof. Since $\mathcal{A}$ and $\mathcal{S}$ are related, conditions of Definition 4.4.3 are satisfied. Hence, we use the denotations of Definition 4.4.3.

For every $\xi = \sigma(\xi_1, \ldots, \xi_k)$ with $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in \mathrm{T}_\Sigma^k$, and $q \in Q$, we define the mapping $\varphi_{\xi,q} : \mathrm{Run}_{\mathcal{A}}^{\mathrm{v}}(q, \xi) \to \mathrm{Red}_{\mathcal{S}}^{\mathrm{v}}(\xi, q)$, for each $\rho \in \mathrm{Run}_{\mathcal{A}}^{\mathrm{v}}(q, \xi)$, as follows: by assuming that $q_i = \varphi(i)$ for each $i \in [k]$ we set

$$\varphi_{\xi,q}(\rho) = \varphi_{\xi_1,q_1}(\rho|_1) \cdots \varphi_{\xi_k,q_k}(\rho|_k)(\sigma(q_1, \ldots, q_k) \to q) \ .$$

Next we show properties of the mapping $\varphi_{\xi,q}$. Firstly, we prove, by induction on the structure of ξ, the following statement:

$$\begin{aligned} &\text{for every } \xi \in \mathrm{T}_\Sigma,\ q \in Q, \text{ and } \rho \in \mathrm{Run}_{\mathcal{A}}^{\mathrm{v}}(q, \xi) : \\ &\text{we have } \mathrm{wt}_{\mathcal{A}}(\xi, \rho) = \mathrm{wt}_{\mathcal{S}}(\varphi_{\xi,q}(\rho)) \ . \end{aligned} \tag{4.16}$$

Induction base: Then there exists $\alpha \in \Sigma^{(0)}$ such that $\xi = \alpha$. Obviously, we have $\rho = (\varepsilon \mapsto q)$ and $\varphi_{\alpha,q}(\rho) = (\alpha \to q)$. Hence, we can calculate as follows:

$$\mathrm{wt}_{\mathcal{A}}(\alpha, \rho) = \delta_0(\varepsilon, \alpha, q) = wt(\alpha \to q) = \mathrm{wt}_{\mathcal{S}}(\varphi_{\alpha,q}(\rho)) \ ,$$

where the second equality follows from the fact that $\mathcal{A}$ and $\mathcal{S}$ are related (*cf.* Definition 4.4.3).

Induction step: Then there exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \dots, \xi_k \in \mathrm{T}_\Sigma$ such that $\xi = \sigma(\xi_1, \dots, \xi_k)$. For each $i \in [k]$, let $q_i = \rho(i)$. Since $\mathcal{A}$ and $\mathcal{S}$ are related, by the definition of $\varphi_{\xi,q}$, the production $p = (\sigma(q_1 \cdots q_k) \to q)$ is in P, and furthermore, for each $i \in [k]$, there exists $r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi_i, q_i)$ such that $\varphi_{\xi_i,q_i}(\rho|_i) = r_i$. Thus, we can calculate as follows:

$$\begin{aligned}
\mathrm{wt}_{\mathcal{A}}(\xi, \rho) &= \Big(\bigotimes_{i=1}^{k} \mathrm{wt}_{\mathcal{A}}(\xi_i, \rho|_i) \Big) \otimes \delta_k(q_1 \cdots q_k, \sigma, q) \\
&= \Big(\bigotimes_{i=1}^{k} \mathrm{wt}_{\mathcal{S}}(\varphi_{\xi_i,q_i}(\rho|_i)) \Big) \otimes wt(p) = \mathrm{wt}_{\mathcal{S}}(\varphi_{\xi,q}(\rho)) \ ,
\end{aligned}$$

where the second equality is due to I.H. and the fact that we have $\delta_k(q_1 \cdots q_k, \sigma, q) = wt(p)$ (*cf.* Definition 4.4.3). This finishes the proof of (4.16). Next we prove, also by induction on the structure of ξ, the following statement:

$$\text{for every } \xi \in \mathrm{T}_\Sigma \text{ and } q \in Q, \text{ the mapping } \varphi_{\xi,q} \text{ is injective} \ . \tag{4.17}$$

Induction base: Then there exists $\alpha \in \Sigma^{(0)}$ such that $\xi = \alpha$. Note that we have $\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \alpha) \subseteq \{\varepsilon \mapsto q\}$. Clearly, since $|\,\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \alpha)| \leq 1$, the mapping $\varphi_{\alpha,q}$ is injective.

Induction step: Then there exist $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \dots, \xi_k \in \mathrm{T}_\Sigma$ such that $\xi = \sigma(\xi_1, \dots, \xi_k)$. Obviously, if $|\,\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi)| \leq 1$, then $\varphi_{\xi,q}$ is injective. Hence, assume that $|\,\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi)| > 1$. Let $\rho_1, \rho_2 \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi)$ such that $\rho_1 \neq \rho_2$. Then there exists $v \in (\mathrm{pos}(\xi) \setminus \{\varepsilon\})$ such that $\rho_1(v) \neq \rho_2(v)$. We proceed by case analysis. If $v = i$ for some $i \in [k]$, then $(\rho_1(1) \cdots \rho_1(k)) \neq (\rho_2(1) \cdots \rho_2(k))$, *i.e.*, $\varphi_{\xi,q}$ is injective. Otherwise, *i.e.*, if $v = iv'$ for some $i \in [k]$ and $v' \in (\mathrm{pos}(\xi_i) \setminus \{\varepsilon\})$, then, since we have $\varphi_{\xi_i,\rho_1(i)} = \varphi_{\xi_i,\rho_2(i)}$, and $\varphi_{\xi_i,\rho_1(i)}$ is injective, we have $\varphi_{\xi_i,\rho_1(i)}(\rho_1|_i) \neq \varphi_{\xi_i,\rho_1(i)}(\rho_2|_i)$, and thus, $\varphi_{\xi,q}(\rho_1) \neq \varphi_{\xi,q}(\rho_2)$. This completes the proof of (4.17). Now we prove, also by induction on the structure of ξ, the following statement:

$$\text{for every } \xi \in \mathrm{T}_\Sigma \text{ and } q \in Q, \text{ the mapping } \varphi_{\xi,q} \text{ is surjective} \ . \tag{4.18}$$

Induction base: Then $\xi = \alpha$ for some $\alpha \in \Sigma^{(0)}$. Since $\mathcal{S}$ is contracting, we have $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\alpha, q) \subseteq \{\alpha \to q\}$. If $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\alpha, q) = \emptyset$, then, since $\mathcal{A}$ and $\mathcal{S}$ are related, we also have $\mathrm{Red}^{\mathrm{v}}_{\mathcal{A}}(q, \alpha) = \emptyset$. Otherwise, *i.e.*, if $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\alpha, q) = \{\alpha \to q\}$, then the production

$p = (\alpha \to q)$ is in P with $wt(p) \neq \mathbb{0}$. Since $\mathcal{A}$ and $\mathcal{S}$ are related, we have $\delta_0(\varepsilon, \alpha, q) \neq \mathbb{0}$ (*cf.* Definition 4.4.3), and thus, we have $\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \alpha) = \{\varepsilon \mapsto q\}$. More precisely, we have $\varphi_{\alpha,q}(\varepsilon \mapsto q) = (\alpha \to q)$ as desired.

Induction step: Then $\xi = \sigma(\xi_1, \dots, \xi_k)$ for some $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \dots, \xi_k \in \mathrm{T}_\Sigma$. Similarly, if $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi, q) = \emptyset$, then, since $\mathcal{A}$ and $\mathcal{S}$ are related, by I.H., we have $\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi) = \emptyset$. Hence, we may assume that $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi, q) \neq \emptyset$. Let $r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi, q)$. Since $\mathcal{S}$ is contracting, there exist $q_i \in Q$ and $r_i \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi_i, q_i)$ for each $i \in [k]$ such that the production $p = (\sigma(q_1, \dots, q_k) \to q)$ is in P with $wt(p) \neq \mathbb{0}$ and $r = r_1 \cdots r_k p$, *i.e.*, we have

$$\xi = \sigma(\xi_1, \dots, \xi_k) \overset{r_1}{\Rightarrow}_{\mathcal{S},\mathrm{dp}} \cdots \overset{r_k}{\Rightarrow}_{\mathcal{S},\mathrm{dp}} \sigma(q_1, \dots, q_k) \overset{p}{\Rightarrow}_{\mathcal{S},\mathrm{dp}} q \enspace .$$

Then, by I.H., for each $i \in [k]$, there exists $\rho_i \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q_i, \xi_i)$ such that $\varphi_{\xi_i,q_i}(\rho_i) = r_i$. Furthermore, since $\mathcal{A}$ and $\mathcal{S}$ are related, we have $\delta_k(q_1 \cdots q_k, \sigma, q) \neq \mathbb{0}$ (*cf.* Definition 4.4.3). Thus, we can consider the run $\rho \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi)$ such that $\rho|_i = \rho_i$ for each $i \in [k]$. Obviously, we have $\varphi_{\xi,q}(\rho) = r$. This completes the proof of (4.18). Moreover, this concludes the proofs of the properties of the mapping $\varphi_{\xi,q}$. Finally, since $\mathcal{A}$ and $\mathcal{S}$ are related (*cf.* Definition 4.4.3), for each $\xi \in \mathrm{T}_\Sigma$, we can calculate as follows:

$$\begin{aligned}
[\![\mathcal{A}]\!](\xi) &= \bigoplus_{\rho \in \mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} = \bigoplus_{q \in Z} \bigoplus_{\rho \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q,\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_q \\
&= \bigoplus_{q \in Z} \bigoplus_{\rho \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q,\xi)} \mathrm{wt}_{\mathcal{S}}(\varphi_{\xi,q}(\rho)) \otimes X_q = \bigoplus_{q \in Z} \bigoplus_{r \in \mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi,q)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_q \\
&= \bigoplus_{r \in \mathrm{Red}^{\mathrm{s}}_{\mathcal{S}}(\xi)} \mathrm{wt}_{\mathcal{S}}(r) \otimes X_{\overset{\Rightarrow}{r}} = [\![\mathcal{S}]\!](\xi) \enspace ,
\end{aligned}$$

where the third equality is due to (4.16); and the fourth one follows from the fact that the mapping $\varphi_{\xi,q}$ is a bijection between the sets $\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi)$ and $\mathrm{Red}^{\mathrm{v}}_{\mathcal{S}}(\xi, q)$ by (4.17) and (4.18). $\square$

In the following theorem we prove the equivalence of wta and wtgrs.

Theorem 4.4.5. cf. *[4, Thm. 34] Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$. Then, for every semiring* B *and* (Σ, B)*-weighted tree language ψ, the following statements hold true.*

1. *If* B *is complete, then ψ is recognizable iff it is r-generated.*
2. *If* B *is computable, then we can construct a* (Σ, B)*-wta $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = \psi$ iff we can construct a finite-reductional* (Σ, B)*-wtgrs $\mathcal{S}$ such that $[\![\mathcal{S}]\!] = \psi$.*

Proof. We first prove Statement 1. Assume that ψ is recognizable. Then there exists a (Σ, B)-wta $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = \psi$. Moreover, we can construct the (Σ, B)-wtgrs

$\mathrm{rel}(\mathcal{A})$ such that $\mathcal{A}$ and $\mathrm{rel}(\mathcal{A})$ are related. Recall that, by Definition 4.4.3, $\mathrm{rel}(\mathcal{A})$ has nullary nonterminal axioms and it is contracting. Thus, by Lemma 4.3.15, $\mathrm{rel}(\mathcal{A})$ is finite-reductional. Finally, since $\mathcal{A}$ and $\mathrm{rel}(\mathcal{A})$ are related, by Lemma 4.4.4, we have $[\![\mathrm{rel}(\mathcal{A})]\!] = [\![\mathcal{A}]\!] = \psi$, *i.e.*, ψ is r-generated.

Now we prove the other direction. Assume that ψ is r-generated. Then there exists a (Σ, B)-wtgrs $\mathcal{S}$ such that $[\![\mathcal{S}]\!] = \psi$. Firstly, by Lemma 4.3.5, we can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ has a single nonterminal axiom and it is r-equivalent to $\mathcal{S}'$. Moreover, if $\mathcal{S}$ is finite-reductional, then $\mathcal{S}'$ is so. Hence, we may assume that $\mathcal{S}$ has a single nonterminal axiom.

By Lemma 4.3.7, we can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ is simple and r-equivalent to $\mathcal{S}$. Since $\mathcal{S}$ has a single nonterminal axiom, $\mathcal{S}'$ also has that. Furthermore, if $\mathcal{S}$ is finite-reductional, then $\mathcal{S}'$ is so. Thus, we may assume that $\mathcal{S}$ has a single nonterminal axiom and it is simple.

By Lemma 4.3.16(1), there exists a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ has a single nonterminal axiom, and furthermore, it is contracting and r-equivalent to $\mathcal{S}$. (Observe that, by Lemma 4.3.15, $\mathcal{S}'$ is finite-reductional.) Therefore, we may assume that $\mathcal{S}$ has a single nonterminal axiom, and it is contracting and finite-reductional.

By Lemma 4.4.1, we can construct a (Σ, B)-wtgrs $\mathcal{S}'$ such that $\mathcal{S}'$ is contracting, and furthermore, it is production complete and r-equivalent to $\mathcal{S}$. Since $\mathcal{S}$ has a single nonterminal axiom, $\mathcal{S}'$ has nullary nonterminal axioms. Moreover, since $\mathcal{S}$ is finite-reductional, by the proof of Lemma 4.4.1, $\mathcal{S}'$ is so. Therefore, we may assume that $\mathcal{S}$ has nullary nonterminal axioms, and it is contracting, finite-reductional, and production complete.

Lastly, we can construct the (Σ, B)-wta $\mathrm{rel}(\mathcal{S})$ such that $\mathrm{rel}(\mathcal{S})$ and $\mathcal{S}$ are related. Then, by Lemma 4.4.4, we have $[\![\mathrm{rel}(\mathcal{S})]\!] = [\![\mathcal{S}]\!] = \psi$, *i.e.*, ψ is recognizable. This concludes the proof of Statement 1.

Now we prove Statement 2. Note that it can be proven in a similar way as Statement 1. Hence, in order to avoid repetitions, here we consider the proof of Statement 1, and address only the differences. In fact, to prove Statement 2, in the proof of Statement 1 we replace the expression "there exists" by the expression "we can construct" three times as follows. Evidently, both of the parts "Assume that ψ is recognizable. Then there exists a (Σ, B)-wta $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = \psi$." and "Assume that ψ is r-generated. Then there exists a (Σ, B)-wtgrs $\mathcal{S}$ such that $[\![\mathcal{S}]\!] = \psi$." can be replaced by the sentences "We can construct a (Σ, B)-wta $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = \psi$." and "We can construct a finite-reductional (Σ, B)-wtgrs $\mathcal{S}$ such that $[\![\mathcal{S}]\!] = \psi$.", respectively (*cf.* the conditions of Statement 2). Finally, in case of Statement 2 the wtgrs $\mathcal{S}$ is finite-reductional, and thus, in case of the application of Lemma 4.3.16, we can replace the expression "there exists" by "can construct" if we extend Lemma 4.3.16(1) with Lemma 4.3.16(2). This finishes the proof of Statement 2. $\square$

In the following example, we show two applications of Theorem 4.4.5. Note that the semiring MinPlus given in Example 2.4.6(5) is both complete and computable.

Example 4.4.6. [4, Ex. 35] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MinPlus})$-wta $\mathcal{A}_\sigma$ constructed in Example 3.1.5. Recall that we have $[\![\mathcal{A}_\sigma]\!] = \#_\sigma$, where $\#_\sigma$ is the $(\Sigma, \mathsf{MinPlus})$-weighted tree language shown in Example 3.1.2.

Then we can construct the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}_\sigma = (S_\sigma, wt, X)$ with the Σ-tgrs $S_\sigma = (\{q^{(0)}\}, \{q\}, \{\ \sigma(qq) \to q\ ,\ \alpha \to q\ \})$, and weights $wt(\sigma(qq) \to q) = 1$ and $wt(\alpha \to q) = X_q = 0$. Obviously, $\mathcal{S}_\sigma$ has a single nonterminal axiom, and it is contracting, finite-reductional, and production complete. Moreover, since $\mathcal{A}_\sigma$ and $\mathcal{S}_\sigma$ are related, by Lemma 4.4.4, we have $[\![\mathcal{S}_\sigma]\!] = [\![\mathcal{A}_\sigma]\!]$. △

Example 4.4.7. [4, Ex. 36] Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MinPlus})$-wtgrs $\mathcal{S}_\mathrm{r}''''$ shown in Example 4.4.2. Recall that $\mathcal{S}_\mathrm{r}'''$ has nullary nonterminal axioms, it is contracting, finite-reductional, and production complete. Also, recall that we have $[\![\mathcal{S}_\mathrm{r}'''']\!] = \#_\sigma$, where $\#_\sigma$ is the $(\Sigma, \mathsf{MinPlus})$-weighted tree language given in Example 3.1.2. Now we can construct the $(\Sigma, \mathsf{MinPlus})$-wta $\mathcal{A} = (N'', \delta, F)$ such that

- for every $A_1 \in \{F_{2,1}, E_{2,\varepsilon}\}$ and $A_2 \in \{F_{2,2}, E_{2,\varepsilon}\}$, we set $\delta_2(A_1A_2, \sigma, F_{2,\varepsilon}) = 0$ and $\delta_2(A_1A_2, \sigma, A) = 1$ for each $A \in (N'' \setminus \{F_{2,\varepsilon}\})$,
- for each pair $(A_1, A_2) \in \{(F_{2,\varepsilon}, F_{2,2}), (F_{2,1}, F_{2,\varepsilon}), (E_{2,\varepsilon}, F_{2,\varepsilon}), (F_{2,\varepsilon}, E_{2,\varepsilon})\}$, we set $\delta_2(A_1A_2, \sigma, F_{2,\varepsilon}) = 1$ and $\delta_2(A_1A_2, \sigma, A) = 2$ for each $A \in (N'' \setminus \{F_{2,\varepsilon}\})$,
- $\delta_2(F_{2,\varepsilon}, F_{2,\varepsilon}, \sigma, F_{2,\varepsilon}) = 2$ and $\delta_2(F_{2,\varepsilon}, F_{2,\varepsilon}, \sigma, A) = 3$ for each $A \in (N'' \setminus \{F_{2,\varepsilon}\})$,
- for each $A \in (N'' \setminus \{E_{2,\varepsilon}, F_{2,\varepsilon}\})$, we set $\delta_0(\varepsilon, \alpha, A) = 0$, and
- every other transition has weight $-\infty$.

Finally, since $\mathcal{A}$ and $\mathcal{S}_\mathrm{r}''''$ are related, by Lemma 4.4.4, we have $[\![\mathcal{A}]\!] = [\![\mathcal{S}_\mathrm{r}'''']\!]$. △

Chapter conclusion. The author of this PhD thesis declares that Theorems 4.2.8, 4.3.4, and 4.4.5 are due to his own work, and those results are published in [4].

Chapter 5

Crisp-determinization of wta

We organize this chapter in the following way. In Section 5.1 we explain why it is worth to study crisp-deterministic wta. In Section 5.2 we introduce the notion of crisp-deterministic wta, define the crisp-determinization problem, and give a sufficient condition for wta to be crisp-determinizable. In Section 5.3 we present two undecidability results related to the crisp-determinization problem. We first show that, in general, it is undecidable whether an arbitrary wta satisfies our sufficient condition or not. Then we go one step further and prove that it is also undecidable whether an arbitrary wta is crisp-determinizable or not. In Section 5.4 we give positive decidability results as we identify two subclasses of wta, for which the crisp-determinization problem is decidable. Finally, in Section 5.5, we show that each of our undecidability and decidability results holds for weighted string automata.

5.1 The problem

A crisp-deterministic wta $\mathcal{A}$ over B has several desirable properties such as $[\![\mathcal{A}]\!]$ has a finite image (called *finite-image property*) or, for each $b \in B$, the set of all trees with weight b under $[\![\mathcal{A}]\!]$ is a recognizable tree language (called *preimage property*). In fact, the class of all crisp-deterministic wta can be characterized using only those two properties *cf.* Lemma 5.3.9 (see also [30, 3]). For further properties of crisp-deterministic wta we refer to [42]. Moreover, it is worth to study crisp-deterministic wta also for the following reason. Fuzzy automata, languages, and grammars have been of interest for a long time *e.g.* [17, 37, 56]; for a survey we refer to [67]. The underlying weight structure of these formal models is some bounded lattice. Recall that, each bounded lattice is a bi-locally finite strong bimonoid (*cf.* Section 2.4). In Section 5.2 we show that each wta over a bi-locally finite strong bimonoid satisfies our sufficient condition to be crisp-determinizable, *i.e.*, advantages of crisp-deterministic wta are available when we investigate such fuzzy formal models.

Fortunately, there are subclasses of all wta for which the crisp-determinization problem is solved positively [21, 30]. However, in those identified subclasses there are only wta over string ranked alphabets, *i.e.*, weighted string automata (for short: wsa) *cf.* [42, Lm. 3.3.3]. One of our aims is to extend some of those positive results to further subclasses of all wta.

Moreover, we deal with decidability questions related to the crisp-determinization problem. In the literature there are some promising partial results regarding the undecidability (decidability) of crisp-determinization. These results justify the relevance of such questions, and create a solid base for further investigations. For instance, each wsa over a finite semiring and over the semiring Nat of natural numbers (*cf.* Example 2.4.6(2)) has the preimage property, and each wsa over a commutative ring which has the finite-image property also has the preimage property [11, 31, 61]. Furthermore, for each wsa over any subsemiring of the rational numbers, the finite-image property is decidable [63] (also *cf.* the classical Burnside property for semigroups [68]). Keeping in mind these existing partial results, our other aim is to prove undecidability (decidability) results related to the crisp-determinization problem. For this, in this chapter we consider selected results presented in [1–3].

5.2 A sufficient condition for crisp-determinization

In this section we give a straightforward generalization of the results given in [21, Sect. 8] from strings to trees. Firstly, we generalize the concept of crisp-deterministic wsa to wta, and introduce the concept of the finite-order property of wta. Interestingly, if a (Σ, B)-wta $\mathcal{A}$ has finite order, then $\mathcal{A}$ is crisp-determinizable (*cf.* Theorem 5.2.8). If, in addition, B is computable, then we can even construct a crisp-deterministic (Σ, B)-wta, which is equivalent to $\mathcal{A}$ (*cf.* Theorem 5.2.12).

For each $b \in B$, we say that b has *finite additive order (in* B*)* if the set

$$\langle\{b\}\rangle_{\{\oplus\}} = \{nb \mid n \in \mathbb{N}\}$$

is finite. Furthermore, for every $B_1, B_2 \subseteq B$, we define

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\} \ .$$

Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. We define $\mathrm{im}(\delta) = \bigcup_{k\in\mathbb{N}} \mathrm{im}(\delta_k)$. Then we say that $\mathcal{A}$ is *crisp-deterministic* [3, p. 9] (also *cf.* [21, Sect. 5]) if it is total and bu deterministic, and $\mathrm{im}(\delta) \subseteq \{\mathbb{0}, \mathbb{1}\}$. Furthermore, we say that $\mathcal{A}$ is *crisp-determinizable* if there exists a crisp-deterministic (Σ, B)-wta $\mathcal{A}'$ such that $\mathcal{A}'$ is equivalent to $\mathcal{A}$.

Let us abbreviate the notation $\langle \mathrm{im}(\delta)\rangle_{\{\otimes\}}$ by $D_{\mathcal{A}}$. Then we say that $\mathcal{A}$ has *finite order (in* B*)* if

- $D_{\mathcal{A}}$ is finite and

- each b in $D_{\mathcal{A}} \otimes \mathrm{im}(F)$ has finite additive order.

In particular, if B is bi-locally finite, then each (Σ, B)-wta has finite order.

In the following example we consider the three wta defined in Examples 3.1.4–3.1.6, and, for each of them, we examine whether it has finite order or not.

Example 5.2.1. Clearly, for the $(\Sigma, \mathsf{MaxPlus})$-wta $\mathcal{A}_{\max}$ given in Example 3.1.4, the set $D_{\mathcal{A}_{\max}} = \langle\{-\infty, 0, 1\}\rangle_{\{+\}} = \mathbb{N}_{-\infty}$ is not finite, and thus, $\mathcal{A}_{\max}$ does not have finite order. Similarly, for the $(\Sigma, \mathsf{MinPlus})$-wta $\mathcal{A}_{\sigma}$ defined in Example 3.1.5, the set $D_{\mathcal{A}_{\sigma}} = \langle\{0, 1\}\rangle_{\{+\}} = \mathbb{N}$ is not finite either, and hence, also $\mathcal{A}_{\sigma}$ does not have finite order. However, the bounded lattice M_3 shown in Example 2.4.7(2) is finite, hence it is bi-locally finite. Thus the (Σ, M_3)-wta $\mathcal{A}_{\mathrm{split}}$ given in Example 3.1.6 has finite order. △

Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then, for each $\xi \in \mathrm{T}_{\Sigma}$ and each $\rho \in \mathrm{Run}_{\mathcal{A}}(\xi)$, the element $\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}$ is in $D_{\mathcal{A}} \otimes \mathrm{im}(F)$. Hence, $[\![\mathcal{A}]\!](\xi)$ is a sum over the finite set $D_{\mathcal{A}} \otimes \mathrm{im}(F)$. Actually, the fact that each element in $D_{\mathcal{A}} \otimes \mathrm{im}(F)$ has a finite additive order guarantees that any sum over $D_{\mathcal{A}} \otimes \mathrm{im}(F)$ is equal to a finite sum over this set. In the rest of this section we formalize that phenomenon. For this, the following notions and results are necessary.

Lemma 5.2.2. Let B be computable and $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then we can compute both $D_{\mathcal{A}}$ and $D_{\mathcal{A}} \otimes \mathrm{im}(F)$.

Proof. Since $\mathcal{A}$ has finite order, both of the sets $D_{\mathcal{A}}$ and $D_{\mathcal{A}} \otimes \mathrm{im}(F)$ are finite. Moreover, since B is computable, for every $b, b' \in B$, we can compute the product $b \otimes b'$.

Firstly, we prove that the set $D_{\mathcal{A}}$ can be computed. For this, we let $D_0 = \mathrm{im}(\delta)$, and $D_{i+1} = D_i \cup (D_i \otimes \mathrm{im}(\delta))$ for each $i \in \mathbb{N}$. Evidently, we have $D_0 \subseteq D_1 \subseteq \ldots \subseteq D_{\mathcal{A}}$. It is easy to see that

$$\text{for each integer } i \in \mathbb{N}\text{: if } D_i = D_{i+1}, \text{ then } D_{i+1} = D_{i+2} \ . \tag{5.1}$$

Furthermore, it is obvious that

$$\text{for every } k \in \mathbb{N}_+ \text{ and } b_1, \ldots, b_k \in \mathrm{im}(\delta) : \quad \text{we have } \Big(\bigotimes_{j=1}^{k} b_j\Big) \in D_{k-1} \ .$$

Since $D_{\mathcal{A}}$ is a finite set, we can find a least number $i_0 \in \mathbb{N}$ such that $D_{i_0} = D_{i_0+1}$, and, by (5.1), we have $D_{i_0} = D_i$ for each $i \in \mathbb{N}$ with $i \geq i_0$. Then it is easy to see that we have $D_{\mathcal{A}} = (\bigcup_{i=1}^{i_0} D_i)$. Since we can compute the set $(\bigcup_{i=1}^{i_0} D_i)$, the set $D_{\mathcal{A}}$ can be computed as well. Moreover, we can compute also the set $D_{\mathcal{A}} \otimes \mathrm{im}(F)$. □

Let $b \in B$ such that b has finite additive order in B. Then there exists a least number $i \in \mathbb{N}_+$ such that $ib = (i + k)b$ for some $k \in \mathbb{N}_+$, and there exists a least

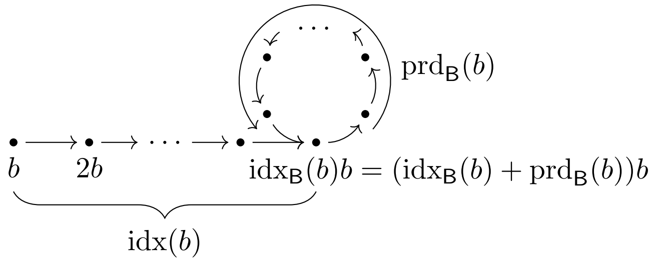


Figure 5.1. *Illustration of the index* $\mathrm{idx}_{\mathsf{B}}(b)$ *and the period* $\mathrm{prd}_{\mathsf{B}}(b)$ *of* b *in* B *(*cf. *[3, Fig. 1])*

number $p \in \mathbb{N}_+$ such that $ib = (i + p)b$. We call i and p the *index (of b in* B*)* and the *period (of b in* B*)*, respectively, and denote them by $\mathrm{idx}_{\mathsf{B}}(b)$ and $\mathrm{prd}_{\mathsf{B}}(b)$, respectively. Moreover, we call $\mathrm{idx}_{\mathsf{B}}(b) + \mathrm{prd}_{\mathsf{B}}(b) - 1$, *i.e.*, the number of elements of $\langle\{b\}\rangle_{\{\oplus\}}$, the *order (of b in* B*)*. Figure 5.1 illustrates the index and the period of b, where each directed arrow means the addition of b.

For each finite subset $K \subseteq \mathbb{N}_+$, we denote the *least common multiple of K* by $\mathrm{lcm}(K)$. Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then we define the integers

$$\mathrm{idx}_{\mathcal{A}} = \max(\{\mathrm{idx}_{\mathsf{B}}(b) \in \mathbb{N}_+ \mid b \text{ is in } D_{\mathcal{A}} \otimes \mathrm{im}(F)\}) \quad \text{and}$$
$$\mathrm{prd}_{\mathcal{A}} = \mathrm{lcm}(\{\mathrm{prd}_{\mathsf{B}}(b) \in \mathbb{N}_+ \mid b \text{ is in } D_{\mathcal{A}} \otimes \mathrm{im}(F)\}) \enspace .$$

Moreover, we define the mapping $J_{\mathcal{A}} : \mathbb{N} \to [0, \mathrm{idx}_{\mathcal{A}} + \mathrm{prd}_{\mathcal{A}} - 1]$, for each $n \in \mathbb{N}$, by

$$J_{\mathcal{A}}(n) = \begin{cases} n & \text{if } n < \mathrm{idx}_{\mathcal{A}} \\ \mathrm{idx}_{\mathcal{A}} + ((n - \mathrm{idx}_{\mathcal{A}}) \mod \mathrm{prd}_{\mathcal{A}}) & \text{if } n \geq \mathrm{idx}_{\mathcal{A}} \end{cases} ,$$

where $(n - \mathrm{idx}_{\mathcal{A}}) \mod \mathrm{prd}_{\mathcal{A}}$ denotes the remainder when $n - \mathrm{idx}_{\mathcal{A}}$ is divided by $\mathrm{prd}_{\mathcal{A}}$. In the first case $J_{\mathcal{A}}(n) < \mathrm{idx}_{\mathcal{A}}$, and in the second case $\mathrm{idx}_{\mathcal{A}} \leq J_{\mathcal{A}}(n) \leq \mathrm{idx}_{\mathcal{A}} + \mathrm{prd}_{\mathcal{A}} - 1$, and thus, $J_{\mathcal{A}}$ is well defined. Note that in both cases $n \equiv_{\mathrm{prd}_{\mathcal{A}}} J_{\mathcal{A}}(n)$, where $\equiv_{\mathrm{prd}_{\mathcal{A}}}$ denotes the congruence relation on the semiring Nat given in Example 2.4.6(3) modulo $\mathrm{prd}_{\mathcal{A}}$. Moreover, for every $n \in \mathbb{N}$ and $b \in (D_{\mathcal{A}} \otimes \mathrm{im}(F))$, we have $nb = J_{\mathcal{A}}(n)(b)$.

Lemma 5.2.3. Let B be computable, and $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then we can compute the integers $\mathrm{idx}_{\mathcal{A}}$ and $\mathrm{prd}_{\mathcal{A}}$.

Proof. By Lemma 5.2.2, we can compute the set $D_{\mathcal{A}} \otimes \mathrm{im}(F)$. Let b be in $D_{\mathcal{A}} \otimes \mathrm{im}(F)$. Since B is computable, for each $n \in \mathbb{N}_+$, we can compute nb. Moreover, since b has finite additive order, there exists a least number n_0 in $\mathbb{N}_+ \setminus \{1\}$ such that $n_0 b = nb$ for some $n \in [n_0 - 1]$. Then we have $\mathrm{idx}_{\mathsf{B}}(b) = n$ and $\mathrm{prd}_{\mathsf{B}}(b) = n_0 - n$. Moreover, since $D_{\mathcal{A}} \otimes \mathrm{im}(F)$ is a finite set, we can compute the integers $\mathrm{idx}_{\mathcal{A}}$ and $\mathrm{prd}_{\mathcal{A}}$. $\square$

Next, for each Σ-tree ξ, we define a partitioning of runs of $\mathcal{A}$ on ξ with respect to the elements in $Q \times D_{\mathcal{A}}$. Formally, let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Since $D_{\mathcal{A}}$ is finite, the set $Q \times D_{\mathcal{A}}$ is also finite. For every $\xi \in \mathrm{T}_\Sigma$ and (q, b) in $Q \times D_{\mathcal{A}}$, we define the *set of all q-runs of $\mathcal{A}$ on ξ with weight b*, denoted by $\mathrm{Run}_{\mathcal{A}}(q, \xi, b)$, as the set

$$\mathrm{Run}_{\mathcal{A}}(q, \xi, b) = \{\rho \in \mathrm{Run}_{\mathcal{A}}(q, \xi) \mid \mathrm{wt}(\xi, \rho) = b\} \ .$$

> *From now on, for each (Σ, B)-wta $\mathcal{A}$, which has finite order, we denote the set $([0, \mathrm{idx}_{\mathcal{A}} + \mathrm{prd}_{\mathcal{A}} - 1])^{(Q \times D_{\mathcal{A}})}$ by $\mathcal{F}_{\mathcal{A}}$.*

Moreover, for each $\xi \in \mathrm{T}_\Sigma$, we define the mappings $n_\xi \in (\mathbb{N}^{(Q \times D_{\mathcal{A}})})$ and $\pi_\xi \in \mathcal{F}_{\mathcal{A}}$, for each (q, b) in $Q \times D_{\mathcal{A}}$, by

$$n_\xi(q, b) = |\,\mathrm{Run}_{\mathcal{A}}(q, \xi, b)| \quad \text{and} \quad \pi_\xi(q, b) = J_{\mathcal{A}}(n_\xi(q, b)) \ .$$

Lemma 5.2.4. Let B be computable and $\mathcal{A}$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then, for each $\xi \in \mathrm{T}_\Sigma$, we can compute the mappings n_ξ and π_ξ.

Proof. Let $\mathcal{A} = (Q, \delta, F)$. Note that, for every $\xi \in \mathrm{T}_\Sigma$ and $q \in Q$, since Q is a finite set, also the set $\mathrm{Run}_{\mathcal{A}}(q, \xi)$ is finite and we can compute it. By Lemma 5.2.2, we can compute the set $D_{\mathcal{A}}$, and thus, we can also compute the set $Q \times D_{\mathcal{A}}$. Furthermore, since B is computable, for every $\xi \in \mathrm{T}_\Sigma$ and (q, b) in $Q \times D_{\mathcal{A}}$, we can compute the set $\mathrm{Run}_{\mathcal{A}}(q, \xi, b)$. Since $Q \times D_{\mathcal{A}}$ is finite, and the set $\mathrm{Run}_{\mathcal{A}}(q, \xi, b)$ can be computed for every $\xi \in \mathrm{T}_\Sigma$ and (q, b) in $Q \times D_{\mathcal{A}}$, we can compute the mapping n_ξ. Moreover, by Lemma 5.2.3, we can compute the integers $\mathrm{idx}_{\mathcal{A}}$ and $\mathrm{prd}_{\mathcal{A}}$, and hence, for every $n \in \mathbb{N}$, the integer $J_{\mathcal{A}}(n)$. Again, since $Q \times D_{\mathcal{A}}$ is finite set, we can also compute the mapping π_ξ. $\square$

In the following definition, for each (Σ, B)-wta $\mathcal{A} = (Q, \delta, F)$ such that $\mathcal{A}$ has finite order, we give a (Σ, B)-wta $\mathcal{A}' = (Q', \delta', F')$ such that $\mathrm{im}(\delta') = \{\mathbb{0}, \mathbb{1}\}$ and $\mathrm{im}(F') = \mathrm{im}([\![\mathcal{A}]\!])$.

Definition 5.2.5. Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Clearly, we have $\{\pi_\xi \mid \xi \in \mathrm{T}_\Sigma\} \subseteq \mathcal{F}_{\mathcal{A}}$, and since $\mathcal{F}_{\mathcal{A}}$ is finite, also $\{\pi_\xi \mid \xi \in \mathrm{T}_\Sigma\}$ is finite. Moreover, the set $\{\pi_\xi \mid \xi \in \mathrm{T}_\Sigma\}$ is not empty obviously. Then we define the (Σ, B)-wta $\mathcal{A}' = (Q', \delta', F')$ where

- $Q' = \{\pi_\xi \mid \xi \in \mathrm{T}_\Sigma\}$,
- for every $k \in \mathbb{N}$, $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$, $\sigma \in \Sigma^{(k)}$, and tree $\xi \in \mathrm{T}_\Sigma$, we set

$$\delta'_k(\pi_{\xi_1} \cdots \pi_{\xi_k}, \sigma, \pi_\xi) = \begin{cases} \mathbb{1} & \text{if } \pi_\xi = \pi_{\sigma(\xi_1, \ldots, \xi_k)} \\ \mathbb{0} & \text{otherwise} \ , \end{cases}$$

- for every $\xi \in \mathrm{T}_\Sigma$, we set

$$F'_{\pi_\xi} = \bigoplus_{(q,b)\in(Q\times D_{\mathcal{A}})} \pi_\xi(q,b)(b \otimes F_q) \enspace .$$

$\triangle$

Next we show that in fact, the (Σ, B)-wta $\mathcal{A}' = (Q', \delta', F')$ given in Definition 5.2.5 is crisp-deterministic (*cf.* Lemma 5.2.7). Since we have $\mathrm{im}(\delta') \subseteq \{\mathbb{0}, \mathbb{1}\}$, it is sufficient to show that $\mathcal{A}'$ is total and bu deterministic. In order to prove Lemma 5.2.7, the following notions, notations, and results (*cf.* Lemma 5.2.6) are necessary.

Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. For each $k \in \mathbb{N}$, we consider $(Q \times D_{\mathcal{A}})^k$ as a set of strings over $Q \times D_{\mathcal{A}}$ of length k. Let $\xi = \sigma(\xi_1, \ldots, \xi_k) \in \mathrm{T}_\Sigma$ with $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$, and $(q, b) \in (Q \times D_{\mathcal{A}})$. Now we define the set $O_\xi(q, b) \subseteq (Q \times D_{\mathcal{A}})^k$ as follows:

for every string $(q_1, b_1) \cdots (q_k, b_k)$ in $(Q \times D_{\mathcal{A}})^k$:
we have $(q_1, b_1) \cdots (q_k, b_k) \in O_\xi(q, b)$
if $\pi_{\xi_i}(q_i, b_i) > 0$ for each $i \in [k]$ and $((\bigotimes_{i=1}^k b_i) \otimes \delta_k(q_1 \cdots q_k, \sigma, q)) = b$.

Lemma 5.2.6. [3, Lm. 7.2] (also *cf.* [21, Lm. 8.1]) Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then, for every $\xi \in \mathrm{T}_\Sigma$ and $(q, b) \in (Q \times D_{\mathcal{A}})$, we have

$$n_\xi(q, b) = \sum_{w\in O_\xi(q,b)} \prod_{i=1}^{\mathrm{len}(w)} n_{\xi|_i}(w(i)) \enspace .$$

Proof. Let $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$ such that $\xi = \sigma(\xi_1, \ldots, \xi_k)$. Recall that we have $O_\xi(q, b) \subseteq (Q \times D_{\mathcal{A}})^k$. Then, for each string $w = (q_1, b_1) \cdots (q_k, b_k)$ in $O_\xi(q, b)$, we define the set

$$\mathrm{Run}_{\mathcal{A}}(w, q, \xi) = \{\rho \in \mathrm{Run}_{\mathcal{A}}(q, \xi) \mid \rho|_i \in \mathrm{Run}_{\mathcal{A}}(q_i, \xi_i, b_i) \text{ for each } i \in [k]\} \enspace .$$

Note that the family $(\mathrm{Run}_{\mathcal{A}}(w, q, \xi) \mid w \in O_\xi(q, b))$ is a partitioning of $\mathrm{Run}_{\mathcal{A}}(q, \xi, b)$. Consequently, we have

$$\begin{aligned} n_\xi(q, b) = |\,\mathrm{Run}_{\mathcal{A}}(q, \xi, b)| &= \sum_{w\in O_\xi(q,b)} |\,\mathrm{Run}_{\mathcal{A}}(w, q, \xi)| \\ &=^{(*)} \sum_{(q_1,b_1)\cdots(q_k,b_k) \text{ in } O_\xi(q,b)} \prod_{i=1}^{k} |\,\mathrm{Run}_{\mathcal{A}}(q_i, \xi_i, b_i)| \\ &= \sum_{(q_1,b_1)\cdots(q_k,b_k) \text{ in } O_\xi(q,b)} \prod_{i=1}^{k} n_{\xi_i}(q_i, b_i) \enspace , \end{aligned}$$

where at $(*)$ we use the fact that $|\operatorname{Run}_{\mathcal{A}}(w,q,\xi)| = \prod_{i=1}^{k} |\operatorname{Run}_{\mathcal{A}}(q_i,\xi_i,b_i)|$ if we have $w = (q_1,b_1)\cdots(q_k,b_k)$. $\square$

Lemma 5.2.7. [3, p. 30–31] Let $\mathcal{A} = (Q,\delta,F)$ be a (Σ,B)-wta such that $\mathcal{A}$ has finite order. Moreover, let $\mathcal{A}' = (Q',\delta',F')$ be the (Σ,B)-wta defined in Definition 5.2.5 obtained from $\mathcal{A}$. Then $\mathcal{A}'$ is a crisp-deterministic (Σ,B)-wta.

Proof. Recall that we have $\operatorname{im}(\delta') \subseteq \{\mathbb{0},\mathbb{1}\}$. Hence, it is sufficient to show that $\mathcal{A}'$ is total and bu deterministic. For this, let $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1,\dots,\xi_k),(\xi'_1,\dots,\xi'_k)$ in $(\mathrm{T}_\Sigma)^k$ such that $\pi_{\xi_i} = \pi_{\xi'_i}$ for each $i \in [k]$. Moreover, we let $\xi = \sigma(\xi_1,\dots,\xi_k)$ and $\xi' = \sigma(\xi'_1,\dots,\xi'_k)$. Obviously, it is sufficient to show that $\pi_\xi = \pi_{\xi'}$. Let $(q,b) \in (Q \times D_{\mathcal{A}})$. Observe that we have $O_\xi(q,b) = O_{\xi'}(q,b)$, and thus, in the rest of this proof we denote that set simply by O. Furthermore, recall that $O \subseteq (Q \times D_{\mathcal{A}})^k$.

Case (a): We have $\pi_\xi(q,b) \le \operatorname{idx}_{\mathcal{A}}$ or $\pi_{\xi'}(q,b) \le \operatorname{idx}_{\mathcal{A}}$. Without loss of generality we may assume that $\pi_\xi(q,b) \le \operatorname{idx}_{\mathcal{A}}$. Evidently, we have $n_\xi(q,b) = \pi_\xi(q,b)$. Let $(q_1,b_1)\cdots(q_k,b_k) \in O$. Clearly, by Lemma 5.2.6, for each $i \in [k]$, we have $n_{\xi|_i}(q_i,b_i) < \operatorname{idx}_{\mathcal{A}}$, and thus,

$$n_{\xi|_i}(q_i,b_i) = \pi_{\xi|_i}(q_i,b_i) = \pi_{\xi'|_i}(q_i,b_i) = n_{\xi'|_i}(q_i,b_i) \enspace ,$$

where the first equality is due to the fact $n_{\xi|_i}(q_i,b_i) < \operatorname{idx}_{\mathcal{A}}$, the second one is due to our assumption, and the last one holds true because $\pi_{\xi'|_i(q_i,b_i)} < \operatorname{idx}_{\mathcal{A}}$. Hence, we can calculate further as follows:

$$\pi_\xi(q,b) = n_\xi(q,b) = \sum_{w\in O}\prod_{i=1}^{k} n_{\xi|_i}(w(i)) = \sum_{w\in O}\prod_{i=1}^{k} n_{\xi'|_i}(w(i)) = n_{\xi'}(q,b) = \pi_{\xi'}(q,b) \enspace ,$$

where the second and the last but one equalities follow from Lemma 5.2.6.

Case (b): Assume now that $\operatorname{idx}_{\mathcal{A}} \le \pi_\xi(q,b), \pi_{\xi'}(q,b) \le \operatorname{idx}_{\mathcal{A}} + \operatorname{prd}_{\mathcal{A}} - 1$. Then, for each $(q_1,b_1)\cdots(q_k,b_k) \in O$ and each $i \in [k]$, we have

$$n_{\xi_i}(q_i,b_i) \equiv_{\operatorname{prd}_{\mathcal{A}}} \pi_{\xi_i}(q_i,b_i) = \pi_{\xi'_i}(q_i,b_i) \equiv_{\operatorname{prd}_{\mathcal{A}}} n_{\xi'_i}(q_i,b_i) \enspace ,$$

where the second equality is due to our assumption. Since $\equiv_{\operatorname{prd}_{\mathcal{A}}}$ is a congruence relation on the semiring Nat of natural numbers (*cf.* Example 2.4.6(2)), we have

$$\begin{aligned}\pi_\xi(q,b) \equiv_{\operatorname{prd}_{\mathcal{A}}} n_\xi(q,b) &= \sum_{w\in O}\prod_{i=1}^{k} n_{\xi_i}(w(i)) \\ &\equiv_{\operatorname{prd}_{\mathcal{A}}} \sum_{w\in O}\prod_{i=1}^{k} n_{\xi'_i}(w(i)) = n_{\xi'}(q,b) \equiv_{\operatorname{prd}_{\mathcal{A}}} \pi_{\xi'}(q,b) \enspace ,\end{aligned}$$

where the equalities are due to Lemma 5.2.6. Since, by our assumption, we have

$\mathrm{idx}_{\mathcal{A}} \leq \pi_{\xi}(q,b), \pi_{\xi'}(q,b) \leq \mathrm{idx}_{\mathcal{A}} + \mathrm{prd}_{\mathcal{A}} - 1$, we conclude that $\pi_{\xi}(q,b) = \pi_{\xi'}(q,b)$. $\square$

The next result shows that, for each (Σ, B)-wta $\mathcal{A}$ such that $\mathcal{A}$ has finite order, the (Σ, B)-wta $\mathcal{A}'$ defined in Definition 5.2.5 obtained from $\mathcal{A}$ is a semantically equivalent crisp-deterministic (Σ, B)-wta.

Theorem 5.2.8. *[3, Thm. 7.3] (also* cf. *[21, Thm. 8.2]) Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$, and B be a strong bimonoid. Moreover, let $\mathcal{A}$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then there exists a (Σ, B)-wta $\mathcal{A}'$ such that $\mathcal{A}'$ is crisp-deterministic and it is equivalent to $\mathcal{A}$.*

Proof. Let $\mathcal{A}'$ be the (Σ, B)-wta obtained by applying Definition 5.2.5 to $\mathcal{A}$. We follow the denotations of Definition 5.2.5, *i.e.*, we have $\mathcal{A} = (Q, \delta, F)$ and $\mathcal{A}' = (Q', \delta', F')$.

By Lemma 5.2.7, $\mathcal{A}'$ is a crisp-deterministic (Σ, B)-wta. Hence, for each $\xi \in \mathrm{T}_{\Sigma}$, there is a unique valid run ρ_{ξ} of $\mathcal{A}'$ on ξ, *i.e.*, we have $\{\rho_{\xi}\} = \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}'}(\xi)$.

Now we prove, by induction on the structure of ξ, the following statement:

$$\text{for each } \xi \in \mathrm{T}_{\Sigma}, \text{ the run } \rho_{\xi} \text{ is a } \pi_{\xi}\text{-run} \; . \tag{5.2}$$

<u>Induction base:</u> Then there exists $\alpha \in \Sigma^{(0)}$ such that $\xi = \alpha$. Then, since $\mathcal{A}'$ is crisp-deterministic, by Definition 5.2.5, we have $\delta'_0(\varepsilon, \alpha, \pi_{\alpha}) = \mathbb{1}$ and $\delta'_0(\varepsilon, \alpha, \pi_{\xi}) = \mathbb{0}$ for each $\pi_{\xi} \in (Q' \setminus \{\pi_{\alpha}\})$ with $\xi \in \mathrm{T}_{\Sigma}$. Furthermore, since ρ_{α} is valid, we have $\rho_{\alpha}(\varepsilon) = \pi_{\alpha}$.

<u>Induction step:</u> Then there exist $k \in \mathbb{N}_{+}$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in \mathrm{T}_{\Sigma}$ such that $\xi = \sigma(\xi_1, \ldots, \xi_k)$. By I.H., for each $i \in [k]$, the run ρ_{ξ_i} is a π_{ξ_i}-run. Then, since $\mathcal{A}'$ is crisp-deterministic, by Definition 5.2.5, we have $\delta'_k(\pi_{\xi_1} \cdots \pi_{\xi_k}, \sigma, \pi_{\xi}) = \mathbb{1}$ and $\delta'_k(\pi_{\xi_1} \cdots \pi_{\xi_k}, \sigma, \pi_{\xi'}) = \mathbb{0}$ for each $\pi_{\xi'} \in (Q' \setminus \{\pi_{\xi}\})$ with $\xi' \in \mathrm{T}_{\Sigma}$. Moreover, since ρ_{ξ} is valid, we have $\rho(\varepsilon) = \pi_{\xi}$ as needed. This completes the proof of (5.2). Finally, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows:

$$\begin{aligned}
[\![\mathcal{A}]\!](\xi) &= \bigoplus_{\rho \in \mathrm{Run}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} = \bigoplus_{(q,b) \text{ in } Q \times D_{\mathcal{A}}} \; \bigoplus_{\rho \in \mathrm{Run}_{\mathcal{A}}(q,\xi,b)} b \otimes F_q \\
&= \bigoplus_{(q,b) \text{ in } Q \times D_{\mathcal{A}}} n_{\xi}(q,b)(b \otimes F_q) = \bigoplus_{(q,b) \text{ in } Q \times D_{\mathcal{A}}} \pi_{\xi}(q,b)(b \otimes F_q) \\
&= F'_{\pi_{\xi}} = \mathrm{wt}_{\mathcal{A}'}(\xi, \rho_{\xi}) \otimes F'_{\pi_{\xi}} = [\![\mathcal{A}']\!](\xi) \;,
\end{aligned}$$

where the second equality easily follows from the fact that the family

$$(\mathrm{Run}_{\mathcal{A}}(q, \xi, b) \mid (q,b) \in Q \times D_{\mathcal{A}})$$

is a partitioning of $\mathrm{Run}_{\mathcal{A}}(\xi)$; the third and the fourth ones are due to the definitions of $n_{\xi}(q,b)$ and $\pi_{\xi}(q,b)$, respectively; the fifth one follows from Definition 5.2.5; the

Algorithm 1: The mapping calc (*cf.* [3, Alg. 2] and [21, Alg. 8.4])

Input: a (Σ, B)-wta $\mathcal{A} = (Q, \delta, F)$ such that B is computable and $\mathcal{A}$ has finite order, $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\pi_{\xi_1}, \ldots, \pi_{\xi_k}) \in (\mathcal{F}_{\mathcal{A}})^k$ for some $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$
Macro: $\xi = \sigma(\xi_1, \ldots, \xi_k)$
Variables: $e : Q \times D_{\mathcal{A}}$; $w : (Q \times D_{\mathcal{A}})^k$; and $n_\xi : \mathbb{N}^{(Q \times D_{\mathcal{A}})}$
Output: $\pi_\xi \in \mathcal{F}_{\mathcal{A}}$

foreach $e \in (Q \times D_{\mathcal{A}})$ **do** $n_\xi(e) \leftarrow 0$
foreach $e \in (Q \times D_{\mathcal{A}})$ **do**
  foreach string $w \in O_\xi(e)$ **do**
    $n_\xi(e) \leftarrow n_\xi(e) + J_{\mathcal{A}}(\prod_{i=1}^{k} \pi_{\xi_i}(w(i)))$
  end
  $\pi_\xi(e) \leftarrow J_{\mathcal{A}}(n_\xi(e))$
end
output π_ξ

sixth one is due to (5.2) and the fact that $\mathrm{im}(\delta') \subseteq \{\mathbb{0}, \mathbb{1}\}$; and the seventh one holds true by Lemma 5.2.7. $\square$

The following result is an immediate consequence of Theorem 5.2.8 and the fact that each wta over a bi-locally finite strong bimonoid has finite order.

Corollary 5.2.9. [3, Cor. 7.5] (also *cf.* [30, Thm. 11]) Let B be bi-locally finite. Then, for each (Σ, B)-wta $\mathcal{A}$, there exists a crisp-deterministic (Σ, B)-wta $\mathcal{A}'$ equivalent to $\mathcal{A}$.

Note that, since the strong bimonoid Stb given in Example 2.4.5(4) is bi-locally finite, by Corollary 5.2.9, each (Σ, Stb)-wta is crisp-determinizable.

Next we prove an effective version of Theorem 5.2.8: If, in addition, B is computable, then, for each (Σ, B)-wta $\mathcal{A}$ such that $\mathcal{A}$ has finite order, we can construct a crisp-deterministic (Σ, B)-wta $\mathcal{A}'$ equivalent to $\mathcal{A}$. Hence, we give a mapping and an algorithm as follows.

By considering Algorithm 1, we note that the mapping calc takes the following data as input: a (Σ, B)-wta $\mathcal{A} = (Q, \delta, F)$ such that B is computable and $\mathcal{A}$ has finite order, $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\pi_{\xi_1}, \ldots, \pi_{\xi_k}) \in (\mathcal{F}_{\mathcal{A}})^k$ for some $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$; and it outputs the mapping $\pi_\xi \in \mathcal{F}_{\mathcal{A}}$ with $\xi = \sigma(\xi_1, \ldots, \xi_k)$. But, compared to the proof of Lemma 5.2.4, calc do not use the family $(\mathrm{Run}_{\mathcal{A}}(q, \xi, b) \mid (q, b) \text{ in } Q \times D_{\mathcal{A}})$ for computing π_ξ. Nevertheless, it can compute π_ξ as the following lemmas prove that.

Lemma 5.2.10. Let B be computable, and $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then, for every $\xi \in \mathrm{T}_\Sigma$ and (q, b) in $Q \times D_{\mathcal{A}}$, we can compute the set $O_\xi(q, b)$.

Proof. By Lemma 5.2.2, we can compute the set $D_{\mathcal{A}}$, and hence, we can compute also the set $Q \times D_{\mathcal{A}}$. Let $\xi = \sigma(\xi_1, \ldots, \xi_k)$ with $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$. Since $Q \times D_{\mathcal{A}}$ is finite, also the set $(Q \times D_{\mathcal{A}})^k$ is finite, and we can compute it.

Let $(q_1, b_1) \cdots (q_k, b_k)$ be a string in $(Q \times D_{\mathcal{A}})^k$. By Lemma 5.2.4, for each $i \in [k]$, we can compute the mapping π_{ξ_i}, and thus, it is decidable whether $\pi_{\xi_i}(q_i, b_i) > 0$ holds true or not. Moreover, since B is computable, we can compute the product $((\bigotimes_{i=1}^k b_i) \otimes \delta_k(q_1 \cdots q_k, \sigma, q))$, and hence, it is decidable whether that product equals b or not. Therefore, it is decidable whether the string $(q_1, b_1) \cdots (q_k, b_k)$ is in $O_\xi(q, b)$ or not. Furthermore, since $(Q \times D_{\mathcal{A}})^k$ is finite, we can compute the set $O_\xi(q, b)$. $\square$

Lemma 5.2.11. [3, Cor. 7.8] Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then, for every $\xi \in \mathrm{T}_\Sigma$ and $e \in (Q \times D_{\mathcal{A}})$, we have

$$\pi_\xi(e) = J_{\mathcal{A}}\Big(\sum_{w \in O_\xi(e)} J_{\mathcal{A}}\Big(\prod_{i=1}^{\mathrm{len}(w)} \pi_{\xi|_i}(w(i)) \Big) \Big) \ .$$

Proof. Obviously, we can calculate as follows

$$\begin{aligned} \pi_\xi(e) = J_{\mathcal{A}}(n_\xi(e)) &= J_{\mathcal{A}}\Big(\sum_{w \in O_\xi(e)} \prod_{i=1}^{\mathrm{len}(w)} n_{\xi|_i}(w(i)) \Big) = J_{\mathcal{A}}\Big(\sum_{w \in O_\xi(e)} J_{\mathcal{A}}\Big(\prod_{i=1}^{\mathrm{len}(w)} n_{\xi|_i}(w(i)) \Big) \Big) \\ &= J_{\mathcal{A}}\Big(\sum_{w \in O_\xi(e)} J_{\mathcal{A}}\Big(\prod_{i=1}^{\mathrm{len}(w)} J_{\mathcal{A}}(n_{\xi|_i}(w(i))) \Big) \Big) = J_{\mathcal{A}}\Big(\sum_{w \in O_\xi(e)} J_{\mathcal{A}}\Big(\prod_{i=1}^{\mathrm{len}(w)} \pi_{\xi|_i}(w(i)) \Big) \Big) \ , \end{aligned}$$

where the second equality is due to Lemma 5.2.6; and the third and the fourth ones follow from the fact that $\equiv_{\mathrm{prd}_{\mathcal{A}}}$ is a congruence relation on the semiring Nat. $\square$

Theorem 5.2.12. *Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$, and* B *be a computable strong bimonoid. Moreover, let $\mathcal{A}$ be a (Σ, B)-wta such that $\mathcal{A}$ has finite order. Then we can construct a (Σ, B)-wta $\mathcal{A}'$ such that $\mathcal{A}'$ is crisp-deterministic and it is equivalent to $\mathcal{A}$.*

Proof. Recall that, by the proof of Theorem 5.2.8, the (Σ, B)-wta $\mathcal{A}'$ defined in Definition 5.2.5 is crisp-deterministic and it is equivalent to $\mathcal{A}$. Thus, here we prove only that we can construct that wta $\mathcal{A}'$. For this, we follow Algorithm 2 and consider its denotations.

Now we consider the family $(g_i \mid i \in \mathbb{N})$ of Σ-hypergraphs $g_i = (Q_i, E_i)$ constructed in lines 1-11. Obviously, we have $g_0 \subseteq g_1 \subseteq \ldots \subseteq g_{\mathcal{A}'}$. Now we prove by case analysis the following statement:

$$\text{for each integer } i \in \mathbb{N}: \ \text{ if } g_i = g_{i+1}, \text{ then } g_{i+1} = g_{i+2} \ . \tag{5.3}$$

Case (a): Let $\pi \in Q_{i+2}$. By Lemma 5.2.11, there exist $k \in I$, $\sigma \in \Sigma^{(k)}$, and $(\pi_1, \ldots, \pi_k) \in (Q_{i+1})^k$ such that $\pi = \mathrm{calc}(\mathcal{A}, k, \sigma, (\pi_1, \ldots, \pi_k))$ (*cf.* the application of

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Algorithm 2: Construction of a crisp-deterministic (Σ, B)-wta $\mathcal{A}'$, which is equivalent to the (Σ, B)-wta $\mathcal{A}$ (*cf.* [3, Alg. 3] and [21, Alg. 8.3])

Input: a (Σ, B)-wta $\mathcal{A} = (Q, \delta, F)$ such that B is computable and $\mathcal{A}$ has finite order
Macro: $I = [0, \mathrm{maxrk}(\Sigma)]$
Variables: $i : \mathbb{N}$; family $(g_i \mid i \in \mathbb{N})$ of Σ-hypergraphs $g_i = (Q_i, E_i)$ with $Q_i : \mathcal{P}(\mathcal{F}_\mathcal{A})$ and $E_i : \mathcal{P}(\bigcup_{k \in I}((Q_i)^k \times \Sigma^{(k)} \times Q_i))$; $k : I$; $\sigma : \Sigma^{(k)}$; $(\pi_1, \ldots, \pi_k) : (\mathcal{F}_\mathcal{A})^k$; $\pi : \mathcal{F}_\mathcal{A}$; $X : \mathcal{P}(\mathcal{F}_\mathcal{A})$; and $Y : \mathcal{P}(\bigcup_{k \in I}(X^k \times \Sigma^{(k)} \times X))$
Output: a crisp-deterministic (Σ, B)-wta $\mathcal{A}' = (Q', \delta', F')$ such that $[\![\mathcal{A}']\!] = [\![\mathcal{A}]\!]$

1 $Q_0 \leftarrow \emptyset$ and $E_0 \leftarrow \emptyset$ % this forms the Σ-hypergraph g_0
2 $i \leftarrow 0$
3 **repeat**
4 $\quad X \leftarrow \emptyset$ and $Y \leftarrow \emptyset$
5 $\quad$ **forevery** $k \in I$, $\sigma \in \Sigma^{(k)}$, and $(\pi_1, \ldots, \pi_k) \in (Q_i)^k$ **do**
6 $\quad\quad \pi \leftarrow \mathrm{calc}(\mathcal{A}, k, \sigma, (\pi_1, \ldots, \pi_k))$ % *cf.* Algorithm 1
7 $\quad\quad X \leftarrow X \cup \{\pi\}$ and $Y \leftarrow Y \cup \{(\pi_1 \cdots \pi_k, \sigma, \pi)\}$
8 $\quad$ **end**
9 $\quad Q_{i+1} \leftarrow Q_i \cup X$ and $E_{i+1} \leftarrow E_i \cup Y$ % this forms the Σ-hypergraph g_{i+1}
10 $\quad i \leftarrow i + 1$
11 **until** $g_i = g_{i-1}$
12 we can construct $\mathcal{A}'$ as follows:
- $Q' = Q_i$,
- $\delta' = (\delta'_k \mid k \in \mathbb{N})$ with $\mathrm{supp}_\mathsf{B}(\delta'_k) = \{(\pi_1 \cdots \pi_k, \sigma, \pi) \mid (\pi_1 \cdots \pi_k, \sigma, \pi) \in E_i\}$ and $\mathrm{im}(\delta'_k) = \{\mathbb{0}, \mathbb{1}\}$, and
- $F'_\pi = \bigoplus_{(q,b) \in (Q \times D_\mathcal{A})} \pi(q, b)(b \otimes F_q)$ for each $\pi \in Q'$.

Algorithm 1 in line 6). Moreover, since, by our assumption, we have $g_i = g_{i+1}$, *i.e.*, $Q_i = Q_{i+1}$, we also have $(\pi_1, \ldots, \pi_k) \in Q_i^k$, and thus, $\pi \in Q_{i+1}$ as desired.

Case (b): Let $(\pi_1 \cdots \pi_k, \sigma, \pi) \in E_{i+2}$ with $k \in I$, $\sigma \in \Sigma^{(k)}$, and $(\pi_1, \ldots, \pi_k) \in (Q_{i+1})^k$, and $\pi \in Q_{i+2}$. Since, by our assumption, we have $g_i = g_{i+1}$, *i.e.*, $Q_i = Q_{i+1}$, we have $(\pi_1 \cdots \pi_k) \in Q_i^k$ as well. Moreover, by Case (a), we have $Q_{i+1} = Q_{i+2}$, *i.e.*, $\pi \in Q_{i+1}$. Thus, we have $(\pi_1 \cdots \pi_k, \sigma, \pi) \in E_{i+1}$. This completes the proof of (5.3). Next we prove, by induction on the structure of ξ, the following statement:

$$\text{for each } \xi \in \mathrm{T}_\Sigma, \text{ we have } \pi_\xi \in Q_{\mathrm{height}(\xi)+1} \quad . \tag{5.4}$$

Induction base: Then there exists $\alpha \in \Sigma^{(0)}$ such that $\xi = \alpha$. Moreover, recall that $\mathrm{height}(\alpha) + 1 = 1$. By Lemma 5.2.11, we have $\pi_\alpha = \mathrm{calc}(\mathcal{A}, 0, \alpha, ())$ (*cf.* the application of Algorithm 1 in line 6), and hence, $\pi_\alpha \in Q_1$.

Induction step: Clearly, there exist $k \in I$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in \mathrm{T}_\Sigma$ such that $\xi = \sigma(\xi_1, \ldots, \xi_k)$. Then, by I.H., for each $j \in [k]$, we have $\pi_{\xi_j} \in Q_{\mathrm{height}(\xi_j)+1}$. Hence, we let $m = (\max(\{\mathrm{height}(\xi_j) \mid j \in [k]\}) + 1)$. Note that we have $\mathrm{height}(\xi) = m$. Since $g_i \subseteq g_m$ for each $i \in [0, m]$, *i.e.*, $Q_i \subseteq Q_m$ for each $i \in [0, m]$, we then have

$\pi_{\xi_1}, \ldots, \pi_{\xi_k} \in Q_m$. Moreover, by Lemma 5.2.11, $\pi_\xi = \mathrm{calc}(\mathcal{A}, k, \sigma, (\pi_{\xi_1}, \ldots, \pi_{\xi_k}))$ (*cf.* the application of Algorithm 1 in line 6), and thus, $\pi_\xi \in Q_{m+1}$. This concludes the proof of (5.4). Now we prove the following statement:

$$\text{there exists a least number } i \in \mathbb{N} \text{ such that } g_i = g_{\mathcal{A}'} \ . \tag{5.5}$$

Recall that we have $g_i \subseteq g_{\mathcal{A}'}$. Hence, it is sufficient to show that $g_{\mathcal{A}'} \subseteq g_i$. Moreover, recall that $g_{\mathcal{A}'} = (Q', \bigcup_{k \in \mathbb{N}} \mathrm{supp}_{\mathsf{B}}(\delta'_k))$. We proceed by case analysis.

Case (a): Let $\pi_\xi \in Q'$ for some $\xi \in \mathrm{T}_\Sigma$. Then, by (5.4), we have $\pi_\xi \in Q_{\mathrm{height}(\xi)+1}$. If $\mathrm{height}(\xi) + 1 \leq i$, then, since $g_j \subseteq g_i$ for each $j \in [0, i]$, *i.e.*, $Q_j \subseteq Q_i$ for each $j \in [0, i]$, we have $\pi_\xi \in Q_i$. Otherwise, if $i < \mathrm{height}(\xi) + 1$, then, by the (repeated) application of (5.3), we have $g_i = g_{\mathrm{height}(\xi)+1}$, *i.e.*, $Q_i = Q_{\mathrm{height}(\xi)+1}$, and thus, we also have $\pi_\xi \in Q_i$.

Case (b): Let $e = (\pi_{\xi_1} \cdots \pi_{\xi_k}, \sigma, \pi_\xi)$ be in $\mathrm{supp}_{\mathsf{B}}(\delta'_k)$ for some $k \in I$, $\sigma \in \Sigma^{(k)}$, and $(\xi_1, \ldots, \xi_k) \in (\mathrm{T}_\Sigma)^k$ such that $\xi = \sigma(\xi_1, \ldots, \xi_k)$. By following the proof of (5.4) and Algorithm 2, we have $e \in E_{\mathrm{height}(\xi)+1}$. If $\mathrm{height}(\xi) + 1 \leq i$, then, since $g_j \subseteq g_i$ for each $j \in [0, i]$, we have $e \in E_i$. Otherwise, if $i < \mathrm{height}(\xi) + 1$, then, by the (repeated) application of (5.3), we have $g_i = g_{\mathrm{height}(\xi)+1}$, *i.e.*, $E_i = E_{\mathrm{height}(\xi)+1}$, and hence, we have $e \in E_i$ as well. This finishes the proof of (5.5). Since, by (5.5), we can compute the Σ-hypergraph g_i, we can compute the Σ-hypergraph $g_{\mathcal{A}'}$ as well, and hence, we can construct $\mathcal{A}'$. This concludes our proof. $\square$

Finally, we give an application of Theorem 5.2.12. Recall that the (Σ, M_3)-wta $\mathcal{A}_{\mathrm{split}}$ given in Example 3.1.6 has finite order (*cf.* Example 5.2.1). Since the bounded lattice M_3 is computable, we can give the wta $\mathcal{A}_{\mathrm{split}}$ to Algorithm 2 as input; and the algorithm outputs a semantically equivalent crisp-deterministic (Σ, M_3)-wta $\mathcal{A}'_{\mathrm{split}}$ as follows.

Example 5.2.13. Let $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$. We consider the (Σ, M_3)-wta $\mathcal{A}_{\mathrm{split}}$ shown in Example 3.1.6 Evidently, since $\mathrm{im}(\delta) = \{a, b, i\} \not\subseteq \{o, i\}$, the wta $\mathcal{A}_{\mathrm{split}}$ is not crisp-deterministic. Moreover, we have

$$D_{\mathcal{A}_{\mathrm{split}}} = \langle \{a, b, i\} \rangle_\wedge = \{o, a, b, i\} = D_{\mathcal{A}_{\mathrm{split}}} \wedge \mathrm{im}(F) \ ,$$

and thus, $(\{q\} \times D_{\mathcal{A}_{\mathrm{split}}}) = \{(q, o), (q, a), (q, b), (q, i)\}$.

Next, for every $\xi \in \mathrm{T}_\Sigma$ and $e \in (\{q\} \times D_{\mathcal{A}_{\mathrm{split}}})$, we calculate the values $n_\xi(e)$ and $\pi_\xi(e)$. Observe that we have $\mathrm{idx}_{\mathcal{A}_{\mathrm{split}}} = \mathrm{prd}_{\mathcal{A}_{\mathrm{split}}} = 1$, and hence, $J_{\mathcal{A}_{\mathrm{split}}}(n) \in \{0, 1\}$ for each $n \in \mathbb{N}$, *i.e.* π_ξ is a mapping from $\{q\} \times D_{\mathcal{A}_{\mathrm{split}}}$ to $\{0, 1\}$ for each $\xi \in \mathrm{T}_\Sigma$.

Now we introduce three notations. We denote by ξ_σ an arbitrary tree $\xi \in \mathrm{T}_\Sigma$ such that $\mathrm{pos}_\sigma(\xi) \neq \emptyset$ and $\mathrm{pos}_\gamma(\xi) = \emptyset$, by ξ_γ an arbitrary tree $\xi \in \mathrm{T}_\Sigma$ such that $\mathrm{pos}_\sigma(\xi) = \emptyset$ and $\mathrm{pos}_\gamma(\xi) \neq \emptyset$, and by $\xi_{\sigma,\gamma}$ an arbitrary tree $\xi \in \mathrm{T}_\Sigma$ such that $\mathrm{pos}_\sigma(\xi) \neq \emptyset$ and

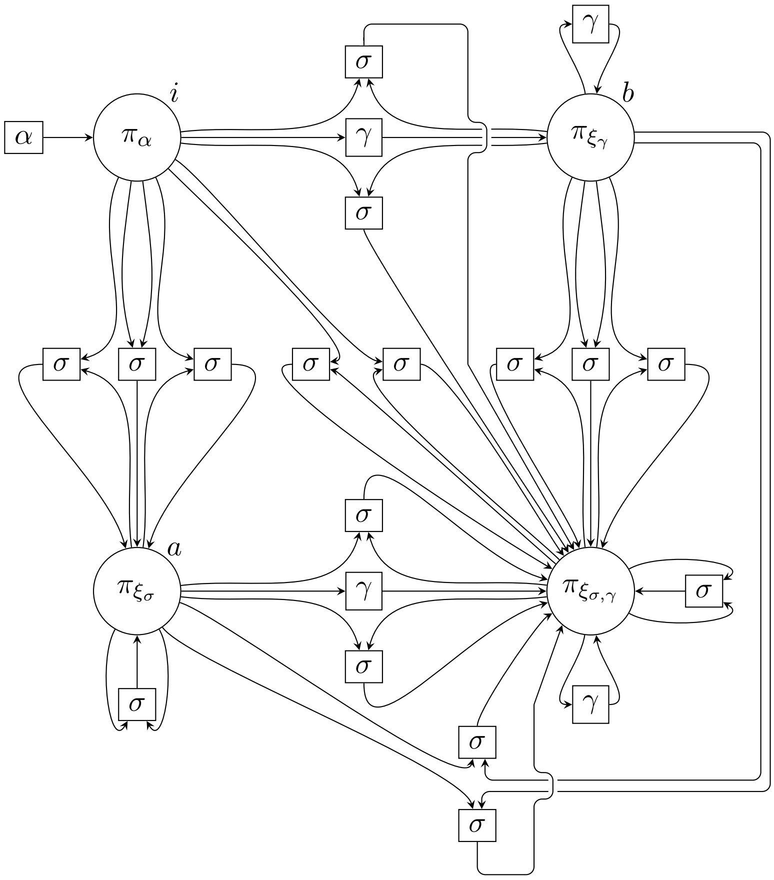


Figure 5.2. *The fta-hypergraph of the crisp-deterministic* (Σ, M_3)*-wta* $\mathcal{A}'_{\text{split}}$ *constructed in Example 5.2.13; note that each depicted transition has weight* i*, and hence, the transition weights are omitted intentionally.*

$\mathrm{pos}_\gamma(\xi) \neq \emptyset$. Evidently, for each $\xi \in \mathrm{T}_\Sigma$, we have either $\xi = \alpha$, or else $\xi = \xi_\sigma$, or else $\xi = \xi_\gamma$, or else $\xi = \xi_{\sigma,\gamma}$, *i.e.*, T_Σ can be partitioned into four sets.

Note that, for each $f \in \{n, \pi\}$, we have

$$f_\alpha(q, i) = f_{\xi_\sigma}(q, a) = f_{\xi_\gamma}(q, b) = f_{\xi_{\sigma,\gamma}}(q, o) = 1 \ ;$$

and, for every mapping $f \in \{n_\xi, \pi_\xi\}$ with $\xi \in \{\alpha, \xi_\sigma, \xi_\gamma, \xi_{\sigma,\gamma}\}$, and each $e \in (\{q\} \times D_{\mathcal{A}_{\text{split}}})$, if $f(e)$ is not listed above, then we have $f(e) = 0$, *e.g.*, $n_\alpha(q, o) = 0$ and $\pi_{\xi_\gamma}(q, a) = 0$, *etc.*

Then we can construct the (Σ, M_3)-wta

$$\mathcal{A}'_{\text{split}} = (\{\pi_\alpha, \pi_{\xi_\sigma}, \pi_{\xi_\gamma}, \pi_{\xi_{\sigma,\gamma}}\}, \delta', F') \ ,$$

where $F'(\pi_\alpha) = o$, $F'(\pi_{\xi_\sigma}) = a$, $F'(\pi_{\xi_\gamma}) = b$, and $F'(\pi_{\xi_{\sigma,\gamma}}) = i$, and the family δ' is as follows:

- $\delta'_0(\varepsilon, \alpha, \pi_\alpha) = i$,
- $\delta'_1(e) = i$ for each $e \in \{(\pi_\alpha, \gamma, \pi_{\xi_\gamma}), (\pi_{\xi_\sigma}, \gamma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_\gamma}, \gamma, \pi_{\xi_\gamma}), (\pi_{\xi_{\sigma,\gamma}}, \gamma, \pi_{\xi_{\sigma,\gamma}})\}$,
- $\delta'_2(e) = i$ for each e in the set

$$\left\{\begin{array}{l}(\pi_\alpha\pi_\alpha, \sigma, \pi_{\xi_\sigma}), (\pi_\alpha\pi_{\xi_\sigma}, \sigma, \pi_{\xi_\sigma}), (\pi_\alpha\pi_{\xi_\gamma}, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_\alpha\pi_{\xi_{\sigma,\gamma}}, \sigma, \pi_{\xi_{\sigma,\gamma}}),\\ (\pi_{\xi_\sigma}\pi_\alpha, \sigma, \pi_{\xi_\sigma}), (\pi_{\xi_\sigma}\pi_{\xi_\sigma}, \sigma, \pi_{\xi_\sigma}), (\pi_{\xi_\sigma}\pi_{\xi_\gamma}, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_\sigma}\pi_{\xi_{\sigma,\gamma}}, \sigma, \pi_{\xi_{\sigma,\gamma}}),\\ (\pi_{\xi_\gamma}\pi_\alpha, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_\gamma}\pi_{\xi_\sigma}, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_\gamma}\pi_{\xi_\gamma}, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_\gamma}, \pi_{\xi_{\sigma,\gamma}}, \sigma, \pi_{\xi_{\sigma,\gamma}}),\\ (\pi_{\xi_{\sigma,\gamma}}\pi_\alpha, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_{\sigma,\gamma}}\pi_{\xi_\sigma}, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_{\sigma,\gamma}}\pi_{\xi_\gamma}, \sigma, \pi_{\xi_{\sigma,\gamma}}), (\pi_{\xi_{\sigma,\gamma}}\pi_{\xi_{\sigma,\gamma}}, \sigma, \pi_{\xi_{\sigma,\gamma}})\end{array}\right\};$$

every other transition in δ' not listed above has weight o. Clearly, $\mathcal{A}'_{\text{split}}$ is crisp-deterministic. Figure 5.2 depicts the fta-hypergraph of $\mathcal{A}'_{\text{split}}$, where each depicted transition has weight i, and hence, transition weights are omitted intentionally. Note that, by Theorem 5.2.8, $\mathcal{A}'_{\text{split}}$ is equivalent to $\mathcal{A}_{\text{split}}$. $\triangle$

5.3 Undecidability of crisp-determinization

In this section we deal with decidability questions related to crisp-determinization of wta. Recall that, by the results of Section 5.2, a wta $\mathcal{A}$ is crisp-determinizable if $\mathcal{A}$ has finite order. Hence, in particular, we are interested in the following ones:
(Q1) Is it decidable for an arbitrary wta $\mathcal{A}$, whether $\mathcal{A}$ has finite order?
(Q2) Is it decidable for an arbitrary wta $\mathcal{A}$, whether $\mathcal{A}$ is crisp-determinizable?
We show that the answer to both questions is negative.

We prove these undecidability results in the way that we reduce each to the finiteness problem of the submonoid T_M generated by an arbitrary Mealy machine M, which is known to be undecidable (*cf.* [44, Thm. 3.13]).

Hence, we first recall Mealy machines with initial states. Then, for each Mealy machine M, we construct a ranked alphabet Σ_M, a computable and idempotent semiring Trans_M, and a $(\Sigma_M, \mathsf{Trans}_M)$-wta $\mathcal{A}_M$ (*cf.* Construction 5.3.4) as follows:

- the states of M become the symbols of Σ_M,
- Trans_M is the semiring of which the carrier set is the finite subsets of the translations in the submonoid T_M generated by M, and its operations are the usual union of sets and the composition extended to sets, and
- the image of $[\![\mathcal{A}_M]\!]$ is bijective to the carrier set $\langle M\rangle$ of the submonoid T_M generated by M (*cf.* Lemma 5.3.5).

Then we recall that translations realized by Mealy machines are closed under composition (*cf.* Lemma 5.3.1), and the equivalence of Mealy machines is decidable (*cf.* Lemma 5.3.2), and show that Trans_M is computable.

Due to its construction, $\mathcal{A}_M$ has finite order if and only if $\mathrm{im}([\![\mathcal{A}_M]\!])$ is finite (*cf.* Lemma 5.3.5). By exploiting the bijectivity of $\mathrm{im}([\![\mathcal{A}_M]\!])$ and the carrier set $\langle M\rangle$ of the submonoid T_M, the wta $\mathcal{A}_M$ has finite order if and only if the submonoid

T_M generated by M is finite (*cf.* Corollary 5.3.6). Thus we answer Question (Q1) negatively (*cf.* Theorem 5.3.7).

Then we proceed with Question (Q2). For this, we first give a characterization of crisp-determinizability (*cf.* Lemma 5.3.9). Here we can give a Σ_M-algebra ATrans_M (*cf.* Construction 5.3.10) to which $[\![\mathcal{A}_M]\!]$ is a Σ_M-algebra homomorphism from the Σ_M-term algebra (*cf.* Lemma 5.3.11). Then we recall a result on algebraic characterization of recognizable tree languages (*cf.* Lemma 5.3.12). By combining these results, we prove an equivalence of the crisp-determinizability of $\mathcal{A}_M$ and the finiteness of the submonoid T_M generated by M (*cf.* Lemma 5.3.13). Finally, we answer Question (Q2) negatively (*cf.* Theorem 5.3.14).

In the rest of this section Γ will denote an arbitrary alphabet.

Mealy machines. For each mapping $\tau : \Gamma^* \to \Gamma^*$, we also say that τ is a *translation over* Γ (for short: Γ-translation, or just translation). A *Mealy machine over* Γ [43, 64] is a quadruple $M = (Q, q_0, \mu, \lambda)$, where
- Q is a finite and nonempty set (*states*),
- $q_0 \in Q$ (*initial state*),
- $\mu : Q \times \Gamma \to Q$ is a mapping (*transition mapping*), and
- $\lambda : Q \times \Gamma \to \Gamma$ is a mapping (*output mapping*).

Let $M = (Q, q_0, \mu, \lambda)$ be a Mealy machine over Γ. As usual, we extended the mappings μ and λ to mappings $\mu^* : Q \times \Gamma^* \to Q$ and $\lambda^* : Q \times \Gamma^* \to \Gamma^*$, respectively, for every $q \in Q$, $w \in \Gamma^*$, and $a \in \Gamma$, as follows:

(i) $\mu^*(q, \varepsilon) = q$ and $\lambda^*(q, \varepsilon) = \varepsilon$ and

(ii) $\mu^*(q, wa) = \mu(\mu^*(q, w), a)$ and $\lambda^*(q, wa) = \lambda^*(q, w)\lambda(\mu^*(q, w), a)$.

The *semantics of* M, denoted by τ_M, is the Γ-translation defined, for each $w \in \Gamma^*$, by $\tau_M(w) = \lambda^*(q_0, w)$. In particular, $\tau_M(\varepsilon) = \varepsilon$. For every Mealy machines M and M' over Γ, we say that M and M' are *equivalent* if $\tau_M = \tau_{M'}$. Furthermore, for each Γ-translation τ, we say that τ is *realizable* if there exists a Mealy machine M such that $\tau = \tau_M$. We denote the *set of all realizable* Γ*-translations* by $\mathsf{Real}(\Gamma)$.

Next we recall that translations realized by Mealy machines over Γ are closed under composition.

Lemma 5.3.1. *cf.* [53, p. 207–208] and [43, Thm. 4.3.8] For every Mealy machines M_1 and M_2 over Γ, we can construct a Mealy machine M over Γ such that $\tau_M = (\tau_{M_2} \circ \tau_{M_1})$. $\square$

Clearly, by Lemma 5.3.1, the set $\mathsf{Real}(\Gamma)$ is closed under the operation $\circ$. Moreover, $\circ$ is an associative binary operation on $\mathsf{Real}(\Gamma)$. Hence, $(\mathsf{Real}(\Gamma), \circ)$ is a semigroup. Finally, since $\mathrm{id}_{\Gamma^*} \in \mathsf{Real}(\Gamma)$, the algebra $(\mathsf{Real}(\Gamma), \circ, \mathrm{id}_{\Gamma^*})$ is a monoid.

Next we show that the monoid $(\mathsf{Real}(\Gamma), \circ, \mathrm{id}_{\Gamma^*})$ is computable. The set $\mathsf{Real}(\Gamma)$ is recursively enumerable as we can enumerate all the Mealy machines of one state over Γ, and then, all the Mealy machines of two states over Γ, and so on. Moreover, by Lemma 5.3.1, the binary operation $\circ$ is computable. Evidently, also the nullary operation id_{Γ^*} is computable. Hence, it remains to recall that the equivalence of Mealy machines is decidable.

Lemma 5.3.2. *cf.* [55, Thm. 3.5] and [43, Thm. 1.7.3] It is decidable, for arbitrary Mealy machines M and M' over Γ, whether M and M' are equivalent. $\square$

Hence, $(\mathsf{Real}(\Gamma), \circ, \mathrm{id}_{\Gamma^*})$ is a computable monoid. Now let $M = (Q, q_0, \mu, \lambda)$ be a Mealy machine over Γ. For each $q \in Q$, we define the Mealy machine $M^q = (Q, q, \mu, \lambda)$ over Γ. Furthermore, for the subset $\{\tau_{M^q} \mid q \in Q\} \subseteq \mathsf{Real}(\Gamma)$, we denote the set $\langle \{\tau_{M^q} \mid q \in Q\}\rangle_{\{\circ, \mathrm{id}_{\Gamma^*}\}}$ by $\langle M \rangle$, *i.e.*, we have

$$\langle M \rangle = \bigcup_{k \in \mathbb{N}} \{\tau_{M^{q_1}} \circ \ldots \circ \tau_{M^{q_k}} \mid (q_1, \ldots, q_k) \in Q^k\} \ , \tag{5.6}$$

where $(\tau_{M^{q_1}} \circ \ldots \circ \tau_{M^{q_k}}) = \mathrm{id}_{\Gamma^*}$ if $k = 0$. Observe that, for each $k \in \mathbb{N}$, the set $\{\tau_{M^{q_1}} \circ \ldots \circ \tau_{M^{q_k}} \mid (q_1, \ldots, q_k) \in Q^k\}$ is in $\mathcal{P}_{\mathrm{fin}}(\langle M \rangle)$. The *submonoid of the monoid* $(\mathsf{Real}(\Gamma), \circ, \mathrm{id}_{\Gamma^*})$ *generated by* M, denoted by T_M, is the submonoid $\mathsf{T}_M = (\langle M \rangle, \circ, \mathrm{id}_{\Gamma^*})$ generated by the set $\{\tau_{M^q} \mid q \in Q\}$. Moreover, we denote the set $\{\{\tau\} \mid \tau \in \langle M \rangle\}$ by $\{\langle M \rangle\}$.

In this section the following undecidability result plays a key role as we prove our undecidability results by reducing them to this one.

Lemma 5.3.3. [44, Thm. 3.13] It is undecidable, for an arbitrary Mealy machine M over Γ, whether the submonoid T_M generated by M is finite. $\square$

Now, from a Mealy machine M over Γ, we can construct the ranked alphabet Σ_M, the semiring Trans_M, and the $(\Sigma_M, \mathsf{Trans}_M)$-wta $\mathcal{A}_M$ as follows.

Construction 5.3.4. Let $M = (Q, q_0, \mu, \lambda)$ be a Mealy machine over Γ. Firstly, we can construct the ranked alphabet Σ_M such that $\Sigma_M = \Sigma_M^{(1)} \cup \Sigma_M^{(0)}$ with $\Sigma_M^{(1)} = Q$ and $\Sigma_M^{(0)} = \{e^{(0)}\}$, where $e \notin Q$ is a new symbol. Evidently, Σ_M is a string ranked alphabet.

Then, inspired by the semiring Lang_Γ given in Example 2.4.6(6), we extend the submonoid T_M generated by M to a semiring as follows. We consider the strong bimonoid

$$\mathsf{Trans}_M = (\mathcal{P}_{\mathrm{fin}}(\langle M \rangle), \cup, \circ, \emptyset, \{\mathrm{id}_{\Gamma^*}\}) \ ,$$

where $\circ$ is extended to sets as usual, *i.e.*, for every $T_1, T_2 \in \mathcal{P}_{\mathrm{fin}}(\langle M \rangle)$, we define $T_2 \circ T_1 = \{\tau_2 \circ \tau_1 \mid \tau_1 \in T_1, \tau_2 \in T_2\}$. Obviously, $\cup$ is idempotent, and $\circ$ is distributive with respect to $\cup$, and hence, Trans_M is an idempotent semiring. Finally, we show that Trans_M is computable as follows. Clearly, by (5.6), $\langle M \rangle$ is a recursively enumerable set because, for each $k \in \mathbb{N}$ and $(q_1, \ldots, q_k) \in Q^k$, we can enumerate the translations $\tau_{M^{q_1}} \circ \ldots \circ \tau_{M^{q_k}}$. Then $\mathcal{P}_{\text{fin}}(\langle M \rangle)$ is recursively enumerable, because the set of finite subsets of each recursively enumerable set is also recursively enumerable. Moreover, by a straightforward generalization of Lemma 5.3.2 to finite sets, $\mathcal{P}_{\text{fin}}(\langle M \rangle)$ is a recursively enumerable set with tests for equality. Observe that both $\cup$ and the extended operation $\circ$ is computable (for the latter *cf.* Lemma 5.3.1). Also, the nullary operations $\emptyset$ and $\{\mathrm{id}_{\Gamma^*}\}$ are computable. Thus, Trans_M is computable. We call Trans_M the *semiring of translations realized by* M.

Finally, we can construct the $(\Sigma_M, \mathsf{Trans}_M)$-wta $\mathcal{A}_M = (\{\perp\}, \delta_M, F_M)$ as follows:

- $\perp$ is a new symbol such that $\perp \notin \Sigma_M$,
- $(\delta_M)_0(\varepsilon, e, \perp) = \{\mathrm{id}_{\Gamma^*}\}$ and $(\delta_M)_1(\perp, q, \perp) = \{\tau_{M^q}\}$ for each $q \in Q$, and
- $(F_M)_\perp = \{\mathrm{id}_{\Gamma^*}\}$.

Observe that $\mathcal{A}_M$ is total and bu deterministic. △

In the following lemma we show that $\mathrm{im}(\llbracket \mathcal{A}_M \rrbracket) = \{\langle M \rangle\}$.

Lemma 5.3.5. *cf.* [3, Lm. 8.2] Let M be a Mealy machine over Γ. Then we have $\mathrm{im}(\llbracket \mathcal{A}_M \rrbracket) = D_{\mathcal{A}_M} = \{\langle M \rangle\}$.

Proof. By following the denotations of Construction 5.3.4, we let $M = (Q, q_0, \mu, \lambda)$ and $\mathcal{A}_M = (\{\perp\}, \delta_M, F_M)$. Since $\mathcal{A}_M$ is total and bu deterministic, for each $\xi \in \mathrm{T}_\Sigma$, there is a unique valid run ρ_ξ of $\mathcal{A}_M$ on ξ, *i.e.*, $\{\rho_\xi\} = \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}_M}(\xi)$.

In the rest of this proof, for every $q \in \Sigma_M^{(1)}$ and $\xi \in \mathrm{T}_{\Sigma_M}$, we abbreviate the tree $q(\xi)$ by $q\xi$. Furthermore, we define the mapping $\mathrm{eval} : \mathrm{T}_{\Sigma_M} \to \mathcal{P}_{\text{fin}}(\langle M \rangle)$ inductively as follows: $\mathrm{eval}(e) = \{\mathrm{id}_{\Gamma^*}\}$, and $\mathrm{eval}(\xi) = \{\tau_{M^q}\} \circ \mathrm{eval}(\xi')$ if $\xi = q\xi'$ for some $q \in Q$ and $\xi' \in \mathrm{T}_{\Sigma_M}$. We first prove, by induction on the structure of ξ, the following statement:

$$\text{for each } \xi \in \mathrm{T}_{\Sigma_M}\text{, we have } \mathrm{wt}(\xi, \rho_\xi) = \mathrm{eval}(\xi) \ . \tag{5.7}$$

<u>Induction base:</u> Then we have $\xi = e$, and thus, we can calculate as follows:

$$\mathrm{wt}(e, \rho_e) = (\delta_M)_0(\varepsilon, e, \perp) = \{\mathrm{id}_{\Gamma^*}\} = \mathrm{eval}(e) \ .$$

<u>Induction step:</u> Then there exist $q \in Q$ and $\xi' \in \mathrm{T}_{\Sigma_M}$ such that $\xi = q\xi'$. Hence, we can calculate in the following way:

$$\mathrm{wt}(\xi, \rho_\xi) = (\delta_M)_1(\perp, q, \perp) \circ \mathrm{wt}(\xi', \rho_{\xi'}) = \{\tau_{M^q}\} \circ \mathrm{eval}(\xi') = \mathrm{eval}(\xi) \ ,$$

where, by I.H. and the fact that $(\delta_M)_1(\perp, q, \perp) = \{\tau_{M^q}\}$, the second equality holds true. This concludes the proof of (5.7). Then, for every $k \in \mathbb{N}$ and $\xi = q_1 \cdots q_k e$ in T_{Σ_M} with $(q_1, \ldots, q_k) \in Q^k$, we have

$$\begin{aligned} \llbracket \mathcal{A}_M \rrbracket(\xi) &= (F_M)_\perp \circ \mathrm{wt}(\xi, \rho_\xi) = (F_M)_\perp \circ \mathrm{eval}(\xi) \\ &= \{\mathrm{id}_{\Gamma^*}\} \circ \{\tau_{M^{q_1}} \circ \ldots \circ \tau_{M^{q_k}} \circ \mathrm{id}_{\Gamma^*}\} = \{\tau_{M^{q_1}} \circ \ldots \circ \tau_{M^{q_k}}\} \ , \end{aligned} \tag{5.8}$$

where the second equality is due to (5.7). Furthermore, we have

$$\begin{aligned}\mathrm{im}([\![\mathcal{A}_M]\!]) &= \bigcup_{k\in\mathbb{N}}\{\{\tau_{M^{q_1}}\circ\ldots\circ\tau_{M^{q_k}}\} \mid (q_1,\ldots,q_k)\in Q^k\}\\ &= \{\langle M\rangle\} = \langle \mathrm{im}(\delta_M)\rangle_{\{\circ\}} = D_{\mathcal{A}_M}\ ,\end{aligned}$$

where the first equality follows from (5.8). □

Then, as a consequence of Lemma 5.3.5, we obtain the following result.

Corollary 5.3.6. Let M be a Mealy machine over Γ. Then $\mathcal{A}_M$ has finite order if and only if the submonoid T_M generated by M is finite.

Proof. Firstly, due to Lemma 5.3.5 and the fact that $\mathrm{im}(F_M) = \{\{\mathrm{id}_{\Gamma^*}\}\}$ by Construction 5.3.4, we have $D_{\mathcal{A}_M} = \{\langle M\rangle\} = (\mathrm{im}(F_M)\circ D_{\mathcal{A}_M})$. Moreover, since Trans_M is idempotent, each $\{\tau\}$ in $\{\langle M\rangle\}$ has finite additive order. Consequently, we have

$$\begin{aligned}\mathcal{A}_M \text{ has finite order} &\iff \text{the set } \{\langle M\rangle\} \text{ is finite}\\ &\iff \text{the submonoid } \mathsf{T}_M \text{ generated by } M \text{ is finite}\ .\end{aligned}$$ □

Now we can show that, in general, the Question (Q1) is undecidable.

Theorem 5.3.7. cf. *[3, Thm. 8.9] It is undecidable, for arbitrary string ranked alphabet Σ, computable and idempotent semiring S, and bottom-up deterministic (Σ,S)-wta $\mathcal{A}$, whether $\mathcal{A}$ has finite order.*

Proof. We prove our statement by contradiction. For this, we assume that it is decidable, for arbitrary string ranked alphabet Σ, computable and idempotent semiring S, and bu deterministic (Σ,S)-wta $\mathcal{A}$, whether $\mathcal{A}$ has finite order.

Now let M be a Mealy machine over Γ. Then, by following Construction 5.3.4, we can construct the string ranked alphabet Σ_M, consider the semiring Trans_M, and can construct the total and bu deterministic $(\Sigma_M,\mathsf{Trans}_M)$-wta $\mathcal{A}_M$. By Corollary 5.3.6, $\mathcal{A}_M$ has finite order if and only if the submonoid T_M generated by M is finite.

Then, by our assumption, it is decidable whether the submonoid T_M generated by M is finite. This contradicts to Lemma 5.3.3, *i.e.*, our assumption is wrong. □

In the rest of this section, we deal with Question (Q2). For this, the following notions and results are crucial. To give a characterization of crisp-determinizability, it is necessary to recall that weighted tree languages recognized by crisp-deterministic wta are closed under sum. For every (Σ,B)-weighted tree languages ψ_1 and ψ_2, the *sum of* ψ_1 *and* ψ_2, denoted by $(\psi_1\oplus\psi_2)$, is the (Σ,B)-weighted tree language $(\psi_1\oplus\psi_2)\colon \mathrm{T}_\Sigma\to B$ defined, for each $\xi\in\mathrm{T}_\Sigma$, by $(\psi_1\oplus\psi_2)(\xi)=\psi_1(\xi)\oplus\psi_2(\xi)$.

Lemma 5.3.8. [42, Thm. 10.4.1(3)] For every two crisp-deterministic (Σ, B)-wta $\mathcal{A}_1$ and $\mathcal{A}_2$, we can construct a crisp-deterministic (Σ, B)-wta $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{A}_1]\!] \oplus [\![\mathcal{A}_2]\!]$. □

The following result gives a characterization of crisp-determinizability.

Lemma 5.3.9. [3, Lm. 5.3] (also *cf.* [30, Lm. 8 and Prop. 9]) Let ψ be a (Σ, B)-weighted tree language. Then the following statements are equivalent.

1. There exists a crisp-deterministic (Σ, B)-wta $\mathcal{A}$ such that $\psi = [\![\mathcal{A}]\!]$.
2. $\mathrm{im}(\psi)$ is finite and, for each $b \in B$, the Σ-tree language $\psi^{-1}(b)$ is recognizable.

Proof. $(1 \Rightarrow 2)$. Let $\mathcal{A} = (Q, \delta, F)$. Clearly, since $\mathcal{A}$ is crisp-deterministic, we have $\mathrm{im}(\psi) \subseteq \mathrm{im}(F)$, *i.e.*, $\mathrm{im}(\psi)$ is finite. Hence, it is sufficient to show that, for each $b \in B$, the Σ-tree language $\psi^{-1}(b)$ is recognizable. Evidently, if $b \notin \mathrm{im}(\psi)$, then $\psi^{-1}(b) = \emptyset$, and thus, $\psi^{-1}(b)$ is recognizable.

Hence, we may assume that $b \in \mathrm{im}(\psi)$. For each $q \in Q$, we can construct the Σ-fta $A_q = (Q, \delta', \{q\})$, where $\delta'_k = \mathrm{supp}_{\mathsf{B}}(\delta_k)$ for each $k \in \mathbb{N}$. Obviously, since $\mathcal{A}$ is crisp-deterministic, the fta A_q is total and bu deterministic. Moreover, for every $\xi \in \mathrm{T}_\Sigma$ and $q \in Q$, we have $\mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(\xi) = \mathrm{Run}^{\mathrm{v}}_{A_q}(\xi)$, and thus, we also have

$$\xi \in \mathrm{L}(A_q) \quad \text{if and only if} \quad \rho_\xi(\varepsilon) = q \ ,$$

where ρ_ξ is the unique valid run of $\mathcal{A}$ on ξ, *i.e.*, we have $\{\rho_\xi\} = \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(\xi)$. Then, for each $\xi \in \mathrm{T}_\Sigma$, we have

$$[\![\mathcal{A}]\!](\xi) = b \quad \text{if and only if} \quad \rho_\xi(\varepsilon) \in F^{-1}(b) \quad \text{if and only if} \quad \xi \in \bigcup_{q \in F^{-1}(b)} \mathrm{L}(A_q) \ .$$

Thus, we have

$$[\![\mathcal{A}]\!]^{-1}(b) = \bigcup_{q \in F^{-1}(b)} \mathrm{L}(A_q) \ ,$$

where, by Lemma 2.3.1, the right-hand side of the equality is a recognizable Σ-tree language. Therefore, the Σ-tree language $[\![\mathcal{A}]\!]^{-1}(b)$ is recognizable.

$(2 \Rightarrow 1)$. Here we generalize the direction $\Rightarrow$ of [30, Lm. 8] from the string case to the tree case. Trivially, we have $\mathrm{im}(\psi) \neq \emptyset$. For this, let $\mathrm{im}(\psi) = \{b_1, \ldots, b_n\}$ for some $n \in \mathbb{N}_+$. Observe that, for each $b \in (B \setminus \mathrm{im}(\psi))$, we have $\varphi^{-1}(b) = \emptyset$, *i.e.*, we do not have to deal with it any further. Moreover, by our assumption, for each $i \in [n]$, the Σ-tree language $\psi^{-1}(b_i)$ is recognizable, *i.e.*, there exists a Σ-fta A_i such that $\mathrm{L}(A_i) = \psi^{-1}(b_i)$. Note that, by Lemma 2.3.2, for each $i \in [n]$, we may assume that A_i is total and bu deterministic.

Let $i \in [n]$. Moreover, let $A_i = (Q_i, \delta_i, F_i)$. Then we can construct the (Σ, B)-wta $\mathcal{A}_i = (Q_i, \delta'_i, F'_i)$ such that

- for each $k \in \mathbb{N}$, we set $\mathrm{supp}_{\mathsf{B}}((\delta'_i)_k) = (\delta_i)_k$ and $\mathrm{im}((\delta'_i)_k) \subseteq \{\mathbb{0}, \mathbb{1}\}$ and

- for each $q \in Q_i$, we set $F'_i(q) = b_i$ if $q \in F_i$, and $F'_i(q) = \mathbb{0}$ otherwise.

Since the fta A_i is total and bu deterministic, and we have $\mathrm{im}((\delta'_i)_k) \subseteq \{\mathbb{0}, \mathbb{1}\}$ for each $k \in \mathbb{N}$, the wta $\mathcal{A}_i$ is crisp-deterministic. Moreover, for each $\xi \in \mathrm{T}_\Sigma$, we have $\mathrm{Run}^{\mathrm{v}}_{A_i}(\xi) = \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}_i}(\xi)$. Consequently, for each $\xi \in \mathrm{T}_\Sigma$, we have

$$\xi \in \mathrm{L}(A_i) \quad \text{if and only if} \quad \rho_\xi(\varepsilon) \in F_i \quad \text{if and only if} \quad [\![\mathcal{A}_i]\!](\xi) = b_i \enspace ,$$

where ρ_ξ is the unique valid run of A_i on ξ, *i.e.*, we have $\{\rho_\xi\} = \mathrm{Run}^{\mathrm{v}}_{A_i}(\xi)$. Furthermore, by Lemma 5.3.8, we can construct a crisp-deterministic (Σ, B)-wta $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = \bigoplus_{i \in [n]} [\![\mathcal{A}_i]\!]$. Then, for each $\xi \in \mathrm{T}_\Sigma$, we have

$$\psi(\xi) = \bigoplus_{i \in I_\xi} b_i = [\![\mathcal{A}]\!](\xi) \enspace ,$$

where $I_\xi = \{i \in [n] \mid \xi \in \varphi^{-1}(b_i)\}$, and the last equivalence follows from Lemma 5.3.8. This concludes our proof. $\square$

Next we give a Σ_M-algebra ATrans_M, of which the carrier set coincides with the carrier set of Trans_M, and of which the operations correspond roughly to the transitions of $\mathcal{A}_M$.

Construction 5.3.10. Let $M = (Q, q_0, \mu, \lambda)$ be a Mealy machine over Γ. We first recall the string ranked alphabet Σ_M given in Construction 5.3.4. Then we consider the Σ_M-algebra

$$\mathsf{ATrans}_M = (\mathcal{P}_{\mathrm{fin}}(\langle M \rangle), \theta_{\mathsf{Trans}_M})$$

such that $\theta_{\mathsf{Trans}_M}(e) = \{\mathrm{id}_{\Gamma^*}\}$ and $\theta_{\mathsf{Trans}_M}(q)(T) = (\{\tau_{M^q}\} \circ T)$ for every $q \in Q$ and $T \in \mathcal{P}_{\mathrm{fin}}(\langle M \rangle)$, where $\circ$ is extended to sets as above (*cf.* Construction 5.3.4). $\triangle$

The following result justifies that $[\![\mathcal{A}_M]\!]$ is a Σ_M-algebra homomorphism.

Lemma 5.3.11. *cf.* [3, Lm. 8.3] Let M be a Mealy machine over Γ. Then $[\![\mathcal{A}_M]\!]$ is a Σ_M-algebra homomorphism from the Σ_M-algebra Term_{Σ_M} to ATrans_M.

Proof. We use the denotations of Construction 5.3.4 and 5.3.10. Recall that we have $\mathsf{Term}_{\Sigma_M} = (\mathrm{T}_{\Sigma_M}, \theta_{\Sigma_M})$ such that $\theta_{\Sigma_M}(e) = e$ and $\theta_{\Sigma_M}(q)(\xi) = q(\xi)$ for every $q \in Q$ and $\xi \in \mathrm{T}_{\Sigma_M}$ (*cf.* Example 2.4.1). Then, by assuming $\psi_M = [\![\mathcal{A}_M]\!]$, we have

$$\psi_M(\theta_{\Sigma_M}(e)) = \psi_M(e) =^{(\dagger)} \{\mathrm{id}_{\Gamma^*}\} = \theta_{\mathsf{Trans}_M}(e) \enspace ,$$

and, for every $q \in Q$ and $\xi \in \mathrm{T}_{\Sigma_M}$, we have

$$\psi_M(\theta_{\Sigma_M}(q)(\xi)) = \psi_M(q(\xi)) =^{(\dagger)} \{\tau_{M^q}\} \circ \psi_M(\xi) = \theta_{\mathsf{Trans}_M}(q)(\psi_M(\xi)),$$

where at $(\dagger)$ we apply (5.8). This completes our proof. $\square$

Now we recall an algebraic characterization of recognizable tree languages as follows.

Lemma 5.3.12. [43, Cor. 2.7.2] For every $L \subseteq \mathrm{T}_\Sigma$, the Σ-tree language L is recognizable if and only if there exist a finite Σ-algebra $\mathsf{A} = (A, \theta)$, a Σ-algebra homomorphism h from Term_Σ to A, and a subset $A' \subseteq A$ such that $L = h^{-1}(A')$. □

Now we prove the equivalence of crisp-determinizability of $\mathcal{A}_M$ and finiteness of the submonoid T_M generated by M.

Lemma 5.3.13. *cf.* [3, Lm. 8.4] Let M be a Mealy machine over Γ. Then the following statements are equivalent.

1. $\mathrm{im}(\llbracket \mathcal{A}_M \rrbracket)$ is finite and $\llbracket \mathcal{A}_M \rrbracket^{-1}(T)$ is recognizable for each $T \in \mathcal{P}_{\text{fin}}(\langle M \rangle)$.
2. The submonoid T_M generated by M is finite.

Proof. ($1 \Rightarrow 2$). By Lemma 5.3.5, our statement trivially holds true.

($2 \Rightarrow 1$). By Lemma 5.3.5, $\mathrm{im}(\llbracket \mathcal{A}_M \rrbracket)$ is finite. Hence, it is sufficient to show that, for each $T \in \mathcal{P}_{\text{fin}}(\langle M \rangle)$, the Σ_M-tree language $\llbracket \mathcal{A}_M \rrbracket^{-1}(T)$ is recognizable. Evidently, for every T in $\mathcal{P}_{\text{fin}}(\langle M \rangle) \setminus \mathrm{im}(\llbracket \mathcal{A}_M \rrbracket)$, we have $\llbracket \mathcal{A}_M \rrbracket^{-1}(T) = \emptyset$, which is clearly recognizable. Hence, we may assume that $T \in \mathrm{im}(\llbracket \mathcal{A}_M \rrbracket)$. Since $\mathrm{im}(\llbracket \mathcal{A}_M \rrbracket)$ is finite, the set $\mathcal{P}_{\text{fin}}(\langle M \rangle)$ is finite as well, *i.e.*, ATrans_M is a finite Σ_M-algebra. Moreover, by Lemma 5.3.11, $\llbracket \mathcal{A}_M \rrbracket$ is a Σ_M-algebra homomorphism from Term_{Σ_M} to ATrans_M. Finally, by Lemma 5.3.12, $\llbracket \mathcal{A}_M \rrbracket^{-1}(\{T\}) = \llbracket \mathcal{A}_M \rrbracket^{-1}(T)$ is recognizable. □

Eventually, we prove that, in general, Question (Q2) is undecidable.

Theorem 5.3.14. cf. *[3, Thm. 8.5] It is undecidable, for arbitrary string ranked alphabet Σ, computable and idempotent semiring S, and bottom-up deterministic (Σ, S)-wta $\mathcal{A}$, whether $\mathcal{A}$ is crisp-determinizable.*

Proof. We prove our statement by contradiction. For this, we assume that it is decidable, for arbitrary string ranked alphabet Σ, computable and idempotent semiring S, and bu deterministic (Σ, S)-wta $\mathcal{A}$, whether $\mathcal{A}$ is crisp-determinizable.

Now let M be a Mealy machine over Γ. By following Construction 5.3.4, we can construct the string ranked alphabet Σ_M, consider the semiring Trans_M, and can construct the total and bu deterministic $(\Sigma_M, \mathsf{Trans}_M)$-wta $\mathcal{A}_M$. Then the following holds true.

$$\begin{aligned}
&\mathcal{A}_M \text{ is crisp-determinizable} \\
&\iff \text{there exists a crisp-deterministic } (\Sigma_M, \mathsf{Trans}_M)\text{-wta } \mathcal{A}'_M \text{ with } \llbracket \mathcal{A}_M \rrbracket = \llbracket \mathcal{A}'_M \rrbracket \\
&\iff \mathrm{im}(\llbracket \mathcal{A}_M \rrbracket) \text{ is finite and } \llbracket \mathcal{A}_M \rrbracket^{-1}(T) \text{ is recognizable for each } T \in \mathcal{P}_{\text{fin}}(\langle M \rangle) \\
&\iff \text{the submonoid } \mathsf{T}_M \text{ generated by } M \text{ is finite,}
\end{aligned}$$

where the second equivalence follows from Lemma 5.3.9, and the last one is due to Lemma 5.3.13. Then, due to our assumption, it is decidable whether the submonoid T_M generated by M is finite. This contradicts to Lemma 5.3.3, *i.e.*, our assumption is wrong. □

5.4 Decidability of crisp-determinization

In this section we identify two subclasses of wta, for which the crisp-determinization problem is decidable. We first introduce the concept of past-finite monotonic strong bimonoid. These particular weight structures have several desirable properties (*cf.* Lemmas 5.4.2 and 5.4.4). Hence, if B is past-finite monotonic, then we can simplify the characterization of crisp-determinizability given in Lemma 5.3.9: for an arbitrary (Σ, B)-wta $\mathcal{A}$, the wta $\mathcal{A}$ is crisp-determinizable if and only if $\mathrm{im}([\![\mathcal{A}]\!])$ is finite.

To characterize finiteness of $\mathrm{im}([\![\mathcal{A}]\!])$, we consider a structural property of $\mathcal{A}$. More precisely, certain Σ-contexts and loops of $\mathcal{A}$ on those Σ-contexts are of interest. If $\mathcal{A}$ has that structural property, then the weight of any run on any Σ-tree corresponds to a weight of a run on a small Σ-tree (*cf.* Lemma 5.4.7).

To formalize a relationship between that structural property of $\mathcal{A}$ and finiteness of $\mathrm{im}([\![\mathcal{A}]\!])$ (*cf.* Lemma 5.4.13), it is crucial that $\mathcal{A}$ has only useful states, *i.e.*, each state of $\mathcal{A}$ is a part of at least one accepting run. However, that normal form of $\mathcal{A}$ can be obtained easily (*cf.* Lemmas 5.4.9 and 5.4.11).

Finally, since it is decidable whether $\mathcal{A}$ has that structural property, we can decide also whether $\mathrm{im}([\![\mathcal{A}]\!])$ is finite (*cf.* Lemma 5.4.14), *i.e.*, whether $\mathcal{A}$ is crisp-determinizable (*cf.* Theorem 5.4.15).

In [15, Def. 12] the concept of *monotonic semiring* is introduced. In the spirit of that definition, we define past-finite monotonic strong bimonoid as follows *cf.* [1, p. 42] and [2, Def. 2.1].

Definition 5.4.1. For every partial ordering $\preceq$ on B, we say that B is *monotonic with respect to* $\preceq$, denoted by $\mathsf{B}_\preceq$, if the following conditions hold true:
(i) for every $b_1, b_2 \in B$, we have $b_1 \preceq b_1 \oplus b_2$, and
(ii) for every b_1, b_2, b_3 in $B \setminus \{\mathbb{0}\}$ with $b_2 \neq \mathbb{1}$, we have $b_1 \otimes b_3 \prec b_1 \otimes b_2 \otimes b_3$.
Let B be monotonic with respect to some partial ordering $\preceq$ on B. We call $\mathsf{B}_\preceq$ *past-finite* if, for each $b \in B$, the set $\mathrm{past}(b) = \{a \in B \mid a \preceq b\}$ is finite. △

In the rest of this section, $\preceq$ *denotes an arbitrary partial ordering on* B *such that* B *is monotonic with respect to* $\preceq$ *if not specified otherwise.*

Note that the notation $\mathsf{B}_\preceq$ is overloaded in the following sense: it denotes (a) the strong bimonoid B and (b) the fact that B is monotonic with respect to $\preceq$.

In the following two lemmas we prove desirable properties of (past-finite) monotonic strong bimonoids.

Lemma 5.4.2. *cf.* [15, Lm. 14] and [42, Lm. 16.2.9] The following statements hold true.

1. We have $\mathbb{0} \preceq b$ for each $b \in B$, and $\mathbb{1} \preceq b$ for each $b \in (B \setminus \{\mathbb{0}\})$.
2. We have $b^0 \prec b^1 \prec b^2 \prec \ldots$ for each $b \in (B \setminus \{\mathbb{0}, \mathbb{1}\})$.
3. $\mathsf{B}_\preceq$ is positive, *i.e.*, it is zero-sum free and zero-divisor free.
4. $\mathsf{B}_\preceq$ is *one-summand free*, *i.e.*, $a \oplus b = \mathbb{1}$ implies $a, b \in \{\mathbb{0}, \mathbb{1}\}$ for every $a, b \in B$.
5. $\mathsf{B}_\preceq$ is *one-product free*, *i.e.*, $a \otimes b = \mathbb{1}$ implies $a = \mathbb{1} = b$ for every $a, b \in B$.

Proof. Statements 1 and 3-5 hold true due to [15, Lm. 14] and the fact that in the proof of [15, Lm. 14] the distributivity laws are not exploited.

Finally, we prove Statement 2. For each $n \in \mathbb{N}$, by Condition (ii) of Definition 5.4.1 for $b_1 = \mathbb{1}$, $b_2 = b$, and $b_3 = b^n$, we have $b^n = \mathbb{1} \otimes b^n \prec \mathbb{1} \otimes b \otimes b^n = b^{n+1}$. Then, since $\preceq$ is transitive, our statement follows. □

Note that, by Lemma 5.4.2(2), if $|B| \geq 2$, then $\mathsf{B}_\preceq$ is not finite. Moreover, there is exactly one monotonic strong bimonoid with finite carrier set: the semiring Boole (*cf.* Example 2.4.6(1)) with its natural order.

Example 5.4.3. Here we show examples and counterexamples of past-finite monotonic strong bimonoids. In order to do that, we consider the semirings given in Example 2.4.6. The semiring Nat is past-finite monotonic with respect to the usual linear ordering $\leq$ on $\mathbb{N}$. Similarly, the semiring $\mathsf{MaxPlus}$ is past-finite monotonic with respect to the usual linear ordering $\leq$ on $\mathbb{N}_{-\infty}$. However, the semiring Int (respectively, $\mathsf{MinPlus}$) is not past-finite monotonic with respect to the usual linear ordering $\leq$ on $\mathbb{Z}$ (respectively, on $\mathbb{N}_\infty$) as $\mathrm{past}(-1)$ (respectively, $\mathrm{past}(\infty)$) is not finite. In fact, in case of the semiring $\mathsf{MinPlus}$, there does not exist a partial ordering $\preceq$ on $\mathbb{N}_\infty$ such that $\mathsf{MinPlus}$ is past-finite monotonic with respect to $\preceq$ (*cf.* [2, Ex. 7.6]).

Moreover, the semiring Lang is past-finite monotonic with respect to the partial ordering $\preceq$, where $\preceq$ is defined, for every $L_1, L_2 \in \mathcal{P}_{\mathrm{fin}}(\Gamma^*)$, by $L_1 \preceq L_2$ if there is an injective mapping $f : L_1 \to L_2$ such that w is a substring[1] of $f(w)$ for each $w \in L_1$ [15]. △

Next we show that, if $\mathsf{B}_\preceq$ is past-finite, then, for every $(\Sigma, \mathsf{B}_\preceq)$-wta $\mathcal{A}$ and $b \in B$, the Σ-tree language $[\![\mathcal{A}]\!]^{-1}(b)$ is recognizable. By Lemma 5.3.9, this implies that $\mathcal{A}$ is crisp-determinizable if and only if $\mathrm{im}([\![\mathcal{A}]\!])$ is finite.

Lemma 5.4.4. [1, Lm. 11] and [2, Thm. 6.10] Let $\mathsf{B}_\preceq$ be past-finite. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathsf{B}_\preceq)$-wta. Then, for each $b \in B$, the Σ-tree language $[\![\mathcal{A}]\!]^{-1}(b)$ is recognizable. If, in addition, $\mathsf{B}_\preceq$ is computable, then, for each $b \in B$, we can construct a Σ-fta A_b such that $\mathrm{L}(A_b) = [\![\mathcal{A}]\!]^{-1}(b)$.

[1] We say that w is a *substring of* $f(w)$ if there exist $v, u \in \Gamma^*$ such that $f(w) = vwu$.

Proof. We put $C = \{a \in B \mid a \not\preceq b\}$. Moreover, let $\sim$ be the equivalence relation on B defined such that $B/{\preceq} = (\{\{a\} \mid a \in \mathrm{past}(b)\} \cup C)$, *i.e.*, its classes are the singleton sets $\{a\}$ for each $a \in \mathrm{past}(b)$, and the set C. Now we prove the following statement:

$$\sim \text{ is a congruence relation on } \mathsf{B}_{\preceq} \ . \tag{5.9}$$

Obviously, it is sufficient to show that C is a congruence class. For this, let $b_1', b_2' \in C$ and $b' \in B$. Since $\mathsf{B}_{\preceq}$ is monotonic, by Condition (i) of Definition 5.4.1, for each $i \in \{1, 2\}$, we have $b_i' \preceq b_i' \oplus b'$, and hence, $(b_i' \oplus b') \in C$. Moreover, if $b' \neq \mathbb{0}$, then, by Condition (ii) of Definition 5.4.1, for each $i \in \{1, 2\}$, we have $b_i' \preceq b_i' \otimes b'$ and $b_i' \preceq b' \otimes b_i'$, and thus, we also have $(b_i' \otimes b') \in C$ and $(b' \otimes b_i') \in C$, respectively. Consequently, C is a congruence class. This concludes the proof of (5.9).

Then, by (5.9), we consider the quotient strong bimonoid

$$\mathsf{B}/{\sim} = (B/{\sim}, \oplus/{\sim}, \otimes/{\sim}, [\mathbb{0}]_\sim, [\mathbb{1}]_\sim)$$

of B modulo $\sim$, where $[b_1]_\sim \oplus /{\sim}\ [b_2]_\sim = [b_1 \oplus b_2]_\sim$ and $[b_1]_\sim \otimes /{\sim}\ [b_2]_\sim = [b_1 \otimes b_2]_\sim$ for every $b_1, b_2 \in B$. Evidently, $\mathsf{B}/{\sim}$ is finite. Moreover, we consider the mapping $h : B \to B/{\sim}$ defined, for each $b' \in B$, by $h(b') = [b']_\sim$. Trivially, h is a strong bimonoid homomorphism from B to $\mathsf{B}/{\sim}$. Then, by Lemma 3.1.7, we can construct a $(\Sigma, \mathsf{B}/{\sim})$-wta $h(\mathcal{A})$ such that $[\![h(\mathcal{A})]\!] = (h \circ [\![\mathcal{A}]\!])$.

Since $\mathsf{B}/{\sim}$ is finite, the wta $h(\mathcal{A})$ has finite order. Hence, by Theorem 5.2.8, there exists a crisp-deterministic $(\Sigma, \mathsf{B}/{\sim})$-wta $h(\mathcal{A})'$ such that $h(\mathcal{A})'$ is equivalent to $h(\mathcal{A})$. Thus, we can calculate as follows:

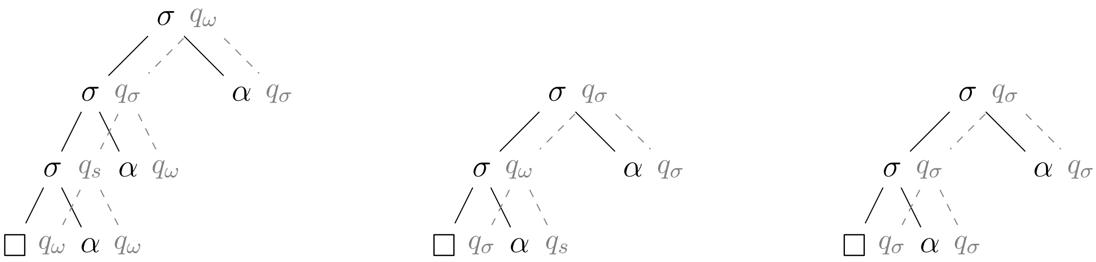
$$[\![\mathcal{A}]\!]^{-1}(b) = (h \circ [\![\mathcal{A}]\!])^{-1}([b]_\sim) = [\![h(\mathcal{A})]\!]^{-1}([b]_\sim) = [\![h(\mathcal{A})']\!]^{-1}([b]_\sim) \ ,$$

where the second equality is due to Lemma 3.1.7, and the last one follows from Theorem 5.2.8. Moreover, since, by Lemma 5.3.9(1 $\Rightarrow$ 2), the Σ-tree language $[\![h(\mathcal{A})']\!]^{-1}([b]_\sim)$ is recognizable, $[\![\mathcal{A}]\!]^{-1}(b)$ is recognizable as well.

Finally, we show that, for each $b \in B$, we can construct a Σ-fta A_b such that $\mathrm{L}(A_b) = [\![\mathcal{A}]\!]^{-1}(b)$. In order to do that, we assume that, in addition, B is computable. Then also $\mathsf{B}/{\sim}$ is computable. Hence, by Theorem 5.2.12, we can construct the crisp-deterministic $(\Sigma, \mathsf{B}/_\sim)$-wta $h(\mathcal{A})'$. Moreover, by following the proof of Lemma 5.3.9(1 $\Rightarrow$ 2), for each $b \in B$, we can construct a Σ-fta A_b such that $\mathrm{L}(A_b) = [\![\mathcal{A}]\!]^{-1}(b)$. □

In the following example we check whether we can apply Lemma 5.4.4 to the three wta which appear in Examples 3.1.4-3.1.6.

Example 5.4.5. Firstly, we consider the $(\Sigma, \mathsf{MaxPlus})$-wta $\mathcal{A}_{\max}$ shown in Example 3.1.4. Recall that, by Example 5.4.3, the semiring $\mathsf{MaxPlus}$ is past-finite mono-



(a) *neither small nor valid loop on the* Σ*-context* c^3

(b) *small but not valid loop on the* Σ*-context* c^2

(c) *small and valid loop on the* Σ*-context* c^2

Figure 5.3. *Loops of the* $(\Sigma, \mathsf{MaxPlus})$*-wta* $\mathcal{A}$ *defined in Example 3.1.4 on some powers of the* Σ*-context* $c = \sigma(\square, \alpha)$ *defined in Example 5.4.6*

tonic with respect to the usual linear ordering $\leq$. Then, by Lemma 5.4.4, for each $n \in \mathbb{N}_{-\infty}$, the Σ-tree language $[\![\mathcal{A}_{\max}]\!]^{-1}(n)$ is recognizable.

But, also by Example 5.4.3, the semiring $\mathsf{MinPlus}$ is not past-finite monotonic with respect to any partial ordering $\preceq$ on $\mathbb{N}_\infty$. Hence, if we consider the $(\Sigma, \mathsf{MinPlus})$-wta $\mathcal{A}_\sigma$ constructed in Example 3.1.5, then we cannot apply Lemma 5.4.4 to show that, for each $n \in \mathbb{N}_\infty$, the Σ-tree language $[\![\mathcal{A}_\sigma]\!]^{-1}(n)$ is recognizable.

Similarly, the bounded lattice M_3 given in Example 2.4.7(2) is not monotonic, because, by Condition (ii) of Definition 5.4.1 for $b_1 = i = b_3$ and $b_2 = a$, we should have $i = i \wedge i \prec i \wedge a \wedge i = a$, but that does not hold true. Thus, for the (Σ, M_3)-wta $\mathcal{A}_{\text{split}}$ shown in Examples 3.1.6, we cannot apply Lemma 5.4.4 either. $\triangle$

Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. For every $c \in \mathrm{T}_\Sigma$, $q \in Q$, and loop $\rho \in \mathrm{Run}_\mathcal{A}(q, c, q)$, we call ρ a *small loop* if $\mathrm{height}(c) \leq |Q|$. Moreover, we say that *small valid loops of* $\mathcal{A}$ *have weight* $\mathbb{1}$ if, for every $c \in \mathrm{T}_\Sigma$, $q \in Q$, and $\rho \in \mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q, c, q)$, we have $\mathrm{wt}(c, \theta) = \mathbb{1}$.

Example 5.4.6. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MaxPlus})$-wta $\mathcal{A}_{\max}$ constructed in Example 3.1.4. Moreover, let $c = \sigma(\square, \alpha)$. Then Figure 5.3 shows some loops of $\mathcal{A}_{\max}$ on c^n for some $n \in \mathbb{N}$ as follows. The loop depicted in Figure 5.3(a) is neither small (as $\mathrm{height}(c^3) \geq |\{q_\sigma, q_\omega, q_s\}|$), nor valid (as $\delta_2(q_\omega q_\omega, \sigma, q_s) = -\infty$). The loop illustrated in Figure 5.3(b) is small but not valid as $\delta_2(q_\sigma q_s, \sigma, q_\omega) = -\infty$. Finally, the loop shown in Figure 5.3(c) is small and valid. $\triangle$

Lemma 5.4.7. [1, Lm. 12] (also *cf.* [2, Lm. 5.5]) Let $\mathcal{A} = (Q, \delta, F)$ be a $(\Sigma, \mathsf{B}_\preceq)$-wta such that small valid loops of $\mathcal{A}$ have weight $\mathbb{1}$. Then, for every $\xi' \in \mathrm{T}_\Sigma$, $q' \in Q$, and $\rho' \in \mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q', \xi')$, there exist $\xi_Q \in \mathrm{T}_\Sigma$ and $\rho_Q \in \mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q', \xi_Q)$ such that $\mathrm{height}(\xi_Q) < |Q|$ and $\mathrm{wt}(\xi', \rho') = \mathrm{wt}(\xi_Q, \rho_Q)$.

Proof. Trivially, if $\mathrm{height}(\xi') < |Q|$, then we let $\xi_Q = \xi'$ and $\rho_Q = \rho'$, and we are done. Hence, we may assume that $\mathrm{height}(\xi') \geq |Q|$. By applying Theorem 3.2.4, there exist $c', c \in \mathrm{C}_\Sigma$, $\xi \in \mathrm{T}_\Sigma$, $q \in Q$, $\theta' \in \mathrm{Run}_\mathcal{A}(q', c', q)$, $\theta \in \mathrm{Run}_\mathcal{A}(q, c, q)$, and $\rho \in \mathrm{Run}_\mathcal{A}(q, \xi)$ such that the conditions mentioned in Theorem 3.2.4 hold true, and, in particular, we have

$$\mathrm{wt}(\xi', \rho') = \mathrm{wt}(c'[c[\xi]], \theta'[\theta[\rho]]) = l_{c',\theta'} \otimes l_{c,\theta} \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c,\theta} \otimes r_{c',\theta'} \quad \text{and}$$
$$\mathrm{wt}(c'[\xi], \theta'[\rho]) = l_{c',\theta'} \otimes \mathrm{wt}(\xi, \rho) \otimes r_{c',\theta'} \ .$$

Observe that, since ρ' is valid, each of the runs θ', θ, and ρ is valid. Then we have

$$(l_{c,\theta} \otimes r_{c,\theta}) = \mathrm{wt}(c, \theta) = \mathbb{1} \ ,$$

where the first equality follows from Lemma 3.2.1, and the second one is due to the fact that small valid loops of $\mathcal{A}$ have weight $\mathbb{1}$. Moreover, since, by Lemma 5.4.2(5), $\mathsf{B}_\preceq$ is one-product free, we have $l_{c,\theta} = r_{c,\theta} = \mathbb{1}$, *i.e.*, $\mathrm{wt}(\xi', \rho') = \mathrm{wt}(c'[\xi], \theta'[\rho])$. Note that we have $\theta'[\rho] \in \mathrm{Run}^{\mathrm{v}}_\mathcal{A}(q', c'[\xi])$ and $\mathrm{size}(c'[\xi]) < \mathrm{size}(\xi')$. If $\mathrm{height}(c'[\xi]) < |Q|$, then we let $\xi_Q = c'[\xi]$ and $\rho_Q = \theta'[\rho]$, and we are done. Otherwise, we continue with the $c'[\xi]$, q', and $\theta'[\rho]$ as before, and after finitely many steps, we obtain the Σ-tree ξ_Q with $\mathrm{height}(\xi_Q) < |Q|$, and the run ρ_Q as required. □

The following notions are crucial to construct a trim wta. For each $B' \subseteq B$, we say that B has *effective tests for* B' if, for every $b \in B$ and $b' \in B'$, we can decide whether $b = b'$. Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. For each $q \in Q$, we call q *useful (in* $\mathcal{A}$*)* if there exist $\xi \in \mathrm{T}_\Sigma$ and $\rho \in \mathrm{Run}^{\mathrm{a}}_\mathcal{A}(\xi)$ such that $q \in \mathrm{im}(\rho)$. We say that $\mathcal{A}$ has *a useful state* if there exists $q \in Q$ such that q is useful. Moreover, $\mathcal{A}$ is said to be *trim* if each of its states is useful. Now, by some examples, we illustrate the notions defined above.

Example 5.4.8. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. Moreover, we consider the $(\Sigma, \mathsf{MaxPlus})$-wta $\mathcal{A}_{\max}$ defined in Example 3.1.4. Let $\xi \in \mathrm{T}_\Sigma$. Recall that, by Example 3.1.4, there are exactly three valid runs of $\mathcal{A}_{\max}$ on ξ. But, out of the three valid runs $\mathcal{A}_{\max}$ on ξ, only two are accepting. Moreover, for each $\rho \in \mathrm{Run}^{\mathrm{a}}_\mathcal{A}(\xi)$, we have $q_s \notin \mathrm{im}(\rho)$. Consequently, q_s is not useful, *i.e.*, $\mathcal{A}_{\max}$ is not trim.

Nevertheless, both the wta $\mathcal{A}_\sigma$ given in Example 3.1.5 and the wta $\mathcal{A}_{\mathrm{split}}$ constructed in Example 3.1.6 are trim *ab ovo*. △

Since deciding whether an arbitrary (Σ, B)-wta $\mathcal{A}$ has a useful state is technical rather than hard if B has effective tests for $\{\mathbb{0}\}$, we just show the following result without its proof.

Lemma 5.4.9. [2, Lm. 4.1] Let B have effective tests for $\{\mathbb{0}\}$. It is decidable, for arbitrary (Σ, B)-wta $\mathcal{A}$, whether $\mathcal{A}$ has a useful state. □

Next we recall unambiguous wta. Let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. We call $\mathcal{A}$ *unambiguous* if, for each $\xi \in \mathrm{T}_\Sigma$, we have $|\mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi)| \leq 1$. If this is the case, then, for each $\xi \in \mathrm{T}_\Sigma$, either (a) $\mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi) = \emptyset$, and thus, $[\![\mathcal{A}]\!](\xi) = \mathbb{0}$, or else (b) there is a unique run ρ in $\mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi)$, *i.e.*, we have $\{\rho\} = \mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi)$, such that $[\![\mathcal{A}]\!](\xi) = \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}$. Obviously, each bu deterministic wta is unambiguous; however, there are easy examples of unambiguous wta, to which there does not exist an equivalent bu deterministic wta *cf.* [57]. Then, by an example, we demonstrate the unambiguous wta.

Example 5.4.10. We consider the wta $\mathcal{A}_{\max}$ shown in Example 3.1.4. As already mentioned, on each tree, there are exactly two accepting runs of $\mathcal{A}_{\max}$, and hence, $\mathcal{A}_{\max}$ is not unambiguous.

This is not the case if we consider either the wta $\mathcal{A}_\sigma$ defined in Example 3.1.5 or the wta $\mathcal{A}_{\mathrm{split}}$ given in Example 3.1.6 as on each tree there is exactly one accepting run of both wta, *i.e.*, both $\mathcal{A}_\sigma$ and $\mathcal{A}_{\mathrm{split}}$ are unambiguous. △

Similarly, since, for an arbitrary (Σ, B)-wta $\mathcal{A}$ with a useful state, constructing a semantically equivalent trim (Σ, B)-wta is technical rather than hard if B has effective tests for $\{\mathbb{0}\}$, we just give also the following result without its proof.

Lemma 5.4.11. [1, Lm. 5] and [2, Thm. 4.2] Let B have effective tests for $\{\mathbb{0}\}$. Moreover, let $\mathcal{A} = (Q, \delta, F)$ be a (Σ, B)-wta. If $\mathcal{A}$ has a useful state, then we can construct a (Σ, B)-wta $\mathcal{A}'$ such that $\mathcal{A}'$ is trim and it is equivalent to $\mathcal{A}$. If, in addition, $\mathcal{A}$ is unambiguous, then $\mathcal{A}'$ is so. □

Here we give an example of the application of Lemma 5.4.11.

Example 5.4.12. Here we continue Example 5.4.8. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. Furthermore, we consider the $(\Sigma, \mathsf{MaxPlus})$-wta $\mathcal{A}_{\max}$ defined in Example 3.1.4. Since q_s is not useful, but the states q_σ and q_ω are so, by Lemma 5.4.11, we can construct the $(\Sigma, \mathsf{MaxPlus})$-wta

$$\mathcal{A}'_{\max} = (\{q_\sigma, q_\omega\}, \delta', F') \ ,$$

where $\delta'_0(\varepsilon, \alpha, q_\sigma) = \delta'_0(\varepsilon, \alpha, q_\omega) = \delta'_2(q_\sigma q_\sigma, \omega, q_\sigma) = \delta'_2(q_\omega q_\omega, \sigma, q_\omega) = 0$, $\delta'_2(q_\sigma q_\sigma, \sigma, q_\sigma) = \delta'_2(q_\omega q_\omega, \omega, q_\omega) = 1$, and every other transition in δ' has weight $-\infty$, and $F'_{q_\sigma} = F'_{q_\omega} = 0$. Trivially, $\mathcal{A}'_{\max}$ is trim. △

Lemma 5.4.13. [1, Thm. 13] (also *cf.* [2, Thm. 7.1]) Let $\mathsf{B}_\preceq$ be past-finite. Moreover, let $\mathcal{A} = (Q, \delta, F)$ be a trim $(\Sigma, \mathsf{B}_\preceq)$-wta such that $\mathsf{B}_\preceq$ is additively locally finite or $\mathcal{A}$ is unambiguous. Then the following statements are equivalent.

1. The set $\mathrm{im}([\![\mathcal{A}]\!])$ is finite.
2. Small valid loops of $\mathcal{A}$ have weight $\mathbb{1}$.

Proof. (1 ⇒ 2). We prove, by contraposition, our statement. In order to do that, assume that there exist $c \in \mathrm{T}_\Sigma$, $q \in Q$, and small $\theta \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, c, q)$ such that $\mathrm{wt}(c, \theta) \neq \mathbb{1}$. Since θ is valid, and since $\mathsf{B}_\preceq$ is zero-divisor free and one-product free by Lemma 5.4.2, we have $\mathbb{1} \prec \mathrm{wt}(c, \theta)$. Moreover, since $\mathcal{A}$ is trim, the state q is useful, and hence, there exist $c' \in \mathrm{T}_\Sigma$, $\xi \in \mathrm{T}_\Sigma$, $q' \in Q$, $\theta' \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q', c, q)$, and $\rho \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, \xi)$ such that $\theta'[\theta[\rho]] \in \mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(c'[c[\xi]])$.

By Lemma 3.2.1, we have $\mathrm{wt}(c, \theta) = l_{c,\theta} \otimes r_{c,\theta}$. Moreover, since $\mathbb{1} \prec \mathrm{wt}(c, \theta)$, we have $\mathbb{1} \prec l_{c,\theta}$ or $\mathbb{1} \prec r_{c,\theta}$. Here we consider the case where $\mathbb{1} \prec l_{c,\theta}$. (Observe that the other case is symmetrical.) Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathrm{wt}(c'[c^n[\xi]], \theta'[\theta^n[\rho]]) &= l_{c',\theta'} \otimes (l_{c,\theta})^n \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c',\theta'} \\
&\prec^{(\dagger)} l_{c',\theta'} \otimes (l_{c,\theta})^{n+1} \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c',\theta'} \\
&\preceq^{(\ddagger)} l_{c',\theta'} \otimes (l_{c,\theta})^{n+1} \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^{n+1} \otimes r_{c',\theta'} \\
&= \mathrm{wt}(c'[c^{n+1}[\xi]], \theta'[\theta^{n+1}[\rho]]) \ ,
\end{aligned}$$

where the first and the last equalities follow from Theorem 3.2.3, at (†) we apply Condition (ii) of Definition 5.4.1 for $b_1 = (l_{c',\theta'} \otimes (l_{c,\theta})^n)$, $b_2 = l_{c,\theta}$, and $b_3 = (\mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n \otimes r_{c',\theta'})$; and at (‡) we apply Condition (ii) of Definition 5.4.1 for $b_1 = (l_{c',\theta'} \otimes (l_{c,\theta})^{n+1} \otimes \mathrm{wt}(\xi, \rho) \otimes (r_{c,\theta})^n)$, $b_2 = r_{c,\theta}$, and $b_3 = r_{c',\theta'}$. Furthermore, since, $\preceq$ is transitive, we have

$$\mathrm{wt}(c'[c^0[\xi]], \theta'[\theta^0[\rho]]) \prec \mathrm{wt}(c'[c^1[\xi]], \theta'[\theta^1[\rho]]) \prec \dots \ . \tag{5.10}$$

Next we construct an infinite sequence $\xi_1, \xi_2, \xi_3, \dots$ of Σ-trees such that the elements $[\![\mathcal{A}]\!](\xi_1), [\![\mathcal{A}]\!](\xi_2), [\![\mathcal{A}]\!](\xi_3), \dots$ in B are pairwise different as follows. We let $\xi_1 = c'[c[\xi]]$. Since $\mathsf{B}_\preceq$ is past-finite, the set $P_1 = \mathrm{past}([\![\mathcal{A}]\!](\xi_1))$ is finite. By (5.10), there exists $n_2 \in \mathbb{N}$ such that $\mathrm{wt}(c'[c^{n_2}[\xi]], \theta'[\theta^{n_2}[\rho]]) \notin P_1$. Thus, we let $\xi_2 = c'[c^{n_2}[\xi]]$ and $\rho_2 = \theta'[\theta^{n_2}[\rho]]$. Since $\rho_2 \in \mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(q', \xi_2)$ and B is monotonic, by Conditions (ii) and (i) of Definition 5.4.1, we have

$$\mathrm{wt}(\xi_2, \rho_2) \preceq \mathrm{wt}(\xi_2, \rho_2) \otimes F_{q'} \preceq \bigoplus_{\rho_2' \in \mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi_2)} \mathrm{wt}(\xi_2, \rho_2') \otimes F_{\rho_2'(\varepsilon)} = [\![\mathcal{A}]\!](\xi_2) \ ,$$

where $F_{q'}$ may be $\mathbb{1}$, and $\mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi_2)$ may equal $\{\rho_2\}$. Consequently, we have $[\![\mathcal{A}]\!](\xi_2) \notin P_1$. Now we put $P_2 = \mathrm{past}([\![\mathcal{A}]\!](\xi_2))$. By (5.10), there exists $n_3 \in \mathbb{N}$ such that $\mathrm{wt}(c'[c^{n_3}[\xi]], \theta'[\theta^{n_3}[\rho]]) \notin (P_1 \cup P_2)$. By continuing this process, we obtain the desired sequence of Σ-trees. Therefore the set $\mathrm{im}([\![\mathcal{A}]\!])$ is not finite.

(2 ⇒ 1). Let $\xi' \in \mathrm{T}_\Sigma$. Recall that $[\![\mathcal{A}]\!](\xi') = \bigoplus_{q' \in Q} \bigoplus_{\rho' \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q', \xi')} \mathrm{wt}(\xi', \rho') \otimes F_{q'}$. Since small valid loops of $\mathcal{A}$ have weight $\mathbb{1}$, by Lemma 5.4.7, for every $q' \in Q$ and $\rho' \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q', \xi')$, there exist a Σ-tree ξ_Q and a run $\rho_Q \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q', \xi_Q)$ such that

$\mathrm{height}(\xi_Q) < |Q|$ and $\mathrm{wt}(\xi', \rho') = \mathrm{wt}(\xi_Q, \rho_Q)$. Hence, we consider the set

$$H = \{\mathrm{wt}(\xi_Q, \rho_Q) \otimes F_{q'} \mid \xi_Q \in \mathrm{T}_\Sigma, \mathrm{height}(\xi_Q) < |Q|, q' \in Q, \rho_Q \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q', \xi_Q)\} \ .$$

Observe that H is a finite set. Now we proceed by case analysis.

$\mathsf{B}_\preceq$ is additively locally finite: Then the set $\langle H \rangle_{\{\oplus, \mathbb{0}\}}$ is finite as well, and, since we have $[\![\mathcal{A}]\!](\xi') \in \langle H \rangle_{\{\oplus, \mathbb{0}\}}$, *i.e.*, $\mathrm{im}([\![\mathcal{A}]\!]) \subseteq \langle H \rangle_{\{\oplus, \mathbb{0}\}}$, the set $\mathrm{im}([\![\mathcal{A}]\!])$ is finite as required.

$\mathcal{A}$ is unambiguous: Then either $[\![\mathcal{A}]\!](\xi') = \mathbb{0}$, or else there is a unique accepting run ρ' of $\mathcal{A}$ on ξ', *i.e.*, we have $\{\rho'\} = \mathrm{Run}^{\mathrm{a}}_{\mathcal{A}}(\xi')$, such that $[\![\mathcal{A}]\!](\xi') = \mathrm{wt}(\xi', \rho') \otimes F_{\rho'(\varepsilon)}$. Thus, we have $\mathrm{im}([\![\mathcal{A}]\!]) \subseteq H$, *i.e.*, the set $\mathrm{im}([\![\mathcal{A}]\!])$ is finite as desired. □

Lemma 5.4.14. [1, Cor. 14] (also *cf.* [2, Thms. 7.5 and 7.7]) Let $\mathsf{B}_\preceq$ be past-finite and have effective tests for $\{\mathbb{1}\}$. It is decidable, for arbitrary trim $(\Sigma, \mathsf{B}_\preceq)$-wta $\mathcal{A}$ such that $\mathsf{B}_\preceq$ is additively locally finite or $\mathcal{A}$ is unambiguous, whether $\mathrm{im}([\![\mathcal{A}]\!])$ is finite.

Proof. Let $\mathcal{A} = (Q, \delta, F)$. By Lemma 5.4.13, we have

$$\mathrm{im}([\![\mathcal{A}]\!]) \text{ is finite} \quad \text{if and only if} \quad \text{small valid loops of } \mathcal{A} \text{ have weight } \mathbb{1} \ .$$

The latter property is decidable for the following reasons. Since there are only finitely many contexts in C_Σ of height less than $|Q|$, there are only finitely many small and valid loops of $\mathcal{A}$. Moreover, for each small and valid loop $\theta \in \mathrm{Run}^{\mathrm{v}}_{\mathcal{A}}(q, c, q)$ for some $c \in \mathrm{C}_\Sigma$ and $q \in Q$, we have $\mathrm{wt}(c, \theta) = \mathbb{1}$ if and only if $\delta_k\big(\theta(v1) \cdots \theta(vk), c(v), \theta(v)\big) = \mathbb{1}$ for every $k \in \mathbb{N}$ and $v \in \mathrm{pos}(c)$ with $c(v) \in \Sigma^{(k)}$, where the right-hand side of the equivalence is decidable as $\mathsf{B}_\preceq$ has effective tests for $\{\mathbb{1}\}$. This concludes our proof. □

Theorem 5.4.15. *[1, Thm. 10] Let Σ be a ranked alphabet such that $\Sigma^{(0)} \neq \emptyset$. Moreover, let $\mathsf{B} = (B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid and $\preceq$ be a partial ordering on B such that B is past-finite monotonic with respect to $\preceq$, and B has effective tests for $\{\mathbb{0}, \mathbb{1}\}$. Then the following statements hold true.*

1. *If, in addition, B is additively locally finite, then it is decidable, for arbitrary (Σ, B)-wta $\mathcal{A}$, whether $\mathcal{A}$ is crisp-determinizable.*
2. *It is decidable, for arbitrary unambiguous (Σ, B)-wta $\mathcal{A}$, whether $\mathcal{A}$ is crisp-determinizable.*

Proof. Let $\mathcal{A}$ be an arbitrary $(\Sigma, \mathsf{B}_\preceq)$-wta such that $\mathsf{B}_\preceq$ is additively locally finite or $\mathcal{A}$ is unambiguous. By Lemma 5.4.9, it is decidable whether $\mathcal{A}$ has a useful state. If the answer is "no", then $\mathrm{im}([\![\mathcal{A}]\!]) = \{\mathbb{0}\}$. Obviously, we can construct a crisp-deterministic $(\Sigma, \mathsf{B}_\preceq)$-wta $\mathcal{A}'$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{A}']\!]$.

Otherwise, *i.e.*, if $\mathcal{A}$ has a useful state, then, by Lemma 5.4.11, we may assume that $\mathcal{A}$ is trim. Then

$$\begin{aligned}
&\mathcal{A} \text{ is crisp-determinizable} \\
&\iff \text{there exists a crisp-deterministic } (\Sigma, \mathsf{B}_{\preceq})\text{-wta } \mathcal{A}' \text{ such that } [\![\mathcal{A}]\!] = [\![\mathcal{A}']\!] \\
&\iff \mathrm{im}([\![\mathcal{A}]\!]) \text{ is finite,}
\end{aligned}$$

where the second equivalence follows from Lemmas 5.3.9 and 5.4.4. Moreover, by Lemma 5.4.14, it is decidable, whether $\mathrm{im}([\![\mathcal{A}]\!])$ is finite. Hence, it is decidable whether $\mathcal{A}$ is crisp-determinizable. This concludes our proof. □

Now we show an application of Theorem 5.4.15 as follows.

Example 5.4.16. Let $\Sigma = \{\sigma^{(2)}, \omega^{(2)}, \alpha^{(0)}\}$. We consider the $(\Sigma, \mathsf{MaxPlus}_{\leq})$-wta $\mathcal{A}_{\max}$ defined in Example 3.1.4. Since, the semiring $\mathsf{MaxPlus}_{\leq}$ is past-finite and additively locally finite, we can apply Theorem 5.4.15(1) to the $(\Sigma, \mathsf{MaxPlus}_{\leq})$-wta $\mathcal{A}_{\max}$. Hence, by Lemma 5.4.11, we first construct the trim $(\Sigma, \mathsf{MaxPlus}_{\leq})$-wta $\mathcal{A}'_{\max}$ such that $\mathcal{A}'_{\max}$ is equivalent to $\mathcal{A}_{\max}$ (*cf.* Example 5.4.12). Then, by considering the context $c = \sigma(\square, \alpha)$ and the (q_σ, q_σ)-run θ of $\mathcal{A}_{\max}$ on c such that $\theta(2) = q_\sigma$, we have a small and valid loop with $\mathbb{1} \prec \mathrm{wt}(c, \theta)$. Consequently, Theorem 5.4.15(1) delivers a "no", *i.e.*, $\mathcal{A}_{\max}$ is not crisp-determinizable. △

Moreover, note that even though the semiring $\mathsf{MinPlus}$ given in Example 2.4.6(4) is additively locally finite, it is not past-finite monotonic by Example 5.4.5, and thus, we cannot apply Theorem 5.4.15(1) to the $(\Sigma, \mathsf{MinPlus})$-wta $\mathcal{A}_\sigma$ shown in Example 3.1.5. Similarly, by Example 5.4.5, also the bounded lattice M_3 is not past-finite monotonic, and hence, we cannot apply Theorem 5.4.15(2) to the unambiguous (Σ, M_3)-wta $\mathcal{A}_{\mathrm{split}}$ constructed in Example 3.1.6.

5.5 Undecidability and decidability results for weighted string automata

Here we recall the concept of weighted string automata and show that each of our undecidability (respectively, decidability) results holds for weighted string automata as well.

Let Γ be a nonempty alphabet. A (Γ, B)-*weighted language* (or just: weighted language) is a B-weighted set $\psi : \Gamma^* \to B$. A *weighted string automaton (over* Γ *and* B*)* (for short: (Γ, B)-wsa, or just: wsa) [33, 70] is a quadruple $\mathcal{A} = (Q, I, \delta, F)$, where

- Q is a finite and nonempty set (*states*) such that $Q \cap \Gamma = \emptyset$,
- $I : Q \to B$ is a mapping (*initial weight mapping*),
- $\delta : Q \times \Gamma \times Q \to B$ is a mapping (*transition mapping*), and

- $F : Q \to B$ is a mapping (*final weight mapping*).

Let $\mathcal{A} = (Q, I, \delta, F)$ be a (Γ, B)-wsa. We define the (run) semantics of $\mathcal{A}$ as follows. Let $w = a_1 \cdots a_n$ be a string in Γ^* with $n \in \mathbb{N}$ and $a_i \in \Gamma$ for each $i \in [n]$. A *run of $\mathcal{A}$ on w* is a string $\rho = q_0 \cdots q_n$ in Q^{n+1}, and the *weight of ρ for w*, denoted by $\mathrm{wt}_{\mathcal{A}}(w, \rho)$, is the element of B defined by

$$\mathrm{wt}_{\mathcal{A}}(w, \rho) = I(q_0) \otimes \delta(q_0, a_1, q_1) \otimes \ldots \otimes \delta(q_{n-1}, a_n, q_n) \otimes F(q_n) \ .$$

Then the *(run) semantics of $\mathcal{A}$* is the weighted language $[\![\mathcal{A}]\!] : \Gamma^* \to B$ defined, for each $w \in \Gamma^*$, by

$$[\![\mathcal{A}]\!](w) = \bigoplus_{\rho \in Q^{\mathrm{len}(w)+1}} \mathrm{wt}_{\mathcal{A}}(w, \rho) \ .$$

In particular, we have $[\![\mathcal{A}]\!](\varepsilon) = \bigoplus_{q \in Q} (I(q) \otimes F(q))$. A (Γ, B)-weighted language ψ is *recognizable* if there exists a (Γ, B)-wsa $\mathcal{A}$ such that $\psi = [\![\mathcal{A}]\!]$.

In [42, Lm. 3.3.3], it is shown that the concept of (Γ, B)-wsa and the concept of (Σ, B)-wta, where Σ is a string ranked alphabet, are essentially the same. In fact, for each (Γ, B)-wsa $\mathcal{A}$, there exist a string ranked alphabet Σ, a bijection $\mathrm{tree} : \Gamma^* \to \mathrm{T}_\Sigma$, and a (Σ, B)-wta $\mathcal{A}_{\mathrm{tree}}$ such that $[\![\mathcal{A}]\!](w) = (\mathcal{A}_{\mathrm{tree}} \circ \mathrm{tree})(w)$ for each $w \in \Gamma^*$. Moreover, the inverse of that statement also holds true.

Since the ranked alphabet in Theorems 5.3.7 and 5.3.14 is a string ranked alphabet, according to the equivalence described above, the corresponding undecidability results hold for weighted string automata. Furthermore, Theorem 5.4.15 holds, in particular, for any string ranked alphabet. Hence, by the above, it also holds for weighted string automata.

Concluding remarks. We consider the $(\Sigma_M, \mathsf{Trans}_M)$-wta $\mathcal{A}_M$ constructed in Construction 5.3.4. Recall that Trans_M is idempotent, *i.e.*, Trans_M is additively locally finite, and $\mathcal{A}_M$ is bu deterministic, *i.e.*, $\mathcal{A}_M$ is unambiguous. Compared to Theorem 5.4.15, Theorem 5.3.14 shows that dropping the condition "B is past-finite monotonic with respect to some partial ordering $\preceq$ on B" results in undecidability.

The author of this PhD thesis declares that his contribution to Theorems 5.2.8, 5.2.12, 5.3.7, 5.3.14, and 5.4.15 is decisive, that Theorems 5.2.8 and 5.4.15 are published in [3] and [1], respectively, and also that Theorems 5.3.7 and 5.3.14 are slightly stronger than [3, Thm. 8.9] and [3, Thm. 8.5], respectively, but are based on the same ideas. Finally, we mention that [2, Thm. 6.6], [5, Thms. 7 and 11], and [2, Thms. 7.5, 7.7, and 7.15] supersede Theorems 5.2.8, 5.3.14, and 5.4.15, respectively, but the contribution of the author to those stronger results is not decisive.

Publications of the author

On the subjects of the thesis

[1] M. Droste, Z. Fülöp, **D. Kószó**, and H. Vogler. "Crisp-Determinization of Weighted Tree Automata over Additively Locally Finite and Past-Finite Monotonic Strong Bimonoids Is Decidable". In: *Descriptional Complexity of Formal Systems (DCFS 2020)*. Ed. by G. Jirásková and G. Pighizzini. Vol. 12442. Lecture Notes in Computer Science. Springer Nature Switzerland, 2020, 39–51.

[2] M. Droste, Z. Fülöp, **D. Kószó**, and H. Vogler. "Finite-image property of weighted tree automata over past-finite monotonic strong bimonoids". In: *Theoretical Computer Science* 919 (2022), 118–143.

[3] Z. Fülöp, **D. Kószó**, and H. Vogler. "Crisp-determinization of weighted tree automata over strong bimonoids". In: *Discrete Mathematics & Theoretical Computer Science* 23(1) (2021), #18.

[4] **D. Kószó**. "Weighted Tree Generating Regular Systems over Strong Bimonoids with Reduction Semantics". In: *Journal of Automata, Languages and Combinatorics* 27(4) (2022), 271–307.

Further related publications

[5] M. Droste, Z. Fülöp, and **D. Kószó**. "Decidability Boundaries for the Finite-Image Property of Weighted Finite Automata". In: *International Journal of Foundations of Computer Science* (To appear).

[6] **D. Kószó**. "Tree generating context-free grammars and regular tree grammars are equivalent". In: *Annales Mathematicae et Informaticae* 56 (2022), 58–70.

Other references

[7] A. Alexandrakis and S. Bozapalidis. “Weighted grammars and Kleene’s theorem”. In: *Information Processing Letters* 24(1) (1987), 1–4.

[8] F. Baader and T. Nipkow. *Term Rewriting and All That*. Cambridge: Cambridge University Press, 1998.

[9] M. Bauderon and B. Courcelle. “Graph expressions and graph rewritings”. In: *Mathematical Systems Theory* 20(1) (1987), 83–127.

[10] J. Berstel and C. Reutenauer. “Recognizable formal power series on trees”. In: *Theoretical Computer Science* 18(2) (1982), 115–148.

[11] J. Berstel and Ch. Reutenauer. *Rational Series and Their Languages*. Vol. 12. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1988.

[12] G. Birkhoff. *Lattice Theory*. 3rd. Vol. 25. Colloquium Publications. American Mathematical Society, 1993.

[13] B. Borchardt. “A pumping lemma and decidability problems for recognizable tree series”. In: *Acta Cybernetica* 16(4) (2004), 509–544.

[14] B. Borchardt. *The Theory of Recognizable Tree Series*. (Ph.D. thesis, 2004, TU Dresden, Germany). Verlag für Wissenschaft und Forschung, 2005.

[15] B. Borchardt, Z. Fülöp, Z. Gazdag, and A. Maletti. “Bounds for tree automata with polynomial costs”. In: *Journal of Automata, Languages and Combinatorics* 10(2–3) (2005), 107–157.

[16] B. Borchardt and H. Vogler. “Determinization of finite state weighted tree automata”. In: *Journal of Automata, Languages and Combinatorics* 8(3) (2003), 417–463.

[17] S. Bozapalidis and O. Louscou-Bozapalidou. “Fuzzy tree language recognizability”. In: *Fuzzy Sets and Systems* 161 (2010), 716–734.

[18] W. S. Brainerd. “Tree Generating Regular Systems”. In: *Information and Control* 14 (1969), 217–231.

[19] M. Büchse, J. May, and H. Vogler. “Determinization of weighted tree automata using factorizations”. In: *Journal of Automata, Languages and Combinatorics* 15(3–4) (2010), 229–254.

[20] S. Burris and H.P. Sankappanavar. *A Course in Universal Algebra*. Ed. by P.R. Halmos, F.W. Gehring, and C.C. Moore. First Edition. Vol. 78. Graduate Texts in Mathematics. Corrected version available at `https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`. New York: Springer-Verlag, 1981.

[21] M. Ćirić, M. Droste, J. Ignjatović, and H. Vogler. "Determinization of weighted finite automata over strong bimonoids". In: *Information Sciences* 180(18) (2010), 3479–3520.

[22] H. Comon, M. Dauchet, R. Gilleron, C. Löding, F. Jacquemard, D. Lugiez, S. Tison, and M. Tommasi. *Tree Automata Techniques and Applications*. Available on: `http://tata.gforge.inria.fr`. 2008.

[23] B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach*. Vol. 138. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2012.

[24] J. Doner. "Tree acceptors and some of their applications". In: *Journal of Computer and System Sciences* 4 (1970), 406–451.

[25] M. Droste and P. Gastin. "Weighted Automata and Weighted Logics". In: *32nd Int. Colloquium Automata, Languages and Programming (ICALP 2005)*. Ed. by L. Caires, G. F. Italiano, L. Monteiro, C. Palamidessi, and M. Yung. Vol. 3580. Lecture Notes in Computer Science. Springer, Berlin, Heidelberg, 2005, 513–525.

[26] M. Droste and P. Gastin. "Weighted automata and weighted logics". In: *Theoretical Computer Science* 380(1–2) (2007), 69–86.

[27] M. Droste and P. Gastin. "Weighted Automata and Weighted Logics". In: *Handbook of Weighted Automata*. Ed. by M. Droste, W. Kuich, and H. Vogler. Monographs in Theoretical Computer Science. An EATCS Series. Springer-Verlag, 2009. Chap. 5.

[28] M. Droste, D. Heusel, and H. Vogler. "Weighted unranked tree automata over tree valuation monoids and their characterization by weighted logics". In: *Algebraic Informatics (CAI 2015)*. Vol. 9270. Lecture Notes in Computer Science. Springer, 2015, 90–102.

[29] M. Droste, Chr. Pech, and H. Vogler. "A Kleene Theorem for Weighted Tree Automata". In: *Theory of Computing Systems* 38 (2005), 1–38.

[30] M. Droste, T. Stüber, and H. Vogler. "Weighted finite automata over strong bimonoids". In: *Information Sciences* 180(1) (2010), 156–166.

[31] M. Droste and H. Vogler. "Weighted Tree Automata and Weighted Logics". In: *Theoretical Computer Science* 366(3) (2006), 228–247.

[32] M. Droste and H. Vogler. “Weighted automata and multi-valued logics over arbitrary bounded lattices”. In: *Theoretical Computer Science* 418 (2012), 14–36.

[33] S. Eilenberg. *Automata, Languages, and Machines – Volume A*. Vol. 59. Pure and Applied Mathematics. Academic Press, 1974.

[34] J. Engelfriet. *Tree automata and tree grammars*. Tech. rep. DAIMI FN-10. see also: arXiv:1510.02036v1 [cs.FL] 7 Oct 2015. Aarhus, Denmark: Inst. of Mathematics, University of Aarhus, Department of Computer Science, 1975.

[35] Z. Ésik and W. Kuich. *Modern Automata Theory*. see: `https://www.dmg.tuwien.ac.at/kuich/mat.pdf`.

[36] Z. Ésik and W. Kuich. “Formal Tree Series”. In: *Journal of Automata, Languages and Combinatorics* 8(2) (2003), 219–285.

[37] Z. Ésik and G. Liu. “Fuzzy tree automata”. In: *Fuzzy Sets and Systems* 158(13) (2007), 1450–1460.

[38] Z. Fülöp. “Local Weighted Tree Languages”. In: *Acta Cybernetica* 22(2) (2015), 393–402.

[39] Z. Fülöp, A. Maletti, and H. Vogler. “A Kleene theorem for weighted tree automata over distributive multioperator monoids”. In: *Theory of Computing Systems* 44 (2009), 455–499.

[40] Z. Fülöp, T. Stüber, and H. Vogler. “A Büchi-like theorem for weighted tree automata over multioperator monoids”. In: *Theory of Computing Systems* 50 (2012), 241–278.

[41] Z. Fülöp and H. Vogler. “Weighted tree automata and tree transducers”. In: *Handbook of Weighted Automata*. Ed. by M. Droste, W. Kuich, and H. Vogler. EATCS Monographs in Theoretical Computer Science. Springer-Verlag, 2009. Chap. 9, 313–403.

[42] Z. Fülöp and H. Vogler. *Weighted Tree Automata – May it be a little more?* see: arXiv:2212.05529v1 [cs.FL] 11 Dec 2022. 2022.

[43] F. Gécseg and M. Steinby. *Tree Automata*. see also: arXiv:1509.06233v1 [cs.FL] 21 Sep 2015. Akadémiai Kiadó, Budapest, 1984.

[44] P. Gillibert. “The finitness problem for automaton semigroups is undecidable”. In: *International Journal of Algebra and Computation* 24(1) (2014), 1–9.

[45] J.S. Golan. *Semirings and their Applications*. Dordrecht: Kluwer Academic Publishers, 1999.

[46] Saul Gorn. “Explicit definitions and linguistic dominoes”. In: *Proceedings of the Systems and Computer Science Conference*. University of Western Ontario, 1965, 77–115.

[47] G. Grätzer. *Universal Algebra*. D. van Nostrand Comp., 1968.

[48] G. Grätzer. *General Lattice Theory*. 2nd. Birkhäuser Verlag, 2003.

[49] A. Habel and H.-J. Kreowski. "Some structural aspects of hypergraph languages generated by hyperedge replacement". In: *Annual Symposium on Theoretical Aspects of Computer Science (STACS 1987)*. Ed. by F.J. Brandenburg, G. Vidal-Naquet, and M. Wirsing. Vol. 247. LNCS. Springer Berlin Heidelberg, 1987, 207–219.

[50] U. Hebisch and H.J. Weinert. *Semirings - Algebraic Theory and Applications in Computer Science*. World Scientific, 1993.

[51] L. Herrmann. "A Medvedev Characterization of Recognizable Tree Series". In: *21th Int. Conf. on Developments in Language Theory (DLT 2017)*. Ed. by É. Charlier, J. Leroy, and M. Rigo. Vol. 10396. Lecture Notes in Computer Science. Springer, Cham, 2017, 210–221.

[52] L. Herrmann. *Weighted Tree Automata with Storage*. Ph.D. Thesis, 2020, TU Dresden, Germany). 2020.

[53] W. M. L. Holcombe. *Algebraic automata theory*. Cambridge University Press, 1982.

[54] W. G. van Hoorn and B. van Oozela. "Fundamental notions in the theory of seminear-rings". In: *Compositio Mathematica* 18 (1967), 65–78.

[55] K. Culik II and A. Salomaa. "On the Decidability of Homomorphism Equivalence for Languages". In: *Journal of Computer and System Sciences* 17 (1978), 163–175.

[56] Y. Inagaki and T. Fukumura. "On the description of fuzzy meaning of context-free languages". In: *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*. Academic Press, New York, 1975, 301–328.

[57] I. Klimann, S. Lombardy, J. Mairesse, and C. Prieur. "Deciding unambiguity and sequentiality from a finitely ambiguous max-plus automaton". In: *Theoretical Computer Science* 327(3) (2004), 349–373.

[58] K. V. Krishna. "Near-Semirings: Theory and Applications". PhD thesis. New Delhi, India: IIT Delhi, 2005.

[59] W. Kuich. "Formal power series over trees". In: *3rd International Conference on Developments in Language Theory (DLT 1997), Thessaloniki, Greece*. Ed. by S. Bozapalidis. Aristotle University of Thessaloniki, 1998, 61–101.

[60] W. Kuich. "Linear systems of equations and automata on distributive multi-operator monoids". In: *Contributions to General Algebra 12 - Proceedings of the 58th Workshop on General Algebra "58. Arbeitstagung Allgemeine Algebra", Vienna University of Technology. June 3-6, 1999*. Ed. by D. Dorninger, G. Eigenthaler, M. Goldstern, H. K. Kaiser, W. More, and W. B. Müller. Verlag Johannes Heyn, 1999, 1–10.

[61] O. Louscou-Bozapalidou. "Some remarks on recognizable tree series". In: *International Journal of Computer Mathematics* 70 (1999), 649–655.

[62] M. Magidor and G. Moran. *Finite Automata over Finite Trees*. Tech. rep. 30. Hebrew University, Jerusalem, 1969.

[63] A. Mandel and I. Simon. "On finite semigroups of matrices". In: *Theoretical Computer Science* 5 (1977), 101–111.

[64] G. H. Mealy. "A Method for Synthesizing Sequential Circuits". In: *The Bell System Technical Journal* 34(5) (1955), 1045–1079.

[65] M. Mohri. "Minimization algorithms for sequential transducers". In: *Theoretical Computer Science* 234(1–2) (2000), 177–201.

[66] D. Radovanović. "Weighted tree automata over strong bimonoids". In: *Novi Sad Journal of Mathematics* 40(3) (2010), 89–108.

[67] G. Rahonis. "Fuzzy languages". In: *Handbook of Weighted Automata*. Ed. by M. Droste, W. Kuich, and H. Vogler. EATCS Monographs in Theoretical Computer Science. Springer-Verlag, 2009. Chap. 12, 481–517.

[68] A. Restivo and C. Reutenauer. "Rational Languages and the Burnside Problem". In: *Theoretical Computer Science* 40 (1985), 13–30.

[69] W. C. Rounds. "Mappings and grammars on trees". In: *Mathematical Systems Theory* 4(3) (1970), 257–287.

[70] M. P. Schützenberger. "On the definition of a family of automata". In: *Information and Control* 4 (1961), 245–270.

[71] J. W. Thatcher. "Characterizing derivation trees of context-free grammars through a generalization of finite-automata theory". In: *Journal of Computer and System Sciences* 1(4) (1967), 317–322.

[72] J. W. Thatcher. "Generalized sequential machine maps". In: *Journal of Computer and System Sciences* 4(4) (1970), 339–367.

[73] J. W. Thatcher and J. B. Wright. "Generalized finite automata theory with an application to a decision problem of second-order logic". In: *Mathematical Systems Theory* 2(1) (1968), 57–81.

Summary

In this PhD thesis we investigate weighted tree automata over strong bimonoids. Fundamentally, a strong bimonoid is a semiring in which the distributivity laws need not hold. Moreover, the model of weighted tree automaton is a natural extension of the model of classical finite-state tree automaton by associating to each possible transition and to each state of a finite-state tree automaton a unique weight, *i.e.*, an element of the underlying strong bimonoid. We mention that, since the strong bimonoid is not necessarily distributive, we cannot apply convenient algebraic methods to study weighted tree automata over strong bimonoids. Thus, we apply only combinatorial approaches for our investigations, and hence, consider only the run semantics of weighted tree automata.

The run semantics of a weighted tree automaton is a mapping from the set of input trees to the carrier set of the underlying strong bimonoid. Each such mapping is called a weighted tree language (or formal power series). In order to calculate the run semantics, we consider runs of the weighted tree automaton. A run is a mapping from the set of positions of an input tree to the set of states of the weighted tree automaton. To calculate the weight of a run, we multiply the transition weights in a fixed order determined by the run using the multiplication operation of the underlying strong bimonoid. Then, to calculate the run semantics of the weighted tree automaton for an input tree, we sum up all weights of runs on that input tree multiplied by the root weight using the addition operation of the underlying strong bimonoid.

In Chapter 1, we give a brief introduction to the theory of finite-state tree automata and to the theory of weighted tree automata. In Chapter 2 we discuss the preliminaries, *i.e.*, the necessary notions, notations, and results of the theory of finite-state tree automata and of the theory of universal algebra.

In Chapter 3, we recall fundamental notions, notations, and results of weighted tree languages and weighted tree automata. Moreover, we present our pumping lemmas for runs of weighted tree automata. With a pumping lemma one can achieve structural implications on small or particular large trees. Since such pumping lemmas already exist for weighted tree automata, the question may arise why we present another pumping lemmas. The existing setting deals with bottom-up deterministic weighted tree automata over semirings and employs initial algebra semantics, but in

our setting we deal with (arbitrary) weighted tree automata over strong bimonoids and employ run semantics. Nevertheless, if we consider the class of all bottom-up deterministic weighted tree automata over semirings, then the two settings coincide.

In Chapter 4 we deal with weighted tree generating regular systems. In the literature, the equivalence of finite-state tree automata and tree generating regular systems is already proven. A tree generating regular system is a particular ground term rewriting system extended with a set of designated trees, also known as axioms. The ground term rewrite relation induced by a tree generating regular system is defined in the standard way. Moreover, the derivation semantics of a tree generating regular system is the set of those input trees, which can be reached from an axiom by applying the transitive and reflexive closure of the induced ground term rewrite relation.

In this chapter we recall the concept of weighted tree generating regular systems such that (a) each weighted tree generating regular system over the Boolean semiring is "equivalent" to a tree generating regular system, and vice versa; and (b) under some mild conditions, each weighted tree generating regular system is equivalent to a weighted tree automata, and vice versa.

Unfortunately, the derivation semantics of tree generating regular systems allows replacements at incomparable positions in an arbitrary order. Since the addition operation of a strong bimonoid is not necessarily idempotent, introducing the concept of weighted tree generating regular systems with the generalized derivation semantics is not suitable for our purposes. More precisely, if we associated a weighted tree generating regular system to a weighted tree automaton and considered the derivation semantics of the weighted tree generating regular system, then more than one computation of the weighted tree generating regular system would correspond to a single run of the weighted tree automaton. Consequently, the derivation semantics of the weighted tree generating regular system and the run semantics of the weighted tree automaton would differ.

In order to avoid that phenomenon, we advocate an alternative, but essentially equivalent semantics, called reduction semantics, for tree generating regular systems and introduce the concept of weighted tree generating regular system only with the generalization of this alternative semantics. The reduction semantics of tree generating regular systems has the following two characteristics:

(i) It is based on the restriction of the induced ground the term rewriting relation. In this case the restriction means that replacements can be performed only at the minimal position with respect to the depth-first post-ordering of positions at which a replacement is possible.
(ii) It reverses the direction of the computation, *i.e.*, the reduction semantics of a tree generating regular system is the set of those input trees, from which can be reached an axiom by applying the transitive and reflexive closure of the

restricted ground term rewrite relation.

Surprisingly, the derivation semantics and the reduction semantics of tree generating regular systems are essentially equivalent as follows. For each tree generating regular system, we can construct another tree generating regular system such that the derivation semantics of the former and the reduction semantics of the latter coincide. Similarly, for each tree generating regular system, we can construct another tree generating regular system such that the reduction semantics of the former and the derivation semantics of the latter coincide. Moreover, introducing the concept of weighted tree generating regular system with the generalized reduction semantics fulfills the aforementioned requirements (a) and (b).

Finally, in Chapter 5, we deal with the crisp-determinization problem of weighted tree automata. We call a weighted tree automaton crisp-deterministic if it is total and bottom-up deterministic, and each of its transitions carries either the additive or the multiplicative unit element of the underlying strong bimonoid; weights different from these unit elements may occur only as final (or root) weights. Moreover, we say that a weighted tree automaton is crisp-determinizable if there exists a run semantically equivalent crisp-deterministic weighted tree automaton. Although, the definition of crisp-deterministic weighted tree automaton is quite restrictive, it is worth to study them, because they have several desirable properties such as they are bottom-up deterministic, their run semantics has a finite image, *etc.* Moreover, each fuzzy tree automaton can be considered as a particular crisp-determinizable weighted tree automaton.

In the literature there is a sufficient condition for crisp-determinization, but only for weighted string automaton. Since each weighted string automaton can be considered as a weighted tree automaton over a particular ranked alphabet, in a straightforward way we generalize that sufficient condition from strings to trees.

Moreover, we prove undecidability (decidability) results regarding the crisp-determinization problem of weighted tree automata. In particular, we show that it is undecidable, for an arbitrary weighted tree automaton, whether that weighted tree automaton satisfies the sufficient condition. Furthermore, also in a straightforward we generalize the characterization of crisp-determinization from strings to trees, and prove that it is undecidable, for an arbitrary weighted tree automaton, whether that weighted tree automaton is crisp-determinizable.

Eventually, we identify two subclasses of all weighted tree automata, for which the crisp-determinization problem is decidable. In order to do that, we advocate a new subclass of all strong bimonoids, called past-finite monotonic strong bimonoids. Basically, a past-finite monotonic strong bimonoid is a strong bimonoid of which the carrier set is a partially ordered set, each element has finitely many predecessors, and the addition and the multiplication operation have a nondecreasing effect. Such weight structures have several desirable properties such they are positive, one-product free, one-summand free *etc.* Furthermore, well known weights structures, like the semiring of natural numbers or the max-plus semiring, belong to this subclass as well. Then, for an arbitrary weighted tree automaton over an additively locally finite and past-finite monotonic strong bimonoid (respectively, an unambiguous weighted tree automaton over a past-finite monotonic strong bimonoid), it is decidable, whether that weighted tree automaton is crisp-determinizable.

Összefoglalás

Ebben a disszertációban erős bimonoidok feletti súlyozott faautomatákkal foglalkozunk. Alapvetően egy erős bimonoid egy olyan félgyűrű, amelyben a disztributivitási azonosságok nem feltétlenül teljesülnek. Továbbá a súlyozott faautomata a véges faautomata természetes kiterjesztése az által, hogy a faautomata minden lehetséges átmenetéhez és minden állapotához egy egyedi súlyt, az erős bimonoid egy elemét, rendeljük. Megemlítjük, hogy mivel az erős bimonoid nem feltétlenül disztributív, ezért nem tudjuk alkalmazni a szokásos algebrai módszereket a súlyozott faautomaták tanulmányozása során. Ezért a vizsgálataink során csak kombinatorikus megközelítéseket használunk, továbbá, csak a súlyozott faautomata futási szemantikájával foglalkozunk.

Egy súlyozott faautomata futási szemantikája egy leképezés a bemenetet alkotó fák halmazából a súlyozott faautomata alatti erős bimonoid tartóhalmazába. Minden ilyen leképezést súlyozott fanyelvnek (vagy formális hatványsornak) nevezünk. A futási szemantika kiszámításához a súlyozott faautomata futásait vesszük alapul. Egy futás egy leképezés egy bemeneti fa pozíciójának halmazából a súlyozott faautomata állapotainak halmazába. Egy futás súlyának kiszámítása során az egyes átmenetek súlyát szorozzuk össze az erős bimonoid szorzás műveletét alkalmazva a futás által meghatározott sorrendben. Egy fa súlyának kiszámításához pedig összeadjuk az erős bimonoid összeadás műveletét alkalmazva a bemeneti fán lévő összes futás súlyát megszorozva a gyökérsúllyal.

Az 1. fejezetben egy rövid bevezetőt adunk a véges faautomaták és a súlyozott faautomaták elméletébe. A 2. fejezetben áttekintjük a véges faautomaták és az univerzális algebra elméletéhez tartozó szükséges fogalmakat, jelöléseket, és eredményeket.

A 3. fejezetben felidézzük a súlyozott fanyelvekhez és a súlyozott faautomatákhoz kapcsolódó fogalmakat, jelöléseket, és eredményeket. Továbbá, megadjuk a súlyozott faautomaták futásaira vonatkozó pumpáló lemmáinkat. Egy pumpáló lemma segítségével lehetőségünk nyílik arra, hogy szerkezeti következtetéseket fogalmazzunk meg kicsi vagy éppen nagyon nagy fákat illetően. Mivel hasonló pumpáló lemmák már léteznek súlyozott faautomatákra, felmerül a kérdés, hogy miért bizonyítunk be újabb pumpáló lemmákat. A meglévő pumpáló lemmák félgyűrű feletti determinisztikus leszálló súlyozott faautomatákra vonatkoznak és az iniciális algebra szemantikát vizsgálják, azonban mi (tetszőleges) erős bimonoid feletti súlyozott faautomatákkal foglalkozunk és a futási szemantikát tanulmányozzuk. Jóllehet, ha a félgyűrű feletti összes determinisztikus leszálló súlyozott faautomaták osztályát nézzük, akkor az eredmények egybeesnek.

A 4. fejezetben súlyozott fageneráló reguláris rendszerekkel foglalkozunk. A szakirodalomban már régóta be van bizonyítva, hogy a véges faautomaták és a fageneráló reguláris rendszerek ekvivalensek. Egy fageneráló reguláris rendszer egy speciális alaptermátíró rendszer kiegészítve kijelölt fák, más néven axiómák, egy halmazával. A fageneráló reguláris rendszer által indukált alaptermátíró relációt a szokásos módon definiáljuk. Továbbá, egy fageneráló reguláris rendszer derivációs szemantikája a bemenetet alkotó fák azon halmaza, amelyek elérhetőek egy axiómából az indukált alaptermátíró reláció reflexív és tranzitív lezártjának alkalmazásával.

Ebben a fejezetben felidézzük a súlyozott fageneráló reguláris rendszerek fogalmát úgy, hogy (a) minden Boole félgyűrű feletti súlyozott fageneráló reguláris rendszer "ekvivalens" legyen egy fageneráló reguláris rendszerrel, és viszont; (b) néhány gyenge feltétel teljesülése esetén, minden súlyozott fageneráló reguláris rendszer ekvivalens legyen egy súlyozott faautomatával, és viszont.

Sajnálatos módon, a fageneráló reguláris rendszer derivációs szemantikája megengedi, hogy összehasonlíthatatlan pozíciók esetén a helyettesítéseket tetszőleges sorrenben végezzük el. Mivel az erős bimonoid összeadás művelete nem feltétlenül idempotens, a súlyozott fageneráló reguláris rendszer fogalmának bevezetése az általánosított derivációs szemantikával nem megfelelő a céljainknak. Pontosabban fogalmazva, ha egy súlyozott faautomatához egy súlyozott fageneráló reguláris rendszert társítunk és vesszük a súlyozott fageneráló reguláris rendszer derivációs szemantikáját, akkor a súlyozott fageneráló reguláris rendszer számításai közül egynél több is megfelelhet a súlyozott faautomata egy futásának. Következésképpen a súlyozott fageneráló reguláris rendszer derivációs szemantikája és a súlyozott faautomata futási szemantikája eltérhet.

Annak érdekében, hogy elkerüljük ezt a jelenséget, egy másik, de az eredetivel ekvivalens szemantika, a redukciós szemantika, bevezetését javasoljuk a fageneráló reguláris rendszerekhez, valamint a súlyozott fageneráló reguláris rendszer fogalmát csak ennek a másik szemantikának az általánosításával vezetjük be. A fageneráló reguláris rendszer redukciós szemantikája a következő két jellegzetességgel rendelkezik:

(i) Az indukált alaptermátíró reláció megszorításán alapszik. Ebben az esetben a megszorítás azt jelenti, hogy a helyettesítések csak a mélységikeresés-utórendezésre nézve minimális lehetséges pozíciókban végezhetőek el.

(ii) Megfordítja a számítás irányát, azaz egy fageneráló reguláris rendszer redukciós szemantikája a bementet alkotó fák azon halmaza, amelyekből elérhető egy axióma a megszorított alaptermátíró reláció reflexív és tranzitív lezártjának alkalmazásával.

Meglepő módon a fageneráló reguláris rendszerek derivációs szemantikája és a redukciós szemantikája lényegében ekvivalens a következőképpen. Minden fageneráló reguláris rendszerhez meg tudunk konstruálni egy másik fageneráló reguláris rendszert úgy hogy az előbbi derivációs szemantikája egybeesik az utóbbi redukciós szemantikájával. Hasonlóképpen, minden fageneráló reguláris rendszerhez meg tudunk konstruálni egy másik fageneráló reguláris rendszert úgy, hogy az előbbi redukciós szemantikája egybeesik az utóbbi derivációs szemantikájával. Továbbá, a súlyozott fageneráló reguláris rendszer fogalmának bevezetése az általánosított redukciós szemantikával kielégíti a korábban említett (a) és (b) követelményeket.

Végül az 5. fejezetben a súlyozott faautomaták egységdeterminizálásával foglalkozunk. Egy súlyozott faautomatát egységdeterminisztikusnak nevezünk, ha totális determinisztikus leszálló, és minden átmenete az erős bimonoid additív vagy multiplikatív egységelemével van súlyozva; az egységelemektől eltérő súlyok csak gyökérsúlyként fordulhatnak elő. Továbbá azt mondjuk, hogy egy súlyozott faautomata egységdeterminizálható, ha létezik vele futási szemantika szerint ekvivalens egységdeterminisztikus súlyozott faautomata. Jóllehet az egységdeterminisztikus súlyozott faautomata definíciója elég korlátozó, mégis megéri tanulmányozni őket, mert számos kívánatos tulajdonsággal rendelkeznek, például az ilyen súlyozott faautomaták egyben determinisztikus leszálló súlyozott faautomaták is, a futási szemantikájuk képe véges, stb. Továbbá, minden fuzzy faautomata felfogható egy speciális egységdeterminizálható súlyozott faautomatának.

A szakirodalomban létezik az egységdeterminizálhatóságra vonatkozó elegendőségi feltétel, de csak súlyozott automatára. Mivel minden súlyozott automata felfogható egy speciális rangolt ábécé feletti súlyozott faautomatának, egyszerűen általánosítjuk az elegendőségi feltételt szavakról fákra.

Továbbá az egységdeterminizálásra vonatkozó eldönthetetlenségi (eldönthetőségi) eredmények bizonyítunk be. Megmutatjuk, hogy tetszőleges súlyozott faautomatára nézve eldönthetetlen, hogy a súlyozott faautomata kielégíti-e az elegedőségi feltételt. Valamint, szintén egyszerűen általánosítjuk az egységdeterminizáció karakterizációját szavakról fákra, és bebizonyítjuk, hogy tetszőleges súlyozott faautomatára nézve eldönthetetlen, hogy a súlyozott faautomata egységdeterminizálható-e.

Legvégül megmutatunk a súlyozott faautomatáknak olyan két részosztályát, amelyekre eldönthető az egységdeterminizálhatóság problémája. Ennek érdekében, az erős bimonoidok egy új részosztályát, a véges-sok-előd monoton erős bimonoidokat, javasoljuk. Egy véges-sok-előd monoton erős bimonoid egy olyan erős bimonoid, amelynek tartóhalmaza egy parciálisan rendezett halmaz, minden elemnek véges sok elődje van, és az összeadás és szorzás műveletek nemcsökkentő hatással bírnak. Az ilyen súlystruktúrák számos kívánatos tulajdonsággal rendelkeznek úgy mint pozitivitás, egyszorzatmentesség, egyösszegmentesség, stb. Továbbá ismert súlystruktúrák, mint például a természetes számok félgyűrűje vagy a max-plusz félgyűrű, is ebbe a részosztályba tartoznak. Ekkor, tetszőleges additívan lokálisan véges és véges-sok-előd monoton erős bimonoid feletti súlyozott faautomatára (valamint véges-sok-előd monoton erős bimonoid feletti egyértelmű súlyozott faautomatára) eldönthető, hogy az adott súlyozott faautomata egységdeterminizálható-e.

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