Global stability for the delayed logistic map

Ph.D. Thesis

János Dudás

Supervisor:
Tibor Krisztin

Doctoral School of Mathematics and Computer Science
Bolyai Institute
University of Szeged

Szeged
2021

Acknowledgment

First, I would like to express my sincere gratitude to my supervisor Tibor Krisztin for his valuable advice, for proposing interesting problems and for his support during my whole research.

I would like to thank László Hatvani, supervisor of my bachelor's and master's thesis, for introducing me to the mysteries of dynamical system. I am also grateful to everyone in the Bolyai Institute - to my teachers and to my colleagues who made my university years memorable.

I would like to express how lucky I am that I had wonderful teachers even before university, who made me love math and taught me that hard work bears fruit. I would like to emphasize my thanks to my elementary school teacher Ilona Fazekas and my high school teacher Tibor Nagy.

Last, but not least, I also owe much gratitude to my family, especially my wife Evelin, my parents and my sister for their everlasting support and trust in me and my work. Without them this thesis would not have been written.

Contents

1 Introduction **1**

2 Global stability for the two-dimensional logistic map **11**

2.1 Preliminaries 11

2.1.1 Dynamics of F_d – general properties 11

2.1.2 Dynamics of F_d for $d = 1$ 14

2.2 Attracting neighborhood with linearization 18

2.3 Transforming to normal form 20

2.3.1 Estimation of the lower-order terms in G, h and h^{-1} 23

2.3.2 The definition of h^{-1} 25

2.3.3 The estimation of R_3 – the higher-order terms in h^{-1} 26

2.3.4 The estimation of R_2 – the higher-order terms in $h^{-1}(G(h))$ 27

2.3.5 The attracting neighborhood 28

2.4 Graph representation 29

2.5 Completion of the proof 32

2.6 The algorithm 33

3 Global stability for the three-dimensional logistic map **37**

3.1 Preliminaries 37

3.2 An attracting neighborhood with linearization 42

3.2.1 Local stability by linearization 44

3.3 A center manifold reduction 46

3.3.1 Attractivity in direction y 50

3.3.2 Conditional invariance in direction y 53

3.4 Transforming to normal form 54

3.4.1 Estimation of the lower-order terms in G, h and h^{-1} 57

3.4.2 The definition of h^{-1} 59

3.4.3 The estimation of R_1 60

3.4.4 The estimation of R_3 – the higher-order terms in h^{-1} 60

3.4.5 The estimation of R_2 – the higher-order terms in $h^{-1}(G(h))$ 62
3.4.6 The dynamics in direction z 63
3.4.7 Combining the y- and z-directional dynamics 64
3.5 Constructing the attracting neighborhood 65
3.6 Computer-assisted part for a fixed a 67
3.7 Computer-assisted part for an interval of a 69

4 Appendix **73**
4.1 The two-dimensional logistic map 73
4.2 The three-dimensional logistic map 75
4.2.1 The eigenvalues 76
4.2.2 The coefficients of $g(z, y)$ 76
4.2.3 The coefficients of the center manifold 76
4.2.4 The fifth-order terms of $\mathcal{N}(\phi(z))$ 77
4.2.5 The lower-order terms of $G(z)$ 79
4.2.6 The coefficients of $h(z)$ 80
4.2.7 The coefficients of $h_0^{-1}(z)$ 80
4.2.8 The lower-order terms of $G(h(w))$ 83
4.2.9 The lower-order terms of $h^{-1}(G(h(w)))$ 85
4.2.10 The Lyapunov-coefficient 88

Summary **89**

Összefoglaló **93**

References **96**

Chapter 1

Introduction

One of the most studied nonlinear maps is the logistic map $[0,1] \ni x \mapsto ax(1-x) \in \mathbb{R}$ with parameter $a > 0$. It is well known (see e.g., [1]) that $x = 0$ is the unique fixed point in $[0,1]$ for $0 < a \leq 1$, and it is globally stable in $[0,1]$. For $1 < a \leq 3$ the nontrivial fixed point $A = 1 - \frac{1}{a}$ is stable and attracts all points of $(0,1)$. At $a = 3$ a period doubling bifurcation takes place, and the fixed point A becomes unstable for $a > 3$. As a increases, there is a sequence of bifurcations, and for some larger value of a, chaotic behavior can be shown.

In 1968 Maynard Smith [2] and in 1971 Levin and May [3] considered the delayed logistic difference equation

$$x_{n+1} = ax_n(1 - x_{n-d}) \tag{1.1}$$

with $a > 0$ and $d \in \mathbb{N}$. This is natural in the context of population models. The size of the subsequent generation of the population depends not only on the size in the previous year but also on the size of the d-year-earlier population. The difference equation (1.1) is equivalent to the $(d+1)$-dimensional map

$$F_d = F_{d,a} : \mathbb{R}^{d+1} \ni u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{d+1} \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ au_{d+1}(1-u_1) \end{pmatrix} \in \mathbb{R}^{d+1}. \tag{1.2}$$

In Chapter 2, based on the article [4] we study the global stability of the logistic equation for $d = 1$. Bartha, Garab and Krisztin in [5, 6] proved analogous results for other second-order difference equations, or equivalently, for other two-dimensional maps. The novelty of [4, 5, 6] is the development of a new method to show sharp results for global stability of fixed points for some two-dimensional maps with a parameter a. It is common in [4, 5, 6] that a supercritical Neimark–Sacker bifurcation takes place at some $a = a_{crit}$.

The Neimark–Sacker bifurcational normal form not only guarantees the existence of an invariant curve around the fixed point for $a > a_{crit}$, but also gives a neighborhood $\mathcal{M}$ around the fixed point for $a \le a_{crit}$ so that $\mathcal{M}$ belongs to the region of attraction of the fixed point. The main achievement of [4, 5, 6] is an explicitly constructed $\mathcal{M}$ which is large enough in the sense that, by using a rigorous computer-assisted technique, it is possible to prove that the iterates of all points outside $\mathcal{M}$ eventually enter $\mathcal{M}$. A relatively large $\mathcal{M}$ can guarantee the success of the computer-assisted part within a reasonable computer time. It is a highly nontrivial result of [4, 5, 6] that starting from the classical Neimark–Sacker bifurcational normal form technique, which was used earlier only for local results, a relatively large attractivity region can be explicitly constructed for $a \le a_{crit}$. In [4, 5, 6] it was essential that the studied systems were two-dimensional.

In Chapter 3, based on the article [7] we extend the method of [4, 5, 6] from two-dimensional to higher-dimensional maps. As the delayed logistic difference equation is interesting in its own right for $d = 2$, we demonstrate on the three-dimensional nonlinear map (1.2) with $d = 2$ how the extension goes to higher dimension. We believe that the main steps of our case study for this specific equation can be followed with natural modifications to handle several other problems, in particular the delayed logistic map for $d > 2$.

For $a \in (0, 1]$, similar to the nondelayed logistic equation, the origin is the unique fixed point of (1.2) in $[0, 1]^{d+1}$. It is elementary to show that the origin is locally stable and $\lim_{n\to\infty} F_d^n(u) = 0$ for every $u \in [0, 1]^{d+1}$. For $a > 1$ a nontrivial fixed point $u_A = (A, A, \dots, A)$ with $A = 1 - \frac{1}{a}$ appears in $[0, 1]^{d+1}$. There is an $a_0 > 1$, depending on d such that this fixed point is locally asymptotically stable for $a \in (1, a_0)$, and unstable for $a > a_0$. A Neimark–Sacker bifurcation takes place at $a = a_0$ (for $d = 1$ see e.g. Example 4.3 in [8]), and there is a stable invariant curve for $a > a_0$ sufficiently close to a_0.

Note that we do not consider the case $a > a_0$. However, near u_A, local information is available from the Neimark–Sacker bifurcation for $a > a_0$ close to a_0. Profound (non-rigorous) numerical studies can be found in [9, 10] for $a > a_0$ and $d = 1$. According to these articles the invariant curve is globally stable for parameter $a > a_0$ close to the critical value a_0. Furthermore, one can see that as a increases, the size of the invariant curve is getting larger. The curve touches the x-axis at about $a = 2.27$, and complicated dynamics occurs. Analogously to the cases $d = 0$ and $d = 1$, it is expected that the dynamics becomes more complex with larger a for $d \ge 2$, too.

For $a \in (1, a_0]$, unlike the nondelayed logistic equation, the set $[0, 1]^{d+1}$ is not invariant under (1.2). Therefore, we consider the map F_d for those $u \in [0, \infty)^{d+1}$ for which all iterates of (1.2) remain in $\mathbb{R}_+^{d+1}$, i.e., $F_d^n(u) \in [0, \infty)^{d+1}$ for every $n \in \mathbb{N}$. Here, F_d^n denotes the n-fold iteration of F_d, i.e., $F_d^0 = \mathrm{id}$ and $F_d^n = F_d\left(F_d^{n-1}\right)$ for $n \in \mathbb{N}$. As we will see for

$a \in (0, a_0]$ the set

$$S_0^d = \Big\{ u \in [0,1]^{d+1} : \; a^k u_{d+1} \prod_{j=1}^{k} (1 - u_j) \leq 1 \quad \text{for} \quad k \in \{1, 2, \dots, d\} \Big\} \tag{1.3}$$

is the largest set, whose points remain in $[0, \infty)^{d+1}$ for all iterates F_d^n with $n \in \mathbb{N}$. Note that S_0^d depends on the parameter a.

Some points of the boundary of S_0^d belong to the attractivity set of the origin, so we define an a-dependent subset of S_0^d.

$$S^d = \Big\{ u \in [0,1)^d \times (0,1) : \; a^k u_{d+1} \prod_{j=1}^{k} (1 - u_j) < 1 \quad \text{for} \quad k \in \{1, 2, \dots, d\} \Big\} \tag{1.4}$$

In the thesis we study the global stability of the nontrivial fixed point u_A for $a \in (1, a_0]$. More precisely, we show that the following conjecture is true for $d \in \{1, 2\}$.

Conjecture 1. *The fixed point u_A is locally stable and $\lim_{n\to\infty} F_d^n(u) = u_A$ for $a \in (1, a_0]$, $u \in S^d$.*

First, note that, here and in the rest of the thesis when we refer to global stability of the logistic map, it needs to be understood as it was stated in Conjecture 1, i.e., it means local stability and attractivity within the set S^d. We emphasize that Conjecture 1 is sharp, that is, global stability holds at the critical parameter value a_0 as well. Conjecture 1 can be formulated so that local stability implies global stability for the fixed point u_A. This is satisfied for several problems, see e.g., [11, 12, 13, 5, 6], but it is not true in general see e.g., [14].

In Chapter 2 we prove Conjecture 1 for $d = 1$. In this case $F_2(u)$ in (1.2) is a two-dimensional map and $a_0 = 2$. The proof of the global stability is a combination of analytical and computer-aided tools. It is based on the method in [5, 6]. We elaborate the analytical part such that it can be easily applied to similar models. Furthermore, a quite important aim is to have a clear picture of the method so that we can prove similar results for higher dimensional models, for example the map (1.2) for $d = 2$, where further difficulties arise.

In Section 2.1 we study the dynamics of (1.2) with purely analytical tools. First, in Subsection 2.1.1 we consider F^d for an arbitrary $d \in \mathbb{N}$. After that, in Subsection 2.1.2 we can prove global stability for $d = 1$ and $a \in \left(1, \frac{3}{2}\right]$.

In the subsequent subsections we construct with analytical tools an attracting neighborhood $\mathcal{M}$ around the nontrivial fixed point u_A. Then with reliable numerical methods we show that every $u \in S^d \setminus \mathcal{M}$ will eventually step into $\mathcal{M}$, that is, there exist an

$n \in \mathbb{N}$ such that $u^n \in \mathcal{M}$, where $u^n = F_2^n(u)$. So these points are also in the region of attraction of the fixed point u_A. Here, reliable means that all possible numerical errors are controlled by using interval arithmetic techniques. Therefore, the computer-assisted part also provides mathematically rigorous statements, see e.g. [15].

In Section 2.2, for smaller parameter values a, i.e., for $a \in (1.5, 1.95]$ we use the linearized map to construct the attracting neighborhood $\mathcal{M}$. However, as we will see it later, this neighborhood shrinks to the fixed point as a tends to the critical value $a_0 = 2$. Therefore, for parameter values a close to a_0 this neighborhood is not big enough for computer use in the second part of the method. Thus, we need another approach too to construct an attracting neighborhood $\mathcal{M}$. In Section 2.3, for parameter values a close to a_0 we use the normal form of the Neimark–Sacker bifurcation. More precisely, with smooth and invertible maps we transform map $F_2(u)$ into its normal form, hereby we obtain an attracting neighborhood $\mathcal{M}$ around the fixed point u_A, whose size is independent of the parameter $a \in [1.95, 2]$.

Since we need the size of the constructed neighborhood $\mathcal{M}$ for computer use, it is not enough to determine only the lower-order terms during the normal form transformation, like we would do in a regular bifurcation analysis. These lower-order terms only assure the existence of such a sufficiently small neighborhood, whose size is not explicitly determined by them. Therefore, it is essential during the transformation to trace the higher-order terms and to estimate them as well as possible, in order to obtain a sufficiently big neighborhood $\mathcal{M}$, since the computational part is more and more compute-intensive and time-consuming, as we get closer to the fixed point.

Sections 2.4 to 2.6 are devoted to the computer-assisted part of our method. We cover S^d with finitely many small squares. Considering these squares as vertices of a graph, we introduce a directed graph, which, to a certain extent, describes the behavior of map $F_2(u)$ on these squares. Therefore, we convert the issue of examining infinitely many points into a finite graph problem, which can be handled by computer. To construct the edges of this graph we use reliable numerical methods in order to handle the rounding errors of the computer. We show with the help of this graph that the iterates of every point from S^d enter the neighborhood $\mathcal{M}$ constructed with analytical tools, completing the proof of Conjecture 1 for $d = 1$.

In Chapter 3 we prove Conjecture 1 for $d = 2$. In this case $F_3(u)$ in (1.2) is a three-dimensional map and $a_0 = \frac{\sqrt{5}+1}{2}$, which can be found also in Table 4.1, where we collected the most frequently used constants from the thesis. With the structure of this section and with the notations we follow Chapter 2 as much as possible, however, there are essential differences since F_3 is a three-dimensional map, so some of the adaptations are highly nontrivial. The novelty of this section is an explicit construction of a relatively large attracting neighborhood of the nontrivial fixed point of the three-dimensional logistic map by using center manifold techniques and the Neimark–Sacker bifurcational normal form.

In Section 3.1 we describe the behavior of $F_3(u)$ in the positive octant. Then, for $a \in \left(1, \frac{4}{3}\right]$ we give a purely analytical proof of the global stability of u_A. Although this analytical proof is straightforward, it is an important part of the proof of Conjecture 1, since for $a > 1$ and close to 1 the fixed point u_A can not be distinguished from the origin by a computer-assisted technique.

The rest of the section is devoted to the case $a \in \left(\frac{4}{3}, a_0\right]$. The basic idea is the same as for the two-dimensional case. First, we analytically construct an attracting neighborhood $\mathcal{M}$ of the fixed point u_A. Then, by applying reliable numerical tools, it is shown that for every $u \in S^d$ the iterates $F_3^n(u)$ eventually enter $\mathcal{M}$. Consequently, all points of S^d belong to the region of attraction of u_A.

In Section 3.2 the map F_3 is transformed to the form

$$H_a : \mathbb{C} \times \mathbb{R} \ni \begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda(a)z + \mathcal{G}_a(z, y) \\ \nu(a)y + \mathcal{H}_a(z, y) \end{pmatrix} \in \mathbb{C} \times \mathbb{R} \tag{1.5}$$

for $a \in \left(\frac{4}{3}, a_0\right]$. Here, $\nu(a) \in \mathbb{R}$ and $\lambda(a) \in \mathbb{C}$ with $\bar{\lambda}(a) \in \mathbb{C}$ are the eigenvalues of the Jacobian matrix of $F_3(u)$ at u_A. The inequality $|\nu(a)| < |\lambda(a)| \leq 1$ holds and $|\lambda(a_0)| = 1$. The nonlinear functions $\mathcal{G}_a(z, y)$ and $\mathcal{H}_a(z, y)$ are smooth functions of $a, z, \bar{z}$ and y, and furthermore, they are $O\left(|(z, y)|^2\right)$ for each fixed $a \in \left(\frac{4}{3}, a_0\right]$.

In Subsection 3.2.1 a standard linearization technique for (1.5) gives an attracting neighborhood of u_A for $a \in \left(\frac{4}{3}, a_0\right)$. When $a \to a_0^-$, the attracting neighborhood obtained via linearization shrinks to the fixed point. Therefore, a different approach is necessary for parameter values close to a_0. In the subsequent subsections, for $a \in \mathcal{I}_0 = [a_0 - 10^{-2}, a_0]$ we adapt the technique from [4, 5] based on the Neimark–Sacker bifurcational normal form. However, we need new ideas, since $F_3(u)$ is three-dimensional, and thus the adaptation of the method from [4] is not that straightforward.

The classical way to study the dynamics of the three-dimensional map F_3 near u_A, or equivalently, the dynamics of H_a near $(0, 0) \in \mathbb{C} \times \mathbb{R}$, for a close to a_0 is as follows. First, a center manifold reduction is carried out, then the map is transformed to its normal form on the center manifold, and finally the attraction property of the center manifold is used. These steps together give local information on the dynamics of (1.5) for a is close to a_0 and (z, y) is close to $(0, 0)$. In particular, for $a \leq a_0$ and a close to a_0, local stability is obtained for the fixed point of (1.5). The major achievement of this subsection is the elaboration of a quantitative version of the above local bifurcation result so that an attracting neighborhood of the fixed point $(0, 0)$ can be explicitly constructed. Sections

3.3–3.5 are devoted to this issue. In Sections 3.6 and 3.7 it turns out that the constructed attracting neighborhood is large enough, and we can handle the remaining points by a rigorous numerical technique.

In Section 3.3 we consider an approximate version of the center manifold reduction. It is well known that for each fixed a there is a local invariant manifold $\mathcal{W}_a^c$ of map (1.5) at $(0,0)$ given by the graph

$$\mathcal{W}_a^c = \{(z,y) \in \mathbb{C} \times \mathbb{R} : \; y = \Phi_a(z), \; |z| < \delta\}$$

of a smooth map $\Phi_a : \{z \in \mathbb{C} : \; |z| < \delta\} \to \mathbb{R}$ with some $\delta = \delta(a) > 0$ and $\Phi_a(0) = 0$, $\Phi_a(z) = O(|z|^2)$. Note that $\Phi_a(\cdot)$ is not complex differentiable, but it is a smooth function of z and $\bar{z}$. The set $\mathcal{W}_a^c$ is a so called generalized center (or center-unstable) manifold of (1.5) at $(0,0)$ corresponding to the leading eigenvalues $\lambda(a), \bar{\lambda}(a)$ (see [16]). The invariance property of $\mathcal{W}_a^c$ means that $H_a(z, \Phi_a(z)) \in \mathcal{W}_a^c$ for $z \in \mathbb{C}$ with small $|z|$. Or equivalently, $\mathcal{N}(\Phi_a(z)) = 0$, where $\mathcal{N}$ is given by

$$\mathcal{N}(\mathcal{F}(z)) = \mathcal{F}\Big(\lambda(a)z + \mathcal{G}_a\big(z, \mathcal{F}(z)\big)\Big) - \nu(a)\mathcal{F}(z) - \mathcal{H}_a\big(z, \mathcal{F}(z)\big),$$

for all maps $\mathcal{F} : \mathbb{C} \to \mathbb{R}$.

The map $\Phi_a(z)$ is problematic concerning its use in quantitative estimations for several reasons. First, it is obtained from a global manifold via modification of the nonlinearity, and its domain can be too small for computational purposes. Furthermore, it is not unique, and there is no explicit formula for it, either. In addition, we need a three-dimensional attracting set around the fixed point for the computer-aided part, and thus a two-dimensional set on the manifold is not sufficient. Therefore, we consider a polynomial approximation of $\Phi_a(z)$, instead. Namely, for each $a \in \mathcal{I}_0$ there is a unique fourth-order (in $z, \bar{z}$) real valued polynomial

$$\phi(z) = \phi_a(z, \bar{z}) = \sum_{\substack{i+j=2 \\ i,j \in \mathbb{N}_0}}^{4} \frac{1}{i!j!} \omega_{ij}(a) z^i \bar{z}^j$$

with

$$\mathcal{N}(\phi(z)) = O(|z|^5).$$

The advantage of $\phi(z)$ comparing to $\Phi_a(z)$ is that it is defined on the whole complex plane, it is unique, and the coefficients $\omega_{ij}(a)$ can be easily determined and estimated. The disadvantage is that the graph of $\phi(z)$ is not locally invariant under H_a any more. However, a particular three-dimensional set T containing the graph of $y = \phi(z)$ behaves

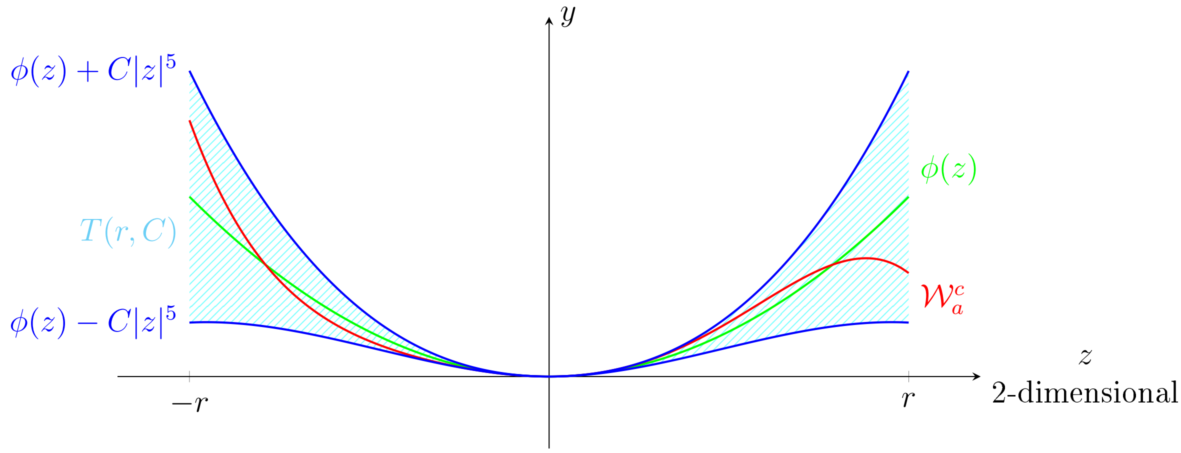


Figure 1.1: The set $T(r, C)$ around the approximation of the center manifold

similarly to a center manifold, and in our work it takes over the role of $\mathcal{W}_a^c$. It will turn out that T is inside the region of attraction of the fixed point.

Define the set

$$T(r, C) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \leq r, \ |y - \phi(z)| \leq C|z|^5\}$$

around $y = \phi(z)$ (see Figure 1.1), where r and C are some positive constants. Note that $T(r, C)$ has a relatively simple form for computational purposes. The term $C|z|^5$ in the definition of $T(r, C)$ guarantees that $T(r, C)$ contains the invariant manifold $\mathcal{W}_a^c$ for small $|z|$. The reason for this special shape of $T(r, C)$, more precisely the term $C|z|^5$, is that the normal form technique needs to be applicable for every $(z, y) \in T(r, C)$.

In Subsections 3.3.1 and 3.3.2 we investigate the y-directional dynamics. We use the property that solutions close to the fixed point decay exponentially to $T(r, C)$ since $|\nu(a)| < |\lambda(a)| \leq 1$. From this it can be shown that $T(r, C)$ is conditionally invariant in direction y for a fixed r, provided that C is sufficiently large. More precisely, for a fixed r and C we show that $H_a(T(r, C)) \subseteq T(\tilde{r}, C)$ with some $\tilde{r} \geq r$. This means that for $(z_0, y_0) \in T(r, C)$ and $(z_1, y_1) = H_a(z_0, y_0) \notin T(r, C)$ we must have $|z_1| > r$, that is, the image under H_a can leave $T(r, C)$ only in direction z.

In Section 3.4 the z-directional dynamics is investigated by using the Neimark–Sacker bifurcational normal form technique from [4]. For every $(z_0, y_0) \in T(r, C)$ the y-coordinate can be written in the form $y_0 = \phi(z_0) + c|z_0|^5$ for some $c \in \mathbb{R}$ with $|c| \leq C$. Thus, for $(z_1, y_1) = H_a(z_0, y_0)$ the z-coordinate is determined by $z_1 = G(z_0)$, where

$$G(z) = G_{a,c}(z, \bar{z}) = \lambda z + \mathcal{G}_a\big(z, \phi(z) + c|z|^5\big). \tag{1.6}$$

For a fixed $a \in \mathcal{I}_0$ and $c \in \mathbb{R}$ with $|c| \leq C$ we can transform (1.6) into a normal form. Namely, a nonlinear invertible map $h : \mathbb{C} \to \mathbb{C}$ can be given such that

$$w \mapsto h^{-1}(G(h(w))) = \lambda w + c_1 w^2 \overline{w} + R_2(w, \overline{w}, a, c),$$

where $c_1 = c_1(a, c)$ is the Lyapunov-coefficient and $R_2(w, \overline{w}, a, c) = O(|w|^4)$. It is important (see [4]) that the transformation h is completely determined by the lower-order terms of (1.6). Because of the special shape of $T(r, C)$, parameter c appears only in the higher-order terms of G, i.e., only in $R_2(w, \overline{w}, a, c)$. Consequently, h is independent of c, and $w = h(z)$ can be considered as a coordinate transformation of the whole set $T(r, C)$.

Applying the normal form method from [4], we obtain

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| < |w|$$

for every sufficiently small $w \neq 0$. This means that in the transformed w–y-coordinate system, the w-coordinate is getting strictly closer and closer to the origin during the iteration, provided that the iterates of (z, y) stay in $T(r, C)$. Combining this fact with the y-directional dynamics we show in Subsection 3.4.7 that $T(\hat{r}, C)$, with some $\hat{r} < r$, is in the region of attraction of the fixed point $(0, 0)$ of (1.5).

However, $T(\hat{r}, C)$ is clearly not a proper neighborhood of the origin in $\mathbb{C} \times \mathbb{R}$. Therefore, in Section 3.5 we define the set

$$\tilde{T}(r, K) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \leq r, \ |\phi(z) - y| \leq K\}$$

for some $r > 0$ and $K > 0$. By using the exponential y-directional attractivity of $T(r, C)$ we show that $\tilde{T}(r, K)$ is in the region of attraction of the fixed point. The neighborhood $\tilde{T}$ of the fixed point is suitable for the computer-assisted part of the proof.

Finally, in Sections 3.6 and 3.7 we describe the computer-assisted part of our method, similar to the two-dimensional case.

Despite the fact that we demonstrate our method only on a specific equation, we believe that it can be applied or extended to other similar maps. For instance the Ricker map (see [5]) and the Pielou map (see [13]) with delay $d = 2$ essentially differ only in that they are not polynomial maps. Hence, only a slight modification would be necessary in the estimations. However, the main question is whether the obtained neighborhood is large enough for the computer-aided part of the method. These two maps along with the logistic map would also be interesting for larger delay, i.e., $d > 2$. We believe that the analytical part could be extended using only natural modifications. However, the computer-aided part can be critical in these cases, since the increasing dimension causes an exponentially growing graph.

We note that the adaptation of the method to continuous models such as ordinary or delayed differential equations is also an open question. We could mention for example the famous Wright problem (see [17]) where local stability implies global stability and a Hopf bifurcation takes place. The proof in [17] is also a combination of purely analytical and reliable numerical tools, however a different approach was used for the analytical part.

It also would be interesting to prove the existence of the unique invariant closed curve for (1.2) around the nontrivial fixed point for parameter values larger than the critical value. However, this question is substantially different from the one studied in this thesis.

Chapter 2

Global stability for the two-dimensional logistic map

2.1 Preliminaries

Throughout the thesis $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}_+$ denote the positive integers, the nonnegative integers and the nonnegative real numbers, respectively. We use also the big O notation in the sense that $f(x) = O(g(x))$ means that there exist positive numbers δ and M such that $|f(x)| \leq Mg(x)$ for $|x| < \delta$.

For symbolic computation we use Wolfram Mathematica v. 11, and for reliable numerical estimation we use interval arithmetic tools of IntLab v. 9 in Matlab 2018.

2.1.1 Dynamics of F_d – general properties

In this subsection we study the dynamics of map (1.2) in $[0,\infty)^{d+1}$ for $d \in \mathbb{N}$ and $a > 0$. Recall the definition of S^d and S_0^d from (1.4) and (1.3). We collect some properties of S^d. First, for $a \in (0, a_0]$ it can be shown that S^d is positively invariant under F_d, that is $F_d(S^d) \subseteq S^d$. Secondly, $F_d^k(u) = 0$ for $u \in S_0^d \setminus S^d$ and $k \in \mathbb{N}$ large enough, i.e., the set $S_0^d \setminus S^d$ is in the region of attraction of the trivial fixed point. Thirdly, for every $u \in [0,\infty)^{d+1} \setminus S_0^d$ there is a $k \in \mathbb{N}$ such that $F_d^k(u) \notin [0,\infty)^{d+1}$, i.e., S_0^d is the largest set for which $F_d^n(u) \in [0,\infty)^{d+1}$ for every $n \in \mathbb{N}_0$. These statements are proven only for $d = 1$ and $d = 2$ in Propositions 5 and 13, respectively. However, the method can be easily generalized for any $d > 2$.

Now, we can turn our attention to dynamics in S^d. For small a the situation is quite simple. It is easy to see that $S^d = S_0^d = [0,1]^{d+1}$ for $0 < a \leq 1$, and the only fixed point in $[0,\infty)^{d+1}$ is the origin. The next proposition easily follows from the fact that $x_{n+1} = ax_n(1 - x_{n-d}) < x_n$, provided that $x_{n-d}, x_n \in (0,1]$ and $0 < a \leq 1$.

Proposition 2. *For every $0 < a \leq 1$ and $u \in [0,1]^{d+1}$ the following holds*

$$\lim_{n\to\infty} F_d^n(u) = 0 \in \mathbb{R}^{d+1}.$$

For $a \in (1, a_0]$ we introduce the sets

$$\begin{aligned} S_-^d &= \big\{u \in S^d : \ u_{d+1} < A, \ \ u_k \leq A, \quad \text{for} \ \ k \in \{1,2,\dots,d\}\big\}, \\ S_+^d &= \big\{u \in S^d : \ u_{d+1} > A, \ \ u_k \geq A, \quad \text{for} \ \ k \in \{1,2,\dots,d\}\big\}. \end{aligned}$$

For given $x_0, x_1, \dots, x_d$ the sequence $(x_n)_{n=0}^{\infty}$, where x_n is defined by (1.1) for $n > d$, corresponds to the $(d+1)$-dimensional sequence $(u^n)_{n=0}^{\infty}$ with

$$u^0 = (x_0, x_1, \dots, x_d), \quad u^n = F_d^n(u^0) = (x_n, x_{n+1}, \dots, x_{n+d}), \quad n \in \mathbb{N}.$$

Proposition 3. *Let $u \in S^d$ be given. If there is an $n_0 \in \mathbb{N}_0$ such that $F_d^n(u) \in S_-^d$ (or $F_d^n(u) \in S_+^d$) for every $n \geq n_0$, then $\lim_{n\to\infty} F_d^n(u) = u_A$.*

Proof. Notice that $u^n \in S_-^d$ implies $x_{n+d} \leq x_{n+d+1}$, since $x_n \leq A$. Consequently, we get a monotone, bounded sequence $(x_n)_{n=n_0+d}^{\infty}$ which converges to some $B \leq A$. Taking the limit of both sides in (1.1), we obtain $B = A$. Consequently, the point u^0 is in the region of attraction of u_A in this case. Similarly, if $(u^n)_{n=0}^{\infty}$ gets stuck in S_+^d, it also converges to the fixed point u_A. □

Now, we assume $1 < a \leq \frac{d+2}{d+1}$, and show that for every $u^0 \in S_-^d$ the sequence $(u^n)_{n=0}^{\infty}$ converges to the nontrivial fixed point u_A.

Proposition 4. *If $a \in \big(1, \frac{d+2}{d+1}\big]$ and $u \in S_-^d$, then $\lim_{n\to\infty} F_d^n(u) = u_A$.*

Proof. First, note that $x_n \leq A$ implies $x_{n+d} \leq x_{n+d+1}$, so the sequence $(x_n)_{n=d}^{\infty}$ is nondecreasing for small $n \geq d$. Using Proposition 3 we can assume that there is an $n_1 > d$ such that $x_k < A$ for $k \in \{d, \dots, n_1 - 1\}$ and $x_{n_1} \geq A$. It can be checked that $(x_n)_{n=d}^{n_1+d}$ is nondecreasing and $u_{n_1} \in S_+^d$. On the other hand $x_n \geq A$ implies $x_{n+d} \geq x_{n+d+1}$, so the sequence $(x_n)_{n=n_1+d}^{\infty}$ is nonincreasing for small $n \geq n_1 + d$. Again, using Proposition 3 we can assume that there is an $n_2 > n_1 + d$ such that $x_k > A$ for $k \in \{n_1 + d, \dots, n_2 - 1\}$ and $x_{n_2} \leq A$. So we obtain $(x_n)_{n=n_1+d}^{n_2+d}$ is nonincreasing and $u_{n_2} \in S_-^d$.

Repeating the argument we get a strictly increasing subsequence $(n_l)_{l=0}^{\infty}$ of $\mathbb{N}_0$ such that $n_0 = 0$, $u^{n_{2k}} \in S_-^d$ and $u^{n_{2k+1}} \in S_+^d$ with

$$\begin{aligned} u^j &\notin S_+^d \quad \text{for} \quad j \in \{n_{2k}, \dots, n_{2k+1} - 1\}, \\ u^j &\notin S_-^d \quad \text{for} \quad j \in \{n_{2k+1}, \dots, n_{2k+2} - 1\}. \end{aligned}$$

Furthermore,

$$(x_j)_{j=n_{2k}+d}^{n_{2k+1}+d} \quad \text{is nondecreasing and} \quad (x_j)_{j=n_{2k+1}+d}^{n_{2k+2}+d} \quad \text{is nonincreasing,} \tag{2.1}$$

and we also have

$$\begin{aligned} x_j &\leq A \quad \text{for} \quad j \in \{n_{2k}, \ldots, n_{2k+1} - 1\}, \\ x_j &\geq A \quad \text{for} \quad j \in \{n_{2k+1}, \ldots, n_{2k+2} - 1\}. \end{aligned} \tag{2.2}$$

Note that $n_{l+1} - n_l > d$ holds.

For the function $t : [0,1] \ni x \mapsto a^d(a-1)(1-x)^{d+1} \in \mathbb{R}$ with $a \in \big(1, \frac{d+2}{d+1}\big]$ we have $t(A) = A$, $t(x) \in [0,1]$ and

$$\begin{aligned} \frac{d}{dx}t(x) &= -(d+1)a^d(a-1)(1-x)^d \leq 0, \\ \frac{d^2}{dx^2}t(x) &= d(d+1)a^d(a-1)(1-x)^{d-1} \geq 0, \\ \frac{d}{dx}t(t(x)) &= (d+1)^2a^{2d}(a-1)^2(1-x)^d\big(1-t(x)\big)^d \geq 0, \\ \frac{d^2}{dx^2}t(t(x)) &= d(d+1)^2a^{2d}(a-1)^2(1-x)^{d-1}\big(1-t(x)\big)^{d-1}\big((d+2)t(x)-1\big). \end{aligned}$$

For $x \in (A,1]$ we obtain $(d+2)t(x) - 1 < (d+2)A - 1 \leq 0$, provided that $a \in \big[1, \frac{d+2}{d+1}\big]$. Hence, $\frac{d^2}{dx^2}t(t(x)) < 0$ for all $x \in (A,1)$. Furthermore, $\frac{d}{dx}t(t(x))\big|_{x=A} = (d+1)^2(a-1)^2 \leq 1$ for $a \in \big[1, \frac{d+2}{d+1}\big]$. Thus, $\frac{d}{dx}t(t(x)) < 1$ for all $x \in (A,1)$. Therefore, the only fixed point of $[A,1] \ni x \mapsto t(t(x)) \in \mathbb{R}$ is A.

Let $(s_k)_{k=0}^{\infty}$ be given by $s_0 = 0$ and $s_k = t^k(s_0)$ for $k \in \mathbb{N}$. Clearly, $(s_{2k})_{k=0}^{\infty}$ is strictly increasing with $s_{2k} < A$ for $k \in \mathbb{N}_0$ and $(s_{2k+1})_{k=0}^{\infty}$ is strictly decreasing with $s_{2k+1} > A$ for $k \in \mathbb{N}_0$. Now, by induction we show that

$$s_{2k} \leq x_j \quad \text{for} \quad j \in \{n_{2k}, \ldots, n_{2k+1} - 1\}, \tag{2.3}$$

$$s_{2k+1} \geq x_j \quad \text{for} \quad j \in \{n_{2k+1}, \ldots, n_{2k+2} - 1\} \tag{2.4}$$

for all $k \in \mathbb{N}_0$. Clearly, (2.3) is true for $k = 0$, since $s_0 = 0$.

Suppose that (2.3) holds for a given $k \in \mathbb{N}_0$. Then using (2.2) we obtain

$$x_{n_{2k+1}+2} = a^{d+1}x_{n_{2k+1}-1}\prod_{l=1}^{d+1}(1 - x_{n_{2k+1}-l}) \leq a^{d+1}A(1 - s_{2k})^{d+1} = s_{2k+1}.$$

Combining this inequality with (2.1), we get that (2.4) holds for the given $k \in \mathbb{N}_0$.

Similarly, (2.4) implies

$$x_{n_{2k+2}+2} = a^{d+1} x_{n_{2k+2}-1} \prod_{l=1}^{d+1} (1 - x_{n_{2k+2}-l}) \geq a^{d+1} A (1 - s_{2k+1})^{d+1} = s_{2k+2}.$$

Using (2.1) again, we get

$$s_{2k+2} \leq x_j \quad \text{for} \quad j \in \{n_{2k+2}, \dots, n_{2k+3} - 1\},$$

which completes the inductive step.

Combining (2.3) and (2.4) with (2.2), it follows that

$$\begin{aligned} x_j &\in [s_{2k}, A] \quad \text{for} \quad j \in \{n_{2k}, \dots, n_{2k+1} - 1\}, \\ x_j &\in [A, s_{2k+1}] \quad \text{for} \quad j \in \{n_{2k+1}, \dots, n_{2k+2} - 1\} \end{aligned}$$

for all $k \in \mathbb{N}_0$. Clearly, $s_k \to A$ implies $x_k \to A$.

As $(s_{2k+1})_{k=0}^{\infty}$ is a decreasing sequence in $[A, 1]$, $s_{2k+3} = t(t(s_{2k+1}))$, and A is the only fixed point of $t(t(x))$ in $[A, 1]$, we obtain $s_{2k+1} \to A$. The continuity of t and $s_{2k+2} = t(s_{2k+1})$ gives $s_{2k} \to A$. □

2.1.2 Dynamics of F_d for $d = 1$

In this subsection we study the equations (1.1) and (1.2) for $d = 1$, so we consider

$$x_{n+1} = a x_n (1 - x_{n-1}),$$

which is equivalent to the two-dimensional map

$$F_2 : \mathbb{R}^2 \ni u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ a u_2 (1 - u_1) \end{pmatrix} \in \mathbb{R}^2. \tag{2.5}$$

Now, we describe the dynamics of map (2.5) in the positive quadrant for $a > 0$.

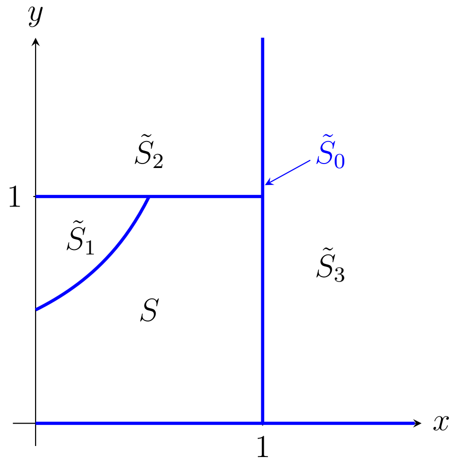


Figure 2.1: The subdivision of the positive quadrant. The union of the blue lines is the set $\tilde{S}_0$ forming the boundaries of the sets $S, \tilde{S}_1, \tilde{S}_2$ and $\tilde{S}_3$.

Introduce the following disjoint sets (see Figure 2.1).

$$
\begin{aligned}
S &= \big\{(u_1, u_2) \in \mathbb{R}^2 : \ x \in [0,1), \quad u_2 \in (0,1), \quad au_2(1-u_1) < 1\big\} \\
\tilde{S}_0 &= \big\{(u_1, 0) \in \mathbb{R}^2 : \ u_1 \geq 0\big\} \cup \big\{(1, u_2) \in \mathbb{R}^2 : \ u_2 > 0\big\} \cup \big\{(u_1, 1) \in \mathbb{R}^2 : \ u_1 \in [0,1)\big\} \\
&\quad \cup \big\{(u_1, u_2) \in \mathbb{R}^2 : \ u_1 \in [0,1) \quad u_2 \in (0,1), \quad au_2(1-u_1) = 1\big\} \\
\tilde{S}_1 &= \big\{(u_1, u_2) \in \mathbb{R}^2 : \ u_1 \geq 0, \quad u_2 \in (0,1), \quad au_2(1-u_1) > 1\big\} \\
\tilde{S}_2 &= \big\{(u_1, u_2) \in \mathbb{R}^2 : \ u_1 \in [0,1), \quad u_2 > 1\big\} \\
\tilde{S}_3 &= \big\{(u_1, u_2) \in \mathbb{R}^2 : \ u_1 > 1, \quad u_2 > 0\big\}
\end{aligned}
$$

Clearly, $\mathbb{R}^2_+ = S \cup \tilde{S}_0 \cup \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3$, moreover S and $S \cup \tilde{S}_0$ correspond to S^1 and S^1_0, respectively, see (1.4) and (1.3). Furthermore, $S = [0,1) \times (0,1)$ and $\tilde{S}_1 = \emptyset$ for $a \in (0,1]$. Recall that $a_0 = 2$ for $d = 1$.

Proposition 5. *For all $a > 0$ we have*

$$F_2^4(\tilde{S}_0) = \{(0,0)\}, \quad F_2(\tilde{S}_1) \subseteq \tilde{S}_2, \quad F_2(\tilde{S}_2) \subseteq \tilde{S}_3, \quad F_2(\tilde{S}_3) \cap \mathbb{R}^2_+ = \emptyset.$$

Furthermore, $F_2(S) \subseteq S$ for $a \in (0,2]$.

Proof. From the definition of $F_2(u)$ it is obvious that $F_2^4(\tilde{S}_0) = \{(0,0)\}$. It is also straightforward to check the relations $F_2(\tilde{S}_1) \subseteq \tilde{S}_2$, $F_2(\tilde{S}_2) \subseteq \tilde{S}_3$ and $F_2(\tilde{S}_3) \cap \mathbb{R}^2_+ = \emptyset$.

For $a \in (0,2]$ and $u \in S$ we have

$$a^2 u_2(1-u_1)(1-u_2) \leq 4 \max_{u_2 \in (0,1)} u_2(1-u_2) \leq 1,$$

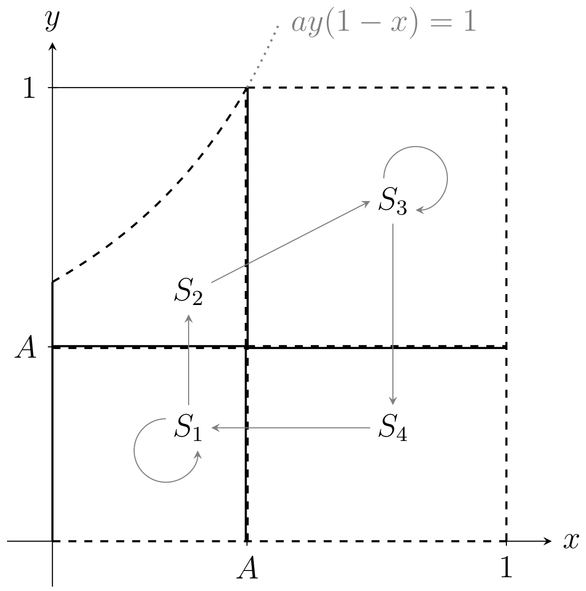


Figure 2.2: The dynamics in $S \setminus u_A$.

which implies the invariance of S. □

Because of Propositions 5 and 2 we can assume $u \in S$ and $a \in (1, 2]$ in the rest of the section. We divide the positive quadrant into four subsets with lines $x = A$, $y = A$, and introduce the following sets.

$$
\begin{aligned}
S_1 &= \{u \in S: \ x \leq A, \quad y < A\} \\
S_2 &= \{u \in S: \ x < A, \quad A \leq y\} \\
S_3 &= \{u \in S: \ A \leq x, \quad A < y\} \\
S_4 &= \{u \in S: \ A < x, \quad y \leq A\}
\end{aligned}
$$

Clearly, $S = \bigcup_{i=1}^{4} S_i \cup \{u_A\}$.

Proposition 6. *Let $a \in (1, a_0]$ and the sequence $(u^n)_{n=0}^{\infty}$ in S be given by $u^0 \neq u_A$ and $u^n = F_2^n(u^0)$, $n \in \mathbb{N}$.*

(i) The sequence $(u^n)_{n=0}^{\infty}$ follows the transition graph given in Figure 2.2.

(ii) If $(u^n)_{n=0}^{\infty}$ does not converge to u_A then there exists an $n_0 \in \mathbb{N}$ such that $u^{n_0} \in S_1$, and $(u^n)_{n=n_0}^{\infty}$ does not eventually stay in S_1 or S_3.

Proof. The transitions between the aforementioned subsets are the following, see Figure 2.2.

- For $u \in S_1$ we obtain $\hat{u} = F_2(u)$ with $\hat{u}_1 < A$, therefore $S_1 \rightarrowtail \{S_1, S_2\}$. That is, $\hat{u} \in S_1$ or $\hat{u} \in S_2$.

- For $u \in S_2$ we have $\hat{u}_1 \geq A$ and $\hat{u}_2 = au_2(1-u_1) > aA(1-A) = A$, so $S_2 \rightarrowtail S_3$.
- For $u \in S_3$ we get $\hat{u}_1 > A$, so $S_3 \rightarrowtail \{S_3, S_4\}$.
- For $u \in S_4$ we obtain $\hat{u}_1 \leq A$ and $\hat{u}_2 = au_2(1-u_1) > aA(1-A) = A$, so $S_4 \rightarrowtail S_1$.

We obtain there is a cycle $S_1 \rightarrowtail S_2 \rightarrowtail S_3 \rightarrowtail S_4 \rightarrowtail S_1$. Possibly, the sequence can get stuck and stay forever in S_1 or S_3, however, in that case Proposition 3 provides the convergence. □

Now we assume $1 < a \leq \frac{3}{2}$ and show that for every $u \in S$ the sequence $(u^n)_{n=0}^{\infty}$ converges to the nontrivial fixed point u_A. Combining this fact with the local asymptotic stability of the fixed point (see Section 2.2), Conjecture 1 is proven for $a \in \left(1, \frac{3}{2}\right]$ and $d = 1$.

Proposition 7. *If $a \in \left(1, \frac{3}{2}\right]$ and $u \in S$, then $\lim_{n\to\infty} F_2^n(u) = u_A$.*

Proof. It is clear from Proposition 6, that we only need to consider the case when the sequence $(u^n)_{n=0}^{\infty}$, where $u_0 = u$ and $u^n = F_2^n(u)$, goes around the fixed point, not getting stuck in S_1 or S_3. Without loss of generality we can assume that $u_0 \in S_1$, and Proposition 4 completes the proof, since $S_1 = S_-^1$. □

In the rest of the section we assume $a \in \left(\frac{3}{2}, 2\right]$. For these parameter values, the above argument does not guarantee convergence for every $u \in S$, but we show, it is enough to consider a subset of S later on.

Proposition 8. *For every $a \in \left(\frac{3}{2}, 2\right]$ the set*

$$\tilde{S} = \{u \in S : \ u_1, u_2 \in [0.072, 0.8]\}$$

is invariant, i.e., $F(\tilde{S}) \subseteq \tilde{S}$. Furthermore, for every $u \in S$, there exists n_0 such that $F_2^n(u) \in \tilde{S}$ for every $n > n_0$.

Proof. We can assume $u^0 \in S_1$ and the sequence $(u^n)_{n=0}^{\infty}$ goes around the fixed point. Since $u \in S$ implies $\hat{u}_2 = au_2(1-u_1) \leq au_2 = a\hat{u}_1$, we can also assume that $u_2 \leq au_1$ for every $u \in S$.

For the upper bound we have to find the maximum of $a^2u_2(1-u_2)(1-u_1)$ assuming $u \in S_1$ and $u_2 \leq au_1$. Since $x \mapsto x(1-x)$ is increasing on $[0, A]$ we are looking for the maximum of $f(x) = a^3x(1-ax)(1-x)$ on $\left[0, \frac{A}{a}\right]$ and $g(x) = (a-1)(1-x)$ on $\left[\frac{A}{a}, A\right]$. The maxima of $g(x)$ and $f(x)$ are $\frac{3}{4}$ and $\frac{4}{3\sqrt{3}}$, respectively, so $\frac{4}{5}$ is an appropriate choice for the upper bound.

Similarly, for every $a \in \left(\frac{3}{2}, 2\right]$ we are looking for the minimum of $a^2u_2(1-u_2)(1-u_1)$ on S_3, assuming $u_1, u_2 \leq \frac{4}{5}$. It is easy to see that this is 0.072. □

We apply this proposition in the computer assisted part of the proof, since it is useful to exclude a small neighborhood of the trivial fixed point $(0,0)$, as we see it later. For more general results on absorbing sets like $\tilde{S}$, the reader is referred to [18].

2.2 Attracting neighborhood with linearization

In this subsection using the linearization of map (2.5) for a fixed parameter $a \in \left(\frac{3}{2}, 2\right)$, we give a neighborhood $\mathcal{M}$ around u_A, which is inside the region of attraction of this fixed point, i.e., $\lim_{n\to\infty} F_2^n(u) = u_A$ for every $u \in \mathcal{M}$.

For each fixed a, translating u_A into $0 \in \mathbb{R}^2$, i.e., introducing the new variable $v = u - u_A$, the shifted version

$$\mathbb{R}^2 \ni v \mapsto F_2(v + u_A) - u_A \in \mathbb{R}^2$$

of (2.5) can be written as

$$\mathbb{R}^2 \ni v \mapsto J_a v + f_a(v) \in \mathbb{R}^2, \tag{2.6}$$

where

$$J_a = \begin{pmatrix} 0 & 1 \\ 1-a & 1 \end{pmatrix}, \quad f_a(v) = \begin{pmatrix} 0 \\ -av_1v_2 \end{pmatrix}.$$

For $a \in \left(\frac{3}{2}, 2\right]$ the eigenvalues of $J(a)$ are $\lambda = \lambda_1(a) = \overline{\lambda_2(a)} = \frac{1+i\sqrt{4a-5}}{2}$ and the corresponding eigenvectors are $q_i(a) = (1, \lambda_i(a))$ for $i \in \{1,2\}$. It is easy to see, that $|\lambda_i(a)| < 1$ for $a \in (1,2)$, $|\lambda_i(2)| = 1$ and $|\lambda_i(a)| > 1$ for $a > 2$ where $i \in \{1,2\}$. Introduce the notation $q = q(a) = q_1(a)$ and denote by $p = p(a)$ the eigenvector of the transposed matrix $J_a^T(a)$ corresponding to $\overline{\lambda}$, normalized to $\langle p, q\rangle = 1$, where $\langle (a_1,a_2),(b_1,b_2)\rangle = \sum_{i=1}^2 \bar{a}_i b_i$ for $(a_1,a_2),(b_1,b_2) \in \mathbb{C}^2$. We obtain $\bar{p} = d(\lambda-1, 1)$, where $d = d(a) = (2\lambda-1)^{-1}$.

Introduce the complex variable $z = \langle p, v\rangle$. The variable v can also be expressed by z, namely,

$$v = qz + \bar{q}\bar{z} = \begin{pmatrix} z + \bar{z} \\ \lambda z + \bar{\lambda}\bar{z} \end{pmatrix}.$$

Moreover, map (2.6) can be written in the form

$$z \mapsto \langle p, J_a v + f_a(v)\rangle = \lambda z + d\, g(z),$$

where $g : \mathbb{C} \to \mathbb{R}$ is the following

$$g(z) = g_a(z, \bar{z}) = -a(z + \bar{z})\left(\lambda(a) z + \overline{\lambda(a)}\bar{z}\right).$$

First we use the map

$$z \mapsto G(z) = \lambda z + d\, g(z) \tag{2.7}$$

without further transformation to construct $\mathcal{M}$.

Proposition 9. *For every $a \in \left(\frac{3}{2}, 2\right)$ define $\varepsilon(a)$ by*

$$\varepsilon(a) = \frac{\sqrt{4a-5}\left(1-\sqrt{a-1}\right)}{a\left(2\sqrt{a-1}+1\right)} \cdot \frac{\sqrt{4a-5}}{\sqrt{a+1}}.$$

Then the set

$$\mathcal{M}(a) = \{u \in S : \ |u_1 - A|, |u_2 - A| \le \varepsilon(a)\}$$

is in the region of attraction of the fixed point u_A of $F_2(u)$.

Proof. At first we show, there exists a $\xi = \xi(a) > 0$ such that $|G(z)| < |z|$ for every $z \in \mathbb{C}$ with $0 < |z| < \xi$. If such a ξ exists, it is clear that the open ball B_ξ° around the origin is invariant and we show that every point of B_ξ° tends to the origin. Let z_0 be an arbitrary point from B_ξ° and consider the nonnegative, strictly decreasing sequence $(|z_n|)_{n=0}^\infty$, where $z_{n+1} = G(z_n)$. This sequence can converge only to a fixed point of the continuous map $r \mapsto \max_{|z|=r} |G(z)|$, which is, inside B_ξ°, solely $r = 0$.

Estimate the right hand side of the map (2.7). Using $|\lambda| = \sqrt{a-1}$, $|d| = \frac{1}{\sqrt{4a-5}}$ and $|g(z)| \le a\,(2|\lambda|+1)\,|z|^2 = a\left(2\sqrt{a-1}+1\right)|z|^2$, we obtain

$$|\lambda z + d\, g(z)| \le |z| \left(\sqrt{a-1} + \frac{a\left(2\sqrt{a-1}+1\right)}{\sqrt{4a-5}} |z| \right) < |z|,$$

for every $z \neq 0$, provided that $|z| < \xi$, where

$$\xi = \frac{\sqrt{4a-5}(1-\sqrt{a-1})}{a\left(2\sqrt{a-1}+1\right)}.$$

To obtain an estimate of $v = (v_1, v_2)$, we use the expression $z = \langle p, v\rangle = d((\lambda - 1)v_1 + v_2)$. For $|v_1|, |v_2| \le \delta$ we obtain

$$|z| = |d| \sqrt{\frac{4a-5}{4} v_1^2 + \left(v_2 - \frac{v_1}{2}\right)^2} \le \delta \frac{\sqrt{a+1}}{\sqrt{4a-5}},$$

therefore, if $\delta \le \xi \frac{\sqrt{4a-5}}{\sqrt{a+1}}$, then $|z| \le \xi$. Set $\varepsilon(a) = \xi \frac{\sqrt{4a-5}}{\sqrt{a+1}}$, then points, whose coordinates satisfy $|v_1|, |v_2| \le \varepsilon(a)$, are in the region of attraction of the fixed point of (2.6), so the proof is complete. □

It is easy to see the set $\mathcal{M}$ shrinks to the fixed point u_A as a tends to 2, since

$\lim_{a\to 2}\varepsilon(a) = 0$. Consequently, close to the critical parameter value, the neighborhood, obtained by linearization is not suitable for reliable numerical methods. In fact, the smaller the neighborhood, the less efficient and more time consuming the numerical part of the proof. Furthermore, the linearization does not provide an attracting neighborhood at the critical parameter value $a = 2$, therefore, we need another approach to construct a neighborhood $\mathcal{M}$ for parameter values close to 2.

In the subsequent subsection we use the normal form of the Neimark–Sacker bifurcation and create a neighborhood whose size is independent of a. Actually, the first method with the linearization become rather compute-intensive at about the parameter range $(1.99, 2)$, but we will use the second technique with the normal form in a bigger parameter range, namely for $a \in (1.95, 2]$. The normal form technique provides a significantly larger neighborhood than the first method can do for parameters close to the critical value, so the second method is more efficient even for $a \in (1.95, 1.99]$, too.

2.3 Transforming to normal form

In this subsection, first we give a general method to construct an attracting neighborhood around a fixed point, which undergoes a supercritical Neimark–Sacker bifurcation at a_0. This neighborhood is suitable for parameters close to the critical value a_0, i.e., for $a \in [a_0 - \beta_0, a_0]$ with some fixed $\beta_0 > 0$. We follow the steps of finding the normal form of the Neimark–Sacker bifurcation, according to Kuznetsov [8].

Suppose that we have a map

$$\mathbb{R}^2 \ni u \mapsto F(u) \in \mathbb{R}^2, \tag{2.8}$$

where $F = F_a$ smoothly depends on the parameter $a \in \mathbb{R}$. Furthermore, we have a fixed point $u^* = u^*(a)$ which undergoes a supercritical Neimark–Sacker bifurcation at a_0. Fix some $\beta_0 > 0$. According to Kuznetsov, if $|\lambda(a)| < 1$ for all $a \in [a_0 - \beta_0, a_0)$, then the map (2.8) can be transformed into the form

$$z \mapsto G(z) = \lambda(a)z + G_2(z, a), \tag{2.9}$$

where $\lambda = \lambda(a)$, $z \in \mathbb{C}$ and $G_2 = O(|z|^2)$ is smooth. (Compare to (2.7).)

We can write the smooth $G(z)$ as a formal Taylor series of complex variables z and $\bar{z}$, i.e.,

$$G(z) = \lambda z + \sum_{\substack{i+j=2 \\ i,j\in\mathbb{N}_0}}^{4} \frac{G_{ij}}{i!\,j!} z^i \bar{z}^j + R_1,$$

where $G_{ij} = G_{ij}(a)$ and $R_1 = R_1(z, \bar{z}, a) = O(|z|^5)$. Then, with smooth and invertible functions we transform the map (2.9) into its normal form

$$w \mapsto \lambda w + c_1 w^2 \overline{w} + R_2, \tag{2.10}$$

where $R_2 = R_2(w, \overline{w}, a) = O\left(|w|^4\right)$ and $c_1 = c_1(a)$ is the Lyapunov-coefficient. If we show that there exists $\rho_0 > 0$ such that for every $w \in \mathbb{C}$ with $0 < |w| < \rho_0$ and $a \in [a_0 - \beta_0, a_0]$

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| < |w| \tag{2.11}$$

holds, then we obtain that $B_{\rho_0} = \{w \in \mathbb{C} : \ |w| < \rho_0\}$ is in the region of attraction of the fixed point 0 of the map (2.10) for every $a \in [a_0 - \beta_0, a_0]$. Since $|\lambda| \leq 1$ for $a \leq a_0$, and the bifurcation is supercritical, i.e., $\operatorname{Re} \frac{c_1(a_0)}{\lambda(a_0)} < 0$, it is easy to see, that inequality (2.11) holds for all sufficiently small ρ_0 and β_0.

Our aim is to obtain an explicit value of ρ_0 for a given β_0. Furthermore, ρ_0 needs to be as big as possible because of the computer assisted part of the proof. Consequently, the estimation of the higher-order terms, i.e., R_2 is the most essential part of the method, just like in the linearized case. Note that, in the end we need to derive a $\{z \in \mathbb{C} : \ |z| < \varepsilon\}$–type neighborhood related to the original map (2.9).

To obtain the normal form, we look for a smooth invertible function $h : \mathbb{C} \to \mathbb{C}$ in a neighborhood of $0 \in \mathbb{C}$ which transforms the map (2.9) with the new coordinate $w = h^{-1}(z)$ into the form

$$w \mapsto h^{-1}(G(h(w))) = \lambda(a)w + c_1(a)w^2\overline{w} + R_2. \tag{2.12}$$

According to Kuznetsov [8] such a function can be found in the form

$$h(w) = w + \frac{h_{20}}{2}w^2 + h_{11}w\overline{w} + \frac{h_{02}}{2}\overline{w}^2 + \frac{h_{30}}{6}w^3 + \frac{h_{12}}{2}w\overline{w}^2 + \frac{h_{03}}{6}\overline{w}^3, \tag{2.13}$$

where $h_{ij} = h_{ij}(a)$. To this transformation we need the nonresonance condition

$$\left(\frac{\lambda(a_0)}{|\lambda(a_0)|}\right)^k \neq 1,$$

for $k \in \{1, 2, 3, 4\}$ and $a \in [a_0 - \beta_0, a_0]$.

Clearly, h has an inverse in a small neighborhood of $0 \in \mathbb{C}$, and h^{-1} can be written in the form

$$h^{-1}(z) = h_0^{-1}(z) + R_3, \tag{2.14}$$

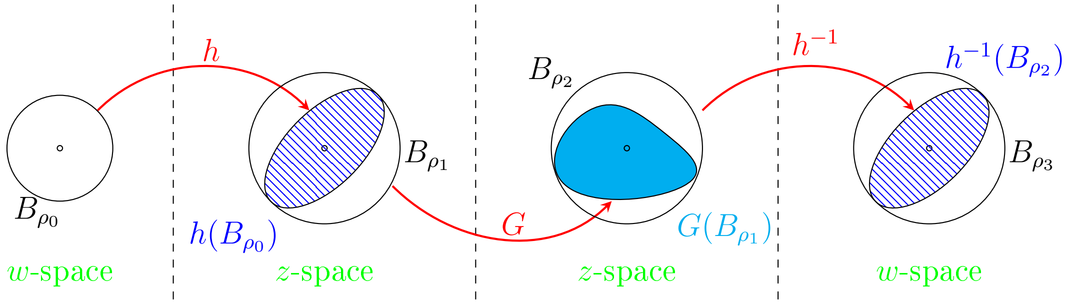


Figure 2.3: The size of the domains of h, G and h^{-1} provided, that $|w| < \rho_0$

where

$$h_0^{-1}(z) = z + \sum_{\substack{i+j=2 \\ i,j\in\mathbb{N}_0}}^{4} \frac{\tilde{h}_{ij}}{i!j!} z^i \bar{z}^j,$$

$R_3 = R_3(z,\bar{z},a) = O(|z|^5)$ and $\tilde{h}_{ij} = \tilde{h}_{ij}(a)$. The coefficient $\tilde{h}_{ij}$ can be obtained by substituting $w = h^{-1}(z)$ into $z = h(w)$ and equating the coefficients of the same type up to fourth-order. The h_{ij} was obtained in a similar manner, namely, we need to choose the coefficients so that the second and third-order terms (apart from $w^2\overline{w}$) of $h^{-1}(G(h(w)))$ are eliminated. The formulas can be found in the Appendix, see Section 4.1. Notice that h_{ij} and consequently $\tilde{h}_{ij}$ depend only on the at most third-order terms of G.

First we will give a finite-order polynomial estimates on the functions G, h and h^{-1}

$$\begin{aligned}
|h(w)| &\le |w| + h_2|w|^2 + h_3|w|^3, \\
|G(z)| &\le |z| + G_2|z|^2 + G_3|z|^3 + G_4|z|^4 + R_{10}|z|^5, \\
\left|h^{-1}(z)\right| &\le |z| + \tilde{h}_2|z|^2 + \tilde{h}_3|z|^3 + \tilde{h}_4|z|^4 + R_{30}|z|^5,
\end{aligned}$$

where the coefficients are independent of a. With them we can give an estimate on R_2, i.e., the higher-order terms of the composition $h^{-1}(G(h(w)))$. Clearly, $h(w)$ is a finite-order polynomial, but generally the Taylor expansions of $G(z)$ and $h^{-1}(z)$ have infinitely many terms. So the at least fifth-order terms are estimated in $R_{10}|z|^5$ and $R_{30}|z|^5$. For the lower-order terms we have explicit formulas and they could be estimated by interval arithmetic. As for the higher-order terms it is essential to be able to say how large can be the moduli of $h(w)$, $G(h(w))$ and $h^{-1}(G(h(w)))$ in (2.12) if $|w| < \rho_0$ is assumed, since the estimate of the remaining terms of a Taylor expansion highly depends on the size of the neighborhood on which it needs to be valid. The radii ρ_1, ρ_2 and ρ_3 must be chosen so that $h(B_{\rho_0}) \subseteq B_{\rho_1}$, $G(B_{\rho_1}) \subseteq B_{\rho_2}$ and $h^{-1}(B_{\rho_2}) \subseteq B_{\rho_3}$ (see Figure 2.3), consequently during the study of G and h^{-1} we can assume that the domains are in B_{ρ_1} and B_{ρ_2} respectively.

After gaining an estimate on R_2 we show that inequality (2.11) holds for $0 < |w| \le \rho_0$.

From this result a neighborhood in the z-plane can easily be obtained. The set $B_\varepsilon = \{z \in \mathbb{C} : |z| \leq \varepsilon\}$ is inside the attracting neighborhood of the fixed point of the map (2.9) if $h^{-1}(B_\varepsilon) \subseteq B_{\rho_0}$, i.e., B_ε is mapped inside the region of attraction of the map (2.10).

Here, we emphasize that for our calculations the only thing we need to know is the at most fourth-order terms of the function $G(z)$ and an $R_{10}|z|^5$–type estimate of the at least fifth-order terms of $G(z)$.

Until this point in the subsection we described the method for a general $F(u)$. Now, we turn our attention to the specific $F_2(u)$ from (2.5).

The main results of this section are the following two propositions. We prove only Proposition 11 as the whole argument can be repeated to get an attracting neighborhood when only $a \in [1.95, 2]$ is assumed. The differences appear only in concrete values in the given estimates. Details of Proposition 10 can be found on our website [19].

Proposition 10. *For all fixed* $a \in [1.95, 2]$*, the set* $\{z \in \mathbb{C} : |z| \leq 0.013\}$ *belongs to the basin of attraction of the fixed point* 0 *of* $G(z)$*.*

Proposition 11. *For all fixed* $a \in [1.995, 2]$*, the set* $\{z \in \mathbb{C} : |z| \leq 0.014\}$ *belongs to the basin of attraction of the fixed point* 0 *of* $G(z)$*.*

Proof. Throughout the proof we suppose $a \in \mathcal{I}_0 = [a_0 - \beta_0, a_0]$, where $\beta_0 = 0.005$ and $a_0 = 2$. In our calculations we use symbolic computation and built-in interval arithmetic tools of Wolfram Mathematica v. 11.

2.3.1 Estimation of the lower-order terms in G, h and h^{-1}

For polynomial estimations we need the following notion in order to handle together the terms of the same order. Let a complex polynomial

$$\mathcal{P}(z, \bar{z}) = \sum_{\substack{i+j=0 \\ i,j \in \mathbb{N}_0}}^{N} p_{ij} z^i \bar{z}^j$$

and a real polynomial

$$\mathcal{P}^{max}(r) = \sum_{n=0}^{N} p_n r^n$$

be given. If for the coefficients the inequality

$$\sum_{\substack{i+j=n \\ i,j \in \mathbb{N}_0}} |p_{ij}| \leq p_n$$

holds for every $n \in \{0, \dots, N\}$ then we say that $\mathcal{P}^{max}(r)$ is an *estimate by order* of $\mathcal{P}(z, \bar{z})$. Clearly, if $\mathcal{P}^{max}(r)$ is an estimate by order of $\mathcal{P}(z, \bar{z})$ then

$$|\mathcal{P}(z, \bar{z})| \leq \mathcal{P}^{max}(|z|) \tag{2.15}$$

for all $z \in \mathbb{C}$. The example $|z^2| < 1 + |z|^4$ shows that (2.15) may hold without being an estimate by order.

We remark that the estimate by order is essentially the same as the method of majorization for analytic functions, and it appears, e.g., in the proof of the Cauchy–Kovalevskaya Theorem (see Section 2.2.2 in [20]).

Throughout this subsection we need to estimate the coefficients of the lower-order terms in G, h and h^{-1} such that these estimates are independent of $a \in \mathcal{I}_0$. We use interval arithmetic tools to compute them for $a \in \mathcal{I}_0$.

In our particular case the function $G(z)$ can be written in the form

$$G(z) = \lambda z + \sum_{\substack{i+j=2 \\ i,j \in \mathbb{N}_0}} \frac{G_{ij}}{i!j!} z^i \bar{z}^j,$$

since G has only at most second-order terms. Furthermore, we use (2.13) and (2.14). The expressions of G_{ij}, h_{ij} and $\tilde{h}_{ij}$ can be determined explicitly (see previous subsection) and can be found in the Appendix, see Section 4.1. We look for constants satisfying the inequalities

$$\sum_{\substack{i+j=2 \\ i,j \in \mathbb{N}_0}} \frac{|G_{ij}|}{i!j!} \leq G_2, \qquad \sum_{\substack{i+j=2 \\ i,j \in \mathbb{N}_0}} \frac{|h_{ij}|}{i!j!} \leq h_2,$$

$$\sum_{\substack{i+j=n \\ i,j \in \mathbb{N}_0}} \frac{|\tilde{h}_{ij}|}{i!j!} \leq \tilde{h}_n \qquad \frac{|h_{30}|}{6} + \frac{|h_{12}|}{2} + \frac{|h_{03}|}{6} \leq h_3,$$

for $a \in \mathcal{I}_0$ and $n \in \{2, 3, 4\}$. With interval arithmetic it can be shown that $G_2 = 3.47$, $h_2 = 2.9$, $h_3 = 4.7$, $\tilde{h}_2 = h_2 = 2.9$, $\tilde{h}_3 = 8.2$ and $\tilde{h}_4 = 30$ satisfy the requirements. From the definition of these constants we obtain the following estimates

$$|G(z)| \leq G^{max}(|z|), \qquad |h(w)| \leq h^{max}(|w|),$$

$$\left|h_0^{-1}(z)\right| \leq \tilde{h}_0^{max}(|z|), \qquad \left|h^{-1}(z)\right| \leq \tilde{h}^{max}(|z|),$$

where

$$
\begin{aligned}
G^{max}(r) &= r + G_2 r^2, \\
h^{max}(r) &= r + h_2 r^2 + h_3 r^3, \\
\tilde{h}_0^{max}(r) &= r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4, \\
\tilde{h}^{max}(r) &= r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + R_{30} r^5,
\end{aligned}
\tag{2.16}
$$

provided that R_{30} satisfies $|R_3| \leq R_{30}|z|^5$ for $z \in B_{\rho_2}$. We will determine R_{30} later. We emphasize that with the polynomials in (2.16) we estimate by order.

From the definition of h_2 and h_3 we also get

$$
|w| - h_2|w|^2 - h_3|w|^3 \leq |h(w)|. \tag{2.17}
$$

Consequently, assuming $|w| \leq \rho$, we can obtain

$$
|w| \leq \eta(\rho)|h(w)| \tag{2.18}
$$

with

$$
\eta(\rho) = \frac{1}{1 - h_2\rho - h_3\rho^2}.
$$

We choose $\rho_0 = 0.015$, $\rho_1 = h^{max}(\rho_0)$, $\rho_2 = G^{max}(\rho_1)$ and from (2.17) it is clear that B_{ρ_2} can not be mapped outside of the circle with radius 0.018, consequently this value is a suitable choice for ρ_3.

2.3.2 The definition of h^{-1}

Set $\delta_1 = \frac{1}{9}$ and $\delta_2 = \frac{1}{16}$. We show that h is injective in $\overline{B}_{\delta_1} \subseteq \mathbb{C}$, and h^{-1} is defined on $\overline{B}_{\delta_2}$. Let $z \in \mathbb{C}$, $a \in \mathcal{I}_0$ be fixed, and denote $h^*_{a,z} : \mathbb{C} \ni w \mapsto w + z - h(w) \in \mathbb{C}$. With this notation $h^*_{a,z} = w$ if and only if $h(w) = z$.

$$
\begin{aligned}
|h^*_{a,z}(w_1) - h^*_{a,z}(w_2)| &= |w_1 - h(w_1) - w_2 + h(w_2)| \\
&\leq |w_1 - w_2| \Big(h_2(|w_1| + |w_2|) + h_3 \left(|w_1|^2 + |w_1| \cdot |w_2| + |w_2|^2 \right) \Big).
\end{aligned}
$$

If $|w_1|, |w_2| \leq \delta_1$ and $|z| \leq \delta_2$, then

$$
|h^*_{a,z}(w_1) - h^*_{a,z}(w_2)| \leq |w_1 - w_2| \left(2\delta_1 h_2 + 3\delta_1^2 h_3 \right)
$$

and

$$
|h^*_{a,z}(w)| \leq \delta_2 + \delta_1^2 h_2 + \delta_1^3 h_3.
$$

It can be checked that $2\delta_1 h_2 + 3\delta_1^2 h_3 < 1$ and $\delta_2 + \delta_1^2 h_2 + \delta_1^3 h_3 < \delta_1$, so the map $h^*_{a,z}$ is a contraction on $\overline{B}_{\delta_1}$. Consequently for every $z \in \overline{B}_{\delta_2}$ there exists a unique $w = w(z) \in \overline{B}_{\delta_1}$ such that $h(w(z)) = z$, i.e., h^{-1} is defined on $\overline{B}_{\delta_2}$.

It is clear, that $\rho_0, \rho_3 < \delta_1$ and $\rho_1, \rho_2 < \delta_2$, where $\rho_0, \rho_1, \rho_2, \rho_3$ were chosen at the end of the previous subsection.

2.3.3 The estimation of R_3 – the higher-order terms in h^{-1}

Now, we turn our attention to the estimation of R_3 in (2.14), which consists of the fifth and higher-order terms of h^{-1}. We need an estimate $|R_3(z)| < R_{30}|z|^5$, assuming $|z| \leq \rho_2$. But first, we give an estimate of type $|R_3(h(w))| < R_{31}|w|^5$, assuming $|w| \leq \rho_3$ (see figure 2.3).

Using the definition of h_0^{-1} and h, we obatin

$$R_3(h(w)) = w - h_0^{-1}(h(w)) = \sum_{\substack{i+j=5 \\ i,j\in\mathbb{N}_0}}^{12} b_{3,ij} w^i \overline{w}^j,$$

where the coefficients $b_{3,ij} = b_{3,ij}(a)$ are complex. Consider the composition

$$\mathcal{R}_3(r) = \tilde{h}_0^{max}\left(h^{max}(r)\right) = \sum_{k=1}^{12} b_{3,k} r^k$$

of the real functions $h^{max}, \tilde{h}_0^{max}$. The coefficients $b_{3,k}$ are independent from a, and are obtained by sybolic computer calculation from the real polynomials h^{max} and $\tilde{h}_0^{max}$. Since we estimated by order with h^{max} and $\tilde{h}_0^{max}$, it is clear that

$$\sum_{\substack{i+j=k \\ i,j\in\mathbb{N}_0}} |b_{3,ij}| \leq b_{3,k}$$

holds for $a \in \mathcal{I}_0$ and $k \in \{5, 6, \ldots, 12\}$. Consequently,

$$|R_3(h(w))| \leq \sum_{\substack{i+j=5 \\ i,j\in\mathbb{N}_0}}^{12} |b_{3,ij}||w|^{i+j} \leq \sum_{k=5}^{12} b_{3,k}|w|^k \leq \sum_{k=5}^{12} b_{3,k}\rho_3^{k-5}|w|^5,$$

provided that $|w| \leq \rho_3$. Using (2.18) we obtain $|w| < \eta(\rho_3)|z|$ and

$$|R_3(z)| \leq \sum_{k=5}^{12} b_{3,k}\rho_3^{k-5}(\eta(\rho_3))^5|z|^5 \leq 1070|z|^5,$$

therefore, $R_{30} = 1070$ is a suitable choice.

2.3.4 The estimation of R_2 – the higher-order terms in $h^{-1}(G(h))$

Now we turn our attention to R_2, which estimates the at least fourth-order terms of $h^{-1}(G(h(w)))$. To obtain a better estimate, we handle the fourth-order terms (R_{24}) and the higher-order ones (R_{25}) separately, i.e., set $R_2 = R_{24} + R_{25}$.

As $h^{-1}(G(h(w)))$ is in normal form, it can be written in the form

$$h^{-1}(G(h(w))) = \lambda w + c_1 w^2 \overline{w} + \sum_{\substack{i+j=4 \\ i,j\in\mathbb{N}_0}} \frac{\beta_{ij}}{i!j!} w^i \overline{w}^j + R_{25},$$

where coefficients $c_1 = c_1(a)$ and $\beta_{ij} = \beta_{ij}(a)$ are complex, and $R_{25} = R_{25}(w, \overline{w}, a, c) = O(|w|^6)$. The fourth-order coefficients β_{ij} can be calculated explicitly and the formulas of $|\beta_{ij}|$ can be found in the Appendix, see Section 4.1. With interval arithmetic it can be shown, that

$$\sum_{\substack{i+j=4 \\ i,j\in\mathbb{N}_0}} \frac{|\beta_{ij}|}{i!j!} \leq 40,$$

consequently $|R_{24}| \leq 40|w|^4$.

As for the higher-order terms, we use h^{max}, G^{max} and $\tilde{h}^{max}$ from (2.16), similarly to the estimation of R_3. Consider the composition

$$\mathcal{R}_2(r) = \tilde{h}^{max}(G^{max}(h^{max}(r))) = \sum_{k=1}^{30} b_{2,k} r^k.$$

Since we estimate by order with the polynomials in (2.16), it is clear, that for $|w| \leq \rho_0$ we have

$$|R_{25}(w)| \leq \sum_{k=5}^{30} b_{2,k}(|w|)\rho_0^{k-4}|w|^4 \leq 90|w|^4.$$

Combining these two results, for $|w| \leq \rho_0$ we obtain

$$|R_2 \leq |R_{24}| + |R_{25}| \leq 130|w|^4,$$

and consequently, $R_{20} = 130$.

2.3.5 The attracting neighborhood

Now, with our previous estimate on R_2 we can finish our proof. Since

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| \leq |w| \left(\left||\lambda| + \tilde{c}_1 |w|^2\right| + R_{20}|w|^3\right),$$

where $\tilde{c}_1 = c_1 \frac{|\lambda|}{\lambda}$, we only need to prove

$$\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3 < 1 \tag{2.19}$$

for every $0 < |w| \leq \rho_0$ and $a \in \mathcal{I}_0$.

To this end, we show that with a suitable $R_4 > 0$ the inequality

$$\left||\lambda| + \tilde{c}_1|w|^2\right| \leq |\lambda| + (\operatorname{Re}\tilde{c}_1)|w|^2 + R_4|w|^3,$$

holds, or equivalently (by squaring the last inequality and idviding it by $|w|^3$)

$$0 \leq 2R_4|\lambda| - (\operatorname{Im}\tilde{c}_1)^2|w| + 2R_4(\operatorname{Re}\tilde{c}_1)|w|^2 + R_4^2|w|^3,$$

for every $|w| \leq \rho_0$ and $a \in \mathcal{I}_0$. For $a \in \mathcal{I}_0$ we obtain with interval arithmetic that $\operatorname{Re}\tilde{c}_1$ and $\operatorname{Im}\tilde{c}_1$ are negative, $|\operatorname{Re}\tilde{c}_1| \leq 2.1$, $|\operatorname{Im}\tilde{c}_1| \leq 3.5$ and $|\lambda| \geq 0.99$. It can be checked that for $|w| \leq \rho_0$ the choice $R_4 = 0.1$ will be suitable. Therefore, the left hand side of the inequality (2.19) can be written in the form

$$\begin{aligned}\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3 &\leq \left(|\lambda| + \operatorname{Re}\tilde{c}_1|w|^2\right) + (R_4 + R_{20})|w|^3 \leq \\ &\leq 1 + \left(\operatorname{Re}\tilde{c}_1 + (R_4 + R_{20})|w|\right)|w|^2,\end{aligned}$$

which is less than 1, provided

$$0 < |w| < \frac{-\operatorname{Re}\tilde{c}_1}{R_4 + R_{20}}.$$

Using the fact that $|\operatorname{Re}\tilde{c}_1| \geq 2$, we obtain

$$\frac{-\operatorname{Re}\tilde{c}_1}{R_4 + R_{20}} > \rho_0,$$

Therefore, inequality (2.19) holds for every $w \in \mathbb{C}$ with $0 < |w| < \rho_0$. Set $\varepsilon_G = 0.014$. From $|h^{-1}(z)| < \tilde{h}^{max}(|z|)$, the inequality $|z| < \varepsilon_G$ implies $|w| = |h^{-1}(z)| < \rho_0$, so the proof of Proposition 11 is complete. □

To obtain a neighborhood in the shifted real coordinate system we use $z = \langle p, v\rangle$, just like in the linearized case. Note that, the set $\{z \in \mathbb{C} : |z| \leq \varepsilon_G\}$ will be transformed into an ellipse-shaped neighborhood in the v-plane.

2.4 Graph representation

In the computer assisted part, we follow the method in [5]. In this subsection for $a \in \left(\frac{3}{2}, 2\right]$ we associate the map (2.5) with a directed graph reflecting the behavior of the map up to a given resolution. Therefore, we can derive properties of our dynamical system through the study of this graph. More precisely our aim is to show with the help of this graph, that every point of $S \setminus \mathcal{M}$ enters the attracting neighborhood $\mathcal{M}$ of the nontrivial fixed point constructed in the previous subsections.

Let D be a subset of $\mathbb{R}^n$. A set $\mathfrak{S}$ is called a *cover* of D if the elements of $\mathfrak{S}$ are subsets of $\mathbb{R}^n$ and $\cup_{\mathfrak{s}\in\mathfrak{S}}\mathfrak{s} \supseteq D$. Let a map $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^n$, a subset $D \subseteq D_f$ and a cover $\mathfrak{S}$ of D be given. The directed graph $G(V, E)$ is called a *graph representation* of f on D with respect to $\mathfrak{S}$ if there exists a bijection $\iota : V \to \mathfrak{S}$ such that

$$f(\iota(u) \cap D) \cap \iota(v) \cap D \neq \emptyset \;\Rightarrow\; (u, v) \in E \tag{2.20}$$

for all $u, v \in V$.

The meaning of the implication (2.20) is the following. If we can get with the map f from an element $\mathfrak{s}_1$ of the cover $\mathfrak{S}$ to an other (possibly the same) element $\mathfrak{s}_2$ of $\mathfrak{S}$, i.e., there exists $x \in \mathfrak{s}_1$ and $y \in \mathfrak{s}_2$ such that $f(x) = y$, then there is an edge between the vertices corresponding to the two sets, more precisely $(u, v) \in E$ for $\mathfrak{s}_1 = \iota(u)$ and $\mathfrak{s}_2 = \iota(v)$. The reverse implication is not necessarily true, namely if there is a directed edge between the vertices u and v, it is not sure there exists $x \in \mathfrak{s}_1$ such that $f(x) \in \mathfrak{s}_2$, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are the corresponding sets to u and v.

It is easy to see that (2.20) can be reformulated as follows. For every $u \in V$

$$f(\iota(u) \cap D) \subseteq \bigcup_{v \in V_u} \iota(v) \cap D, \tag{2.21}$$

where V_u denotes the set of vertices, into which there is an edge from u in graph G. So the sets corresponding to vertices in V_u need to form a cover of the image of $\iota(u)$. From this it can be seen the graph representation can be regarded as some kind of upper estimate of the original map f. The finer the cover is, the better the graph representation approximates the map. Therefore, if we would like to determine the possible location of the image of a point $P \in \mathbb{R}^n$ under f, we can do it with the help of the graph, since $f(P) \in \cup_{v\in V_u}\iota(v) \cap D$ for $P \in \iota(u)$. This means iterating f the point P can move forward only along the edges, i.e., it can move from an element of the cover to an other one only if there is a directed edge between the two vertices corresponding to them. Consequently, we can draw conclusions regarding the possible location of the iterates of a point studying only the graph. In the following we take the liberty to handle the elements of the cover as vertices and vice versa, omitting the use of ι.

The construction of the graph representation in our case is the following. For a fixed $k \in \mathbb{N}$ we divide the unit square $[0,1]^2$ parallel to the sides into small congruent closed squares with side length $r = 2^{-k}$. According to Proposition 8 we only need to consider the squares lying in $\left[\frac{1}{16}, \frac{7}{8}\right]^2$. The cover $\mathfrak{S}$ of S consists of these sets. The small squares correspond to the vertices of the graph. As for the edges, for every small square $\mathfrak{s}$ we construct a rectangle with reliable numerical methods which contains $f(\mathfrak{s})$. If the rectangle intersects the small square $\mathfrak{s}_2$, then there is an edge from $\mathfrak{s}$ to $\mathfrak{s}_2$. It is clear, that this construction satisfies relation (2.21). Note that we considered only that part of the rectangle obtained by the numerical method, which lies inside the square $\left[\frac{1}{16}, \frac{7}{8}\right]^2$, but this is not a restriction, since the studied set $\tilde{S}$ is invariant under the map (2.5), so getting out of the unit square is only the consequence of the numerical method and the 'upper estimate' nature of the graph representation. Note also that, instead of map (2.5) we use the second iterate of it, since the formula is still compact enough not to cause big overestimation in interval arithmetic and it considerably speeds up the calculations.

In the thesis we suppose a graph is always finite. A graph is *strongly connected* if there are uv and vu (directed) paths for every $u \neq v$ vertices of the the graph. We use the following decomposition of a directed graph (see [21]).

Proposition 12. *The vertices of a directed graph can be classified and the classes can be ordered such that*

- *the subgraphs spanned by the classes are strongly connected, and*
- *for every directed edge between these classes, the class of the tail of this edge precedes (in the order) the class of the head of it,*

moreover, the partition above is unique.

The aforementioned classes are called the *strongly connected components* (*SCC*) of the graph. A strongly connected component is called *nonessential* if it consists of one vertex without loop. Otherwise we call it *essential.*

From the graph representation and from Proposition 12 it is clear what happens to an arbitrary point of S during the iteration of f. Starting from a small square containing this point it moves to an other (possibly the same) small square along a directed edge. If we are not in an essential SCC we step out of this small square not returning to here afterwards because of the ordering of the SCCs. If we are in an essential SCC it can happen, that the point stays here forever, or the point steps out of this SCC, but in this case it can not return to this SCC any more.

Since during the partition we obtain finitely many small squares and consequently the graph is finite, it is straightforward that for every point of S there exists an essential strongly connected component, which the point enters during the iteration and never leaves it. So it is true for every $P \in S$ that it enters an essential SCC with finitely many steps and stays here afterwards, therefore, we only need to study the essential SCCs.

Our aim is to show that those essential SCCs, in which the points of S can get stuck, are in the attracting neighborhood $\mathcal{M}$ of the fixed point u_A, which neighborhood was constructed analytically in the previous subsections. It is important to note that, it is possible for some essential SCC that none of the points of S can get stuck here. Actually, this would be the case close to origin, since this fixed point is a saddle; that shows the necessity of Proposition 8 and $\tilde{S}$.

As a next step we refine the partition as follows. We divide the small squares into four smaller squares, that have a side length half as long as before, determine their images with reliable numerical methods and construct the SCCs again. Because of the inclusion isotonicity property ($I_1 \subseteq I_2 \Rightarrow F(I_1) \subseteq F(I_2)$, where F is the interval-extension of f, see [15]) of interval arithmetic if there is an edge between two new small squares, then there must be an edge between their predecessors with the same orientation. We come to the conclusion that during the refinement, an essential SCC can arise only from a former essential SCC, therefore it is really enough to trace merely the essential SCCs. Note that, with the refinements the graph representation becomes a more and more accurate approximation of the represented map, so an essential SCC can fall apart into smaller pieces, and it even can happen that none of the small squares born from a former essential SCC compose a new essential SCC, i.e., this cycle in the graph is only the consequence of the 'upper estimate' nature of the graph representation. We continue these refinement steps, until all the remaining SCCs are inside the region of attraction of the fixed point obtained in the previous subsection. If it occurs in finitely many steps Conjecture 1 is proven for $d = 1$.

Finally, instead of checking after every refinement, whether the remaining SCCs are in the analytically constructed attracting neighborhood $\mathcal{M}$, we can remove all the small squares lying entirely in $\mathcal{M}$ before the first refinement. In that case for a fixed a and $d = 1$, Conjecture 1 will be proven if the set of the new SCCs will be empty after a refinement. We show the correctness of this method.

- If we erase a vertex which is a nonessential SCC, it has no effect at all compared to our former method (when checking after every refinement).

- If we remove a whole essential SCC, it also has no substantial effect because during the checking it always would be in the attracting neighborhood.

- The only significant change happens, when we erase only a proper subgraph of an essential SCC. Consider such an SCC and color blue the vertices we want to remove (and do not remove them yet). Delete the directed edges stemming from a blue vertex, then form the SCCs (referred to as new SCCs later on) of the new graph and order them such that the blue vertices are at the end of the ordering. (It can be done, since there are no edges from colored to uncolored vertices.)

  - An uncolored vertex can be in a new essential SCC; in that case they remain under study after the removal of the blue vertices, just as they would be in the original method.
  - However, if an uncolored vertex is a nonessential SCC it will be erased (as we keep only the essential SSCs), unlike in the method without deleting the vertices of the attracting neighborhood, but this is not a problem because every point of this vertex enters a new SCC or a blue vertex (because of the ordering) in finitely many steps, so this vertex really can be deleted.

Note that the aforementioned method can be regarded as a proof, since the graph problems are finite, so the computer can work on them punctually, moreover the method used during the construction of edges was executed with reliable numerical methods, therefore if we have sufficiently much time, then we could reconstruct by hands the parts which were executed by the computer, and we would come to the same conclusion if our estimates are as good as those of the computer.

2.5 Completion of the proof

In the previous subsections we obtained an attracting neighborhood and then a method to prove the global stability of the nontrivial fixed point for a fixed $a \in [1.5, 2]$. In this subsection we show, how to modify our method to handle not only a single value of $[1.5, 2]$ but a small subinterval of that, instead.

Let $[a] = [a^-, a^+] \subseteq [1.5, 2]$ be a fixed small interval. First, we need a new attracting neighborhood $\mathcal{M}([a])$ such that for every $a \in [a]$, the attracting neighborhood $\mathcal{M}(a)$ contains this set, i.e., $\cap_{a\in[a]}\mathcal{M}(a) \supseteq \mathcal{M}([a])$. To this end, we need to take into consideration the displacement of the fixed point and the change in the size of the neighborhood. Secondly, during the construction of the edges of the graph representation the number a have to be replaced by the interval $[a]$, since (2.5) and consequently its second iterate

	parameter	size of slices	shape of $\mathcal{M}$	parameters of $\mathcal{M}$
$\mathcal{I}_1$	$[1.5, 1.95]$	2^{-10}	rectangle	$7 \cdot 10^{-3}$
$\mathcal{I}_2$	$[1.95, 1.995]$	2^{-13}	ellipse	$\varepsilon_G = 0.0138$
$\mathcal{I}_3$	$[1.995, 2]$	2^{-16}	ellipse	$\varepsilon_G = 0.0146$

Table 2.1: The partition of the parameter range

depends on a. So while studying the image of a small square $\mathfrak{s}$, we need to study it for every $a \in [a]$, i.e., we take a set of small squares during the estimation of the image set such that they cover $F_2^2(\mathfrak{s})$ for every $a \in [a]$.

We divide the interval $[1.5, 2]$ into subintervals $\mathcal{I}_1 = [1.5, 1.95]$, $\mathcal{I}_2 = [1.95, 1.995]$ and $\mathcal{I}_3 = [1.995, 2]$, then divide further these intervals into smaller subintervals with length $2^{-10}, 2^{-13}$, and 2^{-16} respectively (see Table 2.1).

For small intervals in $\mathcal{I}_1$ we use the linearized map and the square-shaped neighborhoods with side length $2\varepsilon(a)$ (Proposition 9). It is easy to see the size of this set and the location of the fixed point also change as a changes. However, it can be shown that $\varepsilon(a) \geq 0.007$ for every $a \in [1, 1.95]$, so considering this value fixed, we need to handle only the displacement of the fixed point.

For small intervals in $\mathcal{I}_2$ and $\mathcal{I}_3$ we use the bifurcation normal form, therefore, the size of the ellipse-shaped neighborhood is fixed (Propositions 10 and 11), so we only need to consider the displacement of the fixed point.

2.6 The algorithm

Table 2.2 shows the pseudo code of the algorithm. The whole program code, and the outputs can be found on link [19].

During the calculation of edges of the graph representation we use the second iterate of the original map (2.5), i.e.,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} a u_2 (1-u_1) \\ a^2 u_2 (1-u_1)(1-u_2) \end{pmatrix}. \tag{2.22}$$

Regarding the examined parameter domain $[a^-, a^+]$ and the sides $\left[x_i^-, x_i^+\right]$ and $\left[y_i^-, y_i^+\right]$ of the squares as intervals, simply, we could use interval arithmetic tools, such as IntLab to compute the image of a small square. However, the map is quite simple, so we can accelerate this method as follows. Notice that $x_2^- = a^- y_0^- \left(1 - x_0^+\right)$, so we only need to force the computer to use a downward rounding in order to guarantee that the obtained

algorithm Log2d
  $V \leftarrow$ the initial partition with $r = 2^{-10}$
  $E \leftarrow$ construct the edges with reliable numerical method
  $C \leftarrow$ determine the SCCs of directed graph $G(V, E)$
  remove vertices of the nonessential SCCs from V
  remove vertices of the SCC at the origin from V **if** possible
  remove the initial attracting neighborhood from V
  repeat
    $V \leftarrow$ refine(V) by setting $r \leftarrow \frac{r}{2}$
    $E \leftarrow$ construct the edges with reliable numerical method
    $C \leftarrow$ determine the SCC of directed graph $G(V, E)$
    remove the nonessential SCCs from V
  until $|V| = \emptyset$
end algorithm

Table 2.2: Algorithm for the two-dimensional logistic map

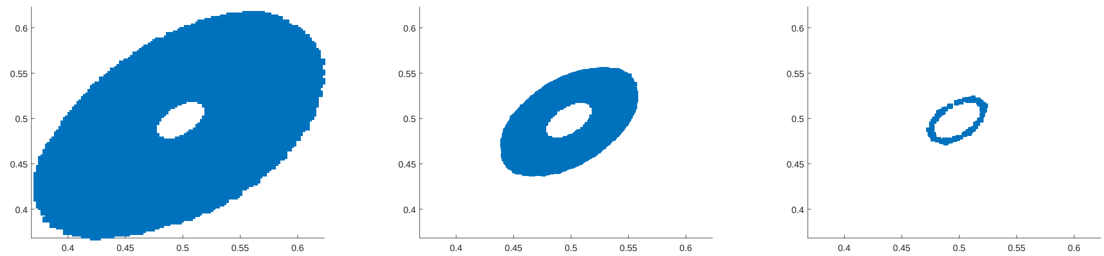


Figure 2.4: The remaining vertices before the first, third and fifth refinements for $[a_-, a_+] = \left[2 - \frac{61}{2^{13}}, 2 - \frac{60}{2^{13}}\right]$

below estimate is really not larger than the possible first coordinates of the image of any point from the initial square. Similarly we can estimate x_2^+, but at this time we use upward rounding. As for the y_2^- and y_2^+, remark that instead of $y_0^-(1 - y_0^+)$ in y_2^- we can use $\min\left\{y_0^-\left(1 - y_0^-\right), y_0^+\left(1 - y_0^+\right)\right\}$ because y_0 denotes the same number in expression (2.22), and the function $x(1-x)$ is monotone on intervals which do not contain $\frac{1}{2}$ in the interior (and it is satisfied in the partition). In y_2^+ we replace the minimum by maximum, after that we proceed just like in the case of x_2.

We implemented our program in MATLAB, and used the built-in *digraph* function to construct the directed graph from the edge list and the *conncomp* function to divide the graph into strongly connected components.

Now, we can run our algorithm with parameters summarized in Table 2.1. As an example, for the parameter slice $\left[2 - \frac{61}{2^{13}}, 2 - \frac{60}{2^{13}}\right]$ we show the evolution of the remaining SCCs during the first 4 iterations on Figure 2.4.

The program ran successfully, therefore we established the nontrivial fixed point is globally attracting for $a \in [1.5, 2]$. Combining this with Proposition 7 and the asymptotic stability for $a \in (1, 2]$ the proof of Conjecture 1 is completed for $d = 1$.

Chapter 3

Global stability for the three-dimensional logistic map

3.1 Preliminaries

In this subsection we study the equations (1.1) and (1.2) for $d = 2$, so we consider

$$x_{n+1} = ax_n(1 - x_{n-2}), \tag{3.1}$$

which is equivalent to the three-dimensional map

$$F_3 : \mathbb{R}^3 \ni u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_3 \\ au_3(1 - u_1) \end{pmatrix} \in \mathbb{R}^3. \tag{3.2}$$

Now, we describe the dynamics of map (3.2) in the positive octant for $a > 0$. Introduce

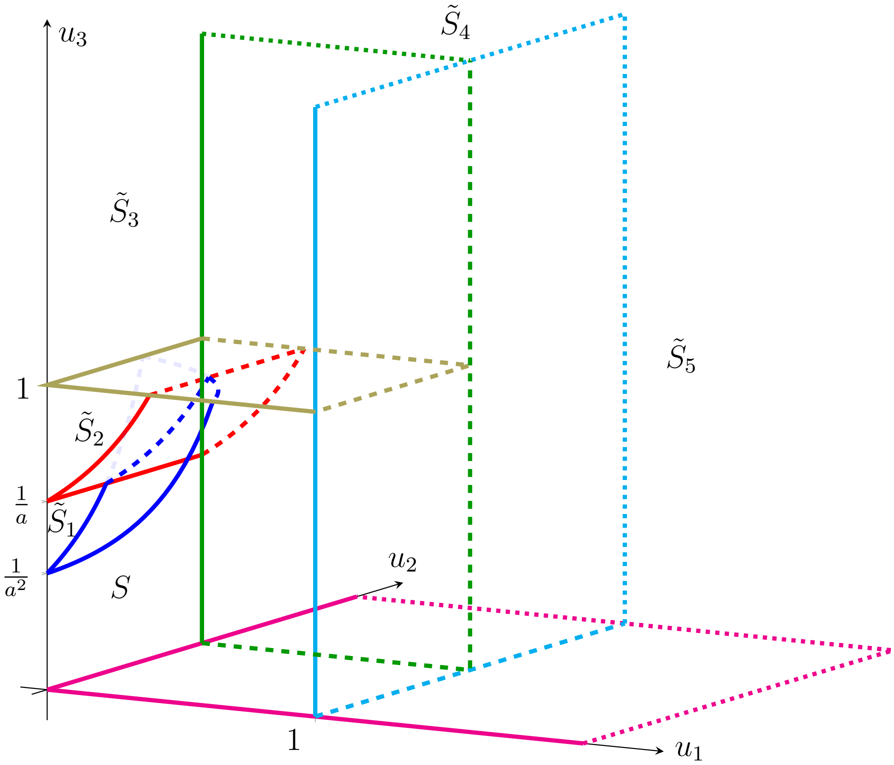


Figure 3.1: The subdivision of the positive octant. The union of the colored surfaces is the set $\tilde{S}_0$ forming the boundaries of the sets $S, \tilde{S}_1, \ldots, \tilde{S}_5$.

the following disjoint sets depending on a (see Figure 3.1).

$$
\begin{aligned}
S &= \left\{u \in \mathbb{R}^3 :\ u_1, u_2 \in [0,1),\quad u_3 \in (0,1),\quad au_3(1-u_1) < 1,\quad a^2u_3(1-u_1)(1-u_2) < 1\right\} \\
\tilde{S}_0 &= \{(u_1, u_2, 0) :\ u_1, u_2 \geq 0\} \\
&\quad \cup \{(1, u_2, u_3) :\ u_2 \geq 0,\quad u_3 > 0\} \\
&\quad \cup \{(u_1, 1, u_3) :\ u_1 \in [0,1),\quad u_3 > 0\} \\
&\quad \cup \{(u_1, u_2, 1) :\ u_1, u_2 \in [0,1)\} \\
&\quad \cup \{(u_1, u_2, u_3) :\ u_1, u_2 \in [0,1),\quad u_3 \in (0,1),\quad au_3(1-u_1) = 1\} \\
&\quad \cup \left\{u \in \mathbb{R}^3 :\ u_1, u_2 \in [0,1),\quad u_3 \in (0,1),\quad au_3(1-u_1) < 1,\quad a^2u_3(1-u_1)(1-u_2) = 1\right\} \\
\tilde{S}_1 &= \left\{u \in \mathbb{R}^3 :\ u_1, u_2 \in [0,1),\quad u_3 \in (0,1),\quad au_3(1-u_1) < 1,\quad a^2u_3(1-u_1)(1-u_2) > 1\right\} \\
\tilde{S}_2 &= \{(u_1, u_2, u_3) :\ u_1, u_2 \in [0,1),\quad u_3 \in (0,1),\quad au_3(1-u_1) > 1\} \\
\tilde{S}_3 &= \{(u_1, u_2, u_3) :\ u_1, u_2 \in [0,1),\quad u_3 > 1\} \\
\tilde{S}_4 &= \{(u_1, u_2, u_3) :\ u_1 \in [0,1),\quad u_2 > 1,\quad u_3 > 0\} \\
\tilde{S}_5 &= \{(u_1, u_2, u_3) :\ u_1 > 1,\quad u_2 \geq 0,\quad u_3 > 0\}
\end{aligned}
$$

Clearly, $\mathbb{R}^3_+ = S \cup \tilde{S}_0 \cup \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3 \cup \tilde{S}_4 \cup \tilde{S}_5$, moreover S and $S \cup \tilde{S}_0$ correspond to S^2 and S^2_0, respectively, see (1.4) and (1.3). Furthermore, $S = [0,1)^2 \times (0,1)$ and $\tilde{S}_1 = \tilde{S}_2 = \emptyset$ for $a \in (0,1]$. Recall that $a_0 = \frac{\sqrt{5}+1}{2}$ for $d = 2$.

Proposition 13. *For all $a > 0$ and $i \in \{1,2,3,4\}$ we have*

$$F_3^7(\tilde{S}_0) = \{(0,0,0)\}\,, \quad F_3(\tilde{S}_5) \cap \mathbb{R}^3_+ = \emptyset \quad \textit{and} \quad F_3(\tilde{S}_i) \subseteq \tilde{S}_{i+1}.$$

Furthermore, $F_3(S) \subseteq S$ for $a \in (1, a_0]$.

Proof. From the definition of $F_3(u)$ it is obvious that $F_3^7(\tilde{S}_0) = \{(0,0,0)\}$. It is also straightforward to check the relations $F_3(\tilde{S}_5) \cap \mathbb{R}^3_+ = \emptyset$ and $F_3(\tilde{S}_i) \subseteq \tilde{S}_{i+1}$ for $i \in \{1,2,3,4\}$.

For the invariance, introduce the notation $\hat{u} = F_3(u)$ with $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$. We only need to check that $\hat{u}$ satisfies the inequality

$$a^2 \hat{u}_3 \prod_{k=1}^{2} (1 - \hat{u}_k) = a^3 u_3 \prod_{k=1}^{3} (1 - u_k) < 1 \tag{3.3}$$

because all the other conditions are guaranteed by the definition of S.

Since the inequality

$$a^2 u_3 \prod_{k=1}^{2} (1 - u_k) < 1$$

holds for $u \in S$ we obtain that

$$a^3 u_3 \prod_{k=1}^{3} (1 - u_k) = a\Big(a^2 u_3 \prod_{k=1}^{2} (1 - u_k)\Big)(1 - u_3) < a(1 - u_3).$$

Consequently, (3.3) holds for $u_3 \geq A$.

For $a \in (0,1]$ the point $u \in S$ satisfies $u_3 > 0 \geq A$, so for $u_3 < A$ we can assume $a \in (1, a_0]$. In this case $A \in \left(0, \frac{1}{2}\right]$ and the map $x \mapsto x(1-x)$ is strictly increasing on $[0, A]$. Thus, for $u_3 < A$ with $a \in (1, a_0]$, we obtain that

$$a^3 u_3 \prod_{k=1}^{3} (1 - u_k) \leq a^3 u_3 (1 - u_3) < a^3 A (1 - A) = a^2 - a.$$

Since $x \mapsto x^2 - x$ is strictly increasing on $[1, \infty)$ and $a_0^2 - a_0 = 1$, we showed that (3.3) holds also for $u_3 < A$. □

Because of Propositions 13 and 2 we can assume that $u \in S$ and $a \in (1, a_0]$ in the rest of the section. We divide S into eight subsets with planes $u_1 = A$, $u_2 = A$, $u_3 = A$, and

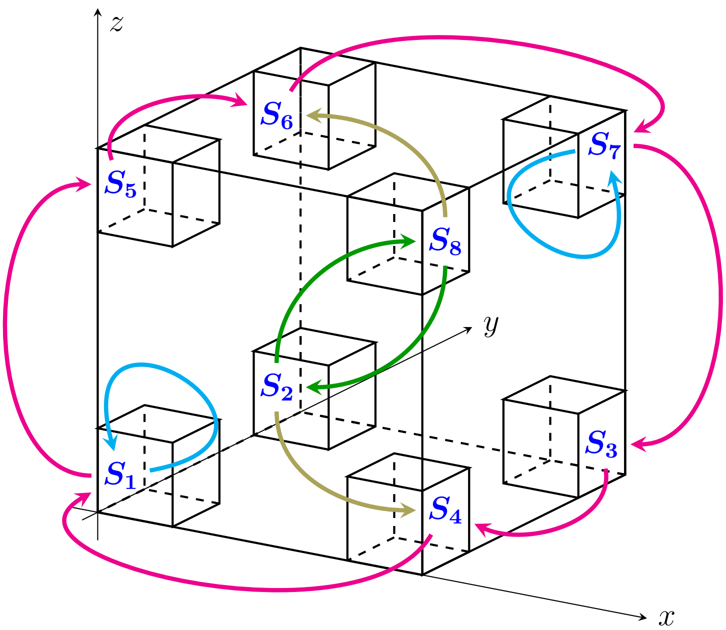


Figure 3.2: The dynamics in $S \setminus u_A$. The sets S_i are symbolized by a smaller cube for the sake of transparency.

introduce the following sets.

$$\begin{aligned}
S_1 &= \{u \in S: \ u_1 \le A, \ \ u_2 \le A, \ \ u_3 < A\}\\
S_2 &= \{u \in S: \ u_1 < A, \ \ u_2 > A, \ \ u_3 < A\}\\
S_3 &= \{u \in S: \ u_1 \ge A, \ \ u_2 > A, \ \ u_3 \le A\}\\
S_4 &= \{u \in S: \ u_1 > A, \ \ u_2 \le A, \ \ u_3 \le A\}\\
S_5 &= \{u \in S: \ u_1 \le A, \ \ u_2 < A, \ \ u_3 \ge A\}\\
S_6 &= \{u \in S: \ u_1 < A, \ \ u_2 \ge A, \ \ u_3 \ge A\}\\
S_7 &= \{u \in S: \ u_1 \ge A, \ \ u_2 \ge A, \ \ u_3 > A\}\\
S_8 &= \{u \in S: \ u_1 > A, \ \ u_2 < A, \ \ u_3 > A\}
\end{aligned}$$

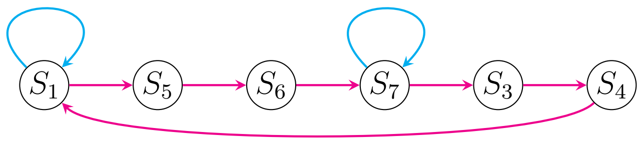
Clearly, $S = \bigcup_{i=1}^{8} S_i \cup \{u_A\}$. For given x_0, x_1, x_2 the sequence $(x_n)_{n=0}^{\infty}$, where x_n is defined by (3.1) for $n > 2$, corresponds to the three-dimensional sequence $(u^n)_{n=0}^{\infty}$ with

$$u^0 = (x_0, x_1, x_2), \quad u^n = F_3^n(u^0) = (x_n, x_{n+1}, x_{n+2}), \quad n \in \mathbb{N}.$$

Proposition 14. *Let $a \in (1, a_0]$ and the sequence $(u^n)_{n=0}^{\infty}$ in S be given by $u^0 \neq u_A$ and $u^n = F_3^n(u^0)$, $n \in \mathbb{N}$.*

(i) The sequence $(u^n)_{n=0}^{\infty}$ *follows the transition graph given in Figure 3.2.*

(ii) If $(u^n)_{n=0}^{\infty}$ *does not converge to* u_A *then there exists an* $n_0 \in \mathbb{N}$ *such that* $u^{n_0} \in S_1$*,* $(u^n)_{n=n_0}^{\infty}$ *follows the transition graph*



and $(u^n)_{n=n_0}^{\infty}$ *does not eventually stay in* S_1 *or* S_7*.*

Proof. In order to show (i), observe the following transitions, see Figure 3.2.

- For $u \in S_1$ we obtain $\hat{u} = F_3(u)$ with $\hat{u}_1 \leq A$ and $\hat{u}_2 < A$. Therefore, $S_1 \rightarrowtail \{S_1, S_5\}$, that is, $\hat{u} \in S_1$ or $\hat{u} \in S_5$.
- For $u \in S_2$ we have $\hat{u}_1 > A$ and $\hat{u}_2 < A$. Thus, $S_2 \rightarrowtail \{S_4, S_8\}$.
- For $u \in S_3$ we get $\hat{u}_1 > A$, $\hat{u}_2 \leq A$ and $\hat{u}_3 = au_3(1 - u_1) \leq aA(1 - A) = A$, so $S_3 \rightarrowtail S_4$.
- For $u \in S_4$ we have $\hat{u}_1, \hat{u}_2 \leq A$ and $\hat{u}_3 < A$. Consequently, $S_4 \rightarrowtail S_1$.
- For $u \in S_5$ we obtain $\hat{u}_1 < A$, $\hat{u}_2 \geq A$ and $\hat{u}_3 \geq A$. Hence, $S_5 \rightarrowtail S_6$.
- For $u \in S_6$ we get $\hat{u}_1, \hat{u}_2 \geq A$ and $\hat{u}_3 > A$. Consequently, $S_6 \rightarrowtail S_7$.
- For $u \in S_7$ we have $\hat{u}_1 \geq A$ and $\hat{u}_2 > A$. Therefore, $S_7 \rightarrowtail \{S_3, S_7\}$.
- For $u \in S_8$ we have $\hat{u}_1 < A$ and $\hat{u}_2 > A$, so $S_8 \rightarrowtail \{S_2, S_6\}$.

We obtain that there is a cycle $\mathcal{C}_1 = (S_1 \rightarrowtail S_5 \rightarrowtail S_6 \rightarrowtail S_7 \rightarrowtail S_3 \rightarrowtail S_4 \rightarrowtail S_1)$. If a sequence $(u^n)_{n=0}^{\infty}$ enters the cycle $\mathcal{C}_1$ then the sequence never leaves $\mathcal{C}_1$. However, during one cycle along $\mathcal{C}_1$, the points of $(u^n)_{n=0}^{\infty}$ can spend more time in S_1 or S_7. Possibly, the sequence can get stuck and stay forever in S_1 or S_7. Furthermore, we have another cycle $\mathcal{C}_2 = (S_2 \rightarrowtail S_8 \rightarrowtail S_2)$. If a sequence steps out of the cycle $\mathcal{C}_2$ then it enters $\mathcal{C}_1$ and never returns to $\mathcal{C}_2$. Consequently, we only need to show that if a sequence $(u^n)_{n=0}^{\infty}$ gets stuck in S_1 or S_7, or in the cycle $\mathcal{C}_2$ then it converges to the fixed point u_A.

First, consider the case, when the sequence $(u^n)_{n=0}^{\infty}$ stays in $\mathcal{C}_2$. We can assume $x_0 < A$. We obtain a sequence $(x_n)_{n=0}^{\infty}$ such that $x_{2k} < A$ and $x_{2k+1} > A$, where $k \in \mathbb{N}_0$. Since

$$A < x_{2k+1} = ax_{2k}(1 - x_{2k-2}) = \frac{1}{1-A}x_{2k}(1 - x_{2k-2}),$$

we get

$$A(1-A) < x_{2k}(1-x_{2k-2}). \tag{3.4}$$

Introduce the function $s(x) = x(1-x)$. Since $s(x)$ is increasing on $\left[0, \frac{1}{2}\right]$ and $0 \leq x_{2k-2} < A < \frac{1}{2}$ we obtain $s(x_{2k-2}) < s(A)$. Combining it with (3.4) we get $x_{2k-2} < x_{2k}$. Consequently, $(x_{2k})_{k=0}^{\infty}$ converges monotonically to some $B \leq A$. Taking the limit of both sides in inequality (3.4), we obtain $A(1-A) \leq B(1-B)$. On the other hand $s(x)$ is increasing on $[0, A]$, so $B(1-B) \leq A(1-A)$, and $B = A$. The odd indexed subsequence also converges to A, since $x_{2k+1} = ax_{2k}(1-x_{2k-2}) \to aA(1-A) = A$. So in this case, the sequence $(u^n)_{n=0}^{\infty}$ converges to the nontrivial fixed point u_A.

If $(u^n)_{n=0}^{\infty}$ gets stuck in S_1 or S_7, then Proposition 3 provides the convergence. □

Now we assume $1 < a \leq \frac{4}{3}$ and show that for every $u^0 \in S$ the sequence $(u^n)_{n=0}^{\infty}$ converges to the nontrivial fixed point u_A. Combining this fact with the local asymptotic stability of the fixed point (see Section 3.2), Conjecture 1 is proven for $a \in \left(1, \frac{4}{3}\right]$ and $d = 2$.

Proposition 15. *If $a \in \left(1, \frac{4}{3}\right]$ and $u \in S$, then $\lim_{n\to\infty} F_3^n(u) = u_A$.*

Proof. It follows from Proposition 14 that we only need to consider the case when the sequence $(u^n)_{n=0}^{\infty}$, where $u_0 = u$ and $u^n = F_3^n(u)$, goes around the fixed point along the cycle $\mathcal{C}_1$, not getting stuck in S_1 or S_7. Without loss of generality we can assume that $u_0 \in S_1$ and Proposition 4 completes the proof, since $S_1 = S_-^2$. □

We remark that the technique of [22, 12] seems to work for (3.1) to prove global stability for $1 < a < \frac{14}{9}$. However, [22, 12] does not apply directly, some additional work is necessary. Note that Proposition 15 is essential in the sense that the case $a \approx 1$ can not be handled in the computer-aided part of the method. The two fixed points can be arbitrarily close to each other, so after some point they can not be handled efficiently by interval arithmetic tools. Moreover, if they get closer to each other, they can not even be distinguished in the floating point system.

In the rest of the section we assume $a \in \left(\frac{4}{3}, a_0\right]$.

3.2 An attracting neighborhood with linearization

For each fixed a, translating u_A into $0 \in \mathbb{R}^3$, i.e., introducing the new variable $v = u - u_A$, the shifted version

$$\mathbb{R}^3 \ni v \mapsto F_3(v + u_A) - u_A \in \mathbb{R}^3$$

of (3.2) can be written as

$$\mathbb{R}^3 \ni v \mapsto J_a v + f_a(v) \in \mathbb{R}^3, \tag{3.5}$$

where

$$J_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-a & 0 & 1 \end{pmatrix}, \quad f_a(v) = \begin{pmatrix} 0 \\ 0 \\ -av_1v_3 \end{pmatrix}.$$

The characteristic polynomial of J_a is

$$P(a,\lambda) = -\lambda^3 + \lambda^2 + 1 - a,$$

and the roots of $P(a,\lambda)$ are $\lambda_i = \lambda_i(a)$, where $i \in \{0,1,2\}$. Since $P(1,\lambda) = -\lambda^2(\lambda-1)$ and $P\left(\frac{31}{27},\lambda\right) = -\left(\lambda+\frac{1}{3}\right)\left(\lambda-\frac{2}{3}\right)^2$, it follows from the graph of $P(a,\lambda)$ that for $a \in \left(1,\frac{31}{27}\right]$ the polynomial $P(a,\lambda)$ has three real roots (counting multiplicity) and $|\lambda_i| < 1$ for $i \in \{0,1,2\}$. For $a > \frac{31}{27}$ the characteristic polynomial has a real root $\nu = \nu(a) = \lambda_0$ and two complex roots $\lambda_1 = \bar{\lambda}_2$. Denote λ the complex root with positive imaginary part, i.e., $\lambda = \lambda(a) = \lambda_1$. Formulas of ν and λ can be found in the Appendix, see Subsection 4.2.1.

From the graph of $P(a,\lambda)$ it follows that $\nu < 0$ and ν is a strictly decreasing function of a for $a > \frac{31}{27}$. Since $P(3,\lambda) = -(\lambda+1)(\lambda-1-i)(\lambda-1+i)$, it is clear that $|\nu| < 1$ for $a \in \left(\frac{31}{27},3\right)$. From Vieta's formulas it follows that

$$\nu + 2\operatorname{Re}\lambda = 1, \qquad 2\nu\operatorname{Re}\lambda + |\lambda|^2 = 0, \qquad \nu|\lambda|^2 = 1-a.$$

From the first two formulas we get

$$|\lambda|^2 = \nu^2 - \nu. \tag{3.6}$$

Combining the facts that $\mathbb{R} \ni s \mapsto s^2 - s \in \mathbb{R}$ is decreasing on $(-\infty,0]$, and $\nu(a)$ is a decreasing function of a, we obtain $|\lambda(a)|$ is a strictly increasing function of a. From (3.6) and the third Vieta's formula, it follows that $|\lambda| = 1$ implies $\nu = \frac{1-\sqrt{5}}{2}$ and $a = a_0$. Thus, we obtain for $a \in (1,a_0)$ that $|\lambda_i| < 1$, $i \in \{0,1,2\}$, i.e., the fixed point u_A of F_3 is locally stable. For $a > a_0$ we get $|\lambda| > 1$, so the fixed point is unstable.

The eigenvectors q_i, corresponding to the eigenvalues λ_i of J_a, are $q_i = q_i(a) = (1,\lambda_i,\lambda_i^2)$, $i \in \{0,1,2\}$. Let Q_a be the matrix whose columns are the eigenvectors q_i, i.e.,

$$Q_a = \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \bar{\lambda} & \nu \\ \lambda^2 & \bar{\lambda}^2 & \nu^2 \end{pmatrix}.$$

For $a > \frac{31}{27}$ the matrix Q_a is invertible and

$$Q_a^{-1} = \begin{pmatrix} d\,(\lambda^2 - \lambda) & d\,(\lambda - 1) & d \\ \bar{d}\,(\bar{\lambda}^2 - \bar{\lambda}) & \bar{d}\,(\bar{\lambda} - 1) & \bar{d} \\ e\,(\nu^2 - \nu) & e\,(\nu - 1) & e \end{pmatrix}, \tag{3.7}$$

where

$$e = e(a) = \frac{1}{3\nu^2 - 2\nu}, \quad d = d(a) = \frac{1}{3\lambda^2 - 2\lambda}.$$

Note that the rows of Q_a^{-1} are the eigenvectors of J_a^T.

Applying the linear transformation

$$\mathcal{Z} = \begin{pmatrix} z \\ \bar{z} \\ y \end{pmatrix} = Q_a^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

the map (3.5) takes the following form

$$\mathcal{Z} \mapsto Q_a^{-1} J_a Q_a \mathcal{Z} + Q_a^{-1} f_a\big(Q_a \mathcal{Z}\big) = \begin{pmatrix} \lambda\, z + d\, g(z, y) \\ \bar{\lambda}\, \bar{z} + \bar{d}\, g(z, y) \\ \nu\, y + e\, g(z, y) \end{pmatrix}, \tag{3.8}$$

where the function $g : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ is

$$g(z, y) = g_a(z, \bar{z}, y) = -a(z + \bar{z} + y)\left(\lambda^2 z + \bar{\lambda}^2 \bar{z} + \nu^2 y\right). \tag{3.9}$$

Clearly, the second component in (3.8) is the complex conjugate of the first one. Therefore, it is sufficient to consider the map

$$H_a : \mathbb{C} \times \mathbb{R} \ni \begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda z + d\, g(z, y) \\ \nu y + e\, g(z, y) \end{pmatrix} \in \mathbb{C} \times \mathbb{R}. \tag{3.10}$$

Remark that for the sake of simplicity, we omit the argument $\bar{z}$, and indicate only the variable z in the subsequent functions. We also emphasize that $g(z, y)$ and $H_a(z, y)$ are smooth functions of $z, \bar{z}$ and y, but they are not necessarily complex differentiable.

3.2.1 Local stability by linearization

First, for a fixed parameter $a \in \left(\frac{4}{3}, a_0\right)$ we use the map (3.10) without further transformation to construct a neighborhood $\mathcal{M}_0(a) \subseteq \mathbb{C} \times \mathbb{R}$, which is inside the region of attraction of the origin, i.e., $\lim_{n \to \infty} H_a^n(z, y) = (0, 0)$ for every $(z, y) \in \mathcal{M}_0(a)$.

Proposition 16. *For every* $a \in \left(\frac{4}{3}, a_0\right)$ *define* $\xi(a)$ *by*

$$\xi(a) = \frac{(1-|\lambda|)}{4a|\lambda|^2(|d|+|e|)}.$$

Then the set

$$\mathcal{M}_0(a) = \{(z,y) \in \mathbb{C} \times \mathbb{R} : \ |z| + |y| < \xi(a)\}$$

is in the region of attraction of the fixed point $(0,0)$ *of* H_a*.*

Proof. Introduce the norm $|(z,y)| = |z| + |y|$ on the space $\mathbb{C} \times \mathbb{R}$. We show that there exists a $\xi = \xi(a) > 0$ such that $|H_a(z,y)| < |(z,y)|$ for every $0 < |(z,y)| < \xi$. If such a ξ exists, it is clear that the open ball B°_ξ around the origin is invariant. We show that every point of B°_ξ tends to the origin. Let (z_0, y_0) be an arbitrary point from B°_ξ and consider the nonnegative, strictly decreasing sequence $(|(z_n, y_n)|)_{n=0}^\infty$, where $(z_{n+1}, y_{n+1}) = H_a(z_n, y_n)$. This sequence can converge only to a fixed point of the continuous map $r \mapsto \max_{|(z,y)|=r} |H_a(z,y)|$, which is only $r = 0$, provided that $r \in [0, \xi)$.

From (3.6) we obtain $|\lambda| > |\nu|$ for $\frac{4}{3} < a < a_0$, so

$$|g(z,y)| \leq 4a|\lambda|^2(|z|+|y|)^2 = 4a|\lambda|^2|(z,y)|^2.$$

Consequently, estimating (3.10) we obtain

$$\begin{aligned} |H_a(z,y)| &\leq |\lambda||(z,y)| + (|d|+|e|)\,|g(z,y)| \\ &\leq |(z,y)|\left(|\lambda| + (|d|+|e|)\,4a|\lambda|^2|(z,y)|\right) < |(z,y)| \end{aligned}$$

for every $(z,y) \neq (0,0)$, provided that $|(z,y)| < \xi(a)$, where

$$\xi(a) = \frac{1-|\lambda|}{4a|\lambda|^2(|d|+|e|)}.$$

Therefore, $\xi(a)$ is a suitable choice. □

First, note that $\mathcal{M}_0(a)$ is in $\mathbb{C} \times \mathbb{R}$. Clearly, this set corresponds to an attracting neighborhood $\mathcal{M}(a) \subseteq \mathbb{R}^3$ around the nontrivial fixed point u_A of (3.2). However, we let this transformation be done in the second, computer-aided part of the proof, in order to obtain better accuracy.

Second, since $\lim_{a \to a_0} |\lambda(a)| = 1$, we obtain that $\mathcal{M}_0(a)$ shrinks to the origin as a tends to a_0. However, the smaller the neighborhood is, the less efficient and more time-consuming the computer-aided part of the proof is. Furthermore, Proposition 16 does not provide at all an attracting neighborhood at the critical parameter value a_0. Consequently, close to a_0 this approach is not suitable for reliable numerical methods, and thus we need to find another way to construct the attracting neighborhood.

3.3 A center manifold reduction

In the subsequent subsections we study the case $a \in \mathcal{I}_0 = [a_0 - 10^{-2}, a_0]$. For $a \in \mathcal{I}_0$ we want to adapt the normal form technique from [4] to create an attracting neighborhood around the nontrivial fixed point of map (3.2). However, $F_3(u)$ is a three-dimensional map, so we need a center manifold reduction first (see [8, 23, 24]).

As we explained in the introduction, a polynomial approximation of the generalized center-unstable manifold will be used here for each $a \in \mathcal{I}_0$. We look for the fourth-order polynomial approximation $\phi(z) = \phi_a(z, \bar{z})$ of the manifold in the form

$$\phi(z) = \sum_{\substack{i+j=2 \\ i,j \in \mathbb{N}_0}}^{4} \frac{1}{i!j!} \omega_{ij} z^i \bar{z}^j, \tag{3.11}$$

where $\omega_{ij} = \omega_{ij}(a)$. Every coefficient ω_{ij} is determined by symbolic computer calculation so that in the expression

$$\mathcal{N}(\phi(z)) = \phi\Big(\lambda z + d\, g\big(z, \phi(z)\big)\Big) - \nu\phi(z) - e\, g\big(z, \phi(z)\big) \tag{3.12}$$

the at most fourth-order terms of $z, \bar{z}$ are eliminated (see [23]), so $\mathcal{N}(\phi(z)) = O\left(|z|^5\right)$. The coefficients ω_{ij} depend smoothly on a and the formulas can be found in the Appendix, see Subsection 4.2.3. In the subsequent subsections we study the dynamics of (3.10) in the set

$$T(r, C) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \leq r, \ |y - \phi(z)| \leq C|z|^5\}.$$

Note that it would suffice to consider a second-order approximation of the generalized center-unstable manifold with a term $C|z|^3$ in $T(r, C)$ in order to obtain an attracting neighborhood. Also in this case C would appear only in the at least fourth-order terms of (1.6) which is of crucial importance. However, this does not provide a sufficiently large neighborhood for the computer-aided part of the method. Indeed, the larger the order of the approximation of the center manifold is, the larger the obtainable initial attracting neighborhood is, since our estimates are more precise. On the other hand, calculations become lengthier as the order of the approximation increases. Furthermore, the growth of attracting neighborhood due to extra orders is diminishing, that is why, we have chosen a fourth-order approximation for $\phi(z)$.

Throughout the section we need an a-independent estimate of some coefficients. First of all, from the characteristic polynomial (see Section 3.2), it is easy to see that

$$|\lambda| \leq 1 \quad \text{and} \quad |\nu| \leq \nu_0 \qquad \text{for } a \in \mathcal{I}_0, \tag{3.13}$$

where $\nu_0 = a_0 - 1$. Furthermore, with interval arithmetic it can be shown that $d_0 = 0.559$ and $e_0 = 0.425$ satisfy the inequalities

$$|d| \leq d_0 \quad \text{and} \quad |e| \leq e_0 \qquad \text{for } a \in \mathcal{I}_0. \tag{3.14}$$

Then, we choose constants ω_k, $k \in \{2, 3, 4\}$ such that the following holds

$$\sum_{\substack{i+j=k \\ i,j \in \mathbb{N}_0}} \frac{|\omega_{ij}|}{i!j!} \leq \omega_k \qquad \text{for } a \in \mathcal{I}_0.$$

Recall that we have already obtained an analytic expression for ω_{ij}, see Subsection 4.2.3. With interval arithmetic we obtain that $\omega_2 = 1.29$, $\omega_3 = 2.193$ and $\omega_4 = 6.233$ are appropriate choices. Let $\phi_k(z)$ denote the k-th-order terms of $\phi(z)$. We obtain

$$|\phi_k(z)| \leq \omega_k r^k \tag{3.15}$$

for $k \in \{2, 3, 4\}$ and $a \in \mathcal{I}_0$. Clearly, (3.15) implies the polynomial estimate

$$|\phi(z)| \leq \phi^{max}(|z|) \tag{3.16}$$

for every $a \in \mathcal{I}_0$, where

$$\phi^{max}(r) = \sum_{k=2}^{4} \omega_k r^k$$

is a real polynomial with positive coefficients, so $\phi^{max}(r)$ is an estimate by order of $\phi(z)$.

Now, we turn our attention to the function $g(z, y)$ from (3.9) and consider the expansion

$$g(z, y) = \sum_{\substack{i+j+k=2 \\ i,j,k \in \mathbb{N}_0}} \frac{g_{ijk}}{i!j!k!} z^i \bar{z}^j y^k, \tag{3.17}$$

where

$$g_{ijk} = g_{ijk}(a) = -a \left(i\lambda^2 + j\bar{\lambda}^2 + k\nu^2 \right).$$

Using interval arithmetics it can be shown that $g_{20} = 4.237$, $g_{11} = 3.805$ and $g_{02} =$

0.6181 satisfy

$$\sum_{\substack{i+j=2\\ i,j\in\mathbb{N}_0}} \frac{|g_{ij0}|}{i!j!} \le g_{20}, \qquad \sum_{\substack{i+j=1\\ i,j\in\mathbb{N}_0}} \frac{|g_{ij1}|}{i!j!} \le g_{11}, \qquad \frac{|g_{002}|}{2!} \le g_{02} \quad \text{for } a \in \mathcal{I}_0.$$

Thus, from (3.9) we obtain

$$|g(z,y)| \le \tilde{g}^{max}(|z|,|y|) \tag{3.18}$$

for every $a \in \mathcal{I}_0$, where

$$\tilde{g}^{max}(r,s) = g_{20}r^2 + g_{11}rs + g_{02}s^2.$$

Later, we need also the composition of functions ϕ and g, so we introduce the eighth-order polynomial

$$g^{max}(r) = \tilde{g}^{max}\big(r, \phi^{max}(r)\big),$$

and for every $a \in \mathcal{I}_0$ we obtain

$$|g(z,\phi(z))| \le g^{max}(|z|). \tag{3.19}$$

It is important that (3.19) is also an estimate by order, since it is a composition of two functions with that property.

After these estimations we can proceed to show the conditional invariance of $T(r,C)$. First of all, $y = \phi(z)$ is just an approximation of the center manifold, so $\mathcal{N}(\phi(z))$ is not zero. Nevertheless, $\mathcal{N}(\phi(z)) = O(|z|^5)$ follows from the choice of coefficients ω_{ij}. In the computer-aided part, for a given $\rho_1 > 0$ we need an explicit $\mathcal{N}(\phi(z)) \le C_0|z|^5$-type estimate for $|z| \le \rho_1$ and $a \in \mathcal{I}_0$, where the constant C_0 is independent of a and z. Recall that the most frequently used constants can be found in Table 4.1.

Proposition 17. *Let $\rho_1 = 0.02234$ and $C_0 = 37.379$. For every $|z| \le \rho_1$ and $a \in \mathcal{I}_0$ the following holds*

$$|\mathcal{N}(\phi(z))| \le C_0|z|^5.$$

Proof. Because of the construction of $\phi(z)$, the fourth and lower-order terms of $z, \bar{z}$ are zero in (3.12). To obtain a better estimate of C_0, we consider the decomposition

$$\mathcal{N}(\phi(z)) = \mathcal{N}_5(z) + \mathcal{N}_{\ge 6}(z),$$

where $\mathcal{N}_5(z)$ and $\mathcal{N}_{\ge 6}(z)$ denote the fifth and the at least sixth-order terms of $\mathcal{N}(\phi(z))$, respectively. Clearly, $\mathcal{N}_5(z)$ can be written in the following form

$$\mathcal{N}_5(z) = \sum_{\substack{i+j=5 \\ i,j\in\mathbb{N}_0}} \frac{N_{ij}}{i!j!} z^i \bar{z}^j,$$

where $N_{ij} = N_{ij}(a)$. Using (3.11) and (3.17) the coefficients N_{ij} can be determined explicitly for $i+j=5$. The formulas can be found in the Appendix, see Subsection 4.2.4. With interval arithmetic it can be shown that $N_5 = 25.094$ satisfies

$$\sum_{\substack{i+j=5 \\ i,j\in\mathbb{N}_0}} \frac{|N_{ij}|}{i!j!} \le N_5 \tag{3.20}$$

for every $a \in \mathcal{I}_0$. From this we obtain $|\mathcal{N}_5(z)| \le N_5 |z|^5$ for $a \in \mathcal{I}_0$.

For the estimation of $\mathcal{N}_{\ge 6}(z)$ we use (3.13), (3.14), (3.16) and (3.19). For every $a \in \mathcal{I}_0$ we get

$$|\mathcal{N}(\phi(z))| \le \phi^{max}\big(r + d_0\, g^{max}(r)\big) + \nu_0\, \phi^{max}(r) + e_0\, g^{max}(r), \tag{3.21}$$

where $r = |z|$. The right hand side of (3.21) is a polynomial, so it can be written in the form

$$\phi^{max}\big(r + d_0\, g^{max}(r)\big) + \nu_0\, \phi^{max}(r) + e_0\, g^{max}(r) = \sum_{k=2}^{32} b_{0,k} r^k.$$

The a-independent real coefficients $b_{0,k}$ can be determined by the real polynomials $\phi^{max}(r)$ and $g^{max}(r)$. The inequality (3.21) is also an estimate by order, since (3.16) and (3.19) also have that property. Furthermore, the lower-order terms of $\mathcal{N}(\phi(z))$ were eliminated, so the following also holds

$$|\mathcal{N}_{\ge 6}(z)| \le \sum_{k=6}^{32} b_{0,k} r^k.$$

Note that this inequality would not be necessarily true without the estimate by order property.

Using $r \le \rho_1$ we obtain

$$|\mathcal{N}_{\ge 6}(z)| \le \left(\sum_{k=6}^{32} b_{0,k} \rho_1^{k-5} \right) |z|^5.$$

It can be shown that $N_6 = 12.285$ satisfies

$$\sum_{k=6}^{32} b_{0,k} \rho_1^{k-5} \le N_6. \tag{3.22}$$

Combining (3.20) and (3.22) we obtain

$$|\mathcal{N}(\phi(z))| \le |\mathcal{N}_5(z)| + |\mathcal{N}_{\ge 6}(z)| \le (N_5 + N_6)|z|^5.$$

So the proposition is proven, since C_0 was chosen such that $N_5 + N_6 \le C_0$. □

In the following corollary we reformulate Proposition 17 in order to obtain a geometrical interpretation of the statement (see Figure 3.3).

Corollary 18. *Let ρ_1 and C_0 be from Proposition 17. For every $|z_0| \le \rho_1$ and $a \in \mathcal{I}_0$ the point $(\hat{z}_0, \hat{y}_0) = H_a(z_0, \phi(z_0))$ satisfies the inequality*

$$|\phi(\hat{z}_0) - \hat{y}_0| \le C_0 |z_0|^5.$$

3.3.1 Attractivity in direction y

Corollary 18 states that for parameter values close to the critical a_0 if we consider a point (z_0, y_0) from the surface $(z, \phi(z))$, i.e., $y_0 = \phi(z_0)$, then the image $(\hat{z}_0, \hat{y}_0)$ remains close to that surface. Now, with the help of Corollary 18 we are able to make a statement about the y-directional behavior of map (3.10). It is well-known (see [24]) that if the fixed point has no eigenvalues moduli greater than one, then the center manifold has an attracting property, i.e., every solution close enough to the fixed point decays exponentially to the center manifold. Based on this idea we prove a similar statement about the approximation $y = \phi(z)$ of the center manifold.

Proposition 19. *Let ρ_1 and C_0 be from Proposition 17. Furthermore, let $\tilde{\rho}_2 = 0.0237$, $\sigma = 2.1 \cdot 10^{-3}$ and $L = 0.66$. For every $|z_0| \le \rho_1$, $|y_0| \le \sigma$ and $a \in \mathcal{I}_0$ the point $(z_1, y_1) = H_a(z_0, y_0)$ satisfies $|z_1| \le \tilde{\rho}_2$ and*

$$|\phi(z_1) - y_1| \le L|\phi(z_0) - y_0| + C_0|z_0|^5.$$

Proof. First, note that σ was chosen large enough such that Proposition 19 can be applied in Propositions 20, 22 and 23, i.e., $\{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le \rho_1, \ |y| \le \sigma\}$ contains the occurring T and $\tilde{T}$. Second, from (3.10), (3.13), (3.14) and (3.18) it is clear that

$$|z_1| \le |z_0| + d_0\, \tilde{g}^{max}(|z_0|, |y_0|)$$

for every $a \in \mathcal{I}_0$. The constant $\tilde{\rho}_2$ was chosen such that

$$\tilde{\rho}_2 \ge \rho_1 + d_0\, \tilde{g}^{max}(\rho_1, \sigma).$$

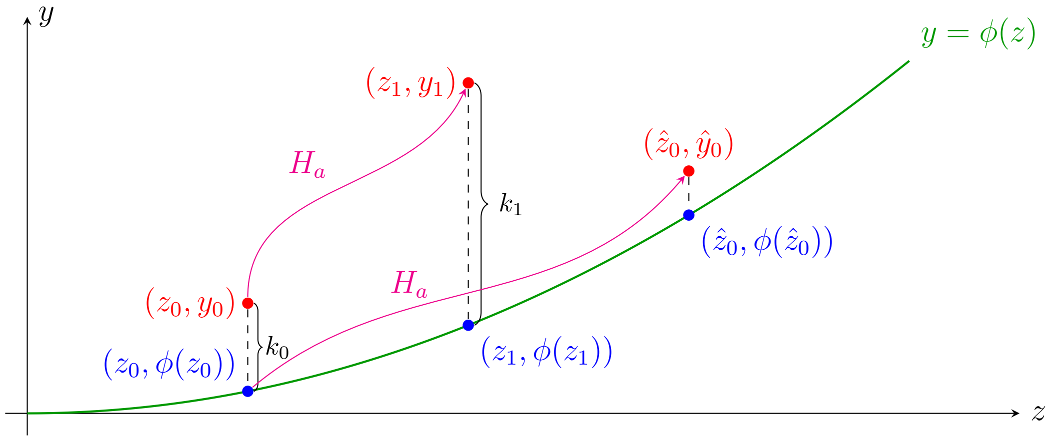


Figure 3.3: The dynamics close to $y = \phi(z)$

Thus, we obtain that $|z_1| \leq \tilde{\rho}_2$ for every $|z_0| \leq \rho_1$ and $|y_0| \leq \sigma$. Similarly, $|\hat{z}_0| \leq \tilde{\rho}_2$ also holds for $(\hat{z}_0, \hat{y}_0) = H_a(z_0, \phi(z_0))$, since $\phi^{max}(\rho_1) \leq \sigma$ (see Corollary 18 and Figure 3.3). Finally, introducing the notation $k_i = y_i - \phi(z_i)$ for $i \in \{0, 1\}$ (see Figure 3.3), the formula to be proven in this proposition can be reformulated into the form

$$|k_1| \leq L|k_0| + C_0|z_0|^5.$$

Using (3.10) and the mean value theorem we obtain

$$\begin{aligned} z_1 &= \lambda z_0 + d\, g\big(z_0, \phi(z_0) + k_0\big) = \hat{z}_0 + d\, k_0\, \partial_2 g\big(z_0, \phi(z_0) + \tilde{k}_0\big), \\ y_1 &= \nu\,\big(\phi(z_0) + k_0\big) + e\, g\big(z_0, \phi(z_0) + k_0\big) = \hat{y}_0 + \nu\, k_0 + e\, k_0\, \partial_2 g\big(z_0, \phi(z_0) + \hat{k}_0\big), \end{aligned} \tag{3.23}$$

where

$$\partial_2 g(z, y) = -a\left((\lambda^2 + \nu^2)z + (\bar{\lambda}^2 + \nu^2)\bar{z} + 2\nu^2 y\right)$$

is the partial derivative of $g(z, y)$ with respect to y, and $\tilde{k}_0, \hat{k}_0$ are some numbers between 0 and k_0. Since $|\phi(z_0) + k_0| \leq \sigma$ and $|\phi(z_0)| \leq \sigma$, we get

$$|\phi(z_0) + \tilde{k}_0| \leq \sigma, \qquad |\phi(z_0) + \hat{k}_0| \leq \sigma. \tag{3.24}$$

From (3.18), we obtain

$$|\partial_2 g(z, y)| \leq g_{11}|z| + 2g_{02}|y|$$

for every $a \in \mathcal{I}_0$. Introduce $\partial_y g$ such that $|\partial_2 g(z, y)| \leq \partial_y g$ holds for every $|z| \leq \rho_1$ and $|y| \leq \sigma$. It can be shown that $\partial_y g = 0.088$ is a suitable choice. Using $\partial_y g$, (3.13), (3.14) and (3.24) we obtain from (3.23) that

$$|z_1 - \hat{z}_0| \le d_0\, \partial_y g\, |k_0|. \tag{3.25}$$

Similarly, from (3.23) we see that

$$|y_1 - \hat{y}_0| \le (\nu_0 + e_0\, \partial_y g)\, |k_0| \tag{3.26}$$

for every $|z_0| \le \rho_1$, $|y_0| \le \sigma$ and $a \in \mathcal{I}_0$.

It is easy to see that

$$|\phi(z_1) - \phi(\hat{z}_0)| \le |z_1 - \hat{z}_0| \partial\phi^{max}(r), \tag{3.27}$$

provided that $|z_1| \le r$ and $|\hat{z}_0| \le r$, where

$$\partial\phi^{max}(r) = 2\omega_2 r + 3\omega_3 r^2 + 4\omega_4 r^3.$$

Since $|z_0| \le \rho_1$ and $|y_0| \le \sigma$ implies $|z_1| \le \tilde{\rho}_2$ and $|\hat{z}_0| \le \tilde{\rho}_2$, we introduce $\partial\phi$ such that

$$\partial\phi^{max}(\tilde{\rho}_2) \le \partial\phi.$$

It can be shown that $\partial\phi = 0.066$ is a proper choice.

Using Corollary 18, (3.25), (3.26), (3.27) and the definition of $\partial\phi$ we get

$$\begin{aligned} |k_1| = |y_1 - \phi(z_1)| &\le |y_1 - \hat{y}_0| + |\hat{y}_0 - \phi(\hat{z}_0)| + |\phi(\hat{z}_0) - \phi(z_1)| \\ &\le (\nu_0 + e_0\, \partial_y g)|k_0| + C_0|z_0|^5 + \partial\phi\, d_0\, \partial_y g|k_0| \\ &= (\nu_0 + e_0\, \partial_y g + \partial\phi\, d_0\, \partial_y g)|k_0| + C_0|z_0|^5 \end{aligned}$$

for every $|z_0| \le \rho_1$, $|y_0| \le \sigma$ and $a \in \mathcal{I}_0$. It can be checked that

$$\nu_0 + e_0\, \partial_y g + \partial\phi\, d_0\, \partial_y g \le L,$$

so the proposition is proven. □

It can be seen that Proposition 19 is slightly weaker than similar statements about center manifolds. Even if a solution remains close to the fixed point, $y = \phi(z)$ does not have a real exponentially attracting property, since we have the extra term $C_0|z_0|^5$ in the estimate. This term originated from the fact that $y = \phi(z)$ is not the center manifold, but only an approximation of it. On the other hand, it is of crucial importance that we explicitly give the neighborhood where the proposition holds, and determine also an explicit value for the parameter L.

3.3.2 Conditional invariance in direction y

Using Proposition 19 we can show the y-directional conditional invariance of $T(\rho_1, C_1)$ for an appropriately chosen $C_1 > C_0$.

Proposition 20. *Let ρ_1 be from Proposition 17 and $C_1 = 7700$. For every $(z_0, y_0) \in T(\rho_1, C_1)$ and $a \in \mathcal{I}_0$ the point $(z_1, y_1) = H_a(z_0, y_0)$ satisfies*

$$|\phi(z_1) - y_1| \leq C_1 |z_1|^5.$$

Proof. Using the notations from Proposition 19 it is clear that $|k_0| = |\phi(z_0) - y_0| \leq C_1 |z_0|^5$, since $(z_0, y_0) \in T(\rho_1, C_1)$. We need to prove the inequality $|k_1| \leq C_1 |z_1|^5$.

Since σ was chosen such that $\phi^{max}(\rho_1) + \rho_1^5 \leq \sigma$, we can apply Proposition 19. We obtain $|z_1| \leq \tilde{\rho}_2$ and

$$|k_1| \leq L|k_0| + C_0 |z_0|^5 \leq (LC_1 + C_0)|z_0|^5.$$

However, we need a $|k_1| \leq C|z_1|^5$-type inequality, so first, we are looking for a constant C_2 such that $|z_0|^5 \leq C_2 |z_1|^5$. From (3.23) we see that

$$|z_1| \geq \lambda_{min}|z_0| - d_0\, \tilde{g}^{max}(|z_0|, |y_0|) \geq \lambda_{min}|z_0| - d_0\, \tilde{g}^{max}\big(|z_0|, \phi^{max}(|z_0|) + C_1 |z_0|^5\big)$$

for every $(z_0, y_0) \in T(\rho_1, C_1)$, and furthermore, $\lambda_{min} = 0.9952$ was chosen so that it satisfies $\lambda_{min} \leq |\lambda|$ for every $a \in \mathcal{I}_0$. Since $g^{max}\big(r, \phi^{max}(r) + C_1 r^5\big) = O(r^2)$, introducing

$$g_0^{max}(r) = \frac{\tilde{g}^{max}\big(r, \phi^{max}(r) + C_1 r^5\big)}{r},$$

where $g_0^{max}(r) = O(r)$, we obtain

$$|z_1| \geq |z_0|\big(\lambda_{min} - d_0\, g_0^{max}(|z_0|)\big) \geq |z_0|\big(\lambda_{min} - d_0\, g_0^{max}(\rho_1)\big)$$

because $|z_0| \leq \rho_1$ and $g_0^{max}(r)$ is a polynomial with positive coefficients.

Since $C_2 = 1.358$ satisfies

$$\frac{1}{\big(\lambda_{min} - d_0\, g_0^{max}(\rho_1)\big)^5} \leq C_2,$$

we conclude $|z_0|^5 \leq C_2|z_1|^5$, and

$$|\phi(z_1) - y_1| = |k_1| \leq (LC_1 + C_0)C_2|z_1|^5.$$

Using the values of C_0, C_1, C_2 and L it can be checked that

$$(LC_1 + C_0)C_2 \leq C_1,$$

so the proposition is proven. □

It follows from Proposition 20 that if $|z_1| \leq \rho_1$, then $(z_1, y_1) \in T(\rho_1, C_1)$. However, at this point we can guarantee only that $|z_1| \leq \tilde{\rho}_2$, and thus

$$H_a(T(\rho_1, C_1)) \subseteq T(\tilde{\rho}_2, C_1).$$

Roughly speaking, for $T(\rho_1, C_1)$ we proved some kind of conditional invariance in direction y under the map (3.10), but we still do not have overall picture about the z-directional dynamics. It will be covered in the subsequent subsection using the bifurcational normal form technique adapted from [4].

Note that it also follows from Proposition 20 that $T(r, C_1)$ is also conditionally invariant in direction y for every $r \leq \rho_1$. Finally, we remark that C_1 is not the smallest value for which Proposition 20 holds; it is about $C \approx 475$. However, we need a relatively thick (in direction y) set for the computer-aided part, and it is easier to construct a real neighborhood around the origin from a $T(r, C)$ with a larger C.

3.4 Transforming to normal form

In the previous subsection we saw that $T(\rho_1, C_1)$ is conditionally invariant in direction y under the map (3.10). Now, using the Neimark–Sacker bifurcational normal form technique from [4] we study the z-directional dynamics. Essentially, we perform a nonlinear transformation on the z-coordinate of the z–y-coordinate system such that (close to the fixed point) in the new coordinate system the transformed coordinate is getting strictly closer to zero by every iteration. In the end, combining this with the y-directional dynamics we obtain that some subset of $T(\rho_1, C_1)$ is in the region of attraction of the fixed point of (3.10).

For every $(z_0, y_0) \in T(\rho_1, C_1)$ the y-coordinate can be written in the form $y_0 = \phi(z_0) + c|z_0|^5$ for some $c \in \mathbb{R}$ with $|c| \leq C_1$. Thus, for $(z_1, y_1) = H_a(z_0, y_0)$ the z-coordinate is determined by $z_1 = G(z_0)$, where

$$G(z) = G_{a,c}(z, \bar{z}) = \lambda z + d\, g\big(z, \phi(z) + c|z|^5\big). \tag{3.28}$$

For every fixed $a \in \mathcal{I}_0$ and c with $|c| \le C_1$ we adapt the normal form technique to (3.28). However, we have an extra parameter c in (3.28) and an additional stable direction in (3.10) compared to [4]. Thus first, before the main result of this subsection we investigate in detail how this c effects the method from [4]. We also discuss the specific shape of $T(r, C)$, which assures that the aforementioned normal form technique can be adapted with minor changes. After that, the proof of Proposition 21 is essentially the same as in [4]. However, we elaborate it for the sake of completeness.

Introducing the sets

$$\tau(c) = \big\{\big(z, \phi(z) + c|z|^5\big) :\ z \in \mathbb{C}\big\}$$
$$\tau(r, c) = \big\{\big(z, \phi(z) + c|z|^5\big) :\ z \in \mathbb{C},\ |z| \le r\big\}$$

we can note that for a fixed $a \in \mathcal{I}_0$ the set $T(\rho_1, C_1)$ is foliated by the sets $\tau(\rho_1, c)$ with $|c| \le C_1$. That is, $T(\rho_1, C_1) = \cup_{|c|\le C_1} \tau(\rho_1, c)$ and $\tau(\rho_1, c) \cap \tau(\rho_1, \hat{c}) = \{(0, 0)\}$ for every $c \ne \hat{c}$ with $|c| \le C_1$ and $|\hat{c}| \le C_1$. Using (3.28) and the normal form technique we consider the dynamics on $\tau(\rho_1, c)$ for a fixed $a \in \mathcal{I}_0$ and $|c| \le C_1$. However, $\tau(\rho_1, c)$ is not invariant under (3.10), more precisely $H_a(\tau(\rho_1, c)) \nsubseteq \tau(c)$ in general. Consequently, proving $|h^{-1}(G(h(w)))| < |w|$ for every a and c separately, could cause a lot of difficulty if there is no connection between the methods for different parameter values. The reason is that the nonlinear transformation of the z-coordinate and the neighborhood on which the transformed map satisfies $|h^{-1}(G(h(w)))| < |w|$ could vary from parameter value to parameter value. Therefore, considering $a \in \mathcal{I}_0$ fixed, our aim is to handle these transformations together in some sense for every c with $|c| \le C_1$.

Following the steps from [4] the function $G(z)$ in (3.28) can be written as a formal Taylor series of complex variables z and $\bar{z}$, i.e.,

$$G(z) = \lambda z + \sum_{\substack{i+j=2 \\ i,j\in\mathbb{N}_0}}^{5} \frac{G_{ij}}{i!j!} z^i \bar{z}^j + R_1, \tag{3.29}$$

where $G_{ij} = G_{ij}(a)$ and $R_1 = R_1(z, \bar{z}, a, c) = O(|z|^6)$. The expression of G_{ij} can be found in the Appendix, see Subsection 4.2.5. Observe that, because of the definition of the set $T(r, C)$, the parameter c appears only in the at least sixth-order terms of G, i.e., G_{ij} is independent of c for $2 \le i + j \le 5$.

It is well known (see, e.g., Lemma 4.7 in [8]) that there is a locally invertible parameterdependent change of the complex coordinate $z = h(w)$ with $h : \mathbb{C} \to \mathbb{C}$ in the form

$$h(w) = w + \frac{h_{20}}{2}w^2 + h_{11}w\overline{w} + \frac{h_{02}}{2}\overline{w}^2 + \frac{h_{30}}{6}w^3 + \frac{h_{12}}{2}w\overline{w}^2 + \frac{h_{03}}{6}\overline{w}^3 \tag{3.30}$$

so that the coefficients $h_{ij} = h_{ij}(a)$ are independent of c, and the map (3.28) is transformed into its normal form

$$w \mapsto h^{-1}(G(h(w))) = \lambda w + c_1 w^2\overline{w} + R_2, \tag{3.31}$$

where $c_1 = c_1(a)$ is the Lyapunov-coefficient and $R_2 = R_2(w, \overline{w}, a, c) = O(|w|^4)$. The inverse h^{-1} of h can be defined in a small neighborhood of $0 \in \mathbb{C}$ in the form $h_0^{-1}(z) + O(|z|^6)$, where

$$h_0^{-1}(z) = z + \sum_{\substack{i+j=2 \\ i,j\in\mathbb{N}_0}}^{5} \frac{\tilde{h}_{ij}}{i!j!} z^i \bar{z}^j \tag{3.32}$$

with c-independent coefficients $\tilde{h}_{ij} = \tilde{h}_{ij}(a)$. The existence of h with the properties above requires the nonresonance condition

$$\left(\frac{\lambda(a)}{|\lambda(a)|}\right)^k \neq 1$$

for $k \in \{1, 2, 3, 4\}$ and $a \in \mathcal{I}_0$, which obviously holds. The coefficients $\tilde{h}_{ij}$ of h_0^{-1} can be obtained from the equation $z = h\left(h_0^{-1}(z)\right) + O(|z|^6)$ by equating the coefficients of the same type up to fifth-order, see Subsection 4.2.7. The coefficient h_{ij} of h are chosen so that the second and third-order terms of $h_0^{-1}(G(h(w)))$ are eliminated (apart from $w^2\overline{w}$), see Subsection 4.2.6. The coefficients h_{ij} and $\tilde{h}_{ij}$ are determined by symbolic computer calculation. As the coefficients G_{ij} with $2 \leq i + j \leq 5$ are independent of c, it is not difficult to see that h_{ij} and $\tilde{h}_{ij}$ are also independent of c.

The inverse h^{-1} will be given in Subsection 3.4.2 as the inverse of the restriction of h to a neighborhood of $0 \in \mathbb{C}$. The main issue here is the construction of

$$h^{-1}(z) = h_0^{-1}(z) + R_3 \tag{3.33}$$

with some $R_3 = R_3(z, \bar{z}, a) = O(|z|^6)$, and an explicit bound for R_3 on its domain uniformly in $a \in \mathcal{I}_0$ and $c \in \mathbb{R}$ with $|c| \leq C_1$.

A key fact is that for each fixed $a \in \mathcal{I}_0$ the transformations h, h^{-1} are the same for all c with $|c| \leq C_1$. Therefore, the transformation $z = h(w)$ is the same for the whole set $T(\rho_1, C_1)$. The c-dependence appears only in R_2 in the transformed map (3.31). The above argument justifies that c, without causing any further difficulties, can only appear in the higher-order (more precisely in the at least third-order) terms in the definition of $T(r, C)$.

Since a supercritical bifurcation takes place at a_0, it can be shown that for every c with $|c| \leq C_1$ and $a \leq a_0$ sufficiently close to a_0 there exists some $\rho_0 = \rho_0(a, c) > 0$ such that for every $w \in \mathbb{C}$ with $0 < |w| \leq \rho_0(a, c)$ the inequality

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| < |w| \tag{3.34}$$

holds. Our aim is to obtain a uniform ρ_0 for all $a \in \mathcal{I}_0$ and $|c| \leq C_1$ such that this ρ_0 is sufficiently large for the rigorous computational part of the method.

Note that if initially we considered a constant neighborhood, similar to (3.46), around $y = \phi(z)$ instead of $T(r, C)$, then additional difficulties would arise to guarantee (3.34) in this neighborhood. Namely, the stable manifold belonging to the fixed point needs to be a subset of the line $w = 0$ after the coordinate transformation, otherwise it is very likely that (3.34) is not satisfied in a neighborhood of the line $w = 0$. Theoretically the transformation could be obtained in this way by the implicit function theorem for instance, but a polynomial transformation (3.30) is not appropriate in general.

The elaboration and combination of the ideas above lead to the following result.

Proposition 21. *Let C_1 be from Proposition 20 and $\varepsilon_G = 0.01976$. For every $a \in \mathcal{I}_0$ the set $T(\varepsilon_G, C_1)$ is in the region of attraction of the fixed point of map* (3.10).

Proof. The rest of this subsection is devoted to the proof of Proposition 21 following the steps of [4]. First, let $a \in \mathcal{I}_0$ and c with $|c| \leq C_1$ be fixed.

3.4.1 Estimation of the lower-order terms in G, h and h^{-1}

Throughout this subsection we need an estimate of the coefficients of the lower-order terms in G, h and h^{-1} such that these estimates are independent of $a \in \mathcal{I}_0$. Using (3.29), (3.30), (3.32) and (3.33) we look for constants satisfying the inequalities

$$\begin{aligned} \sum_{\substack{i+j=n\\ i,j\in\mathbb{N}_0}} \frac{|G_{ij}|}{i!j!} &\leq G_n,\\ \sum_{\substack{i+j=2\\ i,j\in\mathbb{N}_0}} \frac{|h_{ij}|}{i!j!} &\leq h_2,\\ \frac{|h_{30}|}{6}+\frac{|h_{12}|}{2}+\frac{|h_{03}|}{6} &\leq h_3,\\ \sum_{\substack{i+j=n\\ i,j\in\mathbb{N}_0}} \frac{|\tilde{h}_{ij}|}{i!j!} &\leq \tilde{h}_n \end{aligned} \tag{3.35}$$

for every $a \in \mathcal{I}_0$ and $n \in \{2,3,4,5\}$. With interval arithmetic it can be shown that $G_2 = 2.341$, $G_3 = 2.352$, $G_4 = 3.955$, $G_5 = 11.237$, $h_2 = \tilde{h}_2 = 2.93$, $h_3 = 4.976$, $\tilde{h}_3 = 10.353$, $\tilde{h}_4 = 34.796$ and $\tilde{h}_5 = 110.572$ satisfy the requirements. From the definition of these constants we obtain the following finite-order polynomial estimates of functions h and h_0^{-1}

$$|h(w)| \leq h^{max}(|w|), \qquad \left|h_0^{-1}(z)\right| \leq \tilde{h}_0^{max}(|z|), \tag{3.36}$$

where

$$\begin{aligned} h^{max}(r) &= r + h_2 r^2 + h_3 r^3,\\ \tilde{h}_0^{max}(r) &= r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + \tilde{h}_5 r^5. \end{aligned} \tag{3.37}$$

We emphasize that in (3.36) we estimate by order and in (3.37) the coefficients are all independent of a and c.

From the definition of h_2 and h_3 we also get

$$|w| - h_2|w|^2 - h_3|w|^3 \leq |h(w)|. \tag{3.38}$$

Consequently, assuming $|w| \leq \rho$ and $h_2\rho + h_3\rho^3 < 1$ we obtain

$$|w| \leq \eta(\rho)|h(w)| \tag{3.39}$$

with

$$\eta(\rho) = \frac{1}{1 - h_2\rho - h_3\rho^2}.$$

3.4.2 The definition of h^{-1}

Set $\delta_1 = \frac{1}{10}$ and $\delta_2 = \frac{1}{16}$. Our aim is to show that h^{-1} can be defined on $\overline{B}_{\delta_2} \subseteq \mathbb{C}$. For a fixed $a \in \mathcal{I}_0$ and $z \in \mathbb{C}$ define $h^*_{a,z} : \mathbb{C} \ni w \mapsto w + z - h(w) \in \mathbb{C}$. Then there is a $w \in \mathbb{C}$ with $h(w) = z$ if and only if $h^*_{a,z}(w) = w$. From (3.36) and (3.37), for all $z, w_1, w_2 \in \mathbb{C}$ we obtain

$$\begin{aligned}|h^*_{a,z}(w_1) - h^*_{a,z}(w_2)| &= |w_1 - h(w_1) - w_2 + h(w_2)| \\ &\leq |w_1 - w_2| \Big(h_2(|w_1| + |w_2|) + h_3 \left(|w_1|^2 + |w_1| \cdot |w_2| + |w_2|^2 \right) \Big).\end{aligned}$$

If $z \in \overline{B}_{\delta_2}$ and $w_1, w_2 \in \overline{B}_{\delta_1}$, then

$$|h^*_{a,z}(w_1) - h^*_{a,z}(w_2)| \leq |w_1 - w_2| \left(2\delta_1 h_2 + 3\delta_1^2 h_3 \right).$$

In addition, for $z \in \overline{B}_{\delta_2}$ and $w \in \overline{B}_{\delta_1}$ we infer

$$|h^*_{a,z}(w)| \leq \delta_2 + \delta_1^2 h_2 + \delta_1^3 h_3.$$

It can be checked that $2\delta_1 h_2 + 3\delta_1^2 h_3 < 1$ and $\delta_2 + \delta_1^2 h_2 + \delta_1^3 h_3 < \delta_1$. Consequently, for each $a \in \mathcal{I}_0$ and $z \in \overline{B}_{\delta_2}$ we obtain $h^*_{a,z}\left(\overline{B}_{\delta_1}\right) \subseteq \overline{B}_{\delta_1}$, and $h^*_{a,z}$ is a contraction on $\overline{B}_{\delta_1}$. Therefore, for each $z \in \overline{B}_{\delta_2}$ there is a unique $w^*(z) \in \overline{B}_{\delta_1}$ with $h(w^*(z)) = z$, and the relation $h\left(\overline{B}_{\delta_1}\right) \supseteq \overline{B}_{\delta_2}$ follows. Define h^{-1} as $\overline{B}_{\delta_2} \ni z \mapsto w^*(z) \in \mathbb{C}$. Clearly, $h^{-1}\left(\overline{B}_{\delta_2}\right) \subseteq \overline{B}_{\delta_1}$.

In the subsequent subsections it is principal to estimate the moduli of $h(w)$, $G(h(w))$ and $h^{-1}(G(h(w)))$ in (3.31), provided that $|w| \leq \rho_0$ (see Figure 2.3). The magnitude of the remaining term R_2 highly depends on the size of the set, where the estimations are considered. Recall $\rho_1 = 0.02234$ from Proposition 17, and set

$$\rho_0 = 0.021, \quad \rho_2 = 0.02354, \quad \rho_3 = 0.02532. \tag{3.40}$$

Clearly, $\rho_0, \rho_3 \leq \delta_1$ and $\rho_1, \rho_2 \leq \delta_2$. Our aim is to show that

$$h(B_{\rho_0}) \subseteq B_{\rho_1}, \quad G(B_{\rho_1}) \subseteq B_{\rho_2}, \quad h^{-1}(B_{\rho_2}) \subseteq B_{\rho_3}$$

see Figure 2.3. For the first relation we chose ρ_0 such that $h^{max}(\rho_0) \leq \rho_1$ holds. Consequently, during the study of G we can assume that $|z| \leq \rho_1$. The other two relations will be shown in the subsequent subsections.

3.4.3 The estimation of R_1

From (3.28), (3.16) and (3.18) it is straightforward that in order to obtain an estimate of R_1, we need the at least sixth-order terms of the real polynomial

$$\mathcal{R}_1(r,c) = r + d_0\,\tilde{g}^{max}\left(r, \phi^{max}(r) + c\,r^5\right) = \sum_{k=1}^{10} b_{1,k}(c)\,r^k,$$

since we estimated by order in $\tilde{g}^{max}$ and ϕ^{max}. Note that, the coefficients $b_{1,k}$ can be obtained by symbolic computer calculation from the real polynomials $\tilde{g}^{max}$ and ϕ^{max}. Thus, using $|c| \le C_1$ and $|z| = r \le \rho_1$ we obtain

$$|R_1| \le \sum_{k=6}^{10} b_{1,k}(c)|z|^k \le \sum_{k=6}^{10} b_{1,k}(C_1)\rho_1^{k-6}|z|^6 \le 16550|z|^6,$$

i.e., $R_{10} = 16550$. Using (3.35) we get the following finite-order polynomial estimate of G

$$|G(z)| \le G^{max}(|z|) \tag{3.41}$$

with

$$G^{max}(r) = r + G_2 r^2 + G_3 r^3 + G_4 r^4 + G_5 r^5 + R_{10} r^6. \tag{3.42}$$

Note that the coefficients are independent of a and c. Now, it can be checked that $G^{max}(\rho_1) \le \rho_2$ holds. Hence, we get $G(B_{\rho_1}) \subseteq B_{\rho_2}$, so during the study of h^{-1} we can assume that $|z| \le \rho_2$.

3.4.4 The estimation of R_3 – the higher-order terms in h^{-1}

Now, we turn our attention to the estimation of R_3 in (3.33), which consists of the sixth and higher-order terms of h^{-1}. More precisely, R_3 is defined as $B_{\delta_2} \ni z \mapsto h^{-1}(z) - h_0^{-1}(z) \in \mathbb{C}$ and we need an estimate $|R_3(z)| < R_{30}|z|^6$, assuming $|z| \le \rho_2$. First, we give an estimate of type $|R_3(h(w))| \le R_{31}|w|^6$.

Set $\tilde{\rho}_3 = 0.0256$. Combining (3.38) with the facts that $[0, \rho_1] \ni s \mapsto s - h_2 s^2 - h_3 s^3 \in \mathbb{R}$ is strictly increasing, and $\tilde{\rho}_3 - h_2\tilde{\rho}_3^2 - h_3\tilde{\rho}_3^3 > \rho_2$, we can give the a priori estimate $\tilde{\rho}_3$ of $|h^{-1}(B_{\rho_2})|$. That is, $w = h^{-1}(z)$ satisfies $|w| < \tilde{\rho}_3$, provided that $|z| < \rho_2$.

Using the definition of h_0^{-1} and h, we obtain

$$R_3(h(w)) = h^{-1}(h(w)) - h_0^{-1}(h(w)) = w - h_0^{-1}(h(w)) = \sum_{\substack{i+j=6 \\ i,j\in\mathbb{N}_0}}^{15} b_{3,ij} w^i \overline{w}^j,$$

where the coefficients $b_{3,ij} = b_{3,ij}(a)$ are complex. Furthermore, consider the composition

$$\mathcal{R}_3(r) = \tilde{h}_0^{max}\big(h^{max}(r)\big) = \sum_{k=1}^{15} b_{3,k}\, r^k$$

of the real functions h^{max} and $\tilde{h}_0^{max}$. Note that, the coefficients $b_{3,k}$ can be obtained by symbolic computer calculation from the real polynomials h^{max} and $\tilde{h}_0^{max}$. Since in (3.36) we estimated by order, it follows that

$$\sum_{\substack{i+j=k\\ i,j\in\mathbb{N}_0}} |b_{3,ij}| \leq b_{3,k}$$

for every $a \in \mathcal{I}_0$ and $k \in \{6, 7, \dots, 15\}$. Thus,

$$|R_3(h(w))| \leq \sum_{k=6}^{15} b_{3,k}\, |w|^k \leq \sum_{k=6}^{15} b_{3,k}\, \tilde{\rho}_3^{k-6} |w|^6,$$

assuming $|w| \leq \tilde{\rho}_3$. Using (3.39) we get $|w| \leq \eta(\tilde{\rho}_3)|z|$ and

$$|R_3(z)| \leq \sum_{k=6}^{15} b_{3,k}\, \tilde{\rho}_3^{k-6} \big(\eta(\tilde{\rho}_3)\big)^6 |z|^6 \leq 9814|z|^6,$$

i.e., $\tilde{R}_{30} = 9814$. Consequently, we can give the estimation

$$\left|h^{-1}(z)\right| \leq r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + \tilde{h}_5 r^5 + \tilde{R}_{30} r^6,$$

where $|z| = r$. However, using $|z| < \rho_2$ we see that $|h^{-1}(z)| \leq \rho_3$, where ρ_3 was defined in (3.40). It means that h^{-1} maps B_{ρ_2} actually into B_{ρ_3}, i.e., we obtain a better estimate of $|h^{-1}(B_{\rho_2})|$. Thus, repeating the argument above with ρ_3 instead of $\tilde{\rho}_3$ we find a better estimation of the higher-order terms

$$|R_3(z)| \leq \sum_{k=6}^{15} b_{3,k}\, \rho_3^{k-6} \big(\eta(\rho_3)\big)^6 |z|^6 \leq 9744|z|^6,$$

i.e., $R_{30} = 9744$. We obtain

$$\left|h^{-1}(z)\right| \leq \tilde{h}^{max}(|z|) \tag{3.43}$$

with

$$\tilde{h}^{max}(r) = r + \tilde{h}_2 r^2 + \tilde{h}_3 r^3 + \tilde{h}_4 r^4 + \tilde{h}_5 r^5 + R_{30} r^6. \tag{3.44}$$

It can be checked that $\tilde{h}^{max}(\rho_2) \leq \rho_3$ holds with R_{30}, so $h^{-1}(B_{\rho_2}) \subseteq B_{\rho_3}$. Note that the coefficients in (3.44) are independent of a and c, and we estimate by order in (3.43) in the sense that the higher-order terms are estimated together in the term R_{30}.

3.4.5 The estimation of R_2 – the higher-order terms in $h^{-1}(G(h))$

Now, we turn our attention to the estimation of R_2, which denotes the at least fourth-order terms in $h^{-1}(G(h(w)))$. To obtain a better estimate we handle the fourth (R_{24}), the fifth (R_{25}) and the higher-order terms (R_{26}) separately, i.e., set $R_2 = R_{24} + R_{25} + R_{26}$.

As $h^{-1}(G(h(w)))$ is in normal form, it can be written in the form

$$h^{-1}(G(h(w))) = \lambda w + c_1 w^2 \overline{w} + \sum_{\substack{i+j=4 \\ i,j \in \mathbb{N}_0}}^{5} \frac{\beta_{ij}}{i!j!} w^i \overline{w}^j + R_{26},$$

where coefficients $c_1 = c_1(a)$ and $\beta_{ij} = \beta_{ij}(a)$ are complex, and $R_{26} = R_{26}(w, \overline{w}, a, c) = O(|w|^6)$. Note that coefficients β_{ij} are independent of c, as in $T(r, C)$ the parameter c appears only in the term $c|z|^5$. Actually, that is why we chose the fourth-order approximation of the center manifold and the fifth-order term $c|z|^5$ in T. We can explicitly determine even the coefficients of the fourth and fifth-order terms in $h^{-1}(G(h(w)))$. Thus, we can obtain better accuracy during the estimation of the higher-order terms of $h^{-1}(G(h(w)))$, and consequently, we can get a larger ρ_0. The formulas of coefficients β_{ij} for $4 \leq i+j \leq 5$, and the Lyapunov-coefficient c_1 can be found in the Appendix. With interval arithmetic we find that

$$\sum_{\substack{i+j=4 \\ i,j \in \mathbb{N}_0}} \frac{|\beta_{ij}|}{i!j!} \leq 33.549 \qquad \text{and} \qquad \sum_{\substack{i+j=5 \\ i,j \in \mathbb{N}_0}} \frac{|\beta_{ij}|}{i!j!} \leq 148.723.$$

Thus, with $R_{240} = 33.549$ and $R_{250} = 148.723$ we obtain

$$|R_{24}| \leq R_{240}|w|^4 \qquad \text{and} \qquad |R_{25}| \leq R_{250}|w|^5.$$

Using (3.37), (3.42) and (3.44) we get the real polynomial

$$\mathcal{R}_2(r) = \tilde{h}^{max}\big(G^{max}\left(h^{max}(r)\right)\big) = \sum_{k=1}^{108} b_{2,k}\, r^k.$$

Note that, the coefficients $b_{2,k}$ can be obtained by symbolic computer calculation from the real polynomials $\tilde{h}^{max}$, G^{max} and h^{max}. Since in (3.36), (3.41) and (3.43) we estimate by order, we infer

$$|R_{26}| \leq \sum_{k=6}^{108} b_{2,k} |w|^k.$$

Using $|w| \leq \rho_0$ one sees that

$$|R_{26}| \leq \sum_{k=6}^{108} b_{2,k}\rho_0^{k-4}|w|^4 \leq 35.122|w|^4,$$

so $R_{260} = 35.122$.

Combining these three results we obtain

$$|R_2| \leq |R_{24}| + |R_{25}| + |R_{26}| \leq (R_{240} + R_{250}\rho_0 + R_{260})|w|^4 = 71.8|w|^4,$$

and consequently, $R_{20} = 71.8$.

3.4.6 The dynamics in direction z

Now, with our previous estimate of R_2 we can finish our proof. Since

$$\left|\lambda(a)w + c_1(a)w^2\overline{w} + R_2\right| \leq |w|\left(\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3\right),$$

where $\tilde{c}_1 = c_1\frac{|\lambda|}{\lambda}$, we only need to prove

$$\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3 < 1 \tag{3.45}$$

for every $a \in \mathcal{I}_0$ and $w \in \mathbb{C}$ with $0 < |w| \leq \rho_0$.

To this end, we show that with a suitable $R_4 > 0$ the inequality

$$\left||\lambda| + \tilde{c}_1|w|^2\right| \leq |\lambda| + (\operatorname{Re}\tilde{c}_1)|w|^2 + R_4|w|^3$$

holds, or equivalently (by squaring the last inequality and dividing it by $|w|^3$)

$$0 \leq 2R_4|\lambda| - (\operatorname{Im}\tilde{c}_1)^2|w| + 2R_4(\operatorname{Re}\tilde{c}_1)|w|^2 + R_4^2|w|^3$$

for $|w| \leq \rho_0$ and $a \in \mathcal{I}_0$. With interval arithmetic we obtain that $\operatorname{Re}\tilde{c}_1$ and $\operatorname{Im}\tilde{c}_1$ are negative, $|\operatorname{Re}\tilde{c}_1| \leq 1.513$, $|\operatorname{Im}\tilde{c}_1| \leq 2.481$ and $|\lambda| \geq 0.9952$ for $a \in \mathcal{I}_0$. So it can be checked that $R_4 = 0.065$ is a suitable choice, assuming $|w| \leq \rho_0$. Therefore, the left hand side of (3.45) can be written in the following form

$$\begin{aligned}\left||\lambda| + \tilde{c}_1|w|^2\right| + R_{20}|w|^3 &\leq \left(|\lambda| + \operatorname{Re}\tilde{c}_1|w|^2\right) + (R_4 + R_{20})|w|^3 \\ &\leq 1 + \Big(\operatorname{Re}\tilde{c}_1 + (R_4 + R_{20})|w|\Big)|w|^2,\end{aligned}$$

which is less than 1, provided that

$$0 < |w| < \frac{-\operatorname{Re}\tilde{c}_1}{R_4 + R_{20}}.$$

Using the fact that $|\operatorname{Re}\tilde{c}_1| \geq 1.511$ we obtain

$$\rho_0 < \frac{-\operatorname{Re}\tilde{c}_1}{R_4 + R_{20}}.$$

Therefore, the inequality (3.45) holds for all $w \in \mathbb{C}$ with $0 < |w| \leq \rho_0$. Consequently, with the arbitrarily chosen $a \in \mathcal{I}_0$ and $|c| \leq C_1$ we proved (3.34) for all $w \neq 0$ with $|w| \leq \rho_0$.

Note that we obtained also that the bifurcation is supercritical since $\operatorname{Re}\tilde{c}_1(a_0)$ is negative.

3.4.7 Combining the y- and z-directional dynamics

Now, let $a \in \mathcal{I}_0$ be fixed and consider the set

$$T' = \left\{(z, y) \in \mathbb{C} \times \mathbb{R} : \left|h^{-1}(z)\right| \leq \rho_0,\ |y - \phi(z)| \leq C_1|z|^5\right\}.$$

Note that ρ_0, h and h^{-1} are independent of c. Clearly, $T' \subseteq T(\rho_1, C_1)$, since $h^{max}(\rho_0) \leq \rho_1$. Considering an arbitrary point $(z_0, y_0) \in T'$ there exists a $c \in \mathbb{R}$ with $|c| \leq C_1$ such that $(z_0, y_0) \in \tau(\rho_1, c)$. For $(z_1, y_1) = H_a(z_0, y_0)$ the inequality $|h^{-1}(z_1)| \leq \rho_0$ follows from (3.34). For the y-directional dynamics we obtain $|y_1 - \phi(z_1)| \leq C_1|z_1|^5$ from Proposition 20. Combining these two results we obtain that T' is invariant under (3.10). Consequently, $(z_n, y_n) \in T'$ for every $n \in \mathbb{N}_0$, where $(z_{n+1}, y_{n+1}) = H(z_n, y_n)$.

Introduce $w_n = h^{-1}(z_n)$ for $n \in \mathbb{N}_0$. Combining (3.34) with the fact that (z_n, y_n) remains in T' for every $n \in \mathbb{N}_0$, we get that $(|w_n|)_{n=0}^{\infty}$ is strictly decreasing. From the continuity of functions h, G and h^{-1} we obtain $w_n \to 0$, consequently, $z_n \to 0$ also holds. Therefore, T' is in the region of attraction of the trivial fixed point of map (3.10). Note that there exists a $\hat{c}$ with $|\hat{c}| \leq C_1$ such that $(z_1, y_1) \in \tau(\rho_1, \hat{c})$. It does not cause any trouble if $c \neq \hat{c}$, i.e., if the point (z_0, y_0) jumps from $\tau(\rho_1, c)$ to $\tau(\rho_1, \hat{c})$, as maps h and h^{-1} are the same for every $|c| \leq C_1$.

Finally, ε_G was chosen such that $\tilde{h}^{max}(\varepsilon_G) \leq \rho_0$, so $T(\varepsilon_G, C_1) \subseteq T'$ is also in the region of attraction of the fixed point. Although $T(\varepsilon_G, C_1)$ is not invariant, its projection on the z-coordinate is a disc, so it is easier to work with it in the computer-assisted part of the proof. Since a was an arbitrary value from $\mathcal{I}_0$, the proof of Proposition 21 is complete. □

Notice that $T(\varepsilon_G, C_1)$ is not a neighborhood of the origin since the y-directional thickness is zero for $z = 0$. In the following subsection we construct thicker sets around the

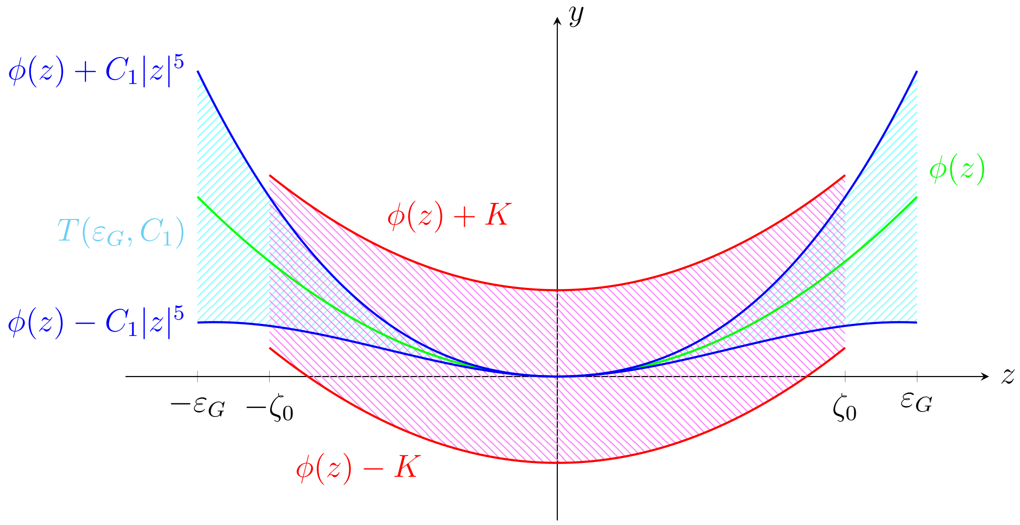


Figure 3.4: The set $\tilde{T}(r, K)$

origin to obtain a proper neighborhood of it.

3.5 Constructing the attracting neighborhood

In the previous subsections we showed that $T(\varepsilon_G, C_1)$ is in the region of attraction of the origin. Based on this set we construct attracting neighborhoods around the fixed point of (3.10).

Introduce the set (see Figure 3.4)

$$\tilde{T}(r, K) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le r, \ |\phi(z) - y| \le K\}. \tag{3.46}$$

Notice that in $\tilde{T}(r, K)$, unlike in $T(r, C)$, the y-directional thickness of the set is independent of the z-coordinate. Using Proposition 19 we can construct an attracting neighborhood around the fixed point.

Proposition 22. *Let* $\zeta_0 = 0.01883$ *and* $K_0 = 2.74 \cdot 10^{-5}$. *The set* $\tilde{T}_0 = \tilde{T}(\zeta_0, K_0)$ *is in the region of attraction of the fixed point of* (3.10).

Proof. Let $(z_0, y_0) \in \tilde{T}_0$ and $(z_1, y_1) = H_a(z_0, y_0)$. The value ζ_0 was chosen such that

$$\zeta_0 + d_0 \tilde{g}^{max}\big(\zeta_0, \phi^{max}(\zeta_0) + K_0\big) \le \varepsilon_G,$$

and therefore, $|z_1| \le \varepsilon_G$ also holds. Recall that in Proposition 19 the constant σ was chosen such that it satisfies $\phi(\zeta_0) + K_0 \le \sigma$. Clearly, $\zeta_0 < \rho_1$, and we can use Proposition

19 to obtain

$$|k_1| \leq L|k_0| + C_0|z_0|^5 \leq LK_0 + C_0|z_0|^5,$$

where $k_i = y_i - \phi(z_i)$ for $i \in \{0, 1\}$.

We chose K_0 such that $K_0 \leq \frac{3}{2}C_1\zeta_0^5$ holds, so we get that $|z_1| > \zeta_0$ implies

$$|k_1| \leq LK_0 + C_0\zeta_0^5 \leq \left(\frac{3}{2}LC_1 + C_0\right)\zeta_0^5 \leq C_1\zeta_0^5 \leq C_1|z_1|^5.$$

Hence, $(z_1, y_1) \in T(\varepsilon_G, C_1)$, i.e., (z_0, y_0) is in the region of attraction of the fixed point.

If $|z_1| \leq \zeta_0$, then it is enough to examine the case $(z_0, y_0) \notin T(\varepsilon_G, C_1)$, otherwise, (z_0, y_0) is clearly inside the region of attraction. Therefore, suppose $|k_0| \geq C_1 z_0^5$. We see that

$$|k_1| \leq L|k_0| + C_0|z_0|^5 \leq \left(L + \frac{C_0}{C_1}\right)|k_0| \leq \frac{2}{3}K_0, \tag{3.47}$$

i.e., $(z_1, y_1) \in \tilde{T}\left(\zeta_0, \frac{2}{3}K_0\right) \subseteq \tilde{T}_0$. Repeating the argument above, we obtain that as long as $(z_n, y_n)_{n=0}^{\infty}$ is outside of $T(\varepsilon_G, C_1)$, the sequence exponentially decays to the approximation $y = \phi(z)$ of the center manifold. Consequently, either there is an $n_0 \in \mathbb{N}$ such that $(z_{n_0}, y_{n_0}) \in T(\varepsilon_G, C_1)$, or $(z_n, y_n) \in \tilde{T}\left(\zeta_0, \left(\frac{2}{3}\right)^n K_0\right) \setminus T(\varepsilon_G, C_1)$ holds for every $n \in \mathbb{N}$. It is easy to see that $\tilde{T}\left(\zeta_0, \left(\frac{2}{3}\right)^n K_0\right) \setminus T(\varepsilon_G, C_1)$ shrinks to the origin, as n tends to infinity. Thus, (z_n, y_n) tends to the fixed point also in this case. Consequently, $\tilde{T}_0$ is also in the region of attraction of the fixed point. □

Now, $\tilde{T}_0$ is a neighborhood of the origin. However, during the computer-aided part of our method we need thicker (in the direction y) neighborhoods.

Proposition 23. *Let* $K_n = \left(\frac{3}{2}\right)^n K_0$ *and*

$$\zeta_1 = 0.01804, \quad \zeta_2 = 0.01731, \quad \zeta_3 = 0.01664, \quad \zeta_4 = 0.01601, \quad \zeta_5 = 0.01543,$$
$$\zeta_6 = 0.01489, \quad \zeta_7 = 0.01437, \quad \zeta_8 = 0.01389, \quad \zeta_9 = 0.01342, \quad \zeta_{10} = 0.01297.$$

Then $\tilde{T}_n = \tilde{T}(\zeta_n, K_n)$ *is in the region of attraction of the origin for* $n \in \{1, 2, \ldots, 10\}$.

Proof. We only need to show that $H_a(\tilde{T}_n \backslash \tilde{T}_{n-1}) \subseteq \tilde{T}_{n-1}$ holds for every $n \in \{1, 2, \ldots, 10\}$. It can be checked that the decreasing sequence $(\zeta_n)_{n=0}^{10}$ satisfies

$$\zeta_n + d_0 g^{max}\big(\zeta_n, \phi^{max}(\zeta_n) + K_n\big) \leq \zeta_{n-1},$$

so the z-direction inclusion is shown. Proposition 19 can be applied, since $\zeta_n \leq \rho_1$ for every $n \in \{1, 2, \ldots, 10\}$, and σ was chosen such that $\phi(\zeta_0) + \left(\frac{3}{2}\right)^{10} K_0 \leq \sigma$. Hence, the y-directional inclusion follows from (3.47). □

Note that for every $n \in \{0, 1, \ldots, 10\}$ the set $\tilde{T}_n$ is an attracting neighborhood of the trivial fixed point of (3.10), and the size of these neighborhoods are independent of $a \in \mathcal{I}_0$. Clearly, it still needs to be transformed, in order to obtain a neighborhood $\mathcal{M}$ of u_A. This transformation is, however, handled by the algorithm, see Section 3.7.

3.6 Computer-assisted part for a fixed a

To study the global behavior of map (3.2) we follow the method from [4] and [5]. In this subsection we show how use it for a fixed $a \in \left[\frac{4}{3}, a_0\right]$. The method is essentially the same as in [4], so we just outline the steps. The only main difference is that the two-dimensional squares are replaced by three-dimensional cubes. The detailed description of the method along with the correctness of that can be found in [4] and in [5].

For a given $a \in \left[\frac{4}{3}, a_0\right]$ we associate the map (3.2) with a directed graph reflecting the behavior of the map up to a given resolution. Using this graph we show that every point of S enters the previously obtained attracting neighborhood $\mathcal{M}$ of the nontrivial fixed point u_A.

The construction of the graph representation in our case is the following. For a fixed $k \in \mathbb{N}$ we divide the unit cube $[0, 1]^3$ parallel to the faces into small congruent closed cubes with side length $r = 2^{-k}$. These small cubes serve as the cover $\mathfrak{S}$ of S and also as the vertices of the graph, too. To determine the edges we construct with interval arithmetic methods (see [15]) a rectangular cuboid for every small cube $\mathfrak{s}_1$ such that this cuboid contains $f(\mathfrak{s}_1)$. So there is an edge from $\mathfrak{s}_1$ to $\mathfrak{s}_2$ if the cuboid constructed for $\mathfrak{s}_1$ intersects the small cube $\mathfrak{s}_2$. Note that instead of map (3.2) we use the third iterate of it, since the formula is still compact enough not to cause big overestimates in interval arithmetic and it considerably speeds up the calculations.

As we saw in Section 2.4, we only need to study the essential SCCs. Our aim is to show that the SCCs are in the attracting neighborhood $\mathcal{M}$ of the fixed point u_A, which neighborhood was constructed analytically in the previous subsections.

It is important to note that it is possible for some essential SCC that in fact none of the points of S can get stuck here. Actually, this can be the case close to the origin. Since this fixed point of map (3.2) is a saddle, the small cube containing this point always has a loop in the graph representation, consequently, it is always an element of an essential SCC. Let us suppose that this SCC is inside the set $[0, \tilde{A}]^3$ for some $\tilde{A} < A$, and a point gets stuck in this SCC, i.e., $(u^{3k})_{k=0}^{\infty}$ is in $[0, \tilde{A}]^3$, (since we considered the third iterate of f). From the transition graph (see Figure 3.2) it follows that $(u^k)_{k=0}^{\infty}$ is also in $[0, \tilde{A}]^3$. Using Proposition 14 we obtain $u^k \to u_A$, which contradicts $\tilde{A} < A$. Consequently, if the essential SCC containing the origin is included in $[0, \tilde{A}]^3$ for some $\tilde{A} < A$, then we can

algorithm Log3d
  $V \leftarrow$ the initial partition with $r = 2^{-10}$
  repeat
    $E \leftarrow$ construct the edges with reliable numerical method
    $C \leftarrow$ determine the SCCs of directed graph $G(V, E)$
    remove vertices of the nonessential SCCs from V
    remove vertices of the SCC at the origin from V **if** possible
    remove the initial attracting neighborhood from V
    $V \leftarrow$ refine the partition by setting $r \leftarrow \frac{r}{2}$
  until $|V| = \emptyset$
end algorithm

Table 3.1: Algorithm for the three-dimensional logistic map

exclude this SCC from our study.

As a next step we refine the partition by dividing the small cubes into eight identical smaller ones. Then we restart the process, i.e., we determine their images with reliable numerical methods and construct the SCCs again. Note that with the refinements the graph representation becomes a more and more accurate approximation of the represented map, so it is likely to appear new nonessential SCCs, which can be excluded from our study. This property slows a little bit down the eightfold increase in the number of vertices caused by the refinement. Finally, instead of checking after every refinement, whether the remaining SCCs are in the analytically constructed attracting neighborhood $\mathcal{M}$, we can simply remove the small cubes lying entirely in $\mathcal{M}$. So for $d = 2$ and a fixed a, Conjecture 1 is proven if the set of the essential SCCs is empty after some refinements. For the correctness of these steps see [4]. Note that in Propositions 16 and 23 we obtained attracting neighborhoods in $\mathbb{C} \times \mathbb{R}$. Therefore, we need a transformation so that we can determine whether a small cube is inside the region of attraction of the fixed point or not.

We also mention here the fact that in $T(r, C)$ and consequently in $\tilde{T}(r, K)$ the constant r can be enlarged only at the expense of C (and K), so we need to find the balance between them. In the optimal case the obtained attracting neighborhood should resemble the shape of the remaining set after some iteration in the computer-aided part of the method. Neither r nor C can be too small. If r is not sufficiently large, we need more refinement steps and the exponentially growing number of vertices causes some difficulties. On the other side, if C (and K) is too small, then the initial attracting neighborhood is too thin in direction y, and we can not remove any small cubes at all for a long time, as the small cubes are larger than the thickness of the attracting neighborhood. Here, we remark that the removal of the cubes in the initial attracting neighborhood considerably reduces the number of the vertices, which results in faster running of the algorithm.

interval	Subinterval length	Proposition	Running time
$\left[\frac{4}{3}, 1.56\right]$	10^{-3}	16	255 sec
$[1.56, 1.6]$	10^{-4}	16	35 min
$[1.6, a_0 - 10^{-2}]$	10^{-5}	16	4 h
$[a_0 - 10^{-2}, 1.616]$	10^{-4}	23	52 min
$[1.616, a_0]$	10^{-5}	23	10 h

Table 3.2: The partition of the parameter range

Table 3.1 shows the pseudo code of the algorithm. The whole program code and the outputs can be found on [25]. Note that the aforementioned method can be regarded as a proof, since the graph problems are finite, so the computer can work on them punctually. Moreover, the method used during the construction of edges was executed with reliable numerical methods. Thus, if we would have sufficient time, then we could reconstruct by hand the parts which were executed by computer, and we would come to the same conclusion if our estimates are as good as the computer's.

3.7 Computer-assisted part for an interval of a

In the previous subsections we obtained an attracting neighborhood and then a method to prove the global stability of the nontrivial fixed point for a fixed $a \in \left[\frac{4}{3}, a_0\right]$. In this subsection we show how to modify our method to handle not only a single value of the interval $\left[\frac{4}{3}, a_0\right]$ but also a small subinterval $[a] = [a_-, a_+]$ of that, instead.

When we replace the single parameter value with an interval, we obtain rougher estimates, as we handle more parameter values together at the same time. Far away from a_0 the convergence is relatively fast, so we can use longer subintervals when we divide the interval $\left[\frac{4}{3}, a_0\right]$ into small intervals (see Table 3.2). Far away from a_0 the algorithm is still fast enough with these rougher estimates. However, close to a_0 the convergence is much slower, so the precision of estimates is more crucial in this case. Therefore, we need to use finer partition close to a_0.

When we apply our method to a small subinterval $[a]$, essentially two modifications need to be done. First, we need to adjust the function (3.2) during the construction of edges in the graph representation. For a given subinterval $[a]$ and a given small cube $\mathfrak{s}$ we consider a set of small cubes such that they cover $F_3^3(\mathfrak{s})$ for every $a \in [a]$. Second, we also need to modify the attracting neighborhood we remove during the algorithm. For a given subinterval $[a]$ the attracting neighborhood must be chosen such that it is inside the region of attraction of the fixed point for every $a \in [a]$. Note that not only the size of the neighborhood but also the location of the fixed point u_A can vary for different values from $[a]$.

For $[a] \subseteq \left[\frac{4}{3}, a_0 - 10^{-2}\right]$ we use the linearized map and Proposition 16. For a given $[a]$ the size of the neighborhood can be chosen as $\min_{a\in[a]} \xi(a)$. For $[a] \subseteq \mathcal{I}_0$ we use the attracting neighborhoods from Proposition 23 which are obtained by the center manifold reduction and the bifurcational normal form. In this case the size of the neighborhood is independent of the choice of $[a]$. In both propositions the neighborhood is given in the $\mathbb{C} \times \mathbb{R}$, but the small cubes are in $\mathbb{R}^3$, and thus we need to transform them first. We accomplish the transformation with computer using interval arithmetic calculation.

For a given $[a]$ and small cube k we use interval arithmetic calculations to determine the new coordinates in the z–y space. First, we need to shift the small cube with the interval version $u_{[A]}$ of u_A. Here, every coordinate of $u_{[A]}$ is an interval containing u_A for every $a \in [a]$, i.e., every coordinate of $u_{[A]}$ is $[A] = \left[1 - \frac{1}{a_-}, 1 - \frac{1}{a_+}\right]$. Then similarly, we apply the interval version $Q_{[a]}^{-1}$ of (3.7) to obtain $[y]$ and $[z]$. Here, every element of Q_a^{-1} is replaced by an interval containing that element of Q_a^{-1} for every $a \in [a]$. Thus, $[y] \subseteq \mathbb{R}$ is an interval and $[z] \subseteq \mathbb{C}$ is a disc such that

$$Q_a^{-1}(u - u_A) \in \{(z, y) : \ z \in [z],\, y \in [y]\}$$

for every $a \in [a]$ and $u \in \mathfrak{s}$.

For $[a] \subseteq \left[\frac{4}{3}, a_0 - 10^{-2}\right]$ we only need to check whether

$$\max_{y\in[y]} |y| + \max_{z\in[z]} |z| \leq \min_{a\in[a]} \xi(a).$$

For $[a] \subseteq \mathcal{I}_0$ first, we need to determine with interval arithmetic the image $[\phi] \subseteq \mathbb{R}$ of $[z]$ under the map $\phi_{[a]}$. Here, $\phi_{[a]}$ means the interval version of ϕ, i.e., every coefficient ω_{ij} is replaced by a disk in the complex plane such that this disk contains $\omega_{ij}(a)$ for every $a \in [a]$. Hence, for every $a \in [a]$ and $z \in [z]$ we have $\phi(z) \in [\phi]$. Then we need to check the inequalities

$$\max_{z\in[z]} |z| \leq \zeta_n, \qquad \max_{x\in[\phi]} |x| - \min_{y\in[y]} |y| \leq K_n, \qquad \max_{y\in[y]} |y| - \min_{x\in[\phi]} |x| \leq K_n$$

for some $n \in \{0, 1, \ldots, 10\}$. Practically, for a given $[a]$ we use larger n at the beginning of the algorithm, since K_n needs to be large enough compared to the refinement of the partition of the unit cube. In this way we can remove a lot of small cubes close to the nontrivial fixed point. Thus, we can reduce the size of the graph which considerably speeds up our algorithm. Later, when the partition is finer, we can use smaller n to obtain a larger (along the z-coordinate) attracting set, which makes our program finish

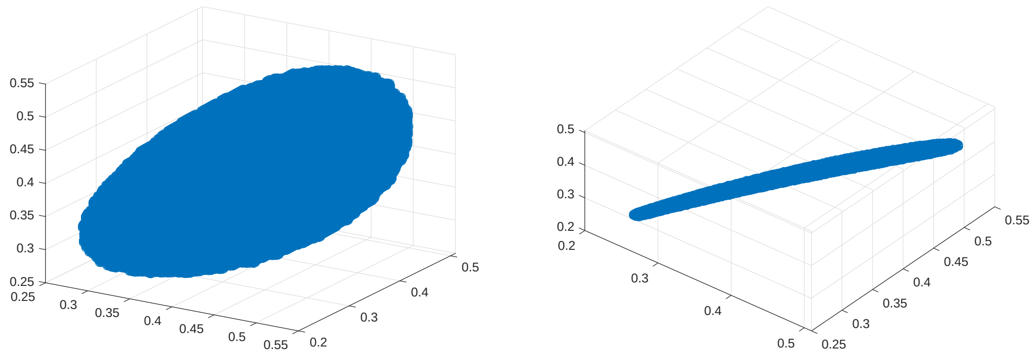


Figure 3.5: The output from two different points of view for $a = 1.612$ after 4 iterations

earlier.

In Figure 3.5 we can see that for $a = 1.612$ after 4 iterations the union of the remaining small cubes are a flat set along the center manifold. The running times can also be found in Table 3.2. It can be observed that close to $a_0 - 10^{-2}$ the first method using the linearization becomes less and less efficient. If our aim had been to reduce the running time, then we could have repeated the second method using the normal form and the center manifold on a larger interval. The calculations in the proofs would have differed only in the specific values. For the sake of example some results can be found on our website see [25].

The program runs successfully, so Conjecture 1 is proven for $d = 2$.

Chapter 4

Appendix

In most cases, for the sake of transparency we do not express that coefficients depend on a, but keep it mind they actually do. Including, but not limited to $G_{ij} = G_{ij}(a)$, $h_{ij} = h_{ij}(a)$ etc.

4.1 The two-dimensional logistic map

The complex eigenvalue with positive real part is

$$\lambda(a) = \frac{1}{2} + \frac{1}{2} i \nabla(a)$$

where $\nabla(a) = \sqrt{4a-5}$.

$$G_{20} = -a + \frac{i}{\nabla(a)}, \qquad G_{11} = \frac{ia}{\nabla(a)}, \qquad G_{02} = a + \frac{i}{\nabla(a)}$$

$$h_{20} = \frac{4a}{4a-5+i\nabla(a)}, \qquad h_{11} = \frac{4ia}{\nabla(a)\left(-i+\nabla(a)\right)^2},$$

$$h_{02} = \frac{a\left(i - \frac{1}{\nabla(a)}\right)}{i - ia + \nabla(a)},$$

$$h_{30} = -\frac{12ia^2}{-2\left(5i+\nabla(a)\right) + a\left(13i - \nabla(a) + 2a\left(-2i+\nabla(a)\right)\right)},$$

$$h_{12} = \frac{16a^2\left(2 - 2i\nabla(a) + a\left(-5+2a-\nabla(a)\right)\right)}{\nabla(a)\left(-i+\nabla(a)\right)^4\left(-7i - 3\nabla(a) + a\left(7i - ia + 2\nabla(a)\right)\right)}$$

$$h_{03} = \frac{96a^2\left(-2+a+i\nabla(a)\right)}{\nabla(a)\left(1+i\nabla(a)\right)\left(a-1+i\nabla(a)\right)\left(12i(a-1) - 7\nabla(a) + \nabla^3(a)\right)}$$

$$\tilde{h}_{20} = -h_{20}, \qquad \tilde{h}_{11} = -h_{11}, \qquad \tilde{h}_{02} = -h_{02}$$

$$\tilde{h}_{30} = 3h_{20}^2 - h_{30} + 3h_{11}\bar{h}_{02}, \qquad \tilde{h}_{21} = 3h_{11}h_{20} + h_{02}\bar{h}_{02} + 2h_{11}\bar{h}_{11}$$

$$\tilde{h}_{12} = 2h_{11}^2 - h_{12} + h_{02}h_{20} + 2h_{02}\bar{h}_{11} + h_{11}\bar{h}_{20}, \qquad \tilde{h}_{03} = 3h_{02}h_{11} - h_{03} + 3h_{02}\bar{h}_{20}$$

$$\tilde{h}_{40} = -15h_{20}^3 + 10h_{20}h_{30} - 30h_{11}h_{20}\bar{h}_{02} - 3h_{02}\bar{h}_{02}^2 + 4h_{11}\bar{h}_{03} - 12h_{11}\bar{h}_{02}\bar{h}_{11}$$

$$\begin{aligned}\tilde{h}_{31} = &-15h_{11}h_{20}^2 + 4h_{11}h_{30} - 12h_{11}^2\bar{h}_{02} + 3h_{12}\bar{h}_{02} - 6h_{02}h_{20}\bar{h}_{02} + h_{02}\bar{h}_{03} \\ &- 12h_{11}h_{20}\bar{h}_{11} - 6h_{02}\bar{h}_{02}\bar{h}_{11} - 6h_{11}\bar{h}_{11}^2 + 3h_{11}\bar{h}_{12} - 3h_{11}\bar{h}_{02}\bar{h}_{20}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{22} = &-12h_{11}^2h_{20} + 3h_{12}h_{20} - 3h_{02}h_{20}^2 + h_{02}h_{30} + h_{03}\bar{h}_{02} - 9h_{02}h_{11}\bar{h}_{02} - 12h_{11}^2\bar{h}_{11} \\ &+ 4h_{12}\bar{h}_{11} - 6h_{02}h_{20}\bar{h}_{11} - 6h_{02}\bar{h}_{11}^2 + 2h_{02}\bar{h}_{12} - 3h_{11}h_{20}\bar{h}_{20} - 3h_{02}\bar{h}_{02}\bar{h}_{20} - 6h_{11}\bar{h}_{11}\bar{h}_{20}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{13} = &-6h_{11}^3 + 6h_{11}h_{12} + h_{03}h_{20} - 9h_{02}h_{11}h_{20} - 3h_{02}^2\bar{h}_{02} + 3h_{03}\bar{h}_{11} - 18h_{02}h_{11}\bar{h}_{11} \\ &- 6h_{11}^2\bar{h}_{20} + 3h_{12}\bar{h}_{20} - 3h_{02}h_{20}\bar{h}_{20} - 12h_{02}\bar{h}_{11}\bar{h}_{20} - 3h_{11}\bar{h}_{20}^2 + h_{11}\bar{h}_{30}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{04} = &\,4h_{03}h_{11} - 12h_{02}h_{11}^2 + 6h_{02}h_{12} - 3h_{02}^2h_{20} - 12h_{02}^2\bar{h}_{11} + 6h_{03}\bar{h}_{20} \\ &- 18h_{02}h_{11}\bar{h}_{20} - 15h_{02}\bar{h}_{20}^2 + 4h_{02}\bar{h}_{30}\end{aligned}$$

$$|\beta_{40}| = \frac{24a^3\sqrt{1 + a + \frac{4}{9(-5+4a)} + \frac{54-10a(3+a)}{9(-9+a(12+(-5+a)a))}}}{4 + a(-6 + a + a^2)}$$

$$|\beta_{31}| = \frac{6a^3\sqrt{4 - 6a + 21a^2 - 242a^3 + 741a^4 - 1035a^5 + 824a^6 - 426a^7 + 148a^8 - 32a^9 + 4a^{10}}}{(-1+a)^2(-5+4a)\sqrt{(-1+a)(1+a)(-4+a(2+a))(-9+a(12+(-5+a)a))}}$$

$$|\beta_{22}| = \frac{4a^3\sqrt{1 + 2a + 10a^2 + 6a^3 + 220a^4 - 434a^5 + 222a^6 - 20a^7 + a^8}}{\sqrt{-5+4a}(-1+a)^3(1+a)(-4+a(2+a))}$$

name	value	first occurrence	context
a_0	$\frac{\sqrt{5}+1}{2}$	Chapter 1	$a \in [1, a_0]$
C_0	37.379	Proposition 17	$\lvert\mathcal{N}(\phi(z))\rvert \leq C_0 \lvert z\rvert^5$
C_1	7700	Propsition 20	$(z, y) \in T(\rho_1, C_1)$
ε_G	0.01976	Proposition 21	$T(\varepsilon_G, C_1)$ - region of attraction
K_0	$2.74 \cdot 10^{-5}$	Proposition 22	$\tilde{T}_0 = \tilde{T}(\zeta_0, K_0)$
L	0.66	Proposition 19	$\lvert\phi(z_1) - y_1\rvert \leq L\lvert\phi(z_0) - y_0\rvert + C_0\lvert z_0\rvert^5$
ρ_0	0.021	(3.40)	$\lvert w\rvert \leq \rho_0$
ρ_1	0.02334	Proposition 17	$(z, y) \in T(\rho_1, C_1)$, $\lvert h(B_{\rho_0})\rvert \leq \rho_1$
$\tilde{\rho}_2$	0.0237	Proposition 19	
ρ_2	0.02354	(3.40)	$\lvert G(B_{\rho_1})\rvert \leq \rho_2$
ρ_3	0.02532	(3.40)	$\lvert h^{-1}(B_{\rho_2})\rvert \leq \rho_3$
σ	$2.1 \cdot 10^{-3}$	Proposition 19	$\lvert y\rvert \leq \sigma$
ζ_0	0.01883	Proposition 22	$\tilde{T}_0 = \tilde{T}(\zeta_0, K_0)$

Table 4.1: The frequently used numerical constants

$$
\begin{aligned}
|\beta_{13}| = 6a^3\big(&- 256 + 64a + 1676a^2 - 2498a^3 - 95a^4 + 2796a^5 \\
&- 2219a^6 + 187a^7 + 730a^8 - 550a^9 + 200a^{10} - 40a^{11} + 4a^{12}\big)^{\frac{1}{2}} \\
&\Big((-1+a)^{\frac{5}{2}}(-5+4a)(-4+a(2+a))\sqrt{(1+a)(-9+a(12+(-5+a)a))}\Big)^{-1}
\end{aligned}
$$

$$
|\beta_{04}| = \frac{24a^3\sqrt{-11+a(-6+a(27+a(-17+4a)))}}{(-4+a(2+a))\sqrt{(-1+a)(-5+4a)(-9+a(12+(-5+a)a))}}
$$

4.2 The three-dimensional logistic map

In Table 4.1 we collected the most important numerical constants with their first occurrences.

4.2.1 The eigenvalues

The complex eigenvalue with positive real part and the real eigenvalue are

$$\lambda(a) = \frac{1}{3} + \frac{1 - i\sqrt{3}}{3 \cdot 2^{\frac{2}{3}} \left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}} + \frac{(1 + i\sqrt{3}) \left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}}{6 \cdot 2^{\frac{1}{3}}},$$

$$\nu(a) = \frac{1}{3} - \frac{2^{\frac{1}{3}}}{3\left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}} - \frac{\left(-29 + 27a + 3\sqrt{3\nabla(a)}\right)^{\frac{1}{3}}}{3 \cdot 2^{\frac{1}{3}}},$$

where $\nabla(a) = 31 - 58a + 27a^2$.

4.2.2 The coefficients of $g(z, y)$

If $a > \frac{31}{27}$, then

$$g_{ijk} = -a\left(i\lambda^2 + j\bar{\lambda}^2 + k\nu^2\right),$$

for $i + j + k = 2$ and $g_{ijk} = 0$ for $i + j + k > 2$.

4.2.3 The coefficients of the center manifold

The coefficients are determined so that in the expression (3.12) the at most fourth-order terms of $z, \bar{z}$ are eliminated, see Section 3.3.

$$\omega_{ji} = \overline{\omega}_{ij}, \qquad \omega_{20} = \frac{g_{200}e}{\lambda^2 - \nu}, \qquad \omega_{11} = \frac{g_{110}e}{\lambda\bar{\lambda} - \nu}$$

$$\omega_{30} = \frac{1}{\lambda^3 - \nu}\Big(e\left(3g_{101}\omega_{20} + g_{300}\right) - 3\lambda g_{200}(\bar{d}\omega_{11} + d\omega_{20})\Big)$$

$$\omega_{21} = \frac{1}{\lambda^2\bar{\lambda} - \nu}\Big(e\left(g_{011}\omega_{20} + 2g_{101}\omega_{11} + g_{210}\right) - \bar{d}\left(\omega_{02}g_{200}\bar{\lambda} + 2\lambda\omega_{11}g_{110}\right)$$
$$- d\left(\omega_{11}g_{200}\bar{\lambda} + 2\lambda g_{110}\omega_{20}\right)\Big)$$

$$\omega_{40} = \frac{1}{\lambda^4 - \nu}\Big(e(3g_{002}\omega_{20}^2 + 4g_{101}\omega_{30} + 6\omega_{20}g_{201} + g_{400}) - 3\bar{d}^2\omega_{02}g_{200}^2 - 2\bar{d}\left(2\lambda\omega_{11}\left(3g_{101}\omega_{20} + g_{300}\right) + 3d\omega_{11}g_{200}^2 + 3\lambda^2 g_{200}\omega_{21}\right) - d\left(4\lambda\omega_{20}\left(3g_{101}\omega_{20} + g_{300}\right) + 3d\omega_{20}g_{200}^2 + 6\lambda^2 g_{200}\omega_{30}\right)\Big)$$

$$\omega_{31} = \frac{-1}{\lambda^3\bar{\lambda} - \nu}\Big(- e\Big(3\omega_{11}\left(g_{002}\omega_{20} + g_{201}\right) + g_{011}\omega_{30} + 3g_{101}\omega_{21} + 3g_{111}\omega_{20} + g_{310}\Big) + d\Big(3\lambda\left(g_{011}\omega_{20}^2 + \omega_{20}\left(2g_{101}\omega_{11} + g_{210}\right) + g_{200}\omega_{21}\bar{\lambda}\right) + \omega_{11}\bar{\lambda}\left(3g_{101}\omega_{20} + g_{300}\right) + 3dg_{110}\omega_{20}g_{200} + 3\lambda^2 g_{110}\omega_{30}\Big) + \bar{d}\Big(3\lambda\left(\omega_{11}\left(g_{011}\omega_{20} + g_{210}\right) + 2g_{101}\omega_{11}^2 + \omega_{12}g_{200}\bar{\lambda}\right) + \omega_{02}\bar{\lambda}\left(3g_{101}\omega_{20} + g_{300}\right) + 6d\omega_{11}g_{110}g_{200} + 3\lambda^2 g_{110}\omega_{21} + 3\bar{d}\omega_{02}g_{110}g_{200}\Big)\Big)$$

$$\omega_{22} = \frac{-1}{\lambda^2\bar{\lambda}^2 - \nu}\Big(- e\Big(\omega_{02}\left(g_{002}\omega_{20} + g_{201}\right) + 2g_{002}\omega_{11}^2 + 2g_{011}\omega_{21} + g_{021}\omega_{20} + 2g_{101}\omega_{12} + 4\omega_{11}g_{111} + g_{220}\Big) + d\Big(2\lambda(2g_{011}\omega_{11}\omega_{20} + \omega_{02}g_{101}\omega_{20} + 2g_{110}\omega_{21}\bar{\lambda} + g_{120}\omega_{20}) + 2\omega_{11}\bar{\lambda}\left(g_{011}\omega_{20} + 2g_{101}\omega_{11} + g_{210}\right) + d\omega_{20}(g_{020}g_{200} + 2g_{110}^2) + \lambda^2 g_{020}\omega_{30} + \omega_{12}g_{200}\bar{\lambda}^2\Big) + \bar{d}\Big(2\lambda\left(2g_{011}\omega_{11}^2 + \omega_{11}\left(\omega_{02}g_{101} + g_{120}\right) + 2g_{110}\omega_{12}\bar{\lambda}\right) + 2\omega_{02}\bar{\lambda}\left(g_{011}\omega_{20} + 2g_{101}\omega_{11} + g_{210}\right) + 2d\omega_{11}\left(g_{020}g_{200} + 2g_{110}^2\right) + \lambda^2 g_{020}\omega_{21} + \omega_{03}g_{200}\bar{\lambda}^2 + \bar{d}\omega_{02}\left(g_{020}g_{200} + 2g_{110}^2\right)\Big)\Big)$$

4.2.4 The fifth-order terms of $\mathcal{N}(\phi(z))$

The fifth-order terms of $\mathcal{N}(\phi(z))$ appear in the proof of Proposition 17. For the definition of $\mathcal{N}(\phi(z))$ see (3.12).

$$N_{ij} = \overline{N}_{ji}$$

$$
\begin{aligned}
N_{50} = 5\Big(&3\bar{d}^2 g_{200}(g_{200}\lambda\omega_{12} + 2g_{101}\omega_{02}\omega_{20}) + 6d\bar{d}g_{200}(2g_{101}\omega_{11}\omega_{20} + g_{200}\lambda\omega_{21}) \\
&+ 3d^2 g_{200}(2g_{101}\omega_{20}^2 + g_{200}\lambda\omega_{30}) - e(2g_{002}\omega_{20}\omega_{30} + g_{101}\omega_{40}) \\
&+ \bar{d}\lambda(3g_{002}\omega_{11}\omega_{20}^2 + 6g_{101}\lambda\omega_{20}\omega_{21} + 4g_{101}\omega_{11}\omega_{30} + 2g_{200}\lambda^2\omega_{31}) \\
&+ d\lambda\big(3g_{002}\omega_{20}^3 + 2g_{101}(2+3\lambda)\omega_{20}\omega_{30} + 2g_{200}\lambda^2\omega_{40}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
N_{41} = {}&3\bar{d}^2\big(g_{200}^2\bar{\lambda}\omega_{03} + 4g_{101}g_{110}\omega_{02}\omega_{20} + 2g_{200}(2g_{101}\omega_{02}\omega_{11} + 2g_{110}\lambda\omega_{12} + g_{011}\omega_{02}\omega_{20})\big) \\
&+ 3d^2\big(4g_{101}\omega_{20}(g_{200}\omega_{11} + g_{110}\omega_{20}) + g_{200}(2g_{011}\omega_{20}^2 + g_{200}\bar{\lambda}\omega_{21} + 4g_{110}\lambda\omega_{30})\big) \\
&+ \bar{d}\Big(3g_{002}\omega_{20}(4\lambda\omega_{11}^2 + \bar{\lambda}\omega_{02}\omega_{20}) \\
&+ 6d\big(4g_{101}\omega_{11}(g_{200}\omega_{11} + g_{110}\omega_{20}) + g_{200}(g_{200}\bar{\lambda}\omega_{12} + 2g_{011}\omega_{11}\omega_{20} + 4g_{110}\lambda\omega_{21})\big) \\
&+ 4g_{101}(3\lambda^2\omega_{11}\omega_{21} + 3\lambda(\bar{\lambda}\omega_{12}\omega_{20} + \omega_{11}\omega_{21}) + \bar{\lambda}\omega_{02}\omega_{30}) \\
&+ 2\lambda(3g_{011}\lambda\omega_{20}\omega_{21} + 3g_{200}\lambda\bar{\lambda}\omega_{22} + 2g_{011}\omega_{11}\omega_{30} + 2g_{110}\lambda^2\omega_{31})\Big) \\
&- e(6g_{002}\omega_{20}\omega_{21} + 4g_{002}\omega_{11}\omega_{30} + 4g_{101}\omega_{31} + g_{011}\omega_{40}) \\
&+ d\Big(3g_{002}(4\lambda + \bar{\lambda})\omega_{11}\omega_{20}^2 + 4g_{101}\big(3\lambda(1+\bar{\lambda})\omega_{20}\omega_{21} + 3\lambda^2\omega_{11}\omega_{30} + \bar{\lambda}\omega_{11}\omega_{30}\big) \\
&+ 2\lambda\big(g_{011}(2+3\lambda)\omega_{20}\omega_{30} + \lambda(3g_{200}\bar{\lambda}\omega_{31} + 2g_{110}\lambda\omega_{40})\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
N_{32} = {} & 3\bar{d}^2\big(2g_{110}g_{200}\bar{\lambda}\omega_{03} + 2g_{011}g_{200}\omega_{02}\omega_{11} + 2g_{110}^2\lambda\omega_{12} + g_{020}g_{200}\lambda\omega_{12} \\
& + 2g_{011}g_{110}\omega_{02}\omega_{20} + g_{101}\omega_{02}(g_{200}\omega_{02} + 4g_{110}\omega_{11} + g_{020}\omega_{20})\big) \\
& + 3d^2\big(g_{101}\omega_{20}(g_{200}\omega_{02} + 4g_{110}\omega_{11} + g_{020}\omega_{20}) + 2g_{011}\omega_{20}(g_{200}\omega_{11} + g_{110}\omega_{20}) \\
& + 2g_{110}g_{200}\bar{\lambda}\omega_{21} + 2g_{110}^2\lambda\omega_{30} + g_{020}g_{200}\lambda\omega_{30}\big) \\
& - e\big(3g_{101}\omega_{22} + g_{002}(3\omega_{12}\omega_{20} + 6\omega_{11}\omega_{21} + \omega_{02}\omega_{30}) + 2g_{011}\omega_{31}\big) \\
& + \bar{d}\Big(6g_{002}\lambda\omega_{11}^3 + 6g_{101}\lambda\omega_{11}\omega_{12} + 12g_{101}\lambda\bar{\lambda}\omega_{11}\omega_{12} + 3g_{200}\lambda\bar{\lambda}^2\omega_{13} + 3g_{101}\bar{\lambda}^2\omega_{03}\omega_{20} \\
& + 3g_{002}\lambda\omega_{02}\omega_{11}\omega_{20} + 6g_{002}\bar{\lambda}\omega_{02}\omega_{11}\omega_{20} + 6g_{011}\lambda\bar{\lambda}\omega_{12}\omega_{20} + 3g_{101}\lambda^2\omega_{02}\omega_{21} \\
& + 6g_{101}\bar{\lambda}\omega_{02}\omega_{21} + 6g_{011}\lambda\omega_{11}\omega_{21} + 6g_{011}\lambda^2\omega_{11}\omega_{21} + 6d\big(2g_{110}g_{200}\bar{\lambda}\omega_{12} \\
& + g_{101}\omega_{11}(g_{200}\omega_{02} + 4g_{110}\omega_{11} + g_{020}\omega_{20}) + 2g_{011}\omega_{11}(g_{200}\omega_{11} + g_{110}\omega_{20}) + 2g_{110}^2\lambda\omega_{21} \\
& + g_{020}g_{200}\lambda\omega_{21}\big) + 6g_{110}\lambda^2\bar{\lambda}\omega_{22} + 2g_{011}\bar{\lambda}\omega_{02}\omega_{30} + g_{020}\lambda^3\omega_{31}\Big) \\
& + d\Big(3g_{002}\omega_{20}(2\lambda\omega_{11}^2 + 2\bar{\lambda}\omega_{11}^2 + \lambda\omega_{02}\omega_{20}) + 6g_{011}\lambda\omega_{20}\omega_{21} \\
& + 6g_{011}\lambda\bar{\lambda}\omega_{20}\omega_{21} + 3g_{200}\lambda\bar{\lambda}^2\omega_{22} + 6g_{011}\lambda^2\omega_{11}\omega_{30} + 2g_{011}\bar{\lambda}\omega_{11}\omega_{30} \\
& + 3g_{101}(2\lambda\omega_{12}\omega_{20} + \bar{\lambda}^2\omega_{12}\omega_{20} + 2\bar{\lambda}\omega_{11}\omega_{21} + 4\lambda\bar{\lambda}\omega_{11}\omega_{21} + \lambda^2\omega_{02}\omega_{30}) \\
& + 6g_{110}\lambda^2\bar{\lambda}\omega_{31} + g_{020}\lambda^3\omega_{40}\Big)
\end{aligned}
$$

4.2.5 The lower-order terms of $G(z)$

For the definition of $G(z)$ and its coefficients see (3.28) and (3.29).

$$\bar{d}\,G_{ij} = d\,\bar{G}_{ji}$$

$$
\begin{aligned}
G_{20} &= -\,2ad\lambda^2, \qquad G_{11} = -ad(\lambda^2 + \bar{\lambda}^2), \qquad G_{30} = -\,6ad(\lambda^2 + \nu^2)\omega_{20}, \\
G_{21} &= -\,ad\big(2\lambda^2\omega_{11} + 2\nu^2\omega_{11} + \bar{\lambda}^2\omega_{20} + \nu^2\omega_{20}\big), \quad G_{40} = -\,ad(6\nu^2\omega_{20}^2 + 4\lambda^2\omega_{30} + 4\nu^2\omega_{30})
\end{aligned}
$$

$$
\begin{aligned}
G_{31} &= -\,ad(6\nu^2\omega_{11}\omega_{20} + 3\lambda^2\omega_{21} + 3\nu^2\omega_{21} + \bar{\lambda}^2\omega_{30} + \nu^2\omega_{30}) \\
G_{22} &= -\,ad(4\nu^2\omega_{11}^2 + 2\lambda^2\omega_{12} + 2\nu^2\omega_{12} + 2\nu^2\omega_{02}\omega_{20} + 2\bar{\lambda}^2\omega_{21} + 2\nu^2\omega_{21})
\end{aligned}
$$

$$
\begin{aligned}
G_{50} &= -ad(20\nu^2\omega_{20}\omega_{30} + 5\lambda^2\omega_{40} + 5\nu^2\omega_{40}) \\
G_{41} &= -ad(12\nu^2\omega_{20}\omega_{21} + 8\nu^2\omega_{11}\omega_{30} + 4\lambda^2\omega_{31} + 4\nu^2\omega_{31} + \bar{\lambda}^2\omega_{40} + \nu^2\omega_{40}) \\
G_{32} &= -ad\big(3\nu^2\omega_{12}\omega_{20} + 2\nu^2\omega_{31} + 3\nu^2(\omega_{12}\omega_{20} + 2\omega_{11}\omega_{21} + \omega_{22}) \\
&\qquad + 3\lambda^2\omega_{22} + 2\nu^2\omega_{02}\omega_{30} + 2\bar{\lambda}^2\omega_{31} + 6\nu^2\omega_{11}\omega_{21}\big)
\end{aligned}
$$

4.2.6 The coefficients of $h(z)$

The coefficients are chosen so that the second and third-order terms of $h_0^{-1}(G(h(w)))$ are eliminated (apart from $w^2\overline{w}$), see Section 3.4. First, the lower-order terms of $h_0^{-1}(z)$ need to be determined as expressions of h_{ij}, see Subsection 4.2.7.

$$
h_{20} = \frac{G_{20}}{(\lambda^2 - \lambda)}, \qquad h_{11} = \frac{G_{11}}{(\lambda\bar{\lambda} - \lambda)}, \qquad h_{02} = \frac{G_{02}}{(\bar{\lambda}^2 - \lambda)}
$$

$$
\begin{aligned}
h_{30} = \frac{1}{\lambda^3 - \lambda}\Big(&3G_{20}h_{20} + 3G_{11}\bar{h}_{02} + G_{30} - 3h_{20}\lambda(\lambda h_{20} + G_{20}) \\
&- 3h_{11}\lambda(\bar{\lambda}\bar{h}_{02} + \bar{G}_{02}) + 3h_{20}^2\lambda^3 + 3h_{11}\bar{h}_{02}\lambda^3\Big)
\end{aligned}
$$

$$
\begin{aligned}
h_{12} = \frac{1}{\lambda\bar{\lambda}^2 - \lambda}\Big(&G_{20}h_{02} + 2G_{02}\bar{h}_{11} + G_{11}\bar{h}_{20} + 2G_{11}h_{11} + G_{12} - h_{20}\lambda(\lambda h_{02} + G_{02}) \\
&- 2h_{02}\bar{\lambda}(\bar{\lambda}\bar{h}_{11} + \bar{G}_{11}) - h_{11}\big(\lambda(\bar{\lambda}\bar{h}_{20} + \bar{G}_{20}) - 2\bar{\lambda}h_{11}(\lambda h_{11} + G_{11})\big) \\
&+ h_{02}h_{20}\bar{\lambda}^2\lambda + 2h_{11}^2\bar{\lambda}^2\lambda + 2h_{02}\bar{h}_{11}\bar{\lambda}^2\lambda + h_{11}\bar{h}_{20}\bar{\lambda}^2\lambda\Big)
\end{aligned}
$$

$$
\begin{aligned}
h_{03} = \frac{1}{\bar{\lambda}^3 - \lambda}\Big(&3G_{02}\bar{h}_{20} + 3G_{11}h_{02} + G_{03} - 3h_{02}\bar{\lambda}(\bar{\lambda}\bar{h}_{20} + \bar{G}_{20}) \\
&- 3h_{11}\bar{\lambda}(\lambda h_{02} + G_{02}) + 3h_{02}\bar{h}_{20}\bar{\lambda}^3 + 3h_{11}h_{02}\bar{\lambda}^3\Big)
\end{aligned}
$$

4.2.7 The coefficients of $h_0^{-1}(z)$

The coefficients are obtained from the equation $z = h\left(h_0^{-1}(z)\right) + O\left(|z|^6\right)$ by equating the coefficients of the same type up to fifth-order, see Section 3.4.

$$
\tilde{h}_{20} = -h_{20}, \qquad \tilde{h}_{11} = -h_{11}, \qquad \tilde{h}_{02} = -h_{02}
$$

$$\tilde{h}_{30} = 3h_{20}^2 - h_{30} + 3h_{11}\bar{h}_{02}, \qquad \tilde{h}_{21} = 3h_{11}h_{20} + h_{02}\bar{h}_{02} + 2h_{11}\bar{h}_{11}$$

$$\tilde{h}_{12} = 2h_{11}^2 - h_{12} + h_{02}h_{20} + 2h_{02}\bar{h}_{11} + h_{11}\bar{h}_{20}, \qquad \tilde{h}_{03} = 3h_{02}h_{11} - h_{03} + 3h_{02}\bar{h}_{20}$$

$$\tilde{h}_{40} = -15h_{20}^3 + 10h_{20}h_{30} - 30h_{11}h_{20}\bar{h}_{02} - 3h_{02}\bar{h}_{02}^2 + 4h_{11}\bar{h}_{03} - 12h_{11}\bar{h}_{02}\bar{h}_{11}$$

$$\begin{aligned}\tilde{h}_{31} = &-15h_{11}h_{20}^2 + 4h_{11}h_{30} - 12h_{11}^2\bar{h}_{02} + 3h_{12}\bar{h}_{02} - 6h_{02}h_{20}\bar{h}_{02} + h_{02}\bar{h}_{03} \\ &- 12h_{11}h_{20}\bar{h}_{11} - 6h_{02}\bar{h}_{02}\bar{h}_{11} - 6h_{11}\bar{h}_{11}^2 + 3h_{11}\bar{h}_{12} - 3h_{11}\bar{h}_{02}\bar{h}_{20}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{22} = &-12h_{11}^2h_{20} + 3h_{12}h_{20} - 3h_{02}h_{20}^2 + h_{02}h_{30} + h_{03}\bar{h}_{02} - 9h_{02}h_{11}\bar{h}_{02} - 12h_{11}^2\bar{h}_{11} \\ &+ 4h_{12}\bar{h}_{11} - 6h_{02}h_{20}\bar{h}_{11} - 6h_{02}\bar{h}_{11}^2 + 2h_{02}\bar{h}_{12} - 3h_{11}h_{20}\bar{h}_{20} - 3h_{02}\bar{h}_{02}\bar{h}_{20} - 6h_{11}\bar{h}_{11}\bar{h}_{20}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{13} = &-6h_{11}^3 + 6h_{11}h_{12} + h_{03}h_{20} - 9h_{02}h_{11}h_{20} - 3h_{02}^2\bar{h}_{02} + 3h_{03}\bar{h}_{11} - 18h_{02}h_{11}\bar{h}_{11} \\ &- 6h_{11}^2\bar{h}_{20} + 3h_{12}\bar{h}_{20} - 3h_{02}h_{20}\bar{h}_{20} - 12h_{02}\bar{h}_{11}\bar{h}_{20} - 3h_{11}\bar{h}_{20}^2 + h_{11}\bar{h}_{30}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{04} = &\,4h_{03}h_{11} - 12h_{02}h_{11}^2 + 6h_{02}h_{12} - 3h_{02}^2h_{20} - 12h_{02}^2\bar{h}_{11} \\ &+ 6h_{03}\bar{h}_{20} - 18h_{02}h_{11}\bar{h}_{20} - 15h_{02}\bar{h}_{20}^2 + 4h_{02}\bar{h}_{30}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{50} = -5\Big(&- 21h_{20}^4 + 4\bar{h}_{03}h_{11}(\bar{h}_{11} + 3h_{20}) + 21h_{20}^2h_{30} - 2h_{30}^2 \\ &- 3\bar{h}_{02}^2(2\bar{h}_{11}h_{02} + \bar{h}_{20}h_{11} + 6h_{11}^2 - h_{12} + 3h_{02}h_{20}) \\ &+ \bar{h}_{02}\big(2\bar{h}_{03}h_{02} - 3h_{11}(4\bar{h}_{11}^2 - 2\bar{h}_{12} + 12\bar{h}_{11}h_{20} + 21h_{20}^2 - 4h_{30})\big)\Big)\end{aligned}$$

$$
\begin{aligned}
\tilde{h}_{41} = &- 24\bar{h}_{11}\bar{h}_{12}h_{11} + 24\bar{h}_{11}^3 h_{11} - \bar{h}_{02}^2(-9\bar{h}_{20}h_{02} + 3h_{03} - 45h_{02}h_{11}) + 60\bar{h}_{11}^2 h_{11}h_{20} \\
&+ 90\bar{h}_{11}h_{11}h_{20}^2 - 2\bar{h}_{03}(4\bar{h}_{11}h_{02} + 2\bar{h}_{20}h_{11} + 10h_{11}^2 - 2h_{12} + 5h_{02}h_{20}) \\
&+ 105h_{11}h_{20}^3 - 30\bar{h}_{12}h_{11}h_{20} - 20\bar{h}_{11}h_{11}h_{30} - 60h_{11}h_{20}h_{30} \\
&+ \bar{h}_{02}\Big(36\bar{h}_{11}^2 h_{02} - 12\bar{h}_{12}h_{02} + 12\bar{h}_{11}(3\bar{h}_{20}h_{11} + 10h_{11}^2 - 2h_{12} + 5h_{02}h_{20}) \\
&+ 5(6\bar{h}_{20}h_{11}h_{20} + 36h_{11}^2 h_{20} - 6h_{12}h_{20} + 9h_{02}h_{20}^2 - 2h_{02}h_{30})\Big)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{32} = &- 9\bar{h}_{12}\bar{h}_{20}h_{11} + 24\bar{h}_{11}^3 h_{02} + 9\bar{h}_{02}^2 h_{02}^2 + 9\bar{h}_{02}\bar{h}_{20}^2 h_{11} - 3\bar{h}_{02}\bar{h}_{30}h_{11} - 24\bar{h}_{12}h_{11}^2 \\
&+ 36\bar{h}_{02}\bar{h}_{20}h_{11}^2 + 60\bar{h}_{02}h_{11}^3 - \bar{h}_{03}(3\bar{h}_{20}h_{02} - h_{03} + 12h_{02}h_{11}) + 6\bar{h}_{12}h_{12} \\
&- 9\bar{h}_{02}\bar{h}_{20}h_{12} - 36\bar{h}_{02}h_{11}h_{12} - 12\bar{h}_{12}h_{02}h_{20} + 18\bar{h}_{02}\bar{h}_{20}h_{02}h_{20} - 6\bar{h}_{02}h_{03}h_{20} \\
&+ 90\bar{h}_{02}h_{02}h_{11}h_{20} + 15\bar{h}_{20}h_{11}h_{20}^2 + 90h_{11}^2 h_{20}^2 - 15h_{12}h_{20}^2 + 15h_{02}h_{20}^3 \\
&+ 18\bar{h}_{11}^2(2\bar{h}_{20}h_{11} + 4h_{11}^2 - h_{12} + 2h_{02}h_{20}) - 4\bar{h}_{20}h_{11}h_{30} - 20h_{11}^2 h_{30} + 4h_{12}h_{30} \\
&- 10h_{02}h_{20}h_{30} - \bar{h}_{11}\Big(18\bar{h}_{12}h_{02} - 9\bar{h}_{02}(4\bar{h}_{20}h_{02} - h_{03} + 12h_{02}h_{11}) \\
&- 36\bar{h}_{20}h_{11}h_{20} - 120h_{11}^2 h_{20} + 24h_{12}h_{20} - 30h_{02}h_{20}^2 + 8h_{02}h_{30}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{23} = &- 4\bar{h}_{02}\bar{h}_{30}h_{02} + 15\bar{h}_{02}\bar{h}_{20}^2 h_{02} - 3\bar{h}_{03}h_{02}^2 - 6\bar{h}_{02}\bar{h}_{20}h_{03} + 54\bar{h}_{02}\bar{h}_{20}h_{02}h_{11} \\
&- 12\bar{h}_{02}h_{03}h_{11} + 72\bar{h}_{02}h_{02}h_{11}^2 - 3\bar{h}_{12}(4\bar{h}_{20}h_{02} - h_{03} + 9h_{02}h_{11}) \\
&+ 12\bar{h}_{11}^2(5\bar{h}_{20}h_{02} - h_{03} + 9h_{02}h_{11}) - 18\bar{h}_{02}h_{02}h_{12} + 18\bar{h}_{02}h_{02}^2 h_{20} \\
&+ 9\bar{h}_{20}^2 h_{11}h_{20} - 3\bar{h}_{30}h_{11}h_{20} + 36\bar{h}_{20}h_{11}^2 h_{20} + 60h_{11}^3 h_{20} - 9\bar{h}_{20}h_{12}h_{20} \\
&- 36h_{11}h_{12}h_{20} + 9\bar{h}_{20}h_{02}h_{20}^2 - 3h_{03}h_{20}^2 + 45h_{02}h_{11}h_{20}^2 - 12h_{02}h_{11}h_{30} \\
&+ \bar{h}_{11}\Big(36\bar{h}_{02}h_{02}^2 + 30\bar{h}_{20}^2 h_{11} - 8\bar{h}_{30}h_{11} + 72h_{11}^3 - 54h_{11}h_{12} - 9h_{03}h_{20} \\
&+ 108h_{02}h_{11}h_{20} + 12\bar{h}_{20}(6h_{11}^2 - 2h_{12} + 3h_{02}h_{20})\Big) - 3\bar{h}_{20}h_{02}h_{30} + h_{03}h_{30}
\end{aligned}
$$

$$\begin{aligned}\tilde{h}_{14} = & - 12\bar{h}_{12}h_{02}^2 + 60\bar{h}_{11}^2h_{02}^2 + 30\bar{h}_{02}\bar{h}_{20}h_{02}^2 - 10\bar{h}_{02}h_{02}h_{03} + 15\bar{h}_{20}^3h_{11} - 10\bar{h}_{20}\bar{h}_{30}h_{11} \\ & + 45\bar{h}_{02}h_{02}^2h_{11} + 30\bar{h}_{20}^2h_{11}^2 - 8\bar{h}_{30}h_{11}^2 + 36\bar{h}_{20}h_{11}^3 + 24h_{11}^4 - 15\bar{h}_{20}^2h_{12} \\ & + 4\bar{h}_{30}h_{12} - 36\bar{h}_{20}h_{11}h_{12} - 36h_{11}^2h_{12} + 6h_{12}^2 + 15\bar{h}_{20}^2h_{02}h_{20} - 4\bar{h}_{30}h_{02}h_{20} \\ & - 6\bar{h}_{20}h_{03}h_{20} + 54\bar{h}_{20}h_{02}h_{11}h_{20} - 12h_{03}h_{11}h_{20} + 72h_{02}h_{11}^2h_{20} \\ & - 18h_{02}h_{12}h_{20} + 9h_{02}^2h_{20}^2 + 2\bar{h}_{11}\Big(45\bar{h}_{20}^2h_{02} - 15\bar{h}_{20}(h_{03} - 6h_{02}h_{11}) \\ & - 2\big(5\bar{h}_{30}h_{02} + 8h_{03}h_{11} - 3h_{02}(12h_{11}^2 - 4h_{12} + 3h_{02}h_{20})\big)\Big) + 3h_{02}^2h_{30}\end{aligned}$$

$$\begin{aligned}\tilde{h}_{05} = -5\Big(& - 21\bar{h}_{20}^3h_{02} - 3\bar{h}_{02}h_{02}^3 - 2\bar{h}_{30}h_{03} + 10\bar{h}_{11}h_{02}h_{03} + 6\bar{h}_{30}h_{02}h_{11} \\ & - 30\bar{h}_{11}h_{02}^2h_{11} + 4h_{03}h_{11}^2 - 12h_{02}h_{11}^3 + 9\bar{h}_{20}^2(h_{03} - 3h_{02}h_{11}) \\ & - 2h_{03}h_{12} + 12h_{02}h_{11}h_{12} + 2h_{02}h_{03}h_{20} - 9h_{02}^2h_{11}h_{20} \\ & + 2\bar{h}_{20}(6\bar{h}_{30}h_{02} - 18\bar{h}_{11}h_{02}^2 + 4h_{03}h_{11} - 12h_{02}h_{11}^2 + 6h_{02}h_{12} - 3h_{02}^2h_{20})\Big)\end{aligned}$$

4.2.8 The lower-order terms of $G(h(w))$

The composition of $G(z)$ and $h(w)$ can be written in the following form

$$G(h(w)) = \sum_{\substack{i+j=1 \\ i,j\in\mathbb{N}_0}}^{5} \frac{\alpha_{ij}}{i!j!} w^i \overline{w}^j + R_5,$$

where $R_5 = R_5(a, w, \overline{w}, c) = O(|w|^6)$ and $\alpha_{ij} = \alpha_{ij}(a)$ is complex.

$$\alpha_{10} = \lambda \qquad \alpha_{20} = G_{20} + h_{20}\lambda$$

$$\alpha_{11} = G_{11} + h_{11}\lambda \qquad \alpha_{02} = G_{02} + h_{02}\lambda$$

$$\begin{aligned}\alpha_{30} &= G_{30} + 3G_{20}h_{20} + h_{30}\lambda + 3G_{11}\bar{h}_{02} \\ \alpha_{21} &= G_{21} + 2G_{20}h_{11} + G_{11}h_{20} + G_{02}\bar{h}_{02} + 2G_{11}\bar{h}_{11} \\ \alpha_{12} &= G_{12} + G_{20}h_{02} + 2G_{11}h_{11} + h_{12}\lambda + 2G_{02}\bar{h}_{11} + G_{11}\bar{h}_{20} \\ \alpha_{03} &= G_{03} + 3G_{11}h_{02} + h_{03}\lambda + 3G_{02}\bar{h}_{20}\end{aligned}$$

$$\alpha_{40} = G_{40} + 6G_{30}h_{20} + 3G_{20}h_{20}^2 + 4G_{20}h_{30} + 6(G_{21} + G_{11}h_{20})\bar{h}_{02} + 3G_{02}\bar{h}_{02}^2 + 4G_{11}\bar{h}_{03}$$

$$\begin{aligned}\alpha_{31} &= G_{31} + 3G_{30}h_{11} + 3G_{21}h_{20} + 3G_{20}h_{11}h_{20} + G_{11}h_{30} + G_{02}\bar{h}_{03} + 3G_{21}\bar{h}_{11} \\ &+ 3G_{11}h_{20}\bar{h}_{11} + 3\bar{h}_{02}(G_{12} + G_{11}h_{11} + G_{02}\bar{h}_{11}) + 3G_{11}\bar{h}_{12}\end{aligned}$$

$$\begin{aligned}\alpha_{22} &= G_{22} + G_{30}h_{02} + 4G_{21}h_{11} + 2G_{20}h_{11}^2 + 2G_{20}h_{12} + G_{12}h_{20} + G_{20}h_{02}h_{20} \\ &+ 4(G_{12} + G_{11}h_{11})\bar{h}_{11} + 2G_{02}\bar{h}_{11}^2 + 2G_{02}\bar{h}_{12} + G_{21}\bar{h}_{20} + G_{11}h_{20}\bar{h}_{20} \\ &+ \bar{h}_{02}(G_{03} + G_{11}h_{02} + G_{02}\bar{h}_{20})\end{aligned}$$

$$\begin{aligned}\alpha_{13} &= G_{13} + 3G_{21}h_{02} + G_{20}h_{03} + 3G_{12}h_{11} + 3G_{20}h_{02}h_{11} + 3G_{11}h_{12} \\ &+ 3(G_{12} + G_{11}h_{11})\bar{h}_{20} + 3\bar{h}_{11}(G_{03} + G_{11}h_{02} + G_{02}\bar{h}_{20}) + G_{11}\bar{h}_{30}\end{aligned}$$

$$\alpha_{04} = G_{04} + 6G_{12}h_{02} + 3G_{20}h_{02}^2 + 4G_{11}h_{03} + 6(G_{03} + G_{11}h_{02})\bar{h}_{20} + 3G_{02}\bar{h}_{20}^2 + 4G_{02}\bar{h}_{30}$$

$$\begin{aligned}\alpha_{50} &= G_{50} + 10G_{40}h_{20} + 15G_{30}h_{20}^2 + 10G_{30}h_{30} + 10G_{20}h_{20}h_{30} + 15G_{12}\bar{h}_{02}^2 \\ &+ 10(G_{21} + G_{11}h_{20})\bar{h}_{03} + 10\bar{h}_{02}(G_{31} + 3G_{21}h_{20} + G_{11}h_{30} + G_{02}\bar{h}_{03})\end{aligned}$$

$$\begin{aligned}\alpha_{41} &= G_{41} + 4G_{40}h_{11} + 6G_{31}h_{20} + 12G_{30}h_{11}h_{20} + 3G_{21}h_{20}^2 + 4G_{21}h_{30} \\ &+ 4G_{20}h_{11}h_{30} + 3G_{03}\bar{h}_{02}^2 + 4G_{31}\bar{h}_{11} + 12G_{21}h_{20}\bar{h}_{11} + 4G_{11}h_{30}\bar{h}_{11} \\ &+ 4\bar{h}_{03}(G_{12} + G_{11}h_{11} + G_{02}\bar{h}_{11}) + 6G_{21}\bar{h}_{12} + 6G_{11}h_{20}\bar{h}_{12} \\ &+ 6\bar{h}_{02}(G_{22} + 2G_{21}h_{11} + G_{12}h_{20} + 2G_{12}\bar{h}_{11} + G_{02}\bar{h}_{12})\end{aligned}$$

$$\begin{aligned}\alpha_{32} = {} & G_{32} + G_{40}h_{02} + 6G_{31}h_{11} + 6G_{30}h_{11}^2 + 3G_{30}h_{12} + 3G_{22}h_{20} + 3G_{30}h_{02}h_{20} \\ & + 6G_{21}h_{11}h_{20} + 3G_{20}h_{12}h_{20} + G_{12}h_{30} + G_{20}h_{02}h_{30} + 6G_{22}\bar{h}_{11} + 12G_{21}h_{11}\bar{h}_{11} \\ & + 6G_{12}h_{20}\bar{h}_{11} + 6G_{12}\bar{h}_{11}^2 + 6G_{12}\bar{h}_{12} + 6G_{11}h_{11}\bar{h}_{12} + 6G_{02}\bar{h}_{11}\bar{h}_{12} \\ & + G_{31}\bar{h}_{20} + 3G_{21}h_{20}\bar{h}_{20} + G_{11}h_{30}\bar{h}_{20} + \bar{h}_{03}(G_{03} + G_{11}h_{02} + G_{02}\bar{h}_{20}) \\ & + 3\bar{h}_{02}(G_{13} + G_{21}h_{02} + 2G_{12}h_{11} + G_{11}h_{12} + 2G_{03}\bar{h}_{11} + G_{12}\bar{h}_{20})\end{aligned}$$

$$\begin{aligned}\alpha_{23} = {} & G_{23} + 3G_{31}h_{02} + G_{30}h_{03} + 6G_{22}h_{11} + 6G_{30}h_{02}h_{11} + 6G_{21}h_{11}^2 + 6G_{21}h_{12} \\ & + 6G_{20}h_{11}h_{12} + G_{13}h_{20} + 3G_{21}h_{02}h_{20} + G_{20}h_{03}h_{20} + 6G_{03}\bar{h}_{11}^2 + 3G_{03}\bar{h}_{12} \\ & + 3G_{11}h_{02}\bar{h}_{12} + 3G_{22}\bar{h}_{20} + 6G_{21}h_{11}\bar{h}_{20} + 3G_{12}h_{20}\bar{h}_{20} + 3G_{02}\bar{h}_{12}\bar{h}_{20} \\ & + 6\bar{h}_{11}(G_{13} + G_{21}h_{02} + 2G_{12}h_{11} + G_{11}h_{12} + G_{12}\bar{h}_{20}) + G_{21}\bar{h}_{30} \\ & + G_{11}h_{20}\bar{h}_{30} + \bar{h}_{02}(G_{04} + 3G_{12}h_{02} + G_{11}h_{03} + 3G_{03}\bar{h}_{20} + G_{02}\bar{h}_{30})\end{aligned}$$

$$\begin{aligned}\alpha_{14} = {} & G_{14} + 6G_{22}h_{02} + 3G_{30}h_{02}^2 + 4G_{21}h_{03} + 4G_{13}h_{11} + 12G_{21}h_{02}h_{11} + 4G_{20}h_{03}h_{11} \\ & + 6G_{12}h_{12} + 6G_{20}h_{02}h_{12} + 6(G_{13} + G_{21}h_{02} + 2G_{12}h_{11} + G_{11}h_{12})\bar{h}_{20} + 3G_{12}\bar{h}_{20}^2 \\ & + 4G_{12}\bar{h}_{30} + 4G_{11}h_{11}\bar{h}_{30} + 4\bar{h}_{11}(G_{04} + 3G_{12}h_{02} + G_{11}h_{03} + 3G_{03}\bar{h}_{20} + G_{02}\bar{h}_{30})\end{aligned}$$

$$\begin{aligned}\alpha_{05} = {} & G_{05} + 10G_{13}h_{02} + 15G_{21}h_{02}^2 + 10G_{12}h_{03} + 10G_{20}h_{02}h_{03} + 15G_{03}\bar{h}_{20}^2 \\ & + 10(G_{03} + G_{11}h_{02})\bar{h}_{30} + 10\bar{h}_{20}(G_{04} + 3G_{12}h_{02} + G_{11}h_{03} + G_{02}\bar{h}_{30})\end{aligned}$$

4.2.9 The lower-order terms of $h^{-1}(G(h(w)))$

See Subsection 3.4.5 for the definition of β_{ij}, $4 \le i + j \le 5$.

$$\begin{aligned}\beta_{40} = {} & \alpha_{40} + 3\alpha_{20}^2\tilde{h}_{20} + 4\alpha_{10}\alpha_{30}\tilde{h}_{20} + 6\alpha_{10}^2\alpha_{20}\tilde{h}_{30} + \alpha_{10}^4\tilde{h}_{40} \\ & + 6(\alpha_{20}\tilde{h}_{11} + \alpha_{10}^2\tilde{h}_{21})\bar{\alpha}_{02} + 3\tilde{h}_{02}\bar{\alpha}_{02}^2 + 4\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{03}\end{aligned}$$

$$\beta_{31} = \alpha_{31} + 3\alpha_{11}\alpha_{20}\tilde{h}_{20} + 3\alpha_{10}\alpha_{21}\tilde{h}_{20} + 3\alpha_{10}^2\alpha_{11}\tilde{h}_{30} \\ + (\alpha_{30}\tilde{h}_{11} + 3\alpha_{10}\alpha_{20}\tilde{h}_{21} + \alpha_{10}^3\tilde{h}_{31} + \tilde{h}_{02}\bar{\alpha}_{03})\bar{\alpha}_{10} + 3\alpha_{20}\tilde{h}_{11}\bar{\alpha}_{11} \\ + 3\alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{11} + 3\bar{\alpha}_{02}(\alpha_{11}\tilde{h}_{11} + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{10} + \tilde{h}_{02}\bar{\alpha}_{11}) + 3\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{12}$$

$$\beta_{22} = \alpha_{22} + 2\alpha_{11}^2\tilde{h}_{20} + 2\alpha_{10}\alpha_{12}\tilde{h}_{20} + \alpha_{02}\alpha_{20}\tilde{h}_{20} + \alpha_{02}\alpha_{10}^2\tilde{h}_{30} + (\alpha_{20}\tilde{h}_{12} + \alpha_{10}^2\tilde{h}_{22})\bar{\alpha}_{10}^2 \\ + 4\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{11} + 2\tilde{h}_{02}\bar{\alpha}_{11}^2 + 2\bar{\alpha}_{10}(\alpha_{21}\tilde{h}_{11} + 2\alpha_{10}\alpha_{11}\tilde{h}_{21} + 2\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11} + \tilde{h}_{02}\bar{\alpha}_{12}) \\ + \alpha_{20}\tilde{h}_{11}\bar{\alpha}_{20} + \alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{20} + \bar{\alpha}_{02}(\alpha_{02}\tilde{h}_{11} + \tilde{h}_{03}\bar{\alpha}_{10}^2 + \tilde{h}_{02}\bar{\alpha}_{20}) + 2\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{21}$$

$$\beta_{13} = \alpha_{13} + \alpha_{03}\alpha_{10}\tilde{h}_{20} + 3\alpha_{02}\alpha_{11}\tilde{h}_{20} + \alpha_{10}\tilde{h}_{13}\bar{\alpha}_{10}^3 + 3\bar{\alpha}_{10}^2(\alpha_{11}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{11}) \\ + 3\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{20} + 3\bar{\alpha}_{11}(\alpha_{02}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{20}) + \alpha_{10}\tilde{h}_{11}\bar{\alpha}_{30} \\ + 3\bar{\alpha}_{10}(\alpha_{12}\tilde{h}_{11} + \alpha_{02}\alpha_{10}\tilde{h}_{21} + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{20} + \tilde{h}_{02}\bar{\alpha}_{21})$$

$$\beta_{04} = \alpha_{04} + 3\alpha_{02}^2\tilde{h}_{20} + \tilde{h}_{04}\bar{\alpha}_{10}^4 + 6\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{20} + 3\tilde{h}_{02}\bar{\alpha}_{20}^2 \\ + 6\bar{\alpha}_{10}^2(\alpha_{02}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{20}) + 4\bar{\alpha}_{10}(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{30})$$

$$\beta_{50} = \alpha_{50} + 10\alpha_{20}\alpha_{30}\tilde{h}_{20} + 5\alpha_{10}\alpha_{40}\tilde{h}_{20} + 15\alpha_{10}\alpha_{20}^2\tilde{h}_{30} + 10\alpha_{10}^2\alpha_{30}\tilde{h}_{30} + 10\alpha_{10}^3\alpha_{20}\tilde{h}_{40} \\ + \alpha_{10}^5\tilde{h}_{50} + 15\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{02}^2 + 10(\alpha_{20}\tilde{h}_{11} + \alpha_{10}^2\tilde{h}_{21})\bar{\alpha}_{03} \\ + 10\bar{\alpha}_{02}(\alpha_{30}\tilde{h}_{11} + 3\alpha_{10}\alpha_{20}\tilde{h}_{21} + \alpha_{10}^3\tilde{h}_{31} + \tilde{h}_{02}\bar{\alpha}_{03}) + 5\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{04}$$

$$\begin{aligned}\beta_{41} = {}& \alpha_{41} + 6\alpha_{20}\alpha_{21}\tilde{h}_{20} + 4\alpha_{11}\alpha_{30}\tilde{h}_{20} + 4\alpha_{10}\alpha_{31}\tilde{h}_{20} + 12\alpha_{10}\alpha_{11}\alpha_{20}\tilde{h}_{30} + 6\alpha_{10}^2\alpha_{21}\tilde{h}_{30}\\
&+ 4\alpha_{10}^3\alpha_{11}\tilde{h}_{40} + \alpha_{40}\tilde{h}_{11}\bar{\alpha}_{10} + 3\alpha_{20}^2\tilde{h}_{21}\bar{\alpha}_{10} + 4\alpha_{10}\alpha_{30}\tilde{h}_{21}\bar{\alpha}_{10} + 6\alpha_{10}^2\alpha_{20}\tilde{h}_{31}\bar{\alpha}_{10}\\
&+ \alpha_{10}^4\tilde{h}_{41}\bar{\alpha}_{10} + 3\tilde{h}_{03}\bar{\alpha}_{02}^2\bar{\alpha}_{10} + \tilde{h}_{02}\bar{\alpha}_{04}\bar{\alpha}_{10} + 4\alpha_{30}\tilde{h}_{11}\bar{\alpha}_{11} + 12\alpha_{10}\alpha_{20}\tilde{h}_{21}\bar{\alpha}_{11}\\
&+ 4\alpha_{10}^3\tilde{h}_{31}\bar{\alpha}_{11} + 4\bar{\alpha}_{03}(\alpha_{11}\tilde{h}_{11} + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{10} + \tilde{h}_{02}\bar{\alpha}_{11}) + 6\alpha_{20}\tilde{h}_{11}\bar{\alpha}_{12}\\
&+ 6\bar{\alpha}_{02}\Big(\alpha_{21}\tilde{h}_{11} + 2\alpha_{10}\alpha_{11}\tilde{h}_{21} + (\alpha_{20}\tilde{h}_{12} + \alpha_{10}^2\tilde{h}_{22})\bar{\alpha}_{10} + 2\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11} + \tilde{h}_{02}\bar{\alpha}_{12}\Big)\\
&+ 6\alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{12} + 4\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{13}\end{aligned}$$

$$\begin{aligned}\beta_{32} = {}& \alpha_{32} + 3\alpha_{12}\alpha_{20}\tilde{h}_{20} + 6\alpha_{11}\alpha_{21}\tilde{h}_{20} + 3\alpha_{10}\alpha_{22}\tilde{h}_{20} + \alpha_{02}\alpha_{30}\tilde{h}_{20} + 6\alpha_{10}\alpha_{11}^2\tilde{h}_{30}\\
&+ 3\alpha_{10}^2\alpha_{12}\tilde{h}_{30} + 3\alpha_{02}\alpha_{10}\alpha_{20}\tilde{h}_{30} + \alpha_{02}\alpha_{10}^3\tilde{h}_{40} + 2\alpha_{31}\tilde{h}_{11}\bar{\alpha}_{10} + 6\alpha_{11}\alpha_{20}\tilde{h}_{21}\bar{\alpha}_{10}\\
&+ 6\alpha_{10}\alpha_{21}\tilde{h}_{21}\bar{\alpha}_{10} + 6\alpha_{10}^2\alpha_{11}\tilde{h}_{31}\bar{\alpha}_{10} + \alpha_{30}\tilde{h}_{12}\bar{\alpha}_{10}^2 + 3\alpha_{10}\alpha_{20}\tilde{h}_{22}\bar{\alpha}_{10}^2\\
&+ \alpha_{10}^3\tilde{h}_{32}\bar{\alpha}_{10}^2 + 6\alpha_{21}\tilde{h}_{11}\bar{\alpha}_{11} + 12\alpha_{10}\alpha_{11}\tilde{h}_{21}\bar{\alpha}_{11} + 6\alpha_{20}\tilde{h}_{12}\bar{\alpha}_{10}\bar{\alpha}_{11}\\
&+ 6\alpha_{10}^2\tilde{h}_{22}\bar{\alpha}_{10}\bar{\alpha}_{11} + 6\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11}^2 + 6\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{12} + 6\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{10}\bar{\alpha}_{12}\\
&+ 6\tilde{h}_{02}\bar{\alpha}_{11}\bar{\alpha}_{12} + 2\tilde{h}_{02}\bar{\alpha}_{10}\bar{\alpha}_{13} + \alpha_{30}\tilde{h}_{11}\bar{\alpha}_{20} + 3\alpha_{10}\alpha_{20}\tilde{h}_{21}\bar{\alpha}_{20} + \alpha_{10}^3\tilde{h}_{31}\bar{\alpha}_{20}\\
&+ \bar{\alpha}_{03}(\alpha_{02}\tilde{h}_{11} + \tilde{h}_{03}\bar{\alpha}_{10}^2 + \tilde{h}_{02}\bar{\alpha}_{20}) + 3\alpha_{20}\tilde{h}_{11}\bar{\alpha}_{21} + 3\alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{21}\\
&+ 3\bar{\alpha}_{02}\Big(\alpha_{12}\tilde{h}_{11} + \alpha_{02}\alpha_{10}\tilde{h}_{21} + \alpha_{10}\tilde{h}_{13}\bar{\alpha}_{10}^2 + 2\bar{\alpha}_{10}(\alpha_{11}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{11})\\
&+ \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{20} + \tilde{h}_{02}\bar{\alpha}_{21}\Big) + 3\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{22}\end{aligned}$$

$$\begin{aligned}\beta_{23} = {}& \alpha_{23} + 6\alpha_{11}\alpha_{12}\tilde{h}_{20} + 2\alpha_{10}\alpha_{13}\tilde{h}_{20} + \alpha_{03}\alpha_{20}\tilde{h}_{20} + 3\alpha_{02}\alpha_{21}\tilde{h}_{20} + \alpha_{03}\alpha_{10}^2\tilde{h}_{30}\\
&+ 6\alpha_{02}\alpha_{10}\alpha_{11}\tilde{h}_{30} + (\alpha_{20}\tilde{h}_{13} + \alpha_{10}^2\tilde{h}_{23})\bar{\alpha}_{10}^3 + 6\alpha_{12}\tilde{h}_{11}\bar{\alpha}_{11} + 6\alpha_{02}\alpha_{10}\tilde{h}_{21}\bar{\alpha}_{11}\\
&+ 3\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{12} + 3\bar{\alpha}_{10}^2(\alpha_{21}\tilde{h}_{12} + 2\alpha_{10}\alpha_{11}\tilde{h}_{22} + 2\alpha_{10}\tilde{h}_{13}\bar{\alpha}_{11} + \tilde{h}_{03}\bar{\alpha}_{12}) + 3\alpha_{21}\tilde{h}_{11}\bar{\alpha}_{20}\\
&+ 6\alpha_{10}\alpha_{11}\tilde{h}_{21}\bar{\alpha}_{20} + 6\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{11}\bar{\alpha}_{20} + 3\tilde{h}_{02}\bar{\alpha}_{12}\bar{\alpha}_{20} + 6\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{21}\\
&+ 6\tilde{h}_{02}\bar{\alpha}_{11}\bar{\alpha}_{21} + 3\bar{\alpha}_{10}(\alpha_{22}\tilde{h}_{11} + 2\alpha_{11}^2\tilde{h}_{21} + 2\alpha_{10}\alpha_{12}\tilde{h}_{21} + \alpha_{02}\alpha_{20}\tilde{h}_{21} + \alpha_{02}\alpha_{10}^2\tilde{h}_{31}\\
&+ 4\alpha_{11}\tilde{h}_{12}\bar{\alpha}_{11} + 2\tilde{h}_{03}\bar{\alpha}_{11}^2 + \alpha_{20}\tilde{h}_{12}\bar{\alpha}_{20} + \alpha_{10}^2\tilde{h}_{22}\bar{\alpha}_{20} + 2\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{21} + \tilde{h}_{02}\bar{\alpha}_{22})\\
&+ \alpha_{20}\tilde{h}_{11}\bar{\alpha}_{30} + 2\alpha_{10}\tilde{h}_{11}\bar{\alpha}_{31} + \alpha_{10}^2\tilde{h}_{21}\bar{\alpha}_{30}\\
&+ \bar{\alpha}_{02}\big(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{04}\bar{\alpha}_{10}^3 + 3\bar{\alpha}_{10}(\alpha_{02}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{20}) + \tilde{h}_{02}\bar{\alpha}_{30}\big)\end{aligned}$$

$$\begin{aligned}\beta_{14} &= \alpha_{14} + \alpha_{04}\alpha_{10}\tilde{h}_{20} + 4\alpha_{03}\alpha_{11}\tilde{h}_{20} + 6\alpha_{02}\alpha_{12}\tilde{h}_{20} + 3\alpha_{02}^2\alpha_{10}\tilde{h}_{30} + \alpha_{10}\tilde{h}_{14}\bar{\alpha}_{10}^4 \\ &+ 4\bar{\alpha}_{10}^3(\alpha_{11}\tilde{h}_{13} + \tilde{h}_{04}\bar{\alpha}_{11}) + 6\alpha_{12}\tilde{h}_{11}\bar{\alpha}_{20} + 6\alpha_{02}\alpha_{10}\tilde{h}_{21}\bar{\alpha}_{20} + 3\alpha_{10}\tilde{h}_{12}\bar{\alpha}_{20}^2 \\ &+ 6\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{21} + 6\tilde{h}_{02}\bar{\alpha}_{20}\bar{\alpha}_{21} + 6\bar{\alpha}_{10}^2(\alpha_{12}\tilde{h}_{12} + \alpha_{02}\alpha_{10}\tilde{h}_{22} + \alpha_{10}\tilde{h}_{13}\bar{\alpha}_{20} + \tilde{h}_{03}\bar{\alpha}_{21}) \\ &+ 4\alpha_{11}\tilde{h}_{11}\bar{\alpha}_{30} + 4\bar{\alpha}_{11}(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{30}) + 4\bar{\alpha}_{10}\big(\alpha_{13}\tilde{h}_{11} + \alpha_{03}\alpha_{10}\tilde{h}_{21} + 3\alpha_{02}\alpha_{11}\tilde{h}_{21} \\ &+ 3\alpha_{11}\tilde{h}_{12}\bar{\alpha}_{20} + 3\bar{\alpha}_{11}(\alpha_{02}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{20}) + \alpha_{10}\tilde{h}_{12}\bar{\alpha}_{30} + \tilde{h}_{02}\bar{\alpha}_{31}\big) + \alpha_{10}\tilde{h}_{11}\bar{\alpha}_{40}\end{aligned}$$

$$\begin{aligned}\beta_{05} &= \alpha_{05} + 10\alpha_{02}\alpha_{03}\tilde{h}_{20} + \tilde{h}_{05}\bar{\alpha}_{10}^5 + 10\bar{\alpha}_{10}^3(\alpha_{02}\tilde{h}_{13} + \tilde{h}_{04}\bar{\alpha}_{20}) + 10\alpha_{02}\tilde{h}_{11}\bar{\alpha}_{30} \\ &+ 10\bar{\alpha}_{20}(\alpha_{03}\tilde{h}_{11} + \tilde{h}_{02}\bar{\alpha}_{30}) + 10\bar{\alpha}_{10}^2(\alpha_{03}\tilde{h}_{12} + \tilde{h}_{03}\bar{\alpha}_{30}) \\ &+ 5\bar{\alpha}_{10}(\alpha_{04}\tilde{h}_{11} + 3\alpha_{02}^2\tilde{h}_{21} + 6\alpha_{02}\tilde{h}_{12}\bar{\alpha}_{20} + 3\tilde{h}_{03}\bar{\alpha}_{20}^2 + \tilde{h}_{02}\bar{\alpha}_{40})\end{aligned}$$

4.2.10 The Lyapunov-coefficient

$$c_1 = \frac{G_{11}G_{20}(2\lambda + \bar{\lambda} - 3)}{2(\lambda^2 - \lambda)(\bar{\lambda} - 1)} + \frac{|G_{11}|^2}{|\lambda|^2 - \bar{\lambda}} + \frac{|G_{02}|^2}{2(\lambda^2 - \bar{\lambda})} + \frac{G_{21}}{2}$$

Summary

In this thesis we are studying the global stability of the delayed logistic difference equation

$$x_{n+1} = ax_n(1 - x_{n-d}),$$

or equivalently the $(d+1)$-dimensional map

$$F_d : \mathbb{R}^{d+1} \ni u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{d+1} \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ au_{d+1}(1-u_1) \end{pmatrix} \in \mathbb{R}^{d+1},$$

where $a > 0$ and $d \in \mathbb{N}$. The thesis is based on papers [4, 7] of the author. Chapter 1 is the introduction where we describe the problem and also outline the main parts of the dissertation. In Chapters 2 and 3 we consider the cases $d = 1$ and $d = 2$, respectively. Finally, in Chapter 4 the lengthy coefficients were collected to preserve the readability of the main part of the thesis.

It is well known that for $a \in (0, 1]$ the origin is the unique fixed point of F_d in $[0, 1]^{d+1}$, which is locally stable and $\lim_{n\to\infty} F_d^n(u) = 0$ for every $u \in [0, 1]^{d+1}$. For $a > 1$ a nontrivial fixed point $u_A = (A, A, \ldots, A)$ with $A = 1 - \frac{1}{a}$ appears in $[0, 1]^{d+1}$. There exists an $a_0 > 1$, depending on d such that this fixed point is locally asymptotically stable for $a \in (1, a_0)$, and unstable for $a > a_0$. At $a = a_0$ a Neimark–Sacker bifurcation takes place. We show that the following conjecture is true for $d \in \{1, 2\}$.

Conjecture. *The fixed point u_A is locally stable and $\lim_{n\to\infty} F_d^n(u) = u_A$ for $a \in (1, a_0]$ and*

$$u \in S^d = \Big\{ u \in [0,1)^d \times (0,1) : \ a^k u_{d+1} \prod_{j=1}^{k} (1 - u_j) < 1 \quad \textit{for} \quad k \in \{1, 2, \ldots, d\} \Big\}.$$

Here, S^d contains exactly those $(x_0, x_1, \ldots, x_d) \in \mathbb{R}_+^{d+1}$ for which $x_n > 0$ for every $n > d$. The conjecture can be formulated so that local stability implies global stability for the fixed point u_A. This is satisfied for several problems, see e.g., [11, 12, 13, 5, 6],

but it is not true in general, see e.g., [14].

In Sections 2.1 and 3.1 we give purely analytical proofs of the conjecture for smaller parameter values, more precisely for $a \in \left(1, \frac{d+2}{d+1}\right]$ with $d = 1$ and $d = 2$, respectively. However, for larger a the proof of the global stability is a combination of analytical and computer-aided tools. First, we construct analytically an attracting neighborhood $\mathcal{M}$ of the nontrivial fixed point u_A. Then, by applying reliable numerical tools, it is shown that for every $u \in S^d$ the iterates $F_d^n(u)$ eventually enter $\mathcal{M}$. Consequently, all points of S^d belong to the region of attraction of u_A. Here, reliable means that all possible numerical errors are controlled by using interval arithmetic techniques. Therefore, the computer-assisted part also provides mathematically rigorous statements.

In Sections 2.2 and 3.2 a standard linearization technique gives an attracting neighborhood of u_A for parameter values further from a_0. However, the attracting neighborhood obtained via linearization shrinks to the fixed point as a tends to a_0. Therefore, for parameter values a close to a_0 this neighborhood is not big enough for computer use in the second part of the method and we need another approach to construct an attracting neighborhood for these parameter values.

In Section 2.3 for a close to a_0 we use the normal form of the Neimark–Sacker bifurcation. More precisely, with smooth and invertible maps, we transform the map into the form

$$w \mapsto \lambda w + c_1 w^2 \overline{w} + R_2,$$

where c_1 is the Lyapunov-coefficient and $R_2 = O\left(|w|^4\right)$. Then we show that there exists a $\rho_0 > 0$ such that

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| < |w|$$

for every $w \in \mathbb{C}$ with $0 < |w| \leq \rho_0$, which guaratees that B_{ρ_0} is inside the attracting neighborhood. Since we need the size of the constructed neighborhood $\mathcal{M}$ for computer use, it is not enough to determine only the lower-order terms during the normal form transformation, like we would do in a regular bifurcation analysis. These lower-order terms only assure the existence of such a sufficiently small neighborhood, whose size is not explicitly determined by them. Therefore, it is essential during the transformation to trace the higher-order terms and to estimate them as well as possible, in order to obtain a sufficiently big neighborhood $\mathcal{M}$.

In Section 3.4 we adapt the Neimark–Sacker bifurcational normal form technique for the case $d = 2$. However, we need new ideas, since $F_3(u)$ is three-dimensional, and thus the adaptation of the method is not that straightforward. The novelty of this section is an explicit construction of a relatively large attracting neighborhood of the nontrivial fixed point of the three-dimensional logistic map by using center manifold techniques and the

Neimark–Sacker bifurcational normal form.

In Section 3.3 we carry out an approximate version of the center manifold reduction. We consider the fourth-order polynomial approximation $\phi(z)$ of the center manifold and the set

$$T(r,C) = \{(z,y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le r, \ |y - \phi(z)| \le C|z|^5\}$$

around $y = \phi(z)$, where r and C are some positive constants.

In Subsections 3.3.1 and 3.3.2 we investigate the y-directional dynamics in $T(r,C)$. Using the property that solutions close to the fixed point decay exponentially to the center manifold we show that $T(r,C)$ is conditionally invariant in direction y. In Section 3.4 the z-directional dynamics in $T(r,C)$ is investigated by using the Neimark–Sacker bifurcational normal form technique. Exploiting the special shape of $T(r,C)$ we can show that in an appropriate coordinate system the transformed z coordinate is strictly decreasing during the iteration, similar to the two-dimensional case. Finally, combining the y- and z-directional dynamics, we obtain that $T(r,C)$ is inside the region of attraction of the fixed point.

However, $T(r,C)$ is clearly not a proper neighborhood of the origin in $\mathbb{C}\times\mathbb{R}$. Therefore, in Section 3.5 we define the set

$$\tilde{T}(\hat{r},K) = \{(z,y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le \hat{r}, \ |\phi(z) - y| \le K\}$$

for some $\hat{r} > 0$ and $K > 0$. By using the exponential y-directional attractivity of $T(r,C)$ we show that $\tilde{T}(\hat{r},K)$ is also in the region of attraction of the fixed point.

In Sections 2.4 to 2.6 and 3.6 to 3.7 we describe the computer-assisted part of our method for $d = 1$ and $d = 2$, respectively. We cover S^d with finitely many $(d+1)$-dimensional small cubes. Considering these cubes as vertices of a graph we introduce a directed graph, which, to a certain extent, describes the behavior of map F_d on these cubes. Therefore, we convert the issue of examining infinitely many points into a finite graph problem, which can be handled by computer. To construct the edges of this graph we use reliable numerical methods in order to handle the rounding errors of the computer. We show with the help of this graph that the iterates of every point from S^d enter the neighborhood constructed before, and the proof of Conjecture is completed for $d \in \{1,2\}$.

Despite the fact that we demonstrate our method only on a specific equation, we believe that it can be applied or extended to other similar maps. For instance the Ricker map (see [5]) and the Pielou map (see [13]) with delay $d = 2$ essentially differ only in that they are not polynomial maps. Hence, only a slight modification would be necessary in the estimations. However, the main question is whether the obtained neighborhood is large enough for the computer-aided part of the method. These two maps along with the logistic map would also be interesting for larger delay, i.e., $d > 2$. We believe that the analytical part could be extended using only natural modifications. However, the computer-aided part can be critical in these cases, since the increasing dimension causes an exponentially growing graph.

It also would be interesting to prove the existence of the unique invariant closed curve around the nontrivial fixed point for parameter values larger than the critical value. However, this question is substantially different from the one studied in this thesis.

Összefoglaló

A disszertációban az

$$x_{n+1} = ax_n(1 - x_{n-d})$$

késleltetett logisztikus differenciaegyenlet globális stabilitását vizsgáljuk az $a > 0$ és $d \in \mathbb{N}$ paraméterek esetén. Az egyenlet ekvivalens módon az

$$F_d : \mathbb{R}^{d+1} \ni u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{d+1} \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ au_{d+1}(1 - u_1) \end{pmatrix} \in \mathbb{R}^{d+1},$$

alakba írható. Jelen mű a szerző [4, 7] cikkein alapul. Az 1. fejezet a bevezetés, ahol ismertetjük a problémakört és felvázoljuk a disszertáció fő lépéseit. A 2. és 3. fejezetben a $d = 1$, illetve $d = 2$ esetekben vizsgáljuk a késleltetett logisztikus leképezés globális stabilitását. Végezetül a 4. fejezetben gyűjtöttük össze az értekezésben előforduló együtthatókat, hogy a terjedelmes kifejezések ne nehezítsék a disszertáció olvashatóságát.

Jól ismert, hogy $a \in (0, 1]$ esetén az F_d függvénynek a $[0, 1]^{d+1}$ halmazon az origó az egyetlen fixpontja, amely lokálisan stabil és $\lim_{n\to\infty} F_d^n(u) = 0$ minden $u \in [0, 1]^{d+1}$ esetén. Amennyiben $a > 1$, úgy egy nemtriviális $u_A = (A, A, \dots, A)$ fixpont jelenik meg $[0, 1]^{d+1}$ halmazban, ahol $A = 1 - \frac{1}{a}$. Továbbá létezik egy d-től függő $a_0 > 1$, úgy, hogy ez a nemtriviális fixpont lokálisan aszimptotikusan stabil, ha $a \in (1, a_0)$, és instabil, ha $a > a_0$. Az $a = a_0$ helyen Neimark–Sacker bifurkáció következik be. Megmutatjuk, hogy a következő sejtés teljesül $d \in \{1, 2\}$ érték esetén.

Sejtés. *Az u_A fixpont lokálisan stabil, továbbá $\lim\limits_{n\to\infty} F_d^n(u) = u_A$ minden $a \in (1, a_0]$ és $u \in S^d$ esetén, ahol*

$$S^d = \Big\{u \in [0, 1)^d \times (0, 1) : \; a^k u_{d+1} \prod_{j=1}^{k} (1 - u_j) < 1 \quad \textit{for} \quad k \in \{1, 2, \dots, d\}\Big\}.$$

A fenti kifejezésben az S^d halmaz pontosan azokat az $(x_0, x_1, \dots, x_d) \in \mathbb{R}_+^{d+1}$ pontokat tartalmazza, melyekre $x_n > 0$ teljesül minden $n > d$ esetén. A sejtés olyan alakba is átfogalmazható, hogy az u_A fixpont lokális stabilitása maga után vonja annak globális stabilitását. Ez a tulajdonság számos esetben teljesül, lásd [11, 12, 13, 5, 6], azonban nem igaz mindig, lásd [14].

A 2.1. és 3.1. szakaszban tisztán analitikus bizonyítást adunk a sejtésre az a paraméter kis értékei esetén, pontosabban az $a \in \left(1, \frac{d+2}{d+1}\right]$ értékekre, ahol $d = 1$, illetve $d = 2$. Ezzel szemben az a_0-hoz közeli $a < a_0$ értékek esetén a bizonyítás analitikus és megbízható, számítógéppel támogatott módszerek ötvözésével történik. Először az u_A fixpont körül analitikus eszközökkel konstruálunk egy $\mathcal{M}$ vonzó tartományt, majd megbízható numerikus módszerekkel megmutatjuk, hogy minden $u \in S^d$ esetén az $F_d^n(u)$ iteráltak előbb-utóbb belelépnek az így kapott $\mathcal{M}$ környezetbe. Következtetésképpen S^d minden pontja az u_A vonzástartományában van. Jelen esetben a megbízható azt jelenti, hogy minden lehetséges numerikus hiba kontrollálva van intervallum aritmetikai technikákkal. Ezáltal a számítógéppel támogatott rész is matematikailag precíz eredményeket szolgáltat.

A 2.2. és 3.2. szakaszban az a_0-tól távolabbi paraméterértékek esetén standard linearizációs technikák segítségével nyerjük az u_A körüli vonzó környezetet. Azonban az így nyert környezet ráhúzódik a fixpontra, ahogy a tart a_0-ba. Emiatt az a_0-hoz közeli paraméterértékek esetén ez a környezet nem lesz elegendően nagy a számítógéppel támogatott rész számára a módszerünk második felében. Így e paraméterértékek esetén egy másik megközelítésre is szükségünk van a vonzó környezet konstrukciójához.

A 2.3. szakaszban a_0-hoz közeli a esetén a Neimark–Sacker bifurkációs normálformát használjuk. Pontosabban sima és invertálható leképezésekkel a

$$w \mapsto \lambda w + c_1 w^2 \overline{w} + R_2$$

alakba transzformáljuk a leképezést, ahol c_1 a Lyapunov-együttható és $R_2 = O\left(|w|^4\right)$. Ezután megmutatjuk, hogy létezik $\rho_0 > 0$ úgy, hogy

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| < |w|$$

minden $w \in \mathbb{C}$-re, ahol $0 < |w| \leq \rho_0$. Ez biztosítja, hogy a $B_{\rho_0} = \{w \in \mathbb{C} : \ |w| \leq \rho_0\}$ halmaz benne van a vonzó környezetben. Mivel a számítógépes részben szükségünk van a konstruált környezet méretére, ezért a normálformás transzformáció során nem elegendő csupán az alacsonyabbrendű tagok együtthatóit meghatározni, ahogy egy szokásos bifurkációs vizsgálat esetén tennénk. Ezek az alacsonyabbrendű tagok csak a létezését biztosítják egy elegendően kis környezetnek, de annak explicit méretét nem szolgáltatják. Emiatt lényeges, hogy a transzformáció során nyomon kövessük a magasabbrendű tagokat, valamint hogy amennyire lehetséges, pontos becslést adjunk rájuk annak érdekében, hogy elegendően nagy $\mathcal{M}$ környezetet kapjunk.

A 3.4. szakaszban a Neimark–Sacker bifurkációs normálformát adaptáljuk a $d = 2$ esetre. Ehhez azonban új ötletekre van szükségünk, ugyanis az $F_3(u)$ egy háromdimenziós függvény, és így a módszer adaptálása nem magától értetődő. E szakasz újdonságát az alkotja, hogy a háromdimenziós logisztikus leképezés nemtriviális fixpontja körül explicit módon konstruáltunk egy viszonylag nagy vonzó környezetet a centrális sokaság és a Neimark–Sacker bifurkációs normálforma segítségével.

A 3.3. szakaszban a centrális sokaságra való redukció egy közelítő változatát hajtjuk végre. Először tekintjük a centrális sokaság $\phi(z)$ negyedrendű polinomiális közelítését, majd az $y = \phi(z)$ körüli

$$T(r, C) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \leq r, \ |y - \phi(z)| \leq C|z|^5\}$$

halmazt, ahol r és C pozitív konstansok.

A 3.3.1. és 3.3.2. alszakaszokban az y irányú dinamikát vizsgáljuk a $T(r, C)$ halmazban. Felhasználva, hogy a fixponthoz közeli megoldások exponenciálisan közelítik meg a centrális sokaságot, megmutatjuk, hogy $T(r, C)$ feltételesen invariáns az y irányban. A 3.4. szakaszban pedig a z irányú dinamikát vizsgáljuk a $T(r, C)$ halmazban a Neimark–Sacker bifurkációs normálforma segítségével. Kihasználva a $T(r, C)$ halmaz speciális alakját, megmutatható, hogy egy megfelelő koordináta-rendszerben a transzformált z koordináta szigorúan csökken a leképezés iterálása során, hasonlóan a kétdimenziós esethez. Végezetül az y és a z irányú dinamika kombinálásával megkaphatjuk, hogy a $T(r, C)$ halmaz benne van a fixpont vonzástartományában.

Világos, hogy $T(r, C)$ nem egy valódi környezete a $\mathbb{C} \times \mathbb{R}$ koordináta-rendszer origójának. Ezért a 3.5. szakaszban bevezetjük a

$$\tilde{T}(\hat{r}, K) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \leq \hat{r}, \ |\phi(z) - y| \leq K\}$$

halmazt adott $\hat{r} > 0$ és $K > 0$ konstansok esetén. Felhasználva a $T(r, C)$ halmaz y irányú exponenciálisan vonzó tulajdonságát megmutatjuk, hogy $\tilde{T}(\hat{r}, K)$ is a fixpont vonzástartományában van.

A 2.4–2.6. és 3.6–3.7. szakaszokban a módszer számítógéppel támogatott részét ismertetjük $d = 1$, illetve $d = 2$ esetben. Az S^d halmazt lefedjük véges sok $(d + 1)$-dimenziós kis kockával. Ezeket a kockákat tekintve a gráf csúcsainak, bevezetünk egy irányított gráfot, amely leírja az F_d leképezés viselkedését ezeken a kis kockákon. Ezáltal az S^d halmaz végtelen sok pontjának vizsgálatát egy véges gráfproblémára vezettük vissza, amely könnyebben kezelhető számítógépes eszközökkel. Az irányított gráf éleinek konstruálásához megbízható numerikus módszereket használunk, hogy a számítógép lebegőpontos ábrázolásából adódó hibákat kezelni tudjuk. E gráf segítségével megmutatjuk, hogy S^d minden pontjának iteráltjai belelépnek a korábban kapott vonzó környezetbe. Bebizonyítva ezzel a sejtést a $d \in \{1, 2\}$ esetekre.

Annak ellenére, hogy módszerünket a logisztikus leképezésen mutattuk be, úgy gondoljuk, hogy könnyedén alkalmazható, illetve kiterjesztető más, hasonló leképezésekre is. Például a Ricker (lásd [5]), valamint a Pielou leképezés (lásd [13]) $d = 2$ késleltetéssel a módszer szempontjából lényegében csak abban különbözik a logisztikus leképezéstől, hogy azzal ellentétben ezek nem polinomiális leképezések. Ennek következtében a módszer könnyen adaptálható; mindössze kis módosítást kell bevezetni becsléseink során. A fő kérdés igazából az, hogy a kapott vonzó környezet elegendően nagy méretű-e a módszer számítógéppel támogatott részéhez, azaz belátható számítási időt és memóriát felhasználva sikeresen le fut-e az algoritmus. A fenti két leképezés a logisztikus leképezéssel együtt érdekes probléma nagyobb késleltetés, azaz $d > 2$ esetén is. Hisszük, hogy az analitikus rész, egészen pontosan a centrális redukció, természetes módon adódó módosításokkal átültethető magasabb dimenzióba is. A kritikus pontot a számítógépes rész jelenti, ugyanis a magasabb dimenzió exponenciálisan növekvő gráfot eredményez.

Ezeken felül érdekes probléma lehet a kritikus értéknél nagyobb paraméterértékek esetén megjelenő egyetlen invariáns zárt görbe létezésének bizonyítása is. Azonban ez a kérdés merőben különbözik a disszertációban vizsgált problémától, így lényegesen több új ötletre van szükségünk ennek megmutatásához.

Bibliography

[1] M. W. Hirsch, S. Smale, and R. L. Devaney. *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. Academic Press, 2004.

[2] J. M. Smith. *Mathematical Ideas in Biology*. Cambridge University Press, 1968.

[3] S. A. Levin and R. M. May. A note on difference-delay equations. *Theoretical Population Biology*, 9(2):178 – 187, 1976.

[4] J. Dudás. Global stability for the 2-dimensional logistic map. *Journal of Difference Equations and Applications*, 25(2):179–201, 2019.

[5] F. A. Bartha, Á. Garab, and T. Krisztin. Local stability implies global stability for the 2-dimensional Ricker map. *Journal of Difference Equations and Applications*, 19(12):2043–2078, 2013.

[6] F. A. Bartha and Á. Garab. Necessary and sufficient condition for the global stability of a delayed discrete-time single neuron model. *Journal of Computational Dynamics*, 1(2):213–232, 2014.

[7] J. Dudás and T. Krisztin. Global stability for the three-dimensional logistic map. *Nonlinearity*, 34(2):894–938, 2021.

[8] Y. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer-Verlag New York, 2004.

[9] D. G. Aronson, M. A. Chory, G. R. Hall, and R. P. McGehee. Bifurcations from an invariant circle for two-parameter families of maps of the plane: a computer-assisted study. *Comm. Math. Phys.*, 83(3):303–354, 1982.

[10] H Sarmah, M. Das, and T. Baishya. Neimark–Sacker bifurcation in delayed logistic map. *International Journal of Applied Mathematics and Statistical Sciences*, 3:19–34, 2014.

[11] E. Liz. Local stability implies global stability in some one-dimensional discrete single-species models. *Discrete and Continuous Dynamical Systems. Series B*, 7(1):191–199, 2007.

[12] E. Liz, V. Tkachenko, and S. Trofimchuk. Global stability in discrete population models with delayed-density dependence. *Mathematical Biosciences*, 199(1):26–37, 2006.

[13] E. Camouzis and G. Ladas. *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures.* Chapman and Hall/CRC, 2008.

[14] V. J. López and E. Parreño. L.a.s. and negative Schwarzian derivative do not imply g.a.s. in Clark's equation. *Journal of Dynamics and Differential Equations*, 28(2):339–374, 2016.

[15] W. Tucker. *Validated Numerics: A Short Introduction to Rigorous Computations.* Princeton University Press, 2011.

[16] T. Krisztin, H-O. Walther, and J. Wu. *Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback.* Fields Institute Monographs, 1999.

[17] J. B. van den Berg and J. Jaquette. A proof of wright's conjecture. *Journal of Differential Equations*, 264(12):7412 – 7462, 2018.

[18] Á. Garab. A note on dissipativity and permanence of delay difference equations. *Electron. J. Qual. Theory Differ. Equ.*, 51:1–12, 2018.

[19] http://www.math.u-szeged.hu/tagok/dudasj/research/Log2D.

[20] M. Renardy and R.C. Rogers. *An Introduction to Partial Differential Equations, Second Edition.* Springer-Verlag, 2004.

[21] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms, and Applications.* Monographs in Mathematics. Springer, 2001.

[22] V. Tkachenko and S. Trofimchuk. Global stability in difference equations satisfying the generalized Yorke condition. *Journal of Mathematical Analysis and Applications*, 303(1):173 – 187, 2005.

[23] S. Wiggins. *Elements of Applied Bifurcation Theory.* Springer-Verlag New York, 2003.

[24] J. Carr. *Applications of Centre Manifold Theory.* Springer-Verlag New York, 1981.

[25] http://www.math.u-szeged.hu/tagok/krisztin/log3dim.html.