On the aggregation of branching processes

Outline of Ph.D. Thesis

by

Fanni Kunosné Nedényi

Supervisors:

Dr. Gyula Pap
professor

and

Dr. Mátyás Barczy
senior research fellow

Doctoral School of Mathematics and Computer Science

Bolyai Institute

University of Szeged, Faculty of Science and Informatics

Szeged, 2020

1 Introduction

The thesis is about the limit behavior of the temporal and contemporaneous aggregates of certain branching processes. Aggregates, also known as partial sums, are well-known to be very important in stochastics. All of the chapters of the thesis deal with this question, either for different processes, different manners regarding the convergence (iterated, simultaneous), or different centralizations.

Chapter 1 contains an introduction where we explain the goal of the thesis, the historical background of the studied topic, along with an overview of the work. In the following paragraph we specify the scheme of aggregation that we use.

The aggregation problem is concerned with the relationship between individual (micro) and aggregate (macro) behavior. In general, we consider independent copies of a stationary branching process, we denote these by $(X_k^{(j)})_{k\in\mathbb{N}}$, $j \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, \dots\}$. We are interested in the limit behavior of the aggregate process $\left(\sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$, as both n, the time parameter, and N, the number of copies tend to infinity in some manner. If we take the limits in an iterated manner, i.e., first n tends to infinity and then N tends to infinity, or vice versa, then the resulting limit theorem is called an iterated one. If both converge to infinity at the same time, then it is called a simultaneous limit theorem. To achieve such limit theorems, we also consider the simple aggregates, $\sum_{k=1}^{n} X_k^{(j)}$, which is called temporal (or time-aggregated), and $\sum_{j=1}^{N} X_k^{(j)}$, which is called contemporaneous (or space-aggregated).

Let us recall the most important antecedents of the work presented in this thesis. The scheme of contemporaneous aggregation of random coefficient autoregressive processes of order 1 (AR(1)) was firstly proposed by the Nobel prize winner Clive W. J. Granger [5] in order to obtain the long memory phenomena in aggregated time series. In a series of papers, Donatas Surgailis and his co-authors studied the aggregation of random coefficient AR(1) processes, where $(X_k^{(j)})_{k\in\mathbb{Z}_+:=\{0,1,\dots\}}$, $j \in \mathbb{N}$, are independent copies of a stationary random coefficient AR(1) process

$$X_k = aX_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

with standardized independent and identically distributed (i.i.d.) innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ and a random coefficient a with values in $(0, 1)$, being independent of $(\varepsilon_k)_{k\in\mathbb{N}}$ and admitting a probability density function of the form

$$\psi(x)(1-x)^{\beta}, \qquad x \in (0,1),$$

where $\beta \in (-1,\infty)$ and ψ is an integrable function on $(0,1)$ having a limit $\lim_{x\uparrow 1}\psi(x) := \psi_1 \in (0,\infty)$. In the paper Pilipauskaitė and Surgailis [7] both iterated and simultaneous limit theorems were presented concerning the limit behavior of the aggregate process $\left(\sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor} X_k^{(j)}\right)_{t\in[0,\infty)}$.

Our aim is to provide such results when branching processes take the place of the random coefficient AR(1) model explained before. These processes are widely applicable as they can model integer-valued phenomena, such as migration and the spreading of contagious diseases like COVID-19. These possible applications are more thoroughly detailed in Chapter 2 of the thesis.

2 Limit theorems for the aggregation of multitype Galton–Watson branching processes with immigration

Chapter 2 is devoted to the investigation of the aggregates of multitype Galton–Watson processes with immigration. The proofs of Chapter 2 are based on the paper Barczy et al. [3].

The p-dimensional process $(\boldsymbol{Y}_k = [Y_{k,1},\ldots,Y_{k,p}]^\top)_{k\in\mathbb{Z}_+}$, where $p\in\mathbb{N}$, is a p-type Galton–Watson branching process with immigration if

$$\boldsymbol{Y}_k = \sum_{\ell=1}^{Y_{k-1,1}} \begin{bmatrix} \xi_{k,\ell}^{(1,1)} \\ \vdots \\ \xi_{k,\ell}^{(1,p)} \end{bmatrix} + \cdots + \sum_{\ell=1}^{Y_{k-1,p}} \begin{bmatrix} \xi_{k,\ell}^{(p,1)} \\ \vdots \\ \xi_{k,\ell}^{(p,p)} \end{bmatrix} + \begin{bmatrix} \varepsilon_k^{(1)} \\ \vdots \\ \varepsilon_k^{(p)} \end{bmatrix} = \sum_{i=1}^{p}\sum_{\ell=1}^{Y_{k-1,i}} \boldsymbol{\xi}_{k,\ell}^{(i)} + \boldsymbol{\varepsilon}_k$$

for every $k\in\mathbb{N}$, where we define $\sum_{\ell=1}^{0} := \boldsymbol{0}$, and $\big\{\boldsymbol{Y}_0, \boldsymbol{\xi}_{k,\ell}^{(i)}, \boldsymbol{\varepsilon}_k : k,\ell\in\mathbb{N}, i\in\{1,\ldots,p\}\big\}$ are independent $\mathbb{Z}_+^p$-valued random vectors. Moreover, for all $i\in\{1,\ldots,p\}$, $\{\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}_{k,\ell}^{(i)} : k,\ell\in\mathbb{N}\}$ and $\{\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_k : k\in\mathbb{N}\}$ consist of identically distributed random vectors, respectively. We suppose that

$$\mathbb{E}(\boldsymbol{\xi}^{(i)}) \in \mathbb{R}_+^p, \quad i\in\{1,\ldots,p\}, \qquad \boldsymbol{m}_{\boldsymbol{\varepsilon}} \in \mathbb{R}_+^p \setminus \{\boldsymbol{0}\},$$
$$\varrho(\boldsymbol{M}_{\boldsymbol{\xi}}) < 1, \qquad \boldsymbol{M}_{\boldsymbol{\xi}} \text{ is primitive},$$

where $\mathbb{R}_+ := [0,\infty)$, $\boldsymbol{m}_{\boldsymbol{\varepsilon}} := \mathbb{E}(\boldsymbol{\varepsilon}) \in \mathbb{R}_+^p$, $\boldsymbol{M}_{\boldsymbol{\xi}} := \mathbb{E}\big(\big[\boldsymbol{\xi}^{(1)},\ldots,\boldsymbol{\xi}^{(p)}\big]\big) \in \mathbb{R}_+^{p\times p}$, and $\varrho(\boldsymbol{M}_{\boldsymbol{\xi}})$ is the spectral radius of the matrix $\boldsymbol{M}_{\boldsymbol{\xi}}$. Then a unique stationary distribution exists. We choose that as the distribution of $\boldsymbol{Y}_0$, resulting that the considered process is strictly stationary. Then we take independent copies, $(\boldsymbol{Y}_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, of this process. For each $N, n \in \mathbb{N}$, we consider the stochastic process $\boldsymbol{S}^{(N,n)} = (\boldsymbol{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$\boldsymbol{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} (\boldsymbol{Y}_k^{(j)} - \mathbb{E}(\boldsymbol{Y}_k^{(j)})), \qquad t \in \mathbb{R}_+.$$

Let us define the matrix

$$\boldsymbol{V} := (V_{i,j})_{i,j=1}^{p} := \left(\boldsymbol{v}_{(i,j)}^\top \begin{bmatrix} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{m}_{\boldsymbol{\varepsilon}} \\ 1 \end{bmatrix} \right)_{i,j=1}^{p} \in \mathbb{R}^{p\times p},$$

where $\boldsymbol{I}_p$ denotes the p-dimensional identity matrix, provided that the covariances

$$\boldsymbol{v}_{(i,j)} := \left[\operatorname{Cov}(\xi^{(1,i)}, \xi^{(1,j)}), \dots, \operatorname{Cov}(\xi^{(p,i)}, \xi^{(p,j)}), \operatorname{Cov}(\varepsilon^{(i)}, \varepsilon^{(j)}) \right]^\top \in \mathbb{R}^{(p+1)\times 1}$$

for $i, j \in \{1, \dots, p\}$, are finite. Now we present the main results of this chapter of the thesis. Note that $\mathcal{D}_{\mathrm{f}}$-lim means the convergence of finite dimensional distributions.

Theorem 2.1. *If all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1, \dots, p\}$, and $\boldsymbol{\varepsilon}$ have finite second moments, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \boldsymbol{S}^{(N,n)} = (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{B},$$

where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a p-dimensional standard Brownian motion.

If all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1, \dots, p\}$, and $\boldsymbol{\varepsilon}$ have finite third moments, then

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \boldsymbol{S}^{(N,n)} = (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{B},$$

where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a p-dimensional standard Brownian motion.

Theorem 2.2. *If all entries of the vectors $\boldsymbol{\xi}^{(i)}$, $i \in \{1, \dots, p\}$, and $\boldsymbol{\varepsilon}$ have finite third moments, then*

$$(nN)^{-\frac{1}{2}} \boldsymbol{S}^{(N,n)} \xrightarrow{\mathcal{D}} (\boldsymbol{I}_p - \boldsymbol{M}_{\boldsymbol{\xi}})^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{B},$$

if both n and N converge to infinity (at any rate), where $\boldsymbol{B} = (\boldsymbol{B}_t)_{t\in\mathbb{R}_+}$ is a standard p-dimensional Brownian motion.

The proofs of these theorems rely on the multidimensional central limit theorem and the functional martingale central limit theorem.

Among others, we also discuss these results in the special case of integer-valued autoregressive processes of order 1 (INAR(1)) with Poisson innovations. These are one-dimensional Galton–Watson processes with immigration where the offsprings have Bernoulli distribution with parameter $\alpha \in (0, 1)$ and the immigrations have Poisson distribution.

3 Iterated limit theorems for the aggregation of randomized INAR(1) processes with Poisson innovations

In Chapters 3 and 4 we consider a certain randomized INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$ with randomized thinning parameter α, given formally by the recursive equation

$$X_k = \sum_{\ell=1}^{X_{k-1}} \xi_{k,\ell} + \varepsilon_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

where $\circ$ is the so-called Steutel and van Harn thinning operator, α is a random variable with values in $(0,1)$ and X_0 is some appropriate random variable. This means that, conditionally on α, the process $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with thinning parameter α, i.e., conditionally on α, the offsprings, $(\xi_{k,\ell})_{k,\ell\in\mathbb{N}}$, have Bernoulli distribution with parameter α. Conditionally on α, the i.i.d. innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ have a Poisson distribution with parameter $\lambda \in (0,\infty)$, and the conditional distribution of the initial value X_0 given α is the unique stationary distribution, namely, a Poisson distribution with parameter $\lambda/(1-\alpha)$. In Chapter 3 we provide a rigorous construction of this process. For the desired iterated and simultaneous limit theorems we assume that the random parameter α admits a mixing distribution having a probability density of the form

$$\psi(x)(1-x)^\beta, \qquad x \in (0,1),$$

where ψ is a function on $(0,1)$ having a limit $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$. Note that necessarily $\beta \in (-1,\infty)$ (otherwise $\int_0^1 \psi(x)(1-x)^\beta \, \mathrm{d}x = \infty$), and the function $(0,1) \ni x \mapsto \psi(x)$ is integrable on $(0,1)$. The Beta distribution is a special case of this form. Certain $\circ$ operators, where the summands are random parameter Bernoulli distributions with a parameter having Beta distribution, appear in catastrophe models. Moreover, Clive W. J. Granger used the square root of a Beta distribution as a mixing distribution for random coefficient AR(1) processes.

Chapter 3 contains an exhaustive list of iterated limit theorems related to the aggregates in multiple manners. In Chapter 3 the proofs are based on the papers Nedényi and Pap [6] and Barczy et al. [1].

We consider three different aggregate processes regarding the centralization: $\widetilde{S}^{(N,n)} := (\widetilde{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$, $S^{(N,n)} := (S_t^{(N,n)})_{t\in\mathbb{R}_+}$ and $\widehat{S}^{(N,n)} :=$

$(\widehat{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ for each $N,n\in\mathbb{N}$, with

$$\widetilde{S}_t^{(N,n)} := \sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)}|\alpha^{(j)})),$$

where we center with the conditional expectation with respect to the random parameter belonging to the corresponding process,

$$S_t^{(N,n)} := \sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(j)} - \mathbb{E}(X_k^{(j)})),$$

where we center with the expectation (note that this only exists for $\beta > 0$), and

$$\widehat{S}_t^{(N,n)} := \sum_{j=1}^{N}\sum_{k=1}^{\lfloor nt\rfloor}\left(X_k^{(j)} - \frac{\sum_{\ell=1}^{n} X_\ell^{(j)}}{n}\right),$$

where we center with the empirical mean of the first n observations for the corresponding process in order to provide a well-applicable, observable alternative. As there are two different approaches to iterated limit theorems ($n\to\infty$ and then $N\to\infty$ or vice versa), and we have different limit theorems for different ranges of the parameter β, this chapter contains many limit theorems.

The next four results are limit theorems for appropriately scaled versions of $\widetilde{S}^{(N,n)}$, first taking the limit $N\to\infty$ and then $n\to\infty$ in the case $\beta\in(-1,1]$.

Theorem 3.1. *If $\beta\in(-1,0)$, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\,\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\, n^{-1}N^{-\frac{1}{2(1+\beta)}}\,\widetilde{S}^{(N,n)} = (V_{2(1+\beta)}t)_{t\in\mathbb{R}_+},$$

where $V_{2(1+\beta)}$ is a symmetric $2(1+\beta)$-stable random variable (not depending on t) with characteristic function

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta V_{2(1+\beta)}}) = \mathrm{e}^{-K_\beta|\theta|^{2(1+\beta)}}, \qquad \theta\in\mathbb{R},$$

where

$$K_\beta := \psi_1\left(\frac{\lambda}{2}\right)^{1+\beta}\frac{\Gamma(-\beta)}{1+\beta}.$$

Theorem 3.2. *If $\beta=0$, then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\,\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\, n^{-1}(N\log N)^{-\frac{1}{2}}\,\widetilde{S}^{(N,n)} = (W_{\lambda\psi_1}t)_{t\in\mathbb{R}_+},$$

where $W_{\lambda\psi_1}$ is a normally distributed random variable with mean zero and with variance $\lambda\psi_1$.

The proofs of Theorems 3.1 and 3.2 rely on a lemma that was developed for this research, which helps us prove the convergence of characteristic functions.

For $\beta \in (0,2)$, let $(\mathcal{B}_{1-\frac{\beta}{2}}(t))_{t\in\mathbb{R}_+}$ denote a fractional Brownian motion with parameter $1-\beta/2$, that is a Gaussian process with zero mean and covariance function

$$\operatorname{Cov}(\mathcal{B}_{1-\frac{\beta}{2}}(t_1), \mathcal{B}_{1-\frac{\beta}{2}}(t_2)) = \frac{t_1^{2-\beta} + t_2^{2-\beta} - |t_2 - t_1|^{2-\beta}}{2}, \qquad t_1, t_2 \in \mathbb{R}_+.$$

Theorem 3.3. *If* $\beta \in (0,1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1+\frac{\beta}{2}} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}} \mathcal{B}_{1-\frac{\beta}{2}}.$$

For the proof of Theorem 3.3 we apply Theorem 4.3 of Beran et al. [4], which is about convergence of partial sums of a Hermite function of a stationary sequence of standard normal random variables.

Theorem 3.4. *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (n\log n)^{-\frac{1}{2}} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sqrt{2\lambda\psi_1} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

For the proof of Theorem 3.4 we use that in case of zero mean Gaussian processes, to prove their convergence in distribution, the convergence of their covariance functions has to be shown.

The next two results are limit theorems for an appropriately scaled version of $\widetilde{S}^{(N,n)}$, first taking the limit $n \to \infty$ and then $N \to \infty$ in the case $\beta \in (-1,1]$.

Theorem 3.5. *If* $\beta \in (-1,1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} N^{-\frac{1}{1+\beta}} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \mathcal{Y}_{1+\beta},$$

where $\mathcal{Y}_{1+\beta} = \big(\mathcal{Y}_{1+\beta}(t) := \sqrt{Y_{(1+\beta)/2}}\, B_t\big)_{t\in\mathbb{R}_+}$, *and* $Y_{(1+\beta)/2}$ *is a positive* $\frac{1+\beta}{2}$*-stable random variable with Laplace transform* $\mathbb{E}(\mathrm{e}^{-\theta Y_{(1+\beta)/2}}) = \mathrm{e}^{-k_\beta \theta^{\frac{1+\beta}{2}}}$, $\theta \in \mathbb{R}_+$, *with*

$$k_\beta := \frac{(2\lambda)^{\frac{1+\beta}{2}}\psi_1}{1+\beta} \Gamma\left(\frac{1-\beta}{2}\right),$$

and $(B_t)_{t\in\mathbb{R}_+}$ *is an independent standard Wiener process. The process* $\mathcal{Y}_{1+\beta}$ *has* $(1+\beta)$*-stable one-dimensional distributions and stationary increments.*

Again, the proof of Theorem 3.5 relies on our lemma mentioned before like those of Theorems 3.1 and 3.2.

Theorem 3.6. *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}}(N\log N)^{-\frac{1}{2}}\, \widetilde{S}^{(N,n)} = \sqrt{\lambda\psi_1}B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

For the proof of Theorem 3.6 we apply Theorem 7.1 of Resnick [8], which is about weak convergence of partial sum processes for a triangular array towards a Lévy process.

Next we show an iterated scaling limit theorem where the order of the iteration can be arbitrary in the case $\beta \in (1,\infty)$. This theorem completes the list of results concerning $\widetilde{S}^{(N,n)}$.

Theorem 3.7. *If* $\beta \in (1,\infty)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}}\widetilde{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}}\widetilde{S}^{(N,n)} = \sigma B,$$

where $\sigma^2 := \lambda\,\mathbb{E}((1+\alpha)(1-\alpha)^{-2})$ *and* $(B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

We note that for $\beta > 1$ the parameter σ is finite. Again, the proof of Theorem 3.7 relies on checking the convergence of the covariance functions of Gaussian processes, and the multidimensional central limit theorem.

Now we present our results for the aggregate process $S^{(N,n)}$. These are the counterparts of Theorems 3.3, 3.4, 3.5, 3.6 and 3.7. Note that the expectation $\mathbb{E}(X_0) = \mathbb{E}\big(\frac{\lambda}{1-\alpha}\big)$ is finite if and only if $\beta > 0$, so Theorems 3.1 and 3.2 can not have counterparts in this sense.

Theorem 3.8. *If* $\beta \in (0,1)$, *then*

$$\begin{aligned}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1}N^{-\frac{1}{1+\beta}}S^{(N,n)} &= \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1}N^{-\frac{1}{1+\beta}}S^{(N,n)}\\ &= \big(Z_{1+\beta}\,t\big)_{t\in\mathbb{R}_+},\end{aligned}$$

where $Z_{1+\beta}$ *is a* $(1+\beta)$*-stable random variable with characteristic function* $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Z_{1+\beta}}) = \mathrm{e}^{-|\theta|^{1+\beta}\omega_\beta(\theta)}$, $\theta\in\mathbb{R}$, *where*

$$\omega_\beta(\theta) := \frac{\psi_1\Gamma(1-\beta)\lambda^{1+\beta}}{-\beta(1+\beta)}\mathrm{e}^{-\mathrm{i}\pi\,\mathrm{sign}(\theta)(1+\beta)/2}, \qquad \theta\in\mathbb{R}.$$

Note that the following theorem was developed for the thesis, it has not appeared in any of our papers.

Theorem 3.9. *If* $\beta = 1$, *then there exists a sequence* $(a_N)_{N\in\mathbb{N}}$ *such that* $\sqrt{N}/a_N = \mathrm{o}(1)$ *as* $N \to \infty$ *(meaning that* $\lim_{N\to\infty} \sqrt{N}/a_N = 0$*) and*

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1} a_N^{-1} S^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} a_N^{-1} S^{(N,n)} = (W\,t)_{t\in\mathbb{R}_+},$$

where W *is a standard normally distributed random variable.*

Theorem 3.10. *If* $\beta \in (1, \infty)$, *then*

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2}} S^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} S^{(N,n)} = (W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}\, t)_{t\in\mathbb{R}_+},$$

where $W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}$ *is a normally distributed random variable with mean zero and with variance* $\lambda^2 \operatorname{Var}((1-\alpha)^{-1})$.

In the proofs of Theorems 3.8, 3.9 and 3.10 we show that some random variables are in the domain of attraction of a stable or normal distribution. Note that in case of Theorems 3.1, 3.2, 3.8, 3.9 and 3.10 the limit processes are lines with random slopes.

Lastly, we present the limit theorems concerning the aggregate process $\widehat{S}^{(N,n)}$. First we observe that

$$\widehat{S}_t^{(N,n)} = \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt\rfloor} \left[X_k^{(j)} - \mathbb{E}(X_k^{(j)} \mid \alpha^{(j)}) - \frac{\sum_{\ell=1}^{n} (X_\ell^{(j)} - \mathbb{E}(X_\ell^{(j)} \mid \alpha^{(j)}))}{n} \right] = \widetilde{S}_t^{(N,n)} - \frac{\lfloor nt\rfloor}{n} \widetilde{S}_1^{(N,n)}$$

for every $t \in \mathbb{R}_+$. Therefore, by Theorems 3.3, 3.5, 3.4, 3.6, and 3.7, using Slutsky's lemma, the following limit theorems hold. Note that the two results of the following corollary which hold when $\beta = 1$ have not been published before.

Corollary 3.11. *If* $\beta \in (0, 1)$, *then*

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1+\frac{\beta}{2}} N^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}} \left(\mathcal{B}_{1-\frac{\beta}{2}}(t) - t\mathcal{B}_{1-\frac{\beta}{2}}(1)\right)_{t\in\mathbb{R}_+},$$

where the process $\mathcal{B}_{1-\frac{\beta}{2}}$ *is given before Theorem 3.3.*

If $\beta \in (-1, 1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} N^{-\frac{1}{1+\beta}} n^{-\frac{1}{2}} \widehat{S}^{(N,n)} = (\mathcal{Y}_{1+\beta}(t) - t\mathcal{Y}_{1+\beta}(1))_{t\in\mathbb{R}_+},$$

where the process $\mathcal{Y}_{1+\beta}$ *is given in Theorem 3.5.*

If $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (n \log n)^{-\frac{1}{2}} N^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sqrt{2\lambda\psi_1}(B_t - tB_1)_{t\in\mathbb{R}_+},$$

moreover,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}} (N \log N)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sqrt{\lambda\psi_1}(B_t - tB_1)_{t\in\mathbb{R}_+},$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

If $\beta \in (1, \infty)$, *then*

$$\begin{aligned}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} &= \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} \\ &= \sigma(B_t - tB_1)_{t\in\mathbb{R}_+},\end{aligned}$$

where σ^2 *is given in Theorem 3.7 and* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

In Corollary 3.11, the limit processes restricted on the time interval $[0, 1]$ are bridges in the sense that they take the same value (namely, 0) at the time points 0 and 1, and especially, in case of $\beta \in (1, \infty)$, it is a Wiener bridge. We note that no counterparts appear for the rest of the theorems because in those cases the limit processes are lines with random slopes, which result the constant zero process in this alternative case. In case of $\beta \in (-1, 0]$, by applying some smaller scaling factors, one could try to achieve a non-degenerate weak limit of $\widehat{S}^{(N,n)}$ by first taking the limit $N \to \infty$ and then that of $n \to \infty$.

To sum up the results of this chapter, let us point out that the scaling of the processes becomes heavier as the parameter β decreases, since the finiteness of the first and second moments of the stationary distribution of the randomized INAR(1) process depends on this parameter. Also, it is interesting that in most of the cases the two different orders of iteration result in significantly different limit theorems as the scaling factors and limit processes differ.

4 Simultaneous limit theorems for the aggregation of randomized INAR(1) processes with Poisson innovations

In Chapter 4 two simultaneous limit theorems are presented. The proofs of Chapter 4 are based on the paper Barczy et al. [2] that has been submitted to a journal.

Contrary to the iterated limit theorems, the list of the simultaneous ones is not complete, some cases remain for future work. We have the following two simultaneous results. To represent the connection between the two parameters n and N, we will use a sequence $N_n, n \in \mathbb{N}$, instead of the parameter N, and the theorems will be given as n tends to infinity, which will always imply that $N_n, n \in \mathbb{N}$, tends to infinity as well.

Theorem 4.1. *If* $\beta \in (-1, 0)$, *then*

$$n^{-1} N_n^{-\frac{1}{2(1+\beta)}} \widetilde{S}^{(N_n,n)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} (V_{2(1+\beta)} t)_{t \in \mathbb{R}_+}$$

as $n \to \infty$ *and* $N_n^{\frac{-\beta}{1+\beta}} n^{-1} \to \infty$, *where* $V_{2(1+\beta)}$ *is a symmetric* $2(1+\beta)$*-stable random variable (not depending on* t*) with characteristic function*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta V_{2(1+\beta)}}) = \mathrm{e}^{-K_\beta |\theta|^{2(1+\beta)}}, \qquad \theta \in \mathbb{R},$$

where $K_\beta = \psi_1 (\frac{\lambda}{2})^{1+\beta} \frac{\Gamma(-\beta)}{1+\beta}$.

We note that Theorem 4.1 can be considered as a counterpart of Theorem 3.1. The scaling factors and the limit processes coincide in these two theorems.

Theorem 4.2. *If* $\beta = 0$, *then*

$$n^{-1} (N_n \log N_n)^{-\frac{1}{2}} \widetilde{S}^{(N_n,n)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} (W_{\lambda\psi_1} t)_{t \in \mathbb{R}_+}$$

as $n \to \infty$ *and* $(\log N_n)^2 n^{-1} \to \infty$, *where* $W_{\lambda\psi_1}$ *has a normal distribution with mean* 0 *and variance* $\lambda\psi_1$.

We note that Theorem 4.2 can be considered as a counterpart of Theorem 3.2. The scaling factors and the limit processes coincide in these two theorems.

The proofs of these limit theorems are based on our lemma developed for this research, which is a counterpart of the one applied for the iterated limit theorems. We plan to handle the remaining cases as future work which will require a different proof technique.

References

[1] M. Barczy, F. Nedényi, and G. Pap. Iterated limits for aggregation of randomized INAR(1) processes with Poisson innovations. *J. Math. Anal. Appl.*, 451(1):524–543, 2017.

[2] M. Barczy, F. Nedényi, and G. Pap. On simultaneous limits for aggregation of stationary randomized INAR(1) processes with Poisson innovations. arXiv: 2001.07127, 2020.

[3] M. Barczy, F. K. Nedényi, and G. Pap. On aggregation of multitype Galton-Watson branching processes with immigration. *Mod. Stoch. Theory Appl.*, 5(1):53–79, 2018.

[4] J. Beran, Y. Feng, S. Ghosh, and R. Kulik. *Long-memory processes. Probabilistic properties and statistical methods.* Springer, Heidelberg, 2013.

[5] C. W. J. Granger. Long memory relationships and the aggregation of dynamic models. *J. Econometrics*, 14(2):227–238, 1980.

[6] F. Nedényi and G. Pap. Iterated scaling limits for aggregation of random coefficient AR(1) and INAR(1) processes. *Statist. Probab. Lett.*, 118:16–23, 2016.

[7] V. Pilipauskaitė and D. Surgailis. Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. *Stochastic Process. Appl.*, 124(2):1011–1035, 2014.

[8] S. I. Resnick. *Heavy-tail phenomena.* Springer, New York, 2007.