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On the angle sum of lines

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Abstract. What is the maximum of the sum of the pairwise (non-obtuse) angles formed by n lines in the Euclidean 3-space? This question was posed by Fejes Tóth in (Acta Math Acad Sci Hung 10:13–19, 1959). Fejes Tóth solved the problem for $n \leq 6$, and proved the asymptotic upper bound $n^2\pi/5$ as $n \to \infty$. He conjectured that the maximum is asymptotically equal to $n^2\pi/6$ as $n \to \infty$. The main result of this paper is an upper bound on the sum of the angles of n lines in the Euclidean 3-space that is asymptotically equal to $3n^2\pi/16$ as $n \to \infty$.

Mathematics Subject Classification. 52C35.

Keywords. Angle sum of lines, Upper bound.

1. Introduction. Consider n lines in the d-dimensional Euclidean space $\mathbb{R}^d$ which all pass through the origin o. What is the maximum $S(n,d)$ of the sum of the pairwise (non-obtuse) angles formed by the lines? This question was raised by Fejes Tóth in [3] for $d = 3$. For general d, the problem is formulated, for example, in [5].

The conjectured maximum of the angle sum is attained by the following configuration: Let $n = k \cdot d + m$ $(0 \leq m < d)$, and denote by $x_1, \ldots, x_d$ the axes of a Cartesian coordinate system in $\mathbb{R}^d$. Take $k+1$ copies of each one of the axes $x_1, \ldots, x_m$, and take k copies of each one of the axes $x_{m+1}, \ldots, x_d$. The sum of the pairwise angles in this configuration is

$$\left[\frac{d(d-1)k^2}{2} + mk(d-1) + \frac{m(m-1)}{2}\right]\frac{\pi}{2}.$$

Fejes Tóth stated this conjecture only for $d = 3$, however, it is quite natural to extend it to any d (see [5]). To the best of our knowledge, this problem is unsolved for $d \geq 3$.

In the case $d = 3$, Fejes Tóth [3] proved the conjecture for $n \leq 6$. He determined $S(n,3)$ for $n \leq 5$ by direct calculation, and he obtained $S(6,3)$

using the recursive upper bound $S(n,3) \leq nS(n-1,3)/(n-2)$ and the precise value of $S(5,3)$, see p. 19 in [3]. The recursive upper bound and $S(6,3)$ together yield that $S(n,3) \leq n(n-1)\pi/5$ for all n. We further note that Fejes Tóth's recursive upper bound on $S(n,3)$ also holds for $S(n,d)$, that is, $S(n,d) \leq nS(n-1,d)/(n-2)$ for any meaningful n and d.

Our main result is summarized in the following theorem.

Theorem 1.1. *Let $l_1,\dots,l_n$ be lines in $\mathbb{R}^3$ which all pass through the origin. If we denote by φ_{ij} the angle formed by l_i and l_j, then*

$$\sum_{1\leq i<j\leq n} \varphi_{ij} \leq \begin{cases} \frac{3}{2}k^2 \cdot \frac{\pi}{2}, & \textit{if } n=2k, \\ \frac{3}{2}k(k+1)\cdot \frac{\pi}{2}, & \textit{if } n=2k+1. \end{cases}$$

We note that the conjectured maximum for $d=3$ is asymptotically equal to $n^2\pi/6$ as $n\to\infty$. The upper bound in Theorem 1.1 is asymptotically $3n^2\pi/16$ as $n\to\infty$, so it improves on Fejes Tóth's bound which is $n^2\pi/5$ as $n\to\infty$. We also note that if one could prove that $S(8,3)$ is equal to the conjectured value, then combining it with Fejes Tóth's recursive upper bound on $S(n,3)$, one would obtain an upper bound on $S(n,3)$ that is asymptotically equal to the one in Theorem 1.1.

We mention that the corresponding problem in which we seek the maximum of the sum of the angles of n rays emanating from the origin of $\mathbb{R}^d$ is solved for any d and n. This problem was also posed in the same paper of Fejes Tóth [3] for $d=3$. The 3-dimensional problem was fully solved as of 1965, see [3,4,7–9]. The proof of Nielsen [8] uses a projection averaging argument. We note that this argument can be modified so as to obtain a solution of the general case of the problem for every n and d. Our proof of Theorem 1.1 also uses this projection averaging idea, however, the details are much more intricate.

2. The planar case. Before we prove Theorem 1.1, we solve the problem in the plane. This result is probably known [5], however, we were unable to find any other reference, thus, we decided to include a short proof for the sake of completeness. The related problem for rays in the plane was analysed by Jiang [6] in 2008. He reproved the known upper bound and gave a full description of the extremal configurations in terms of balanced configurations of vectors.

We say that a line l' is to the right of l if l' is obtained from l by a rotation about the origin with angle α, where $-\pi/2<\alpha<0$. Similarly, if $0<\alpha<\pi/2$, then l' is to the left of l. If $l'=l$ or l' is perpendicular to l, then l' is neither to the left nor to the right of l. We say that a configuration $l_1,\dots,l_n$ of n lines is *balanced* if for any line $l\neq l_1,\dots,l_n$ the number of lines to the left of l and the number of lines to the right of l differ by at most 1. We remark that Jiang defined balanced systems of vectors in [6]. Our definition of a balanced configuration of lines is similar to but not the same as that of Jiang.

Theorem 2.1. *Let $l_1,\dots,l_n$ be lines in $\mathbb{R}^2$ which all pass through the origin. If we denote by φ_{ij} the angle formed by l_i and l_j, then*

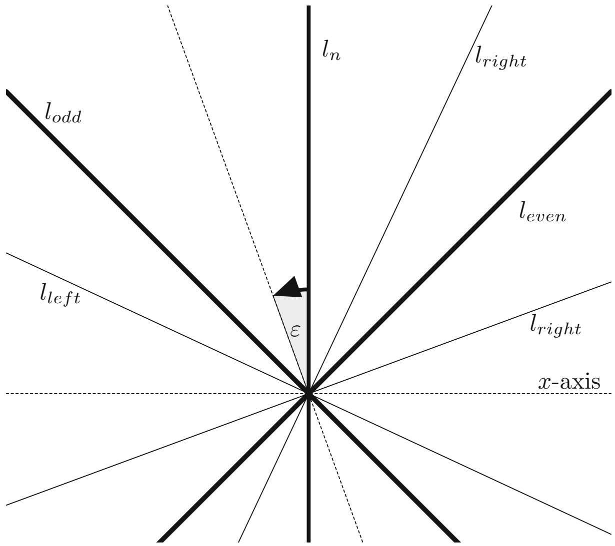

FIGURE 1. Rotating l_1

$$\sum_{1\leq i<j\leq n} \varphi_{ij} \leq \begin{cases} k^2 \cdot \frac{\pi}{2}, & \text{if } n = 2k, \\ k(k+1) \cdot \frac{\pi}{2}, & \text{if } n = 2k+1. \end{cases}$$

Equality holds if, and only if, $l_1, \dots, l_n$ *is balanced.*

Proof. The idea of the proof is similar to that of Jiang [6, Theorem 1]. Note that a simple compactness argument guarantees that the maximum of the angle sum exists, and it is attained by some configuration.

Observe that if l and l' are two perpendicular lines and l'' is an arbitrary third line, then the angle sum determined by l, l', and l'' is always π. This implies that if we have a perpendicular pair in a configuration of lines, then the pair can be freely rotated about the origin while the total sum of the angles remains unchanged.

Let $k = \lfloor n/2 \rfloor$, then $n = 2k$ or $n = 2k+1$. We are going to show that any configuration of n lines can be continuously transformed into a configuration that is the disjoint union of k perpendicular pairs (and possibly one remaining line in arbitrary position) such that the angle sum does not decrease during the transformation. This clearly proves Theorem 2.1.

Assume that $(l_1, l_2), \dots, (l_{2m-1}, l_{2m})$, $m < k$ is a maximal set of pairwise disjoint perpendicular pairs in $l_1, \dots, l_n$. During the transformation we will keep each already existing perpendicular pair. By the above observation, we may disregard these pairs as the angle sum of $l_1, \dots, l_n$ is independent of their positions.

Let l_n be vertical (it coincides with the y-axis), see Fig. 1. By symmetry, we may clearly assume that there are at least as many lines to the right of l_n as to the left. The case $l_{2m+1} = l_{2m+2} = \cdots = l_n$ being obvious, we may assume that there is at least one line to the right of l_n.

Observe that rotating l_n by a small positive angle $\varepsilon > 0$, the sum of the angles in $l_1, \ldots, l_n$ does not decrease. Thus, we may rotate l_n until it becomes perpendicular to a line on its right-hand side. In this way, we have created a new perpendicular pair that is disjoint from $(l_1, l_2), \ldots, (l_{2m-1}, l_{2m})$. This completes the proof of Theorem 2.1.

We only sketch the analysis of the equality case. It is clear that if equality holds then the configuration of lines must be balanced. One can see that if $n = 2k$, then a balanced configuration of n lines consists of k pairs of perpendicular lines. On the other hand, a balanced configuration of $n = 2k+1$ lines, similarly as in the proof of the inequality, can be continuously transformed into a disjoint union of k perpendicular pairs of lines and one remaining line in arbitrary position such that the angle sum does not change during the transformation. The details are left to the reader. This yields that for a fixed n the angle sum is the same in any balanced configuration of n lines. This finishes the proof of the equality case. $\square$

3. Proof of Theorem 1.1. Let S^2 be the unit sphere of $\mathbb{R}^3$ centred at the origin. We denote the Euclidean scalar product by $\langle\cdot,\cdot\rangle$ and the induced norm by $|\cdot|$. For $\mathbf{u}, \mathbf{v} \in S^2$, we introduce $\mathbf{v}^{\mathbf{u}} = (\mathbf{u} \times \mathbf{v}) \times \mathbf{u}$, which is the component of $\mathbf{v}$ perpendicular to $\mathbf{u}$. Let $\mathbf{v}_1, \mathbf{v}_2 \in S^2$, and let $\varphi = \angle(\mathbf{v_1}, \mathbf{v_2})$ denote the angle formed by $\mathbf{v}_1, \mathbf{v}_2$. Introduce $\varphi^{\mathbf{u}} = \varphi^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2)$ for the angle formed by $\mathbf{v}_1^{\mathbf{u}}$ and $\mathbf{v}_2^{\mathbf{u}}$, and write

$$\varphi_*^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2) := \min\{\varphi^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2), \pi - \varphi^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2)\}.$$

Let

$$I(\mathbf{v}_1, \mathbf{v}_2) = I(\varphi) := \frac{1}{4\pi} \int_{S^2} \varphi_*^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2) \mathrm{d}\mathbf{u},$$

where the integration is with respect to the spherical Lebesgue measure. We will use the following lemma of Fáry [2].

Lemma 3.1. (*Fáry, Lemme 1. on p. 133 in* [2])

$$\varphi = \frac{1}{4\pi} \int_{S^2} \varphi^{\mathbf{u}} \mathrm{d}\mathbf{u} \quad \text{for any} \quad 0 \leq \varphi \leq \pi.$$

We start the proof of Theorem 1.1 with two lemmas. The main aim of these lemmas is to verify that $I(\varphi) \geq 2\varphi/3$ for $0 \leq \varphi \leq \pi/2$. From that fact Theorem 1.1 follows quickly through an integral averaging argument. As a first step, we calculate the exact values of $I(\varphi)$ at the endpoints of the interval at $\varphi = 0$ and $\varphi = \pi/2$.

Lemma 3.2. *With the notation introduced above,*

$$I(0) = 0 \quad \text{and} \quad I(\pi/2) = \pi/3.$$

Proof. The statement $I(0) = 0$ is clearly true, so we need to calculate $I(\pi/2)$ only. Let $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$ and define $A = \{(x, y, z) \in S^2 \mid xy \leq 0\}$,

$A^C = \{(x, y, z) \in S^2 \mid xy > 0\}$, and $A^C_+ = \{(x, y, z) \in S^2 \mid xy > 0, x > 0\}$. Then the following holds

$$\begin{aligned} I(\pi/2) &= \frac{1}{4\pi}\int\limits_{S^2} \varphi^{\mathbf{u}}_*(\mathbf{v}_1, \mathbf{v}_2)\mathrm{d}\mathbf{u} = \frac{1}{4\pi}\int\limits_{A} \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} + \frac{1}{4\pi}\int\limits_{A^C} \pi - \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} \\ &= \frac{1}{4\pi}\int\limits_{S^2} \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} - \frac{1}{4\pi}\int\limits_{A^C} \pi - 2\varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} \\ &= \frac{\pi}{2} + \frac{1}{4\pi}\int\limits_{A^C} \pi\mathrm{d}\mathbf{u} - 2\cdot\frac{1}{4\pi}\int\limits_{A^C} \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} \\ &= \pi - 4\cdot\frac{1}{4\pi}\int\limits_{A^C_+} \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} \end{aligned}$$

using Lemma 3.1. Obviously, it is enough to show that

$$\int\limits_{A^C_+} \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} = \frac{2\pi^2}{3}.$$

Introduce the following spherical coordinates

$$\mathbf{u} = \mathbf{u}(\theta, \psi) = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta),$$

where $0 \le \theta \le \pi$ and $0 \le \psi \le 2\pi$. It is easily seen that

$$\begin{aligned} \varphi^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2) &= \arccos\frac{\langle(\mathbf{u}\times\mathbf{v}_1)\times\mathbf{u}, (\mathbf{u}\times\mathbf{v}_2)\times\mathbf{u}\rangle}{|(\mathbf{u}\times\mathbf{v}_1)\times\mathbf{u}|\cdot|(\mathbf{u}\times\mathbf{v}_2)\times\mathbf{u}|} \\ &= \arccos\frac{\langle\mathbf{u}\times\mathbf{v}_1, \mathbf{u}\times\mathbf{v}_2\rangle}{|\mathbf{u}\times\mathbf{v}_1|\cdot|\mathbf{u}\times\mathbf{v}_2|}. \end{aligned}$$

Straightforward calculations yield that $\mathbf{u}\times\mathbf{v}_1 = (0, \cos\theta, -\sin\theta\sin\psi)$ and $\mathbf{u}\times\mathbf{v}_2 = (-\cos\theta, 0, \sin\theta\cos\psi)$, and hence

$$\begin{aligned} \langle\mathbf{u}\times\mathbf{v}_1, \mathbf{u}\times\mathbf{v}_2\rangle &= -\sin^2\theta\sin\psi\cos\psi, \\ |\mathbf{u}\times\mathbf{v}_1|\cdot|\mathbf{u}\times\mathbf{v}_2| &= \sqrt{\cos^2\theta + \sin^4\theta\sin^2\psi\cos^2\psi}. \end{aligned}$$

Thus

$$\begin{aligned} \int\limits_{A^C_+} \varphi^{\mathbf{u}}\mathrm{d}\mathbf{u} &= \int\limits_0^{\pi}\int\limits_0^{\pi/2} \arccos\frac{-\sin^2\theta\sin\psi\cos\psi}{\sqrt{\cos^2\theta + \sin^4\theta\sin^2\psi\cos^2\psi}}\cdot\sin\theta\mathrm{d}\psi\mathrm{d}\theta \\ &= 2\cdot\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\left(\pi - \arctan\frac{\cos\theta}{\sin^2\theta\sin\psi\cos\psi}\right)\cdot\sin\theta\mathrm{d}\psi\mathrm{d}\theta \\ &= \pi^2 - 2\int\limits_0^{\pi/2}\int\limits_0^{\pi/2} \arctan\frac{\cos\theta}{\sin^2\theta\sin\psi\cos\psi}\cdot\sin\theta\mathrm{d}\theta\mathrm{d}\psi. \end{aligned} \tag{1}$$

The inner integral in (1) can be directly calculated as follows. Let

$$\begin{aligned} g(\theta,\psi) =& \frac{1}{2}\tan\psi\cdot\ln\left(2\cos(2\theta)\cos(2\psi)+2\cos(2\theta)-2\cos(2\psi)+6\right)\\ &+\frac{1}{2}\cot\psi\cdot\ln\left(-2\cos(2\theta)\cos(2\psi)+2\cos(2\theta)+2\cos(2\psi)+6\right)\\ &-\cos\theta\cdot\arctan\frac{\cos\theta}{\sin^2\theta\sin\psi\cos\psi}. \end{aligned}$$

One can check by a tedious but straightforward calculation that

$$\frac{\partial g(\theta,\psi)}{\partial\theta} = \arctan\frac{\cos\theta}{\sin^2\theta\sin\psi\cos\psi}\cdot\sin\theta.$$

Now, for a fixed $0<\psi<\pi/2$, we obtain

$$\begin{aligned} &\int_0^{\pi/2}\arctan\frac{\cos\theta}{\sin^2\theta\sin\psi\cos\psi}\cdot\sin\theta\mathrm{d}\theta\\ &\quad=\frac{1}{2}\tan\psi\cdot\ln\left(\cos(\pi-2\psi)+\cos(\pi+2\psi)-2\cos(2\psi)+4\right)\\ &\qquad+\frac{1}{2}\cot\psi\cdot\ln\left(-\cos(\pi-2\psi)-\cos(\pi+2\psi)+2\cos(2\psi)+4\right)\\ &\qquad-\left[\frac{1}{2}\tan\psi\cdot\ln\left(\cos(-2\psi)+\cos(2\psi)-2\cos(2\psi)+8\right)\right.\\ &\qquad\left.+\frac{1}{2}\cot\psi\cdot\ln\left(-\cos(-2\psi)-\cos(2\psi)+2\cos(2\psi)+8\right)-\pi/2\right]\\ &\quad=\frac{1}{2}\tan\psi\cdot\ln(4(1-\cos(2\psi)))+\frac{1}{2}\cot\psi\cdot\ln(4(1+\cos(2\psi)))\\ &\qquad+\pi/2-\frac{\ln 8}{2}(\tan\psi+\cot\psi)\\ &\quad=\frac{1}{2}(\pi+\tan\psi\ln(\sin^2\psi)+\cot\psi\ln(\cos^2\psi))\\ &\quad=\frac{\pi}{2}+\tan\psi\ln(\sin\psi)+\cot\psi\ln(\cos\psi). \end{aligned}$$

We turn to the outer integral in (1).

$$\begin{aligned} &\int_0^{\pi/2}\int_0^{\pi/2}\arctan\frac{\cos\theta}{\sin^2\theta\sin\psi\cos\psi}\cdot\sin\theta\mathrm{d}\theta\mathrm{d}\psi\\ &\qquad=\int_0^{\pi/2}\frac{\pi}{2}+\tan\psi\ln(\sin\psi)+\cot\psi\ln(\cos\psi)\mathrm{d}\psi\\ &\qquad=\frac{\pi^2}{4}+\int_0^{\pi/2}\tan\psi\ln(\sin\psi)\mathrm{d}\psi+\int_0^{\pi/2}\cot\psi\ln(\cos\psi)\mathrm{d}\psi. \end{aligned}$$

Using the substitution $u = \sin\psi$ in the first integral and $u = \cos\psi$ in the second integral, we obtain that

$$\int_0^{\pi/2} \tan\psi \ln(\sin\psi)\mathrm{d}\psi = \int_0^{\pi/2} \cot\psi \ln(\cos\psi)\mathrm{d}\psi = \int_0^1 \frac{u\ln u}{1-u^2}\mathrm{d}u.$$

Integration by parts gives

$$\int_0^1 \frac{u\ln u}{1-u^2}\mathrm{d}u = \left.\frac{-\ln u \ln(1-u^2)}{2}\right|_0^1 + \frac{1}{2}\int_0^1 \frac{\ln(1-u^2)}{u}\mathrm{d}u,$$

where $\left.\frac{-\ln u \ln(1-u^2)}{2}\right|_0^1 = 0$ by L'Hospital's rule. Now, the substitution $x = u^2$ yields

$$\frac{1}{2}\int_0^1 \frac{\ln(1-u^2)}{u}\mathrm{d}u = \frac{1}{4}\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x = \frac{-1}{4}\int_0^1 \frac{\mathrm{Li}_1(x)}{x}\mathrm{d}x$$
$$= \frac{-1}{4}\mathrm{Li}_2(1) = \frac{-\pi^2}{24},$$

where in the last two steps we used the polylogarithm functions $\mathrm{Li}_s(z)$ and their well-known properties. For more information on the polylogarithm functions, we refer to [10]. This finishes the proof of Lemma 3.2. □

Lemma 3.3. *The function $I(\varphi)$ is concave on $[0, \pi/2]$, and*

$$I(\varphi) \geq 2\varphi/3 \quad \textit{for} \quad 0 \leq \varphi \leq \pi/2. \tag{2}$$

Before we turn to the proof of Lemma 3.3, for the sake of completeness, we recall some definitions and a theorem from [1].

The function $f : [a, b] \to \mathbb{R}$ is *superadditive* on $[a, b]$ if for any positive $h < b-a$ and $x \in [a, b-h]$, $f(a+h) - f(a) \leq f(x+h) - f(x)$, cf. Definition 2.2 on p. 61 in [1]. We call f *locally superadditive* on $[a, b]$ if for every $x_0 \in [a, b]$, there exist arbitrarily small neighborhoods of x_0 on which f is superadditive, cf. Definition 2.3 on p. 62 in [1].

Theorem 3.1 (Bruckner, Theorem 3.1 on p. 62 in [1]). *Let f be locally superadditive and differentiable on an interval $[a, b]$, with the derivative f' continuous almost everywhere in $[a, b]$. Then f is convex.*

Proof of Lemma 3.3. Obviously, $I(\varphi)$ is a continuously differentiable function of φ on $[0, \pi/2]$.

Fix $0 \leq \alpha \leq \beta \leq \pi/2$, a small $0 \leq \delta \leq \pi/2 - \beta$, and a vector $\mathbf{u} \in S^2$. Let $\angle(\cdot, \cdot)$ denote the angle formed by two vectors. Choose four coplanar vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w} \in S^2$ such that $\angle(\mathbf{w}_1, \mathbf{w}_2) = \alpha$, $\angle(\mathbf{w}_1, \mathbf{w}_3) = \beta$, $\angle(\mathbf{w}_1, \mathbf{w}) = \delta$, $\angle(\mathbf{w}, \mathbf{w}_2) = \alpha + \delta$, and $\angle(\mathbf{w}, \mathbf{w}_3) = \beta + \delta$, see Fig. 2. As before, we use the abbreviations $\alpha^{\mathbf{u}} = \alpha^{\mathbf{u}}(\mathbf{w}_1, \mathbf{w}_2)$ and $\alpha_*^{\mathbf{u}} = \alpha_*^{\mathbf{u}}(\mathbf{w}_1, \mathbf{w}_2)$, and similarly for the other angles.

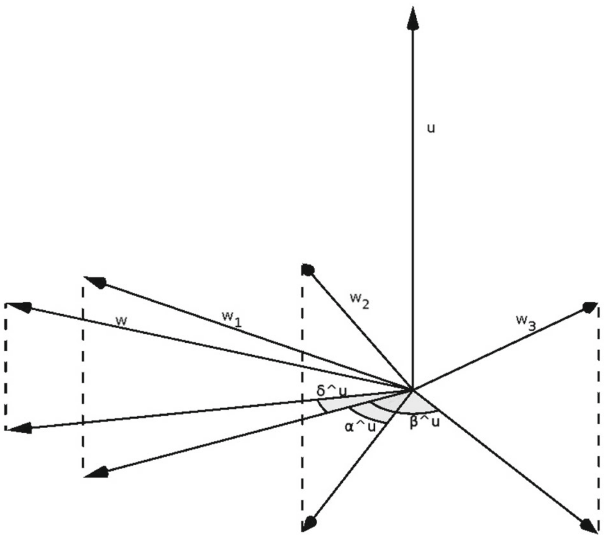

Figure 2. The projection of the angles

We claim that

$$(\alpha+\delta)^{\mathbf{u}}_* - \alpha^{\mathbf{u}}_* \geq (\beta+\delta)^{\mathbf{u}}_* - \beta^{\mathbf{u}}_*. \tag{3}$$

To prove (3), we write the left-hand side, and, respectively, the right-hand side as follows:

$$(\alpha+\delta)^{\mathbf{u}}_* - \alpha^{\mathbf{u}}_* = \begin{cases} -\delta^{\mathbf{u}}, & \text{if } \alpha^{\mathbf{u}} > \pi/2, \\ \pi - 2\alpha^{\mathbf{u}} - \delta^{\mathbf{u}}, & \text{if } \alpha^{\mathbf{u}} \leq \pi/2 \text{ and } (\alpha+\delta)^{\mathbf{u}} > \pi/2, \\ \delta^{\mathbf{u}}, & \text{if } (\alpha+\delta)^{\mathbf{u}} \leq \pi/2, \end{cases} \tag{4}$$

and

$$(\beta+\delta)^{\mathbf{u}}_* - \beta^{\mathbf{u}}_* = \begin{cases} -\delta^{\mathbf{u}}, & \text{if } \beta^{\mathbf{u}} > \pi/2, \\ \pi - 2\beta^{\mathbf{u}} - \delta^{\mathbf{u}}, & \text{if } \beta^{\mathbf{u}} \leq \pi/2 \text{ and } (\beta+\delta)^{\mathbf{u}} > \pi/2, \\ \delta^{\mathbf{u}}, & \text{if } (\beta+\delta)^{\mathbf{u}} \leq \pi/2. \end{cases} \tag{5}$$

To show (3), we consider three cases as in (4). If $\alpha^{\mathbf{u}} > \pi/2$, then $\beta^{\mathbf{u}} > \pi/2$, and equality holds in (3). If $\alpha^{\mathbf{u}} \leq \pi/2$ and $(\alpha+\delta)^{\mathbf{u}} > \pi/2$, then $(\beta+\delta)^{\mathbf{u}} > \pi/2$, and either the first or the second case applies in (5). Now, $\pi - 2\alpha^{\mathbf{u}} - \delta^{\mathbf{u}} \geq -\delta^{\mathbf{u}}$ is equivalent to $\alpha^{\mathbf{u}} \leq \pi/2$, thus it holds true. Also, from $\alpha^{\mathbf{u}} \leq \beta^{\mathbf{u}}$, it follows that $\pi - 2\alpha^{\mathbf{u}} - \delta^{\mathbf{u}} \geq \pi - 2\beta^{\mathbf{u}} - \delta^{\mathbf{u}}$, as claimed. The only case that remains to be checked is when $(\alpha+\delta)^{\mathbf{u}} \leq \pi/2$, and thus $(\alpha+\delta)^{\mathbf{u}}_* - \alpha^{\mathbf{u}}_* = \delta^{\mathbf{u}}$. If, in (5), the first or the third case applies, then the inequality in (3) clearly holds. Thus, we only need to consider the case when $(\beta+\delta)^{\mathbf{u}} > \pi/2$. Then $\delta^{\mathbf{u}} > \pi - 2\beta^{\mathbf{u}} - \delta^{\mathbf{u}}$, which finishes the proof of (3).

Since (3) holds true for any unit vector $\mathbf{u} \in S^2$, it follows that for any $0 \leq \alpha \leq \beta \leq \pi/2$, and $0 \leq \delta \leq \pi/2 - \beta$, we have

$$I(\alpha+\delta) - I(\alpha) \geq I(\beta+\delta) - I(\beta). \tag{6}$$

Hence $-I$ is superadditive on any subinterval of $[0, \pi/2]$, and thus it satisfies all the conditions of Theorem 3.1 on the interval $[0, \pi/2]$. It follows that $-I$ is convex, and so I is concave, as stated. Finally, the inequality (2) is a simple consequence of Lemma 3.2 and of the concavity of I. This completes the proof of Lemma 3.3. □

Proof of Theorem 1.1. Consider the lines $l_1, \dots, l_n$, and a vector $\mathbf{u} \in S^2$. Let S be the plane through the origin with normal vector $\mathbf{u}$, and let l_i' denote the orthogonal projection of the line l_i onto S. We denote by $\varphi_{ij}^{\mathbf{u}}$ the (non-obtuse) angle formed by l_i' and l_j'. Applying (2), we obtain that

$$\begin{aligned}\frac{1}{4\pi}\int_{S^2} \sum_{1\le i<j\le n} \varphi_{ij}^{\mathbf{u}} \mathrm{d}u &= \sum_{1\le i<j\le n} \frac{1}{4\pi}\int_{S^2} \varphi_{ij}^{\mathbf{u}} \mathrm{d}u \\ &\ge \sum_{1\le i<j\le n} 2\varphi_{ij}/3 = \frac{2}{3}\sum_{1\le i<j\le n} \varphi_{ij}.\end{aligned}$$

Therefore, there exists a $\mathbf{u}_0 \in S^2$ with the property

$$\sum_{1\le i<j\le n} \varphi_{ij}^{\mathbf{u}_0} \ge \frac{2}{3}\sum_{1\le i<j\le n} \varphi_{ij}.$$

Finally, Theorem 2.1 implies that

$$\sum_{1\le i<j\le n} \varphi_{ij}^{\mathbf{u}_0} \le \begin{cases} k^2 \cdot \frac{\pi}{2}, & \text{if } n = 2k, \\ k(k+1) \cdot \frac{\pi}{2}, & \text{if } n = 2k+1, \end{cases}$$

which completes the proof of Theorem 1.1. □

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COVERING THE SPHERE BY EQUAL ZONES

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Abstract. A zone of half-width w on the unit sphere S^2 in Euclidean 3-space is the parallel domain of radius w of a great circle. L. Fejes Tóth raised the following question in [6]: what is the minimal w_n such that one can cover S^2 with n zones of half-width w_n? This question can be considered as a spherical relative of the famous plank problem of Tarski. We prove lower bounds for the minimum half-width w_n for all $n \geqq 5$.

1. Introduction

Let S^2 denote the unit sphere in 3-dimensional Euclidean space $\mathbb{R}^3$ centred at the origin o. The spherical distance $d_s(x,y)$ of two points $x, y \in S^2$ is defined as the length of a (shorter) geodesic arc connecting x and y on S^2, or equivalently, the central angle $\angle xoy$ spanned by x and y. Following L. Fejes Tóth [6], a *zone* Z of half-width w in S^2 is the parallel domain of radius w of a great circle C, that is,

$$Z(C,w) := \left\{ x \in S^2 \mid d_s(x,C) \leqq w \right\}.$$

We call C the central great circle of Z. In this paper, we investigate the following problem.

Problem 1 (L. Fejes Tóth [6]). *For a given n, find the smallest number w_n such that one can cover S^2 with n zones of half-width w_n. Find also the optimal configurations of zones that realize the optimal coverings.*

We note that in the same paper [6] L. Fejes Tóth also asked the analogous question with not necessarily congruent zones, and conjectured that the sum of the half-widths of the zones that can cover S^2 is always at least $\pi/2$.

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Furthermore, L. Fejes Tóth [6] posed the question: what is the minimum of the sum of the half-widths of n (not necessarily congruent) zones that can cover a spherically convex disc on S^2? These questions are similar to the classical plank problem of Tarski, see for example Bezdek [1] for a recent survey on this topic.

L. Fejes Tóth formulated the following conjecture:

Conjecture 1 (L. Fejes Tóth [6]). *For $n \geqq 1$, $w_n = \pi/(2n)$.*

It is clear that $w_n \leqq \pi/(2n)$ since n zones of half-width $\pi/(2n)$, whose central great circles all pass through a pair of antipodal points of S^2 and which are distributed evenly, cover S^2. On the other hand, as the zones must cover S^2, the sum of their areas must be at least (actually, greater than) 4π, that is, $w_n > \arcsin(1/n)$.

Rosta [13] proved that $w_3 = \pi/6$, and that the unique optimal configuration consists of three zones whose central great circles pass through two antipodal points of S^2 and are distributed evenly. Linhart [9] showed that $w_4 = \pi/8$, and the unique optimal configuration is similar to the one for $n = 3$. To the best of our knowledge, no further results about this problem have been achieved to date and thus L. Fejes Tóth's conjecture remains open.

The paper is organized as follows. In Section 2, we determine the area of the intersection of two congruent zones as a function of their half-widths and the angle of their central great circles under some suitable restrictions. In Section 3, we use the currently known best upper bounds for the maximum of the minimal pairwise spherical distances of n points in S^2 to estimate from above the contribution of a zone in an optimal covering. Adding up these estimated contributions, we obtain a lower bound for w_n, which is the main result of our paper, and it is stated in Theorem 1. Finally, we calculate the numerical values of the established lower bound for some specific n.

2. Intersection of two zones

We start with the following simple observation. Consider two zones Z_1 and Z_2 of half-width w whose central great circles make an angle α. If $\alpha \geqq 2w$, then the intersection of Z_1 and Z_2 is the union of two disjoint congruent spherical domains. These domains are symmetric to each other with respect to o, and they resemble a rhombus which is bounded by four small circular arcs of equal (spherical) length. If $\alpha \leqq 2w$, then the intersection is a connected, band-like domain. Let $2F(w, \alpha)$ denote the area of $Z_1 \cap Z_2$.

Lemma 1. *Let $0 \leqq w \leqq \pi/4$ and $2w \leqq \alpha \leqq \pi/2$. Then*

$$F(w, \alpha) = 4 \sin w \arcsin\left(\frac{1 - \cos\alpha}{\cot w \sin\alpha}\right) + 4 \sin w \arcsin\left(\frac{1 + \cos\alpha}{\cot w \sin\alpha}\right) \tag{1}$$

$$- 2\arccos\left(\frac{\cos\alpha - \sin^2 w}{\cos^2 w}\right) - 2\arccos\left(\frac{-\cos\alpha - \sin^2 w}{\cos^2 w}\right) + 2\pi.$$

Moreover, $F(w,\alpha)$ is a monotonically decreasing function of α in the interval $[0,\pi/2]$.

PROOF. First, we prove (1). Let Z_1 be the zone of half-width w whose central great circle C_1 is the intersection of the xy-plane with S^2. Let c_1 and c_3 denote the small circles which bound Z_1 such that c_1 is contained in the closed half-space $z \geqq 0$.

Let Z_2 be the zone of half-width w whose central great circle C_2 is the intersection of S^2 with the plane which contains the y-axis and which makes an angle α with the xy-plane as shown in Fig. 1. Let c_2 and c_4 be the small circles bounding Z_2, cf. Fig. 1.

The intersection $Z_1 \cap Z_2$ is the union of two connected components R_1 and R_2. Assume that R_1 is contained in the closed half-space $y \leqq 0$. Let c_i', $i = 1,\dots,4$ denote the arc of c_i that bounds R_1. Observe that $c_1',\dots,c_4'$ are of equal length; we denote their common arc length by $l(w,\alpha)$. The radii of $c_1,\dots,c_4$ are all equal to $\cos w$.

Assume that the boundary ∂R_1 of R_1 is oriented such that the small circular arcs follow each other in the cyclic order c_1', c_2', c_3', c_4'. For $i \in \{1,\dots,4\}$, let $\varphi_i(w,\alpha)$ denote the turning angle of ∂R_1 at the intersection point of c_i' and c_{i+1}' with the convention that $c_5 = c_1$. Notice that the signed geodesic curvature of ∂R_1 (in its smooth points) is equal to $-\tan w$.

By the Gauss–Bonnet Theorem it holds that

$$F(w,\alpha) = 2\pi + 4\tan w \cdot l(w,\alpha) - \sum_{i=1}^{4} \varphi_i(w,\alpha).$$

Next, we calculate the $\varphi_i(w,\alpha)$. Note that $\varphi_i(w,\alpha) = \varphi_{i+2}(w,\alpha)$ for $i = 1,2$.

Let Π_1 be the plane whose normal vector is $u_1 = (0,0,1)$ and contains the point $(0,0,\sin w)$. Let Π_2 be the plane which we get by rotating Π_1 around the y-axis by angle α so its normal vector is $u_2 = (-\sin\alpha, 0, \cos\alpha)$, see Fig. 1. Note that $S^2 \cap \Pi_1 = c_1$ and $S^2 \cap \Pi_2 = c_2$.

$$\Pi_1 : \; z = \sin w, \quad \Pi_2 : \; -x\sin\alpha + z\cos\alpha = \sin w$$

Now let $L_1 = \Pi_1 \cap \Pi_2$ and $L_1 \cap S^2 = \{l_1, l_1'\}$, such that l_1 has negative y-coordinate. Then

$$l_1 = \Big(\sin w(\cot\alpha - \csc\alpha), -\sqrt{1 - \sin^2 w\big(1 + (\cot\alpha - \csc\alpha)^2\big)}, \sin w\Big).$$

Let Π be the plane that is tangent to S^2 in l_1, and let $E_1 = \Pi_1 \cap \Pi$ and $E_2 = \Pi_2 \cap \Pi$. Then φ_1 is one of the angles made by E_1 and E_2. Let $v_1 =$

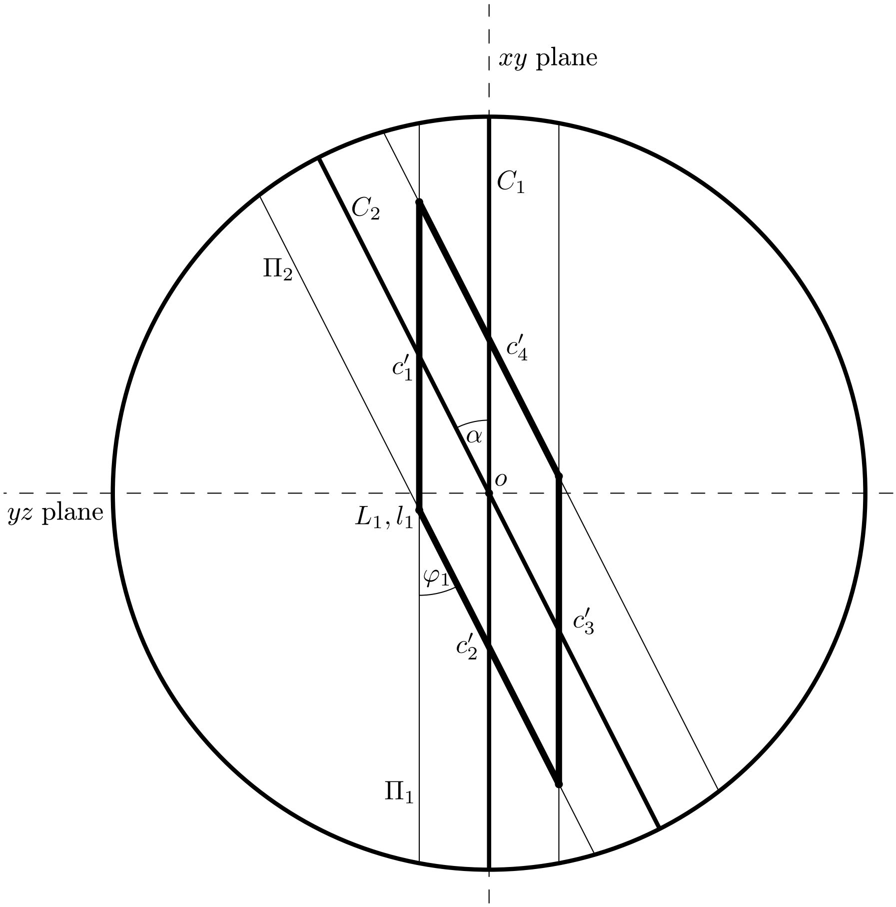

Fig. 1: Orthogonal projection onto the xz plane

$l_1 \times u_1$ and $v_2 = l_1 \times u_2$. Then v_1 and v_2 are vectors parallel to E_1 and E_2, respectively, such that their orientations agree with that of ∂R_1.

$$v_1 = \Big(-\sqrt{1 - \sin^2 w\big(1 + (\cot\alpha - \csc\alpha)^2\big)}, -\sin w(\cot\alpha - \csc\alpha), 0\Big)$$

$$v_2 = \Big(-\cos\alpha\sqrt{1 - \sin^2 w\big(1 + (\cot\alpha - \csc\alpha)^2\big)}, -\cos\alpha\sin w(\cot\alpha - \csc\alpha)$$

$$-\sin\alpha\sin w, -\sin\alpha\sqrt{1 - \sin^2 w\big(1 + (\cot\alpha - \csc\alpha)^2\big)}\ \Big).$$

We only need to calculate the lengths of v_1 and v_2 and their scalar product. By routine calculations we obtain

$$\varphi_1 = \arccos \frac{\langle v_1, v_2\rangle}{|v_1|\,|v_2|} = \arccos\left(\frac{\cos\alpha - \sin^2 w}{\cos^2 w}\right).$$

The angle φ_2 can be evaluated similarly; one only needs to write $\pi - \alpha$ in place of α in the above calculations. Then

$$\varphi_2 = \arccos\left(\frac{-\cos\alpha - \sin^2 w}{\cos^2 w}\right).$$

To finish the calculation, we need to find $l(w,\alpha)$. Let $l_i := c_i' \cap c_{i+1}'$ for $i = 1,\dots,4$ with $c_5' = c_1'$. Let d_i, $i = 1,\dots,4$ be the absolute value of the y-coordinate of l_i. Simple trigonometry shows that

$$d_1 = \frac{1-\cos\alpha}{\cot w \sin\alpha}, \quad \text{and} \quad d_4 = \frac{1+\cos\alpha}{\cot w \sin\alpha}.$$

Then the length of c_1' is equal to the following

$$l(w,\alpha) = \cos w \arcsin d_1 + \cos w \arcsin d_4 \tag{2}$$

$$= \cos w \arcsin\left(\frac{1-\cos\alpha}{\cot w \sin\alpha}\right) + \cos w \arcsin\left(\frac{1+\cos\alpha}{\cot w \sin\alpha}\right).$$

In summary,

$$F(w,\alpha) = 2\pi + 4\sin w \arcsin\big(\tan w(\csc\alpha + \cot\alpha)\big) \tag{3}$$

$$+ 4\sin w \arcsin\big(\tan w(\csc\alpha - \cot\alpha)\big)$$

$$-2\arccos\left(\frac{\cos\alpha - \sin^2 w}{\cos^2 w}\right) - 2\arccos\left(\frac{\cos\alpha + \sin^2 w}{-\cos^2 w}\right)$$

Finally, we prove that F is monotonically decreasing in α. This is obvious in the interval $[0, 2w]$.

Let $\alpha \geqq 2w$ and $\varepsilon > 0$ be sufficiently small with $\alpha + \varepsilon \leqq \pi/2$. Consider the spherical "rhombus" R_1^* which is obtained as the intersection of Z_1 and another zone Z_2^* of half-width w whose central great circle C_2^* is the intersection of S^2 with the plane which contains the y-axis and which makes an angle $\alpha + \varepsilon$ with the xy-plane, similarly as for Z_2 above. Let F_1 be the area of $R_1 \setminus R_1^*$ and F_1^* be the area of $R_1^* \setminus R_1$. For the monotonicity of $F(w,\alpha)$ in α, we only need to show that $F_1 > F_1^*$.

The region $R_1 \setminus R_1^*$ consists of two disjoint congruent connected domains (in fact, two triangular regions bounded by arcs of small circles). Note that one such region, say P, is fully contained in the positive hemisphere of S^2 ($z \geqq 0$), and the other region is contained in the negative hemisphere ($z \leqq 0$). Similarly, let Q be the one of the two connected, congruent and disjoint regions whose union is $R_1^* \setminus R_1$ and which has a common (boundary) point with P. Let $q = P \cap Q$, then q has positive z-coordinate. It easily follows from the position of q that the arc $c_2 \cap Q$ is longer than $c_2 \cap P$, and, similarly, $c_2^* \cap Q$ is longer than $c_2^* \cap P$, so the area of Q is larger than the area of P, which completes the proof of the lemma. □

Remark 1. Let Z_1 and Z_2 be two zones of half-width $w \in (0, \pi/4]$ which make an angle α. Then it is clear that the area of $Z_1 \cup Z_2$ is a monotonically increasing function of α for $\alpha \in [0, 2w]$.

3. A lower bound for w_n

For an integer $n \geqq 3$, let d_n denote the maximum of the minimal pairwise (spherical) distances of n points on the unit sphere S^2. Finding d_n is a long-standing problem of discrete geometry which goes back to the Dutch botanist Tammes [15]. As of now, the exact value of d_n is only known in the following cases.

n	d_n	
3	$2\pi/3$	L. Fejes Tóth [7]
4	1.91063	L. Fejes Tóth [7]
5	$\pi/2$	Schütte, van der Waerden [14]
6	$\pi/2$	L. Fejes Tóth [7]
7	1.35908	Schütte, van der Waerden [14]
8	1.30653	Schütte, van der Waerden [14]
9	1.23096	Schütte, van der Waerden [14]
10	1.15448	Danzer [4]
11	1.10715	Danzer [4]
12	1.10715	L. Fejes Tóth [7]
13	0.99722	Musin, Tarasov [10]
14	0.97164	Musin, Tarasov [11]
24	0.76255	Robinson [12]

Table 1: Known (approximate) values of d_n

Alternate proofs were given by Hárs [8] for the case $n = 10$, and by Böröczky [2] for the case $n = 11$.

For $n \geqq 3$, L. Fejes Tóth [5] proved the following upper estimate

$$d_n \leqq \tilde{\delta}_n := \arccos\left(\frac{\cot^2\left(\frac{n}{n-2}\frac{\pi}{6}\right) - 1}{2}\right), \tag{4}$$

where equality holds exactly in the cases $n = 3, 4, 6, 12$ (see the table above). Moreover, $\lim_{n\to\infty} \tilde{\delta}_n / d_n = 1$, that is, $\tilde{\delta}_n$ provides an exact asymptotic upper bound for d_n as $n \to \infty$.

Robinson [12] improved the upper estimate (4) of L. Fejes Tóth as follows. Assume that the pairwise distances between the n points on the sphere are all at least a where $0 < a < \arctan 2$. Let $\Delta_1(a)$ denote the area and $\tilde{\alpha}$ the internal angle of an equilateral spherical triangle with side length a, and denote by $\Delta_2(a)$ the area of a spherical triangle with two sides of length a making an angle of $2\pi - 4\tilde{\alpha}$. Let δ_n be the unique solution of the equation $4n\Delta_1(a) + (2n-12)\Delta_2(a) - 12\pi = 0$. Then (cf. [12]) $d_n \leqq \delta_n \leqq \tilde{\delta}_n$ for $n \geqq 13$.

Let $d_n^* := \min\{\pi/2, d_n\}$ for $n \geqq 2$, and let

$$\delta_n^* := \begin{cases} d_n^* & \text{for } 3 \leqq n \leqq 14 \text{ and } n = 24, \\ \delta_n & \text{otherwise.} \end{cases} \tag{5}$$

We will also need a lower bound on d_n for our argument. We note that, for example, van der Waerden [16] proved a non-trivial lower bound on d_n, however, for our purposes the following simpler bound is sufficient. Set $\varrho_n := \arccos(1 - 2/n)$, and consider a maximal (saturated) set of points $p_1, \dots, p_m$ on the unit sphere S^2 such that their pairwise spherical distances are at least ϱ_n. By maximality it follows that the spherical circular discs (spherical caps) of radius ϱ_n centered at $p_1, \dots, p_m$ cover S^2. As the (spherical) area of such a cap is $4\pi/n$, we obtain that $m \cdot 4\pi/n \geqq 4\pi$, that is, $m \geqq n$, which implies that $\varrho_n := \arccos(1 - 2/n) \leqq d_n$. As $x \leqq \arccos(1 - x^2/2)$ for $0 \leqq x \leqq 1$, the following inequality is immediate:

$$\frac{2}{\sqrt{n}} \leqq d_n^* \leqq \delta_n^*. \tag{6}$$

For $0 \leqq \alpha \leqq \pi/2$ and $n \geqq 3$ we introduce $f(w, \alpha) = 4\pi \sin w - 2F(w, \alpha)$ and

$$G(w, n) = 4\pi \sin w + \sum_{i=2}^{n} f(w, \delta_{2i}^*).$$

LEMMA 2. *For a fixed $n \geqq 3$, the function $G(w, n)$ is continuous and monotonically increasing in w in the interval $[0, \delta_{2n}^*/3]$. Furthermore, $G(0, n) = 0$ and $G(\delta_{2n}^*/3, n) \geqq 4\pi$.*

Proof. The continuity of G and that $G(0,n)=0$ are obvious. First we show that the function $f(w,\alpha)$ is monotonically increasing in w for $0 \leqq w \leqq \alpha/3$. This clearly implies that $G(w,n)$ is also monotonically increasing in the interval stated in the lemma. As $n \geqq 3$, we may and do assume that $w \leqq \delta_6^*/3 = \pi/6$.

Note that $f(w,\alpha)$ is the area of a zone of half-width w minus the area of its intersection with a second zone of half-width w whose central great circle makes an angle α with the central great circle of the first zone. With the same notations as in the proof of Lemma 1, it is clear that for sufficiently small $\Delta w > 0$, the quantity $f(w+\Delta w,\alpha) - f(w,\alpha)$ is (approximately) proportional to $2l(c_1) - 4l(c_1') - 4l(c_2') = 2\big(l(c_1) - 4l(c_1')\big)$. Notice that, for a fixed $w \in [0,\pi/4]$, the function $l(c_1') = l(w,\alpha)$ is monotonically decreasing in α for $\alpha \in [2w,\pi/2]$. Thus, using $3w \leqq \alpha$,

$$l(c_1) - 4l(c_1') \geqq l(c_1) - 4l(w,3w)$$

$$= 4\cos w \left(\frac{\pi}{2} - \arcsin\left(\frac{1-\cos(3w)}{\cot w \sin(3w)}\right) - \arcsin\left(\frac{1+\cos(3w)}{\cot w \sin(3w)}\right)\right).$$

One can check that if $w \in (0,\pi/6]$, then both arguments in the above arcsin functions take on values in $[0,2/3]$. By the monotonicity and convexity of arcsin, we obtain that

$$\arcsin\left(\frac{1-\cos(3w)}{\cot w \sin(3w)}\right) + \arcsin\left(\frac{1+\cos(3w)}{\cot w \sin(3w)}\right) \leqq \arcsin(2/3)\frac{3\tan w}{\sin(3w)}$$

$$\leqq \arcsin(2/3)\frac{3\tan(\pi/6)}{\sin(\pi/2)} < \frac{\pi}{2},$$

which shows the monotonicity of $G(w,n)$.

Finally, we show that $G(\delta_{2n}^*/3,n) \geqq 4\pi$. For $n \leqq 24$, this statement can be checked by direct calculation, thus we may assume $n \geqq 25$. Using the definitions of G and f, and Lemma 1, we obtain that

$$G\left(\frac{\delta_{2n}^*}{3},n\right) = n\cdot 4\pi \sin\frac{\delta_{2n}^*}{3} - 2\cdot\sum_{i=2}^{n} F\left(\frac{\delta_{2n}^*}{3},\delta_{2i}^*\right)$$

$$\geqq 4n\pi \sin\frac{\delta_{2n}^*}{3} - 2\sum_{i=2}^{n} F\left(\frac{\delta_{2n}^*}{3},\delta_{2n}^*\right)$$

$$= 4n\pi \sin\frac{\delta_{2n}^*}{3} - 2(n-1)F\left(\frac{\delta_{2n}^*}{3},\delta_{2n}^*\right)$$

$$\geqq 4n\pi \sin\frac{\delta_{2n}^*}{3} - 2(n-1)F\left(\frac{\delta_{2n}^*}{3}, \frac{2\delta_{2n}^*}{3}\right). \tag{7}$$

Note that $\delta_{2n}^* = \delta_{2n}$ for $n \geqq 25$. Elementary trigonometry yields that

$$F\left(\frac{\alpha}{2}, \alpha\right) = 4\sin\frac{\alpha}{2}\arcsin\left(\tan^2\frac{\alpha}{2}\right) + 2\pi\sin\frac{\alpha}{2} - 2\arccos\left(1 - 2\tan^2\frac{\alpha}{2}\right).$$

Thus (7) is equal to

$$4\pi\sin\frac{\delta_{2n}}{3}$$
$$+ 4(n-1)\left(\arccos\left(1 - 2\tan^2\frac{\delta_{2n}}{3}\right) - 2\sin\frac{\delta_{2n}}{3}\arcsin\left(\tan^2\frac{\delta_{2n}}{3}\right)\right).$$

As $n \geqq 25$, we have that $0 < \delta_{2n} < 0.75$. Using that $\cos x \geqq 1 - x^2/2$ for $x \in [0, \pi/2]$, we obtain that

$$\arccos\left(1 - 2\tan^2\frac{\delta_{2n}}{3}\right) \geqq 2\tan\frac{\delta_{2n}}{3}.$$

Similarly, as for $0 < x < 0.16$ we have that $x < 1.01\sin x$, we obtain that

$$2\sin\frac{\delta_{2n}}{3}\arcsin\left(\tan^2\frac{\delta_{2n}}{3}\right) < 2.02\tan^3\frac{\delta_{2n}}{3}.$$

Finally, using that $x - 1.01x^3 > x - 1.01 \cdot 0.4^2 \cdot x > 0.8x$ for $0 < x < 0.4$, we obtain that (7) can be estimated from below as follows

$$G\left(\frac{\delta_{2n}^*}{3}, n\right) \geqq 6.4(n-1)\tan\frac{\delta_{2n}}{3} > 2.1(n-1)\delta_{2n}.$$

By (6) we know that $\delta_{2n} > \sqrt{2}/\sqrt{n}$, and thus the proof of Lemma 2 is complete. $\square$

Now, we are ready to state our main theorem.

Theorem 1. *For $n \geqq 3$, let w_n^* denote the unique solution of the equation $G(w, n) = 4\pi$ in the interval $[0, \delta_{2n}^*/3]$. Then* $\arcsin(1/n) < w_n^* \leqq w_n$.

Proof. Let $Z_i(w_n, C_i), i = 1, \ldots, n$ be zones that form a minimal covering of S^2 with respect to w. For $i \in \{1, \ldots, n\}$, let p_i be one of the poles of C_i and let $p_{n+i} = -p_i$. Then there exist two points $p_{i_1}, p_{j_1} \in \{p_1, \ldots, p_{2n}\}$ with $i_1 < j_1$ and $j_1 \neq n + i_1$ (that is, p_{i_1} and p_{j_1} are poles of two different great circles) such that $d_s(p_{i_1}, p_{j_1}) \leqq d_{2n}^*$. Observe that the area of the part of Z_{i_1} that is not covered by any Z_k with $i_1 \neq k$ is at most $f(w, \delta^*_{2n})$ by Lemma 1, inequality (6) and Remark 1. Now, remove Z_{i_1} from the covering and repeat the argument for the remaining zones. Note that in the last step of the process there is only one zone left Z_{i_n}, so the area of the part of Z_{i_n} not covered by any other zone is $4\pi \sin w$.

If for $k = 1, \dots, n$ we add the areas of Z_{i_k} not covered by any Z_{i_l} for $l > k$, then the sum is obviously bounded from above by $G(w, n)$. Since $Z_1, \dots, Z_n$ cover S^2, therefore $G(w, n) \geqq 4\pi$, which shows that $w^*_n \leqq w_n$. It is also clear form the argument that $\arcsin(1/n) < w^*_n$. This finishes the proof of Theorem 1. □

4. Concluding remarks

Remark 2. Instead of Robinson's bound δ_n, one may use the original bound $\tilde{\delta}_n$ of L. Fejes Tóth, and prove Theorem 1, obtaining a lower bound $\tilde{w}^*_n$ for w_n. Clearly, this bound is slightly weaker than w^*_n, that is, $\tilde{w}^*_n \leqq w^*_n \leqq w_n$. However, we note that, thanks to the explicit formula (4), $\tilde{w}^*_n$ can be computed more easily than w^*_n. The difference between w^*_n and $\tilde{w}^*_n$ is shown in Table 2 for some specific values of n.

We also mention that for certain values of n Robinson's upper bound has been improved, see for example Böröczky and Szabó [3] for the cases $n = 15, 16, 17$. These stronger upper bounds, if included in the calculations, would provide only a very small improvement on w^*_n, so we decided to use only the known solutions of the Tammes problem and Robinson's general upper bound.

Remark 3. We note that the analogous question to Problem 1 can be raised in higher dimensions as well. A zone $Z = Z(C, w)$ of half-width w on the unit sphere S^{d-1} of the d-dimensional Euclidean space $\mathbb{R}^d$ is the parallel domain of radius w of a great sphere C. What is the minimal $w(d, n)$ such that one can cover S^{d-1} with n zones of half-width $w(d, n)$, and what configurations realize the optimal coverings? We do not wish to formulate a conjecture about this problem, instead, we note the following simple fact. For $d \geqq 4$, $w(d, 3) = \pi/6$. One can see this the following way. Let $Z_i = Z(C_i, w)$, $i = 1, 2, 3$ be three zones that cover S^{d-1}. Assume that $C_i = S^{d-1} \cap H_i$ for $i = 1, 2, 3$ where H_i is a hyperplane. Let $L = \cap_i H_i$. Then L is a linear subspace of $\mathbb{R}^d$, and $\dim L \geqq d - 3$. Let $L^\perp$ denote the linear subspace of $\mathbb{R}^d$ which is the orthogonal complement of L. Clearly, $L^\perp \cap S^{d-1} = S^j$, where $j \leqq 2$. If $\dim L^\perp = 1$, then $w = \pi/2$. So we may assume that $\dim L^\perp = 2$ or 3. Notice that the zones $Z_i, i = 1, 2, 3$ cover S^{d-1} if and only if the zones $Z'_i = Z_i \cap (L^\perp \cap S^{d-1}), i = 1, 2, 3$ cover $L^\perp \cap S^{d-1} = S^j$. We note also that the half-widths of $Z'_i, i = 1, 2, 3$ are all equal to w. Now, if $j = 1$, then it is

n	$\arcsin(1/n)$	$\tilde{w}_n^*$	w_n^*	$\pi/(2n)$
5	0.20135	0.22983	0.22983	0.31415
6	0.16744	0.18732	0.18732	0.26179
7	0.14334	0.15824	0.15824	0.22439
8	0.12532	0.13692	0.13692	0.19634
9	0.11134	0.12063	0.12067	0.17453
10	0.10016	0.10782	0.10787	0.15707
11	0.09103	0.09748	0.09753	0.14279
12	0.08343	0.08895	0.08899	0.13089
13	0.07699	0.08179	0.08183	0.12083
14	0.07148	0.07569	0.07573	0.11219
15	0.06671	0.07044	0.07048	0.10471
16	0.06254	0.06587	0.06591	0.09817
17	0.05885	0.06185	0.06189	0.09239
18	0.05558	0.05830	0.05833	0.08726
19	0.05265	0.05513	0.05516	0.08267
20	0.05002	0.05229	0.05232	0.07853
21	0.04763	0.04972	0.04975	0.07479
22	0.04547	0.04740	0.04743	0.07139
23	0.04349	0.04528	0.04531	0.06829
24	0.04167	0.04335	0.04337	0.06544
25	0.04001	0.04157	0.04159	0.06283
50	0.02000	0.02050	0.02051	0.03141
100	0.01000	0.01016	0.01017	0.01570

Table 2: Bounds for w_n

clear that $w \geqq \pi/6$ by elementary geometry, and if $j = 2$, then by Rosta's result [13], it holds that $w \geqq w_3 = \pi/6$. Finally, in both cases, $w = \pi/6$ suffices to cover S^{d-1}.

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ON THE MULTIPLICITY OF ARRANGEMENTS OF CONGRUENT ZONES ON THE SPHERE

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Abstract. Consider an arrangement of n congruent zones on the d-dimensional unit sphere S^{d-1}, where a zone is the intersection of an origin symmetric Euclidean plank with S^{d-1}. We prove that, for sufficiently large n, it is possible to arrange n congruent zones of suitable width on S^{d-1} such that no point belongs to more than a constant number of zones, where the constant depends only on the dimension and the width of the zones. Furthermore, we also show that it is possible to cover S^{d-1} by n congruent zones such that each point of S^{d-1} belongs to at most $A_d \ln n$ zones, where the A_d is a constant that depends only on d. This extends the corresponding 3-dimensional result of Frankl, Nagy and Naszódi [8]. Moreover, we also examine coverings of S^{d-1} with congruent zones under the condition that each point of the sphere belongs to the interior of at most $d-1$ zones.

1. Introduction and Results

A *plank* in the Euclidean d-space $\mathbb{R}^d$ is a closed region bounded by two parallel hyperplanes. The width of a plank is the distance between its bounding hyperplanes. The famous plank problem of Tarski [15] seeks the minimum total width of n planks that can cover a convex body K (a compact convex set with non-empty interior).

In this paper we consider a spherical variant of the plank problem, which originates from L. Fejes Tóth [6]. Following Fejes Tóth, we call the parallel domain of spherical radius $w/2$ of a great sphere C on the d-dimensional unit sphere S^{d-1} a *spherical zone*, or zone for short. C is the central great sphere of the zone and w is its (spherical) width. For positive integers $d \geq 3$ and n, let $w(d,n)$ denote the smallest number such that the union of n zones of width $w(d,n)$ can cover S^{d-1}. Fejes Tóth asked in [6] the exact value of $w(3,n)$. He conjectured that in the optimal configuration the central great circles of the zones all go through an antipodal pair of points and they are distributed equally, so in this case $w(d,n) = \pi/n$. The conjecture of Fejes Tóth was verified for $n = 3$ (Rosta [14]) and $n = 4$ (Linhart [12]). Fodor, Vígh and Zarnócz [7] gave a lower bound for $w(3,n)$ that is valid for all n. Recently, Jiang and Polyanskii [10] completely solved L. Fejes Tóth's conjecture by proving for all d, that to cover S^{d-1} by n (not necessarily congruent) zones, the total width of the zones must be at least π, and that the optimal configuration is essentially the same as conjectured by L. Fejes Tóth.

Here, we examine arrangements of congruent zones on S^{d-1} from the point of view of multiplicity. The multiplicity of an arrangement is the maximal number of zones with nonempty intersection. We seek to minimize the multiplicity for given d and n as a function of the common width of the zones. It is clear that for $n \geq d$, the multiplicity of any arrangement with n congruent zones is at least d and at most n. Notice that in the Fejes Tóth configuration the multiplicity is exactly n, that is, maximal.

In particular, if $d = 3$ and $n \geq 3$, then the multiplicity of any covering is at least 3. Our first result is a very slight strengthening of this simple fact for the case when $n \geq 4$.

Theorem 1. *Let $n \geq 1$ be an integer, and let S^2 be covered by the union of n congruent zones. If each point of S^2 belongs to the interior of at most two zones, then $n \leq 3$. Moreover, if $n = 3$, then the three congruent zones are pairwise orthogonal.*

Note that Theorem 1 does not imply that the multiplicity of a covering of S^2 with $n \geq 4$ congruent zones would have to be larger than 3. In fact, one can cover S^2 with 4 zones such that the multiplicity is 3. For this, consider three zones whose central great circles pass through a pair of antipodal points (North and South Poles) and are distributed evenly. Let the central great circle of the fourth zone be the Equator. The common width can be chosen in such a way that there is no point contained in more than three zones. Also, one can arrange five zones such that the multiplicity is still 3. We start with the previously given four zones, and take another copy of the zone whose central great circle is the Equator. Now slightly tilt these two zones. It is not difficult to see that the multiplicity of the resulting configuration is 3. The details are left to the reader.

We further note, see Remark 1, that the statement of Theorem 1 can probably be extended to all $d \geq 3$. In particular, it certainly holds for $3 \leq d \leq 100$.

Now, we turn to the question of finding upper bounds on the multiplicity of arrangements of zones on S^{d-1}. Let $\alpha : \mathbb{N} \to (0, 1]$ be a positive real function with $\lim_{n\to\infty} \alpha(n) = 0$. For a positive integer $d \geq 3$, let $m_d = \sqrt{2\pi d} + 1$. Let $k : \mathbb{N} \to \mathbb{N}$ be a function that satisfies the limit condition

$$\limsup_{n\to\infty} \alpha(n)^{-(d-1)} \left(\frac{e\, C_d^*\, n\, \alpha(n)}{k(n)} \right)^{k(n)} = \beta < 1, \tag{1}$$

where

$$C_d^* = \frac{4(m_d + 1)(d-1)\kappa_{d-1}}{d\kappa_d}.$$

Theorem 2. *For each positive integer $d \geq 3$, and any real function $\alpha(n)$ described above, for sufficiently large n, there exists an arrangement of n zones of spherical half-width $m_d\alpha(n)$ on S^{d-1} such that no point of S^{d-1} belongs to more than $k(n)$ zones.*

The following statement provides an upper bound on the multiplicity of coverings of the d-dimensional unit sphere by n congruent zones.

Theorem 3. *For each positive integer $d \geq 3$, there exists a positive constant A_d such that for sufficiently large n, there is a covering of S^{d-1} by n zones of half-width $m_d \frac{\ln n}{n}$ such that no point of S^{d-1} belongs to more than $A_d \ln n$ zones.*

Below we list some interesting special cases of Theorem 3 according to the size of the function $\alpha(n)$.

Corollary 1. *With the same hypotheses as in Theorem 2, the following statements hold.*

i) *If* $\alpha(n) = n^{-(1+\delta)}$ *for some* $\delta > 0$, *then* $k(n) = const.$. *Moreover, if* $\delta > d-1$, *then* $k(n) = d$.
ii) *If* $\alpha(n) = \frac{1}{n}$, *then* $k(n) = B_d \frac{\ln n}{\ln \ln n}$ *for some suitable constant* B_d.

We note that Theorem 3 and an implicit version of Theorem 2 were proved by Frankl, Nagy and Naszódi for the case $d = 3$, see [8, Theorem 1.5 and Theorem 1.6] and also the proof of Theorem 1.5 therein. They provided two independent proofs, one of which is a probabilistic argument and the other one uses the concept of VC-dimension. We further add that the weaker upper bound of $O(\sqrt{n})$ on the minimum multiplicity of coverings of S^2 was posed as an exercise in the 2015 Miklós Schweitzer Mathematical Competition [11] by A. Bezdek, F. Fodor, V. Vígh and T. Zarnócz (cf. Exercise 7).

Our proofs of Theorems 2 and 3 are based on the probabilistic argument of Frankl, Nagy and Naszódi [8], which we modified in such a way that it works in all dimensions. In the course of the proof we also give an upper estimate for the constant A_d whose order of magnitude is $O(d)$.

Obviously, there is a big gap between the lower and upper bounds for the multiplicity of coverings of S^{d-1} by congruent zones. At this time, it is an open problem if the minimum multiplicity of coverings of S^{d-1} by n congruent zones is bounded or not, and it also remains unknown whether the multiplicity is monotonic in n, see the corresponding conjectures of Frankl, Nagy and Naszódi on S^2 in [8, Conjectures 4.2 and 4.4].

The multiplicity of coverings of $\mathbb{R}^d$ and S^d by convex bodies have already been investigated. In their classical paper, Erdős and Rogers [4] proved, using a probabilistic argument, that $\mathbb{R}^d$ ($d \geq 3$) can be covered by translates of a given convex body such that the density of the covering is less than $d \log d + d \log \log d + 4n$ and no point of $\mathbb{R}^d$ belongs to more than $e(d \log d + d \log \log d + 4n)$ translates. Later, Füredi and Kang [9] gave a different proof of the result of Erdős and Rogers using John ellipsoids and the Lovász Local Lemma. Böröczky and Wintsche [3] showed that for $d \geq 3$ and $0 < \varphi < \pi/2$, S^d can be covered by spherical caps of radius φ such that the multiplicity of the covering is at most $400 d \ln d$.

2. Proofs

2.1. **Proof of Theorem 1.** Assume that $n \geq 3$ and S^2 is covered by n congruent zones such that no point of S^2 belongs to the interior of more than two zones. Then the n central great circles of the zones divide S^2 into convex spherical polygons. As no three such great circles can pass through a point of S^2, every such polygon has at least three sides.

In contrast to the Euclidean plane, the incircle of every convex spherical polygon is uniquely determined. The inradius of each such polygon is less than or equal to the half-width of the zones.

We will use the following lemma.

Lemma 1. *Every convex spherical polygon with* $k > 3$ *sides and inradius* r *contains a point* P *whose distance from at least three sides is less than* r.

Proof. Denote the incircle by C and denote its centre by O.
Case 1. There are at least three sides tangent to the incircle C.

Among the tangent sides there are two, say e and f, which are not adjacent on the boundary of C. The extensions of e and f form a spherical 2-gon. Start moving the centre O along the diagonal of this 2-gon towards its closest endpoint. Then the distance of O from the extended sides e and f continuously decrease and O eventually gets arbitrarily close to an additional side. When this happens O is closer than r to at least three sides.

Case 2. There are exactly two sides tangent to the incircle C.

Let e and f be the only two sides tangent to the incircle C. Consider again the 2-gon whose sides are the extensions of e and f. Notice that C is also the incircle of this 2-gon. Thus, moving O along the diagonal towards either of the two endpoints continuously decreases the distance of O from the extended sides e and f. At least one of the directions will take O arbitrarily close to an additional side. When this happens O is, again, closer than r to at least three sides. □

Lemma 1 yields immediately that each spherical polygon determined by the n central great circles of the zones is a spherical triangle. The vertices and sides of these triangular domains form a planar graph G on S^2. The number v of vertices is $2\binom{n}{2}$, and the number of edges is $2n(n-1)$. By Euler's formula, the number f of faces (the number of spherical triangles) is

$$f = e + 2 - v = n^2 - n + 2.$$

Furthermore, the degree of each vertex is four, thus $4v = 3f$, which yields that

$$n^2 - n - 6 = 0.$$

The only positive root of the above quadratic equation is $n = 3$.

Let $n = 3$, and assume that the central great circles of two zones intersect in the North and South poles of S^2. The part of S^2 not covered by these two zones is the union of two or four spherical 2-gons bounded by small circular arcs that are parts of the boundaries of the zones. If the uncovered part consists of only two such 2-gons, then there must be a point of S^2 which belongs to the interior of all three zones. As the vertices of the uncovered 2-gons that are on the same hemisphere (say the Northern one) must be on one of the bounding small circles of the third zone, they must be coplanar. This is only satisfied when the first two zones are perpendicular. This finishes the proof of Theorem 1.

Remark 1. Consider now n congruent zones on S^{d-1} such that no point belongs to the interior of more than $d-1$ zones. Then the central great spheres of the zones divide S^{d-1} into convex spherical polytopes similar to the 3-dimensional case. We note that the argument of Lemma 1 can be generalized to arbitrary d, only one has to consider $d-1$ cases instead of two. Thus, the central great spheres of the zones divide S^{d-1} into spherical simplices.

Now, a similar combinatorial analysis can be carried out, with the help of the Euler-Poincaré formula, as in S^2. Let $f_{i,d}(n)$ denote the number of i-dimensional faces determined by the central great spheres of the n zones for $d \geq 3$ and $n \geq d-1$. We use the conventions: $f_{-1,d}(n) = 1$ and $f_{d,d}(n) = 1$. As we have seen in the proof of Lemma 1, $f_{0,3} = 2\binom{n}{2}$, $f_{1,3}(n) = 2n(n-1)$, and $f_{2,3} = n^2 - n + 2$.

Then we have the following recursions for $f_{i,d}(n)$ when $d \geq 4$:

$$\begin{aligned} f_{0,d}(n) &= 2\binom{n}{d-1}, \\ f_{i,d}(n) &= \frac{n}{d-i-1} f_{i,d-1}(n-1) \quad (1 \leq i \leq d-2), \\ f_{d-1,d}(n) &= \frac{2}{d} f_{d-2,d}(n). \end{aligned}$$

As the n central great spheres are in general position, a vertex is incident with exactly $d-1$ of them, which explains the formula for $f_{0,d}(n)$. Since the cells are simplices, counting its facets one gets the identity $2f_{d-2,d}(n) = df_{d-1,d}(n)$. Finally, if $1 \leq i \leq d-2$, then consider a fixed central great sphere. The other central great spheres intersect the chosen one in $n-1$ great spheres (of one less dimension) that are in general position. Taking into account that we have n great spheres and that an i-dimensional face is incident with exactly $d-i-1$ great spheres, one gets the second formula above.

Now, for a fixed d, using the Euler–Poincaré formula, $\sum_{i=-1}^{d}(-1)^{d+1} f_{i,d}(n) = 0$ this holds as we have a triangulation of S^{d-1} into simplices one can obtain a polynomial equation $p(d,n) = 0$ of degree at most $d-1$ in n. When $n = d$, then n pairwise orthogonal congruent zones satisfy all conditions, thus, $n = d$ is always a root of $p(d,n)$. In particular, for $4 \leq d \leq 6$, the reduced forms of $p(d,n)$ in which the coefficient of n^{d-1} is 1 are the following

$$\begin{aligned} p(4,n) &= (n-4)(n+1)n, \\ p(5,n) &= (n-5)(n^3 - n^2 - 2n - 8), \\ p(6,n) &= (n-6)(n-2)(n-1)^2 n. \end{aligned}$$

Thus, if $d = 4$ or 6, then $n = d$ is the largest root that satisfies our conditions. In the case $d = 5$ one can check that $p(5,d)$ has two complex roots and two real roots, one real root is 5 and the other one is smaller than 5.

We can now formulate the following conjecture.

Conjecture 1. *Let $d \geq 3$ and $n \geq 1$ be integers, and let S^{d-1} be covered by the union of n congruent zones. If each point of S^{d-1} belongs to the interior of at most $d-1$ zones, then $n \leq d-1$. Moreover, if $n = d$, then the d congruent zones are pairwise orthogonal*

By Theorem 1 and the above argument we have proved the first statement of Conjecture 1 for $3 \leq d \leq 6$. If $n = d$, then the orthogonality of the zones can be proved essentially the same way as in the proof of Theorem 1. Furthermore, we have computed the roots of $p(d,n)$ for $7 \leq d \leq 100$ by computer (numerically) and observed than in each case the largest real root is $n = d$, which supports our conjecture.

Finally we note that computer calculations suggest that in the case when $d \geq 6$ is even,

$$p(d,n) = (n-d)(n-d+5)\prod_{i=0}^{d-4}(n-i).$$

2.2. **Proof of Theorem 2.** For two points $P, Q \in S^{d-1}$, their spherical distance is the length of the shorter unit-radius circular arc on S^{d-1} that connects them. We denote the spherical distance by $d_S(P,Q)$.

Let $0 < \omega \leq \pi/2$. We say that the points $P_1, \dots, P_m \in S^{d-1}$ form a *saturated set* for ω if the spherical distances $d_S(P_i, P_j) \geq \omega$ for all $i \neq j$ and no more points can be added such that this property holds. Investigating the dependence of m on d and ω is a classical topic in the theory of packing and covering; for a detailed overview of known results in this direction see, for example, the survey paper by Fejes Tóth and Kuperberg [5].

It is clear that m is of the same order of magnitude as $\omega^{-(d-1)}$. In the next lemma, we prove a somewhat more precise statement. Although the content of the lemma is well-known, we give a proof because we need inequalities for m with exact constants in subsequent arguments, and also for the sake of completeness. Let κ_d denote the volume of the d-dimensional unit ball B^d.

Lemma 2. *Let $0 < \varepsilon < 1$. Then there exists $0 < \omega_0 \leq \pi/2$ depending on ε with the following property. Let $0 < \omega < \omega_0$, and let $P_1, \dots, P_m$ be a saturated point set for ω. Then*

$$(1+\varepsilon)^{-1}\frac{d\kappa_d}{\kappa_{d-1}}\omega^{-(d-1)} \leq m \leq (1+\varepsilon)\frac{8^{\frac{d-1}{2}}d\kappa_d}{\kappa_{d-1}}\omega^{-(d-1)}.$$

Proof. The following formula is known for the surface area $S(t)$ of a cap of height t of S^{d-1}, cf. [2, formula (3.4) on p. 796],

$$\lim_{t\to 0+} S(t)\, t^{-\frac{d-1}{2}} = 2^{\frac{d-1}{2}}\kappa_{d-1}.$$

Therefore, there exists $0 < t_0 = t_0(\varepsilon)$ such that for all $0 < t < t_0$ it holds that

$$(1+\varepsilon)^{-1}2^{\frac{d-1}{2}}\kappa_{d-1} \leq S(t)\, t^{-\frac{d-1}{2}} \leq (1+\varepsilon)2^{\frac{d-1}{2}}\kappa_{d-1}.$$

Furthermore, let $0 < \omega_0 = \omega_0(\varepsilon)$ be such that $t_0 = 1 - \cos\omega_0$.

The spherical caps of (spherical) radius $\omega/2$ centred at $P_1, \dots, P_m$ form a packing on S^{d-1}, and the spherical caps of radius ω form a covering of S^{d-1}. In view of the above inequalities for the surface area of caps, we obtain that for $0 < \omega < \omega_0$ it holds that

$$m(1+\varepsilon)^{-1}\, 2^{\frac{d-1}{2}}\kappa_{d-1}\left(1-\cos\frac{\omega}{2}\right)^{\frac{d-1}{2}} \leq d\kappa_d \leq m(1+\varepsilon)2^{\frac{d-1}{2}}\kappa_{d-1}(1-\cos\omega)^{\frac{d-1}{2}}.$$

By simple rearrangement we get that

$$(1+\varepsilon)^{-1}\frac{d\kappa_d}{2^{\frac{d-1}{2}}\kappa_{d-1}(1-\cos\omega)^{\frac{d-1}{2}}} \leq m \leq (1+\varepsilon)\frac{d\kappa_d}{2^{\frac{d-1}{2}}\kappa_{d-1}\left(1-\cos\frac{\omega}{2}\right)^{\frac{d-1}{2}}}.$$

Now, we use that for $0 < x < 1$, it holds that $x^2/4 < 1 - \cos x < x^2/2$, which follow simply from the Taylor series of $\cos x$, and obtain the desired inequalities

$$(1+\varepsilon)^{-1}\frac{d\kappa_d}{\kappa_{d-1}}\,\omega^{-(d-1)} \leq m \leq (1+\varepsilon)\frac{8^{\frac{d-1}{2}}d\kappa_d}{\kappa_{d-1}}\,\omega^{-(d-1)}.$$

□

We denote a spherical zone of (spherical) half-width t by $\Pi(t)$. Since, for small t, it holds that

$$2(d-1)\kappa_{d-1}\,\sin t < S(\Pi(t)) < 2(d-1)\kappa_{d-1}\, t,$$

it follows that

$$\lim_{t\to 0^+} S(\Pi(t))\cdot t^{-1} = 2(d-1)\kappa_{d-1}.$$

Let $\varepsilon > 0$. Then there exists $t_1 = t_1(\varepsilon) > 0$ such that for $0 < t < t_1$ the following holds

$$(1+\varepsilon)^{-1}2(d-1)\kappa_{d-1}\, t \le S(\Pi(t)) \le (1+\varepsilon)2(d-1)\kappa_{d-1}\, t.$$

Let $\alpha(n)$ be a given positive function with $\lim_{n\to\infty}\alpha(n) = 0$. From now on, we fix $\varepsilon = 1$, set $m_d = \sqrt{2\pi d} + 1$, and assume n to be sufficiently large.

Let $Q_1, \ldots, Q_m$ be a saturated set of points on S^{d-1} such that $d_S(Q_i, Q_j) \ge \alpha(n)/2$ for any $i \ne j$. It follows from Lemma 1 that

$$\begin{aligned} m &\le 2\frac{8^{\frac{d-1}{2}} d\kappa_d}{\kappa_{d-1}}(\alpha(n)/2)^{-(d-1)} \\ &= 2\frac{2^{\frac{d-1}{2}} d\kappa_d}{\kappa_{d-1}}\alpha(n)^{-(d-1)} \\ &= c_d\, \alpha(n)^{-(d-1)}. \end{aligned}$$

Consider n independent random points from S^{d-1} chosen according to the uniform probability distribution and consider the corresponding spherical zones $\Pi_1, \ldots, \Pi_n$ of (spherical) half-width $m_d\alpha(n)$ whose poles are these points. Furthermore, let Π_i^-, Π_i^+ be the corresponding planks of half-width $(m_d-1)\alpha(n)$ and $(m_d+1)\alpha(n)$, respectively.

Now, we are going to estimate the probability of the event that there exists a point p on S^{d-1} which belongs to at least $k = k(n)$ zones. The probability that a point $p \in S^{d-1}$ belongs to a spherical plank Π_i^+ can be estimated from above as follows.

$$\mathbb{P}(p \in \Pi_i^+) \le \frac{4(m_d+1)(d-1)\kappa_{d-1}}{d\kappa_d}\,\alpha(n) = C_d^*\,\alpha(n).$$

Note that $C_d^* = O(d)$ as $d \to \infty$.

Then

$$\begin{aligned} &\mathbb{P}(\exists p \in \Pi_{i_1}\cap\cdots\cap\Pi_{i_k} :\ \text{for some } 1 \le i_1 < \ldots < i_k \le n) \\ &\le \mathbb{P}(\exists Q_j \in \Pi_{i_1}^+\cap\cdots\cap\Pi_{i_k}^+ :\ \text{for some } 1 \le i_1 < \ldots < i_k \le n) \\ &\le m\cdot\mathbb{P}(Q_1 \in \Pi_{i_1}^+\cap\cdots\cap\Pi_{i_k}^+ :\ \text{for some } 1 \le i_1 < \ldots < i_k \le n) \\ &\le m\cdot\binom{n}{k(n)}\left(C_d^*\,\alpha(n)\right)^{k(n)} \\ &\le c_d\,\alpha(n)^{-(d-1)}\binom{n}{k(n)}\left(C_d^*\,\alpha(n)\right)^{k(n)} \end{aligned}$$

An application of the Stirling-formula (cf. Page 10 of [8]) yields that

$$\binom{n}{k} \le C\frac{n^n}{k^k(n-k)^{n-k}} \tag{2}$$

for some suitable constant $C > 0$.

Then applying (2) we get that

$$
\begin{aligned}
& c_d\ \alpha(n)^{-(d-1)} \binom{n}{k(n)} (C_d^*\ \alpha(n))^{k(n)} \\
\leq & c_d\ \alpha(n)^{-(d-1)} \cdot C \frac{n^n (n-k(n))^{k(n)}}{(k(n))^{k(n)-n}} (C_d^*\ \alpha(n))^{k(n)} \\
\leq & \tilde{c_d}\ \alpha(n)^{k(n)-d+1} \left(\frac{n}{k(n)}\right)^{k(n)} (e \cdot C_d^*)^{k(n)} \\
= & \tilde{c_d}\ \alpha(n)^{-(d-1)} \left(\frac{e\ C_d^*\ n\ \alpha(n)}{k(n)}\right)^{k(n)}. \qquad (3)
\end{aligned}
$$

By (1) we obtain

$$\limsup_{n\to\infty} \mathbb{P}(\exists p \in \Pi_{i_1} \cap \cdots \cap \Pi_{i_k} :\ \text{for some } 1 \leq i_1 < \ldots < i_k \leq n) < 1,$$

therefore the probability of the event that no point of S^{d-1} belongs to at least $k(n)$ zones is positive for sufficiently large n. This finishes the proof of Theorem 2.

2.3. **Proof of Theorem 3.** Let $\alpha(n) = \frac{\ln n}{n}$, and let $k(n) = A_d \ln n$, where A_d be a suitable positive constant that satisfies the following equation

$$\left(\frac{C_d^*}{x}\right)^x = e^{-d-x}.$$

Then

$$(1) = \lim_{n\to\infty} \tilde{c_d} \frac{n^{d-1}}{(\ln n)^{d-1}} \cdot n^{A_d} \left(\frac{C_d^*}{A_d}\right)^{A_d \ln n} = 0. \qquad (4)$$

Furthermore, in this case the probability that an arbitrary fixed point p of S^{d-1} is in Π_i^- (for a fixed i) is

$$\mathbb{P}(p \in \Pi_i^-) \geq 2^{-1} \cdot \frac{2(d-1)\kappa_{d-1}}{d\kappa_d} \cdot (m_d - 1)\alpha(n).$$

Using the inequality $\frac{\kappa_{d-1}}{d\kappa_d} > \frac{1}{\sqrt{2\pi d}}$ (cf. Lemma 1 in [1]), we obtain that

$$\mathbb{P}(p \in \Pi_i^-) \geq \frac{(m_d-1)(d-1)}{\sqrt{2\pi d}} \cdot \frac{\ln n}{n} = (d-1)\frac{\ln n}{n}$$

Thus, the probability that $\cup_1^n \Pi_i$ does not cover S^{d-1} satisfies

$$
\begin{aligned}
\mathbb{P}(S^{d-1} \not\subseteq \cup_1^n \Pi_i) &\leq \mathbb{P}(\exists Q_j \notin \cup_1^n \Pi_i^-) \\
&\leq m \cdot \mathbb{P}(Q_1 \notin \cup_1^n \Pi_i^-) \\
&\leq c_d \left(\frac{n}{\ln n}\right)^{d-1} \cdot \left(1 - (d-1)\frac{\ln n}{n}\right)^n \\
&\leq 2c_d \left(\frac{1}{\ln n}\right)^{d-1}
\end{aligned}
$$

for a sufficiently large n. Therefore

$$\lim_{n\to\infty} \mathbb{P}(S^{d-1} \not\subseteq \cup_1^n \Pi_i) = 0. \qquad (5)$$

Thus, taking into account (4) and (5), the probability of the event that all S^{d-1} is covered by the zones and no point of S^{d-1} belongs to more than $A_d \ln n$ zones is positive for sufficiently large n. This finishes the proof of Theorem 3.

Remark 2. We note that $A_d = O(d)$ as $d \to \infty$. Clearly, A_d can be lowered slightly by taking into account all the factors of (4).

Remark 3. We further note that one can obtain the result of Theorem 3 with the help of Theorem 1.6 of [8] using the VC-dimension of hypergraphs; for more details we refer to the discussion in [8] after Theorem 1.6. However, as this alternate proof is less geometric in nature, we decided to describe the more direct probabilistic proof of Theorem 3. We leave the proof of Theorem 3 that uses the VC-dimension to the interested reader. Furthermore, the direct probabilistic argument provides an explicit estimate of the involved constant A_d, as well.

2.4. **Proof of Corollary 1.** Let $\alpha(n) = \frac{1}{n^{1+\delta}}$ for some $\delta > 0$. If $k = k(n) > (d-1)/\delta + d - 1$, then

$$\begin{aligned}\limsup_{n\to\infty} \alpha(n)^{-(d-1)} \left(\frac{e\, C_d^*\, n\, \alpha(n)}{k(n)}\right)^{k(n)} &= \lim_{n\to\infty} n^{(1+\delta)(d-1)} \left(\frac{e\, C_d^*\, n^{-\delta}}{k}\right)^{k} \\ &= \lim_{n\to\infty} n^{(1+\delta)(d-1)-\delta k} = 0.\end{aligned}$$

This means that in this case, for sufficiently large n, it can be guaranteed that one can arrange n zones of half-width $m_d\alpha(n)$ on S^{d-1} such that no point belongs to more than $k = const.$ zones, and the value of k only depends on d and δ. Moreover, if $\delta > d-1$, then $k = d$ suffices. Of course, in this case the zones cannot cover S^{d-1}. This proves i) of Corollary 1.

Now, let $\alpha(n) = \frac{1}{n}$, and let $k(n) = B_d \frac{\ln n}{\ln\ln n}$, where $B_d > \max\{e\, C_d^*, d-1\}$ is a positive constant. Then

$$\begin{aligned}\limsup_{n\to\infty} \alpha(n)^{-(d-1)} \left(\frac{e\, C_d^*\, n\, \alpha(n)}{k(n)}\right)^{k(n)} &= \lim_{n\to\infty} n^{d-1} \left(\frac{e\, C_d^*\, \ln\ln n}{B_d\, \ln n}\right)^{B_d \frac{\ln n}{\ln\ln n}} \\ &\le \lim_{n\to\infty} \left(\frac{n^{\frac{(d-1)\ln\ln n}{B_d \ln n}} \ln\ln n}{\ln n}\right)^{B_d \frac{\ln n}{\ln\ln n}} = 0,\end{aligned}$$

as

$$\begin{aligned}&\lim_{n\to\infty} \frac{n^{\frac{(d-1)\ln\ln n}{B_d \ln n}} \ln\ln n}{\ln n} \\ &= \lim_{n\to\infty} \exp\left(\frac{d-1}{B_d} \ln\ln n + \ln\ln\ln n - \ln\ln n\right) = 0.\end{aligned}$$

This finishes the proof of part ii) of Corollary 1. The above statement is interesting because $\alpha(n) = \frac{1}{n}$ is the smallest order of magnitude for the half-width of the zones for which one can possibly have a covering.

Remark 4. We note that the $d = 3$ special case of part ii) of Corollary 1 was explicitly proved by Frankl, Nagy and Naszódi in [8] (cf. Theorem 4.1) in a slightly different form both by the probabilistic method and using VC-dimension. We also note that the general d-dimensional statement of part ii) of Corollary 1 may also be proved from Theorem 1.6 of [8].

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ON THE VOLUME BOUND IN THE DVORETZKY–ROGERS LEMMA

FERENC FODOR, MÁRTON NASZÓDI, AND TAMÁS ZARNÓCZ

Abstract. The classical Dvoretzky–Rogers lemma provides a deterministic algorithm by which, from any set of isotropic vectors in Euclidean d-space, one can select a subset of d vectors whose determinant is not too small. Subsequently, Pelczyński and Szarek improved this lower bound by a factor depending on the dimension and the number of vectors.

Pivovarov, on the other hand, determined the expectation of the square of the volume of parallelotopes spanned by d independent random vectors in $\mathbb{R}^d$, each one chosen according to an isotropic measure. We extend Pivovarov's result to a class of more general probability measures, which yields that the volume bound in the Dvoretzky–Rogers lemma is, in fact, equal to the expectation of the squared volume of random parallelotopes spanned by isotropic vectors. This allows us to give a probabilistic proof of the improvement of Pelczyński and Szarek, and provide a lower bound for the probability that the volume of such a random parallelotope is large.

1. Introduction

Given a set of isotropic vectors in Euclidean d-space $\mathbb{R}^d$ (see definition below), the *Dvoretzky–Rogers lemma* states that one may select a subset of d "well spread out" vectors. As a consequence, the determinant of these d vectors is at least $\sqrt{d!/d^d}$. This selection is deterministic: we start with an arbitrary element of the set, and then select more vectors one-by-one in a certain greedy manner.

Pivovarov [Piv10, Lemma 3, p. 49], on the other hand, chooses d vectors randomly and then computes the expectation of the square of the resulting determinant. In this note, we extend Pivovarov's result to a wider class of measures, and apply this extension to obtain the improved lower bound of Pelczyński and Szarek, cf. [PS91]Proposition 2.1, on the maximum of the volume of parallelotopes spanned by d vectors from the support of the measure. Thus, we give a probabilistic interpretation of the volume bound in the Dvoretzky–Rogers lemma.

We denote the Euclidean scalar product by $\langle \cdot, \cdot \rangle$, the induced norm by $|\cdot|$. We use the usual notation B^d for the unit ball of $\mathbb{R}^d$ centered at the origin o, and S^{d-1} for its boundary $\mathrm{bd}B^d$. We call a compact convex set $K \subset \mathbb{R}^d$ with non-empty interior a *convex body*. For detailed information on the properties of convex bodies, we refer to the books by Gruber [Gru07] and Schneider [Sch14].

Let Id_d be the identity map on $\mathbb{R}^d$. For $u, v \in \mathbb{R}^d$, let $u \otimes v : \mathbb{R}^d \to \mathbb{R}^d$ denote the *tensor product* of u and v, that is, $(u \otimes v)(x) = \langle v, x \rangle u$ for any $x \in \mathbb{R}^d$. Note that when $u \in S^{d-1}$ is a unit vector, $u \otimes u$ is the orthogonal projection to the linear subspace spanned by u.

For two functions $f(n), g(n)$, we use the notation $f(n) \sim g(n)$ (as $n \to \infty$) if $\lim_{n\to\infty} f(n)/g(n) = 1$.

An *isotropic measure* is a probability measure μ on $\mathbb{R}^d$ with the following two properties.

$$(1) \qquad \int_{\mathbb{R}^d} x \otimes x \, \mathrm{d}\mu(x) = \mathrm{Id}_d,$$

and the center of mass of μ is at the origin, that is,

$$(2) \qquad \int_{\mathbb{R}^d} x \, \mathrm{d}\mu(x) = 0.$$

Pivovarov [Piv10] proved the following statement about the volume of random parallelotopes spanned by d independent, isotropic vectors.

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Lemma 1 (Pivovarov [Piv10], Lemma 3)**.** *Let $x_1, \dots, x_d$ be independent random vectors distributed according to the isotropic measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$. Assume that $x_1, \dots, x_d$ are linearly independent with probability 1. Then*

$$\mathbb{E}([\det(x_1, \dots, x_d)]^2) = d!. \tag{3}$$

We note that Lutwak, Yang and Zhang in [LYZ04, §2] established similar results for the case of discrete isotropic measures, which could also be used to prove the volumetric bounds in Theorem 2, see, for example, [LYZ04, formula (2.5) on page 167].

We extend Lemma 1 to a more general class of measures in the following way.

Lemma 2. *Let $x_1, \dots, x_d$ be independent random vectors distributed according to the measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$ satisfying* (1)*. Assume that $\mu_i(\{0\}) = 0$ for i=1,..., d. Then* (3) *holds.*

We provide a simple and direct proof of Lemma 2 in Section 2.

Lemmas 1 and 2 yield the value of the second moment of the volume of random parallelotopes with isotropic generating vectors. On the other hand, Milman and Pajor [MP, §3.7] gave a lower bound for the p-th moment (with $0 < p < 2$) of this volume in the case when the generating vectors are selected according to the uniform distribution from an isotropic and origin-symmetric convex body; for more general results, cf. [BGVV14, §3.5.1]. All of the previously mentioned results hold in *expectation.*

As a different approach, we mention Pivovarov's work [Piv10], where lower bounds on the volume of a random parallelotope are shown to hold *with high probability* under the assumption that the measures are log-concave.

For more information on properties of random parallelotopes, and random polytopes in general, we refer to the book by Schneider and Weil [SW08], the survey by Schneider [Sch], and the references therein.

In this paper, our primary, geometric motivation in studying isotropic measures is the following celebrated theorem of John [Joh48], which we state in the refined form obtained by Ball [Bal92] (see also [Bal97]).

Theorem 1. *Let K be a convex body in $\mathbb{R}^d$. Then there exists a unique ellipsoid of maximal volume contained in K. Moreover, this maximal volume ellipsoid is the d-dimensional unit ball B^d if and only if there exist vectors $u_1, \dots, u_m \in \mathrm{bd}K \cap S^{d-1}$ and (positive) real numbers $c_1, \dots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = \mathrm{Id}_d, \tag{4}$$

and

$$\sum_{i=1}^{m} c_i u_i = 0. \tag{5}$$

Note that taking the trace in (4) yields $\sum_{i=1}^{m} c_i = d$. Thus, the Borel measure μ_K on $\sqrt{d}S^{d-1}$ with $\mathrm{supp}\mu_K = \{\sqrt{d}u_1, \dots, \sqrt{d}u_m\}$ and $\mu_K(\{\sqrt{d}u_i\}) = c_i/d$ $(i = 1, \dots, m)$ is a discrete isotropic measure.

If a finite system of unit vectors $u_1, \dots, u_m$ in $\mathbb{R}^d$, together with a set of positive weights $c_1, \dots, c_m$ satisfies (4) and (5), then we say that it forms a *John decomposition of the identity.* For each convex body K, there exists an affine image K' of K for which the maximal volume ellipsoid contained in K' is B^d, and K' is unique up to orthogonal transformations of $\mathbb{R}^d$.

The classical lemma of Dvoretzky and Rogers [DR50] states that in a John decomposition of the identity, one can always find d vectors such that the selected vectors are not too far from an orthonormal system.

Lemma 3 (Dvoretzky–Rogers lemma [DR50])**.** *Let $u_1, \dots, u_m \in S^{d-1}$ and $c_1, \dots, c_m > 0$ such that* (4) *holds. Then there exists an orthonormal basis $b_1, \dots, b_d$ of $\mathbb{R}^d$ and a subset $\{x_1, \dots, x_d\} \subset \{u_1, \dots, u_m\}$ with $x_j \in \mathrm{lin}\{b_1, \dots, b_j\}$ and*

$$\sqrt{\frac{d-j-1}{d}} \leq \langle x_j, b_j \rangle \leq 1 \tag{6}$$

for $j = 1, \dots, d$.

Consider the parallelotope P spanned by the selected d vectors $x_1, \dots, x_d$. The volume of P is bounded from below by

$$(\mathrm{Vol}(P))^2 = [\det(x_1, \dots, x_d)]^2 \geq \frac{d!}{d^d}. \tag{7}$$

Our study of (7) is motivated in part by the recent proof [Nas16] of a conjecture of Bárány, Katchalski and Pach, where this bound is heavily relied on.

The main results of this paper are the following two statements. Theorem 2 is essentialy the same as Proposition 2.1 of Pelczyński and Szarek [PS91], however, here we give a probabilistic proof and interpretation. In Theorem 2 (ii) and (iii), we also note that when m is small the improvement on the original Dvoretzky–Rogers bound is larger.

Theorem 2. *Let $u_1, \dots, u_m \in S^{d-1}$ be unit vectors satisfying (4) with some $c_1, \dots, c_m > 0$. Then there is a subset $\{x_1, \dots, x_d\} \subset \{u_1, \dots, u_m\}$ with*

$$[\det(x_1, \dots, x_d)]^2 \geq \gamma(d, \overline{m}) \cdot \frac{d!}{d^d},$$

where $\gamma(d, \overline{m}) = \frac{\overline{m}^d}{d!} \binom{\overline{m}}{d}^{-1}$, and $\overline{m} = \min\{m, d(d+1)/2\}$.

Moreover, for $\gamma(d, \overline{m})$, we have

- *(i) $\gamma(d, \overline{m}) \geq \gamma(d, d(d+1)/2) \geq 3/2$ for any $d \geq 2$ and $m \geq d$. And $\gamma(d, d(d+1)/2)$ is monotonically increasing, and $\lim_{d \to \infty} \gamma(d, d(d+1)/2) = e$.*
- *(ii) Fix a $c > 1$, and consider the case when $m \leq cd$ with $c \geq 1 + 1/d$. Then*

$$\gamma(d, m) \geq \gamma(d, \lceil cd \rceil) \sim \sqrt{\frac{c-1}{c}} \left(\frac{c-1}{c} \right)^{(c-1)d} e^d, \quad \text{as } d \to \infty.$$

- *(iii) Fix an integer $k \geq 1$, and consider the case when $m \leq d + k$. Then*

$$\gamma(d, m) \geq \gamma(d, d+k) \sim \frac{k! e^k}{\sqrt{2\pi}} \frac{e^d}{(d+k)^{k+1/2}}, \text{ as } d \to \infty.$$

We note that in (ii) and (iii), the improvements are exponentially large in d as d tends to infinity.

The following statement provides a lower bound on the probability that d independent, identically distributed random vectors selected from $\{u_1, \dots, u_m\}$ according to the distribution determined by the weights $\{c_1, \dots, c_m\}$ has large volume.

Proposition 1. *Let $\lambda \in (0, 1)$. With the notations and assumptions of Theorem 2, if we choose the vectors $x_1, \dots, x_d$ independently according to the distribution $\mathbb{P}(x_\ell = u_i) = c_i/d$ for each $\ell = 1, \dots, d$ and $i = 1, \dots, m$, then with probability at least $(1 - \lambda)e^{-d}$, we have that*

$$[\det(x_1, \dots, x_d)]^2 \geq \lambda \gamma(d, \overline{m}) \cdot \frac{d!}{d^d}.$$

The geometric interpretation of Theorem 2 is the following. If K is a convex polytope with n facets, and B^d is the maximal volume ellipsoid in K, then the number of contact points $u_1, \dots, u_m$ in John's theorem is at most $m \leq n$. Thus, Theorem 2 yields a simplex in K of not too small volume, with one vertex at the origin.

In particular, consider $k = 1$ in Theorem 2 (iii), that is, when K is the regular simplex whose inscribed ball is B^d. Then the John decomposition of the identity determined by K consists of $d + 1$ unit vectors that determine the vertices of a regular d-simplex inscribed in B^d, which we denote by Δ_d, and note that $\mathrm{Vol}(\Delta_d) = (d+1)^{\frac{d+1}{2}} / (d^{d/2} d!)$. Clearly, in this John decomposition of the identity, the volume of the simplex determined by any d of the vectors $u_1, \dots, u_{d+1}$ is

$$\mathrm{Vol}(\Delta_d)/(d+1) = \frac{(d+1)^{\frac{d-1}{2}}}{d^{d/2} d!}. \tag{8}$$

By Theorem 2, we obtain that

$$\max[\det(u_{i_1}, \dots, u_{i_d})]^2 \geq \frac{(d+1)^{d-1}}{d!} \cdot \frac{d!}{d^d} = \frac{(d+1)^{d-1}}{d^d},$$

which yields the same bound for the largest volume simplex as the right-hand-side of (8). Thus, Theorem 2 is sharp in this case.

We will use the following theorem in our argument.

Theorem 3 ([Joh48, Peł90, Bal92, GS05]). *If a set of unit vectors satisfies* (4) *(resp.,* (4) *and* (5)*) with some positive scalars c_i', then a subset of m elements also satisfies* (4) *(resp.,* (4) *and* (5)*) with some positive scalars c_i, where*

$$d+1 \leq m \leq d(d+1)/2 \tag{9}$$

(resp., $d+1 \leq m \leq d(d+3)/2$).

In Section 4, we outline a proof of Theorem 3 for two reasons. First, we will use the part when only (4) is assumed, which is only implicitly present in [GS05]. Second, in [GS05], the result is described in terms of the contact points of a convex body with its maximal volume ellipsoid, that is, in the context of John's theorem. We, on the other hand, would like to give a presentation where the linear algebraic fact and its use in convex geometry are separated. Nevertheless, our proof is very close to the one given in [GS05].

2. Proof of Lemma 2

The idea of the proof is to slightly rotate each distribution so that the probability that the d vectors are linearly independent is 1. Then we may apply Pivovarov's lemma, and use a limit argument as the d rotations each tend to the identity.

Let $A_1, \ldots, A_d$ be matrices in $SO(d)$ chosen independently of each other and of the x_is according to the unique Haar probability measure on $SO(d)$. Fix an arbitrary non-zero unit vector e in $\mathbb{R}^d$. Note that $A_i x_i/|x_i|$ and $A_i e$ have the same distribution: both are uniformly chosen points of the unit sphere according to the uniform probability distribution on S^{d-1}. A bit more is true: the joint distribution of $A_1x_1/|x_1|, \ldots, A_dx_d/|x_d|$ and the joint distribution of $A_1e, \ldots, A_de$ are the same: they are independently chosen, uniformly distributed points on the unit sphere. It follows that

$$\mathbb{P}(A_1x_1, \ldots, A_dx_d \text{ are lin. indep.}) = \mathbb{P}(A_1e, \ldots, A_de \text{ are lin. indep.}) = 1.$$

Denote the Haar measure on $Z := SO(d)^d$ by ν. Thus, we have

$$\begin{aligned}1 = \mathbb{P}(A_1x_1, \ldots, A_dx_d \text{ are lin. indep.}) =\\ \int_Z \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{1}_{\{A_1x_1,\ldots,A_dx_d \text{ are lin. indep.}\}}(x_1, \ldots, x_d, A_1, \ldots, A_d)\\ \mathrm{d}\mu_1(x_1) \ldots \mathrm{d}\mu_d(x_d)\, \mathrm{d}\nu(A_1, \ldots, A_d)\\ = \int_Z \mathbb{P}(A_1x_1, \ldots, A_dx_d \text{ are lin. indep.} \,|\, A_1, \ldots, A_d)\, \mathrm{d}\nu(A_1, \ldots, A_d),\end{aligned}$$

where $\mathbb{1}$ denotes the indicator function.

Thus,

$$1 = \mathbb{P}\Big[\mathbb{P}(A_1x_1, \ldots, A_dx_d \text{ are lin. indep.} \,|\, A_1, \ldots, A_d) = 1\Big]. \tag{10}$$

We call a d-tuple $(A_1, \ldots, A_d) \in Z$ 'good' if $A_1x_1, \ldots, A_dx_d$ are linearly independent with probability 1. In (10), we obtained that the set of not good elements of Z is of measure zero.

Thus, we may choose a sequence $(A_1^{(j)}, A_2^{(j)}, \ldots, A_d^{(j)})$, $j = 1, 2, \ldots$ in Z, such that $\|A_i^{(j)} - \mathrm{Id}_d\| < 1/j$ for all i and j, and $(A_1^{(j)}, \ldots, A_d^{(j)})$ is good for each j.

Note that for any j,

$$\Big[\det\Big(A_1^{(j)}x_1, \ldots, A_d^{(j)}x_d\Big)\Big]^2 \leq |A_1^{(j)}x_1|^2 |A_2^{(j)}x_2|^2 \ldots |A_d^{(j)}x_d|^2, \tag{11}$$

and

$$\mathbb{E}\left[|A_1^{(j)}x_1|^2 |A_2^{(j)}x_2|^2 \ldots |A_d^{(j)}x_d|^2\right] = d^d. \tag{12}$$

We conclude that

$$\mathbb{E}\left([\det(x_1,\ldots,x_d)]^2\right) =$$

$$\mathbb{E}\left(\left[\det \lim_{j\to\infty}\left(A_1^{(j)}x_1,\ldots,A_d^{(j)}x_d\right)\right]^2\right) \stackrel{(a)}{=}$$

$$\mathbb{E}\left(\left[\lim_{j\to\infty}\det\left(A_1^{(j)}x_1,\ldots,A_d^{(j)}x_d\right)\right]^2\right) \stackrel{(b)}{=}$$

$$\lim_{j\to\infty}\mathbb{E}\left(\left[\det\left(A_1^{(j)}x_1,\ldots,A_d^{(j)}x_d\right)\right]^2\right),$$

where, in (a), we use that the determinant is continuous. In (b), Lebesgue's Dominated Convergence Theorem may be applied by (11) and (12).

Fix j and let $y_1 = A_1^{(j)}x_1,\ldots,y_d = A_d^{(j)}x_d$. In order to emphasize that the assumption (2) is not needed, and also for completeness, we repeat Pivovarov's argument. For $k = 1,\ldots,d-1$, let P_k denote the orthogonal projection of $\mathbb{R}^d$ onto the linear subspace $\operatorname{span}\{y_1,\ldots,y_k\}^\perp$. Thus,

(13) $$|\det(y_1,\ldots,y_d)| = |y_1||P_1y_2|\cdots|P_{d-1}y_d|.$$

Note that with probability 1, $\operatorname{rank}P_k = d-k$. It follows from (1) that $\mathbb{E}|P_ky_{k+1}|^2 = d-k$. Fubini's Theorem applied to (13) completes the proof of Lemma 2.

3. Proofs of Theorem 2 and Proposition 1

Let $u_1,\ldots u_m \in S^{d-1}$ be a set of vectors satisfying (4) with some positive weights $c_1,\ldots,c_m$. We set the probability of each vector u_i, $i = 1,\ldots m$ as $p_i = c_i/d$, and obtain a discrete probability distribution.

Let $u_{i_1},\ldots,u_{i_d}$ be independent random vectors from the set $u_1,\ldots,u_m$ chosen (with possible repetitions) according to the above probability distribution.

By Lemma 2, we have that

$$\mathbb{E}\left([\det(u_{i_1},\ldots,u_{i_d})]^2\right) = \frac{d!}{d^d}.$$

Since the probability that the random vectors $u_{i_1},\ldots,u_{i_d}$ are linearly dependent is positive,

$$\max[\det(u_{i_1},\ldots,u_{i_d})]^2 > \frac{d!}{d^d}.$$

Our goal is to quantify this inequality by bounding from below the probability that the determinant is 0. Let

$$M^2 := \max[\det(u_{i_1},\ldots,u_{i_d})]^2.$$

Note that if an element of $\{u_1,\ldots,u_m\}$ is selected at least twice, then $\det(u_{i_1},\ldots,u_{i_d}) = 0$. Thus,

$$\mathbb{E}\left([\det(u_{i_1},\ldots,u_{i_d})]^2\right) \le M^2P_1,$$

where P_1 denotes the probability that all indices are pairwise distinct. Therefore,

$$M^2 \ge \frac{d!}{d^d}\cdot\frac{1}{P_1}.$$

Note that P_1 is a degree d elementary symmetric function of the variables $p_1,\ldots,p_m$. Furthermore, $p_1+\ldots+p_m = 1$ and $p_i \ge 0$ for all $i = 1,\ldots,m$. It can easily be seen (using Lagrange multipliers, or by induction on m) that for fixed m and d, the maximum of P_1 is attained when $p_1 = \ldots = p_m = 1/m$. Thus,

$$P_1 \le d!\binom{m}{d}\frac{1}{m^d}.$$

In summary,

$$M^2 \ge \frac{d!}{d^d}\cdot\frac{m^d}{d!}\binom{m}{d}^{-1}.$$

First, we note that $\gamma(d,m) := \frac{m^d}{d!}\binom{m}{d}^{-1}$ is decreasing in m. Thus, by (9), we may assume that m is as large as possible, that is, $m = \frac{d(d+1)}{2}$ proving the first part of Theorem 2.

3.1. **Proof of Theorem 2 (i).** Let $\gamma(d) := \gamma\left(d, d(d+1)/2\right)$. We show that $\gamma(d)$ is increasing in d.

With the notation $m := d(d+1)/2$, we note that $(d+1)(d+2)/2 = m+d+1$. Thus,

$$\frac{\gamma(d+1)}{\gamma(d)} = \frac{(m+d+1)^{d+1} m \cdots (m-d+1)}{m^d (m+d+1)\cdots(m+1)} = \frac{(m+d+1)^d}{m^d}\cdot\frac{m\cdots(m-d+1)}{(m+d)\cdots(m+1)}$$

Thus, we need to show that

$$1+\frac{d+1}{m} > \sqrt[d]{\left(1+\frac{d}{m}\right)\left(1+\frac{d}{m-1}\right)\cdots\left(1+\frac{d}{m-d+1}\right)},$$

which, by the AM/GM inequality follows, if

$$1+\frac{d+1}{m} \geq 1+d\frac{\frac{1}{m}+\frac{1}{m-1}+\ldots+\frac{1}{m-d+1}}{d},$$

which is equivalent to

$$\frac{d}{m} \geq \frac{1}{m-1}+\frac{1}{m-2}+\ldots+\frac{1}{m-d+1}.$$

For this to hold, it is sufficient to show that for every integer or half of an integer $1 \leq i \leq d/2$, we have that

$$\frac{2d}{(d-1)m} \geq \frac{1}{m-i}+\frac{1}{m-d+i}. \tag{14}$$

After substituting $m = d(d+1)/2$, it is easy to see that (14) holds.
Finally, $\lim_{d\to\infty}\gamma(d) = e$ follows from Stirling's formula.

3.2. **Proof of Theorem 2 (ii) and (iii).** Stirling's formula yields both claims.

3.3. **Proof of Proposition 1.** Let X denote the random variable $X := [\det(x_1,\ldots,x_d)]^2$, $E := \mathbb{E}(X) = \frac{d!}{d^d}$, and $q := \mathbb{P}\left(X \geq \frac{\lambda E}{P_1}\right)$, where, as in the proof of Theorem 2, $P_1 := \mathbb{P}(x_1,\ldots,x_d$ are pairwise distinct).

In the proof of Theorem 2, we established

$$P_1 \leq (\gamma(d,\overline{m}))^{-1}, \text{ and thus, } q \leq \mathbb{P}\left([\det(x_1,\ldots,x_d)]^2 \geq \lambda\gamma(d,\overline{m})\cdot\frac{d!}{d^d}\right). \tag{15}$$

Using the fact that X is at most one, we have

$$E \leq \frac{\lambda E}{P_1}\mathbb{P}\left(X < \frac{\lambda E}{P_1} \text{ and } x_1,\ldots,x_d \text{ are pairwise distinct}\right) + \mathbb{P}\left(X \geq \frac{\lambda E}{P_1}\right).$$

That is, $E \leq \frac{\lambda E}{P_1}(P_1 - q) + q$, and thus, by (15)

$$q \geq \frac{(1-\lambda)E}{1-\frac{\lambda E}{P_1}} \geq \frac{(1-\lambda)d!}{d^d - \lambda\gamma(d,\overline{m})d!} \geq (1-\lambda)e^{-d},$$

completing the proof of Proposition 1.

4. Proof of Theorem 3

First, observe that (4) holds with some positive scalars c_i, if and only if, the matrix Id_d/d is in the convex hull of the set $\mathcal{A} = \{v_i \otimes v_i :\ i = 1,\ldots,m\}$ in the real vector space of $d\times d$ matrices. The set $\mathcal{A}$ is contained in the subspace of symmetric matrices with trace 1, which is of dimension $d(d+1)/2-1$. Carathéodory's theorem [Sch14, Theorem 1.1.4] now yields the desired upper bound on m.

In the case when both (4) and (5) are assumed, we lift our vectors into $\mathbb{R}^{d+1}$ as follows. Let $\hat{v}_i = \sqrt{\frac{d}{d+1}}(v_i, 1/\sqrt{d}) \in \mathbb{R}^{d+1}$. It is easy to check that $|\hat{v}_i| = 1$, and that (4) holds for the vectors $\hat{v}_i$ with some positive scalars $\hat{c}_i$ if, and only if, (4) and (5) hold for the vectors v_i with scalars $c_i = \frac{d}{d+1}\hat{c}_i$. Now, $\hat{v}_i \otimes \hat{v}_i$, $i = 1,\ldots,m$ are symmetric $(d+1)\times(d+1)$ matrices of trace one, and their $(d+1,d+1)$th entry is $1/(d+1)$.

The dimension of this subspace of $\mathbb{R}^{(d+1)\times(d+1)}$ is $d(d+3)/2-1$, thus, again, by Carathéodory's theorem, the proof is complete.

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