FUZZY REASONING MODELS AND FUZZY TRUTH VALUE BASED INFERENCE

Zsolt Gera

Doctoral School in Mathematics and Computer Science
Department of Computer Algorithms and Artificial Intelligence
University of Szeged
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Introduction

During the last decades fuzzy sets and fuzzy logic have been interesting fields for research with increasing popularity linking computer science, mathematics, and engineering. Today, type-2 fuzzy sets are a hot topic, partly due to the increasing computational capacity of computers. However, there is still not a single, superior fuzzy logic or fuzzy reasoning method, although there are numerous competing theories.

The main contribution of the thesis is that it approaches fuzzy reasoning from three different angles. First, by creating a new, hybrid fuzzy rule-learning model with classical inference methods. Second, by introducing a new reasoning method, with intuitive and practical properties. And third, by supervising the so-called fuzzy truth value-based reasoning model, and showing new ways to represent and calculate with fuzzy operations such as conjunctions and implications.

During my research I was guided by pragmatism. Even the more theoretical results were created with usability in mind. This view is reflected in every chapter: let it be a simplification of a complex formula, or a new, simple method of reasoning.

The thesis is organized as follows. After a brief overview of fuzzy sets and operators in Chapter 1, Chapter 2 introduces the squashing function, a parameterized family of monotone functions. It approximates the so-called cut function, which appears in piecewise-linear membership functions and the Łukasiewicz operator family. The purpose of the approximation is to have a function with a continuous gradient. The results of Chapter 2 has been published in [31, 32].

Chapter 3 shows the application of squashing functions. A hybrid, genetic algorithm and gradient based local optimization framework is introduced, to extract fuzzy rules from input and output sample data. The membership functions of these rules are compound squashing functions with a continuous gradient. The results of Chapter 3 has been published in [33].

Chapter 4 investigates certain important facets of fuzzy reasoning. Traditionally, the convex membership functions of fuzzy sets are decomposed into left- and right-hand sides and the calculations are done on the two sides separately. In this view, it is sufficient to use monotonic functions in the inference schemes. This Chapter shows the classical Compositional Rule of Inference with a sigmoid-like function, especially the squashing function. Based on the conclusions, a new method, the Membership Driven Inference reasoning scheme is introduced and its efficient computation is shown. The results of Chapter 4 has been published in [45].

The last two chapters deal with type-2 fuzzy sets. Chapter 5 shows techniques to reduce the computational complexity of type-2 logical operations by choosing appropriate membership functions and operators. The results of Chapter 5 has been published in [46].

Chapter 6 investigates type-2 fuzzy implication operators, which are essential for type-2 fuzzy inference systems. The results of Chapter 6 has been published in [47].

Acknowledgements

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Chapter 1

Introduction to fuzzy sets and operators

The concept of fuzzy sets was introduced by Zadeh [96] in 1965 as a generalization of classical sets. While in classical set theory an element is either a member of a set or not, fuzzy sets allow graded memberships of elements. Zadeh generalized the $\{0, 1\}$-valued (or, in other words crisp) characteristic function of classical sets to the unit interval. This way, objects may belong to a set to any degree between 0 and 1. The framework of fuzzy sets fully contains the framework of crisp sets. As a consequence of the generalization, fuzzy sets only possess a part of crisp set properties.

The theory of fuzzy sets have always had a dual view. On the one hand, it can be regarded as a generalization of classical set theory. On the other hand fuzzy theory can be interpreted as a multiple-valued logic, with generalized logical values and operations. As the thesis focuses on fuzzy logic and reasoning, it adopts the latter view.

Since the introduction of fuzzy sets, many extensions and generalizations have appeared. One such generalization is type-2 fuzzy sets. Type-2 fuzzy sets extend the concept of classical (type-1) fuzzy sets in the following way. While the latter assign a specific membership degree to each element on its domain, a type-2 fuzzy set has fuzzy membership degrees. These are fuzzy subsets of the unit interval, i.e. fuzzy truth values. This way, the imprecision of membership values can be incorporated into the type-2 fuzzy framework. A very important special class of type-2 fuzzy sets are interval-valued fuzzy sets, where the membership values of a fuzzy set are intervals.

Type-2 fuzzy sets became a very active field of research in the past years. As type-1 fuzzy systems and theories have been extended to type-2 ones, it became clear that the added complexity in computation is worth it.

1.1 Basic Notions

Let the unit interval be denoted by $\mathcal{I}$. Fuzzy sets are functions $U \to \mathcal{I}$, where U is the universe of discourse. The characteristic function of a fuzzy set is called its membership function, which uniquely represents the fuzzy set.

The most elementary operations of classical logic are negation, conjunction, disjunction and implication. Fuzzy logical operations are the extensions of these two-valued operations to the unit interval.

Definition 1.1. *A negation is an order reversing automorphism of* $\mathcal{I}$*.*

Note, that this definition implies that the negation of 0 is 1 and vice versa, i.e. compatibility with crisp logic is preserved. A more restrictive class of negations is generally used.

Definition 1.2. *A strong negation (denoted by* $'$*) is an involutive order reversing automorphism of* $\mathcal{I}$*.*

Involutivness means that for any $x \in \mathcal{I}$ $x'' = x$, i.e. an involutive function applied twice gives the identity mapping. The simplest and most widely used negation is $x' = 1 - x$. The thesis, if not stated otherwise, deals only with strong negations.

The fuzzy conjunction and disjunction operations are represented by triangular norms (t-norms for short) and triangular conorms (t-conorms for short). The term t-norm was introduced by Menger [63] in the theory of probabilistic metric spaces. Later, Trillas and Höhle were the first to suggest the use of general t-norms and t-conorms for the intersection and union of fuzzy sets.

Definition 1.3. *A t-norm is a binary operation* $\triangle : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ *that is commutative, associative, increasing in each variable, and has unit element* 1*.*

Definition 1.4. *A t-conorm is a binary operation* $\bigtriangledown : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ *that is commutative, associative, increasing in each variable, and has unit element* 0*.*

A fuzzy implication is a generalization of classical implication operations. Its definition retains only the most necessary properties of implications.

Definition 1.5. *Fuzzy implications are two-place functions* $\rhd : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ *which fulfill the boundary conditions according to the boolean implication, and antitone in the first and monotone in the second argument.*

Fuzzy coimplications $\lhd$ are dual to fuzzy implications by a strong negation according to

$$x \lhd y = \left(x' \rhd y'\right)'.$$

The definition of fuzzy implication is very unrestrictive. By considering additional conditions, the class of fuzzy implication can be further restricted. Such conditions are derived from similar boolean identities. According to [35], the following conditions are the most important ones (see [85, 44, 92, 13]):

1. $1 \rhd x = x$.
2. $x \rhd (y \rhd z) = y \rhd (x \rhd z)$.
3. $x \rhd y$ if and only if $x \le y$.
4. $x \rhd 0$ is a strong negation.
5. $x \rhd y \ge y$.
6. $x \rhd x = 1$.
7. $x \rhd y = y' \rhd x'$ for a strong negation $'$.

8. $\rhd$ is a continuous function.

Residuation is a fundamental concept in multiple-valued logic. In fuzzy logic, residuals of t-norms (and t-conorms) are implications (coimplications).

Definition 1.6. *A fuzzy implication $\rhd$ is the residual implication of a t-norm $\triangle$ if*

$$x \rhd y = \bigvee_{(x \,\triangle\, z) \leq y} z.$$

Definition 1.7. *A fuzzy coimplication $\lhd$ is the residual coimplication of a t-conorm $\triangledown$ if*

$$x \lhd y = \bigwedge_{(x \,\triangledown\, z) \geq y} z.$$

1.2 The Representation of Fuzzy Operators

The representation theorem of Trillas [84] states that any strong negation $'$ can be expressed as

$$x' = \theta^{-1}\left(1 - \theta(x)\right), \tag{1.1}$$

where θ is an automorphism of the unit interval. It is called the generator function of the negation. An alternative representation theorem of strong negations was first proved by Dombi [29].

Theorem 1.8. *Let $' : \mathcal{I} \to \mathcal{I}$ be a continuous function. It is a strong negation if and only if there exists a continuous and strictly monotone function $\varphi : \mathcal{I} \to [-\infty, \infty]$ with $\varphi(\nu) = 0$, $\nu \in]0,1[$ such that for all $x \in \mathcal{I}$*

$$x' = \varphi^{-1}(-\varphi(x)). \tag{1.2}$$

Here ν is called the fixed point of the negation, i.e. for which $\nu' = \nu$.

Theorem 1.8 states that all strong negations can be expressed as $\varphi^{-1}(-\varphi(x))$ with a suitable φ generator function. In this formula the negation's fixed point is implicitly hidden in the generator function. The next representation theorem of strong negations explicitly contains its fixed point i.e. neutral value [29].

Theorem 1.9. *Let $' : \mathcal{I} \to \mathcal{I}$ be a continuous function, then the following are equivalent:*

- *The function $'$ is a strong negation with fixed point ν.*
- *There exists a continuous and strictly monotone function $\varphi : \mathcal{I} \to [-\infty, \infty]$ and $\nu \in]0,1[$ such that for all $x \in \mathcal{I}$*

$$x' = \varphi^{-1}(2\varphi(\nu) - \varphi(x)). \tag{1.3}$$

The works of Abel [1], Aczél [3, 4, 5] and Ling [57] on associative functions and abstract semigroups established the fundamental results on the generator functional representation of certain t-norms and t-conorms. The key to characterize such t-norms and t-conorms is the Archimedean property.

Definition 1.10. *A t-norm $\triangle$ (resp. a t-conorm $\bigtriangledown$) is Archimedean if $x\triangle^{(n)}x \to 0$ (resp. $x\bigtriangledown^{(n)}x \to 1$) as $n \to \infty$, where $x\triangle^{(n)}x = x\triangle \dots \triangle x$ n times.*

Theorem 1.11. *A t-norm $\triangle$ is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function $\varphi : \mathcal{I} \to [0, \infty]$, with $\varphi(1) = 0$ such that*

$$x\triangle y = \varphi^{(-1)} \left(\varphi(x) + \varphi(y)\right), \tag{1.4}$$

where $\varphi^{(-1)}$ is the pseudoinverse of f defined by

$$\varphi^{(-1)}(x) = \begin{cases} \varphi^{-1}(x), & if x \leq \varphi(0) \\ 0, & otherwise \end{cases} \tag{1.5}$$

The class of continuous and Archimedean operators can be further divided into nilpotent and strict operators.

Definition 1.12. *A continuous t-norm $\triangle$ (resp. t-conorm $\bigtriangledown$) is nilpotent if $\exists x, y \in]0, 1[$ such that $x\triangle y = 0$ (resp. $x\bigtriangledown y = 1$).*

A t-norm or a t-conorm is strict if it is continuous and strictly increasing on $]0, 1[^2$.

Note, that the two notions are exclusive. For example, suppose $\exists x_0, y_0$ such that $x_0\triangle y_0 = 0$, then it also holds for all $x < x_0$ and $y < y_0$ because of the monotonicity of t-norms.

There are three representative t-norms: the minimum $x\triangle_M y = x \wedge y$, the product $x\triangle_P y = xy$, and Łukasiewicz t-norm $x\triangle_W y = (x + y - 1) \vee 0$. All strict t-norms are isomorphic to the product [75], and all nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [70].

Analogously, the three representative t-conorms are the maximum ($x\bigtriangledown_M y = x \vee y$), the algebraic sum ($x\bigtriangledown_P y = x + y - xy$) and Łukasiewicz t-conorm ($x\bigtriangledown_W y = (x + y) \wedge 1$).

Residual fuzzy implications can be represented by

$$x \triangleright y = \varphi^{-1} \left((\varphi(y) - \varphi(x)) \vee 0 \right), \tag{1.6}$$

where φ is the generator function of an Archimedean t-norm.

Chapter 2

The Squashing function

The construction and the interpretation of fuzzy membership functions have always been a crucial question of fuzzy set theory. Bilgic and Türksen gave a comprehensive overview of the most relevant interpretations in [35]. For the construction of membership functions Dombi [30] had an axiomatic point of view, Civanlar and Trussel [24] used statistical data, Bagis [9], Denna et al. [27], Karaboga [54] applied tabu search. However, most fuzzy applications use piecewise linear membership functions because of their easy handling, for example in embedded fuzzy control applications where the limited computational resources does not allow the use of complicated membership functions. In other areas where the model parameters are learned by a gradient based optimization method, they can not be used because the lack of continuous derivatives. For example to fine tune a fuzzy control system by a simple gradient based technique it is required that the membership functions are differentiable for every input. There are numerous papers dealing with the concept of fuzzy set approximation and membership function differentiability (see for example [16], [49], [73]).

The Łukasiewicz (or nilpotent) operator class (see e.g. [2, 50, 23]) is commonly used for various purposes (see e.g. [21, 22]). In the formulation of this well known operator family the cut function (denoted by $[\cdot]$) plays an important role. We can get the cut function from x by taking the maximum of 0 and x and then taking the minimum of the result and 1.

Definition 2.1. *Let the cut function be*

$$[x] = \min(\max(0, x), 1) = \begin{cases} 0, & \text{if} \quad x \le 0 \\ x, & \text{if} \quad 0 < x < 1 \\ 1, & \text{if} \quad 1 \le x \end{cases}$$

Let the generalized cut function be

$$[x]_{a,b} = [(x-a)/(b-a)] = \begin{cases} 0, & \text{if} \quad x \le a \\ \frac{x-a}{b-a}, & \text{if} \quad a < x < b \\ 1, & \text{if} \quad b \le x \end{cases}$$

where $a, b \in \mathbb{R}$ and $a < b$.

In neural networks terminology this cut function is called saturating linear transfer function. All nilpotent operators are constructed using the cut function. The formulas of the nilpotent conjunction, disjunction, implication and negation are the following:

$$x \triangle_W y = [x + y - 1], \tag{2.1}$$

$$x \bigtriangledown_W y = [x + y], \tag{2.2}$$

$$x \triangleright_W y = [1 - x + y], \tag{2.3}$$

$$x' = 1 - x, \tag{2.4}$$

where $x, y \in \mathcal{I}$. The truth tables of the former three can be seen on Fig. 2.1.

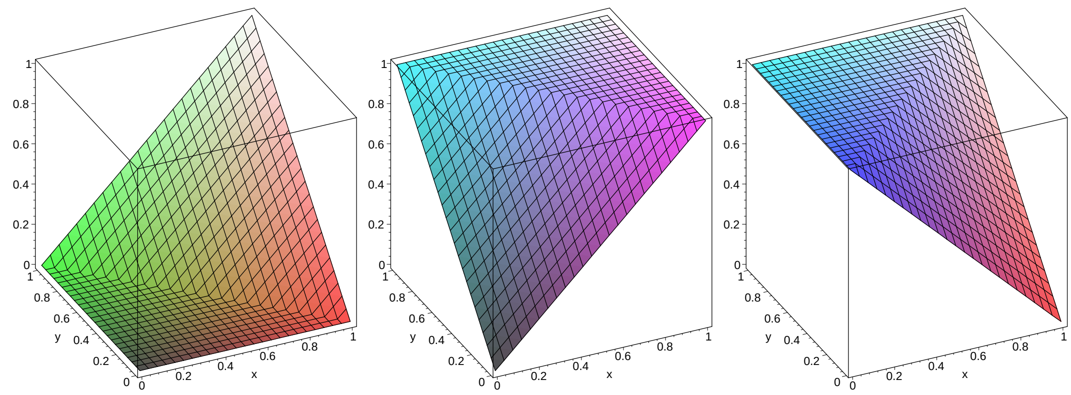


Figure 2.1: The truth tables of the nilpotent conjunction, disjunction and implication

We will refer to triangular and trapezoidal membership functions as piecewise linear membership functions. They are very common in fuzzy control because of their easy handling. The generalized cut function can be used to describe piecewise linear membership functions. Generally a trapezoidal membership function can be constructed as the conjunction of two generalized cut functions as

$$[x]_{a,b} \triangle_W ([x]_{c,d})' = [[x]_{a,b} + 1 - [x]_{c,d} - 1] \tag{2.5}$$

$$= [[x]_{a,b} - [x]_{c,d}], \tag{2.6}$$

where a, b, c, d are real numbers and $a < b \leq c < d$. As a special case, if $b = c$ then we get a triangular membership function. For an example of the general case see Fig. 2.2.

The Łukasiewicz operator family has good theoretical properties. These are for example the law of non-contradiction (that is the conjunction of a variable and its negation is always zero) and the law of excluded middle (that is the disjunction of a variable and its negation is always one) both hold, and the residual and material implications coincide. These properties make these operators widely used in fuzzy logic and the closest one to Boolean logic. Besides these good theoretical properties this operator family does not have a continuous gradient. So for example classical gradient based optimization techniques are impossible with Łukasiewicz operators. The root of this problem is the shape of the cut function itself.

A solution to above mentioned problem is a continuously differentiable approximation of the cut function, which can be seen on Fig. 2.3. In this chapter we'll construct such an approximating function by means of sigmoid functions. The reason for choosing the sigmoid function was that this function has a very important role in many areas. It is

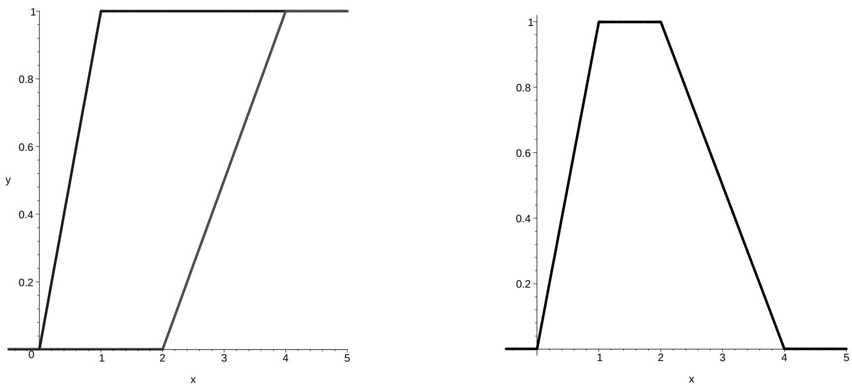


Figure 2.2: On the left there are two generalized cut functions. On the right: a trapezoidal membership function constructed as the conjunction of the former two, with a negation applied to the right one. Its parameters are: $a = 0, b = 1, c = 2, d = 4$

frequently used in artificial neural networks ([19]), optimization methods, economical and biological models ([56]).

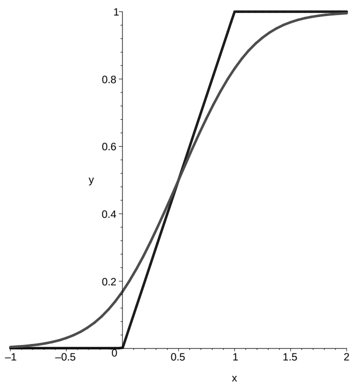


Figure 2.3: The cut function and its approximation

2.1 The sigmoid function

The sigmoid function (see Fig. 2.4) is defined as

$$\sigma_d^{(\beta)}(x) = \frac{1}{1 + e^{-\beta(x-d)}} \tag{2.7}$$

where the lower index d is omitted if it is 0.

Let us examine some of its properties which will be useful later:

- its derivative can be expressed by itself (see Fig. 2.5):

$$\frac{\partial \sigma_d^{(\beta)}(x)}{\partial x} = \beta \sigma_d^{(\beta)}(x) \left(1 - \sigma_d^{(\beta)}(x)\right) \tag{2.8}$$

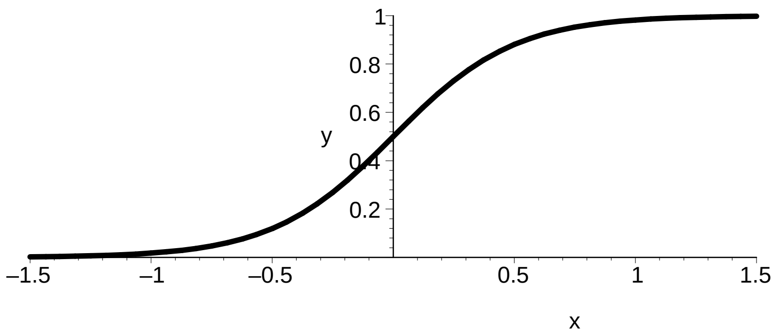


Figure 2.4: The sigmoid function, with parameters $d = 0$ and $\beta = 4$

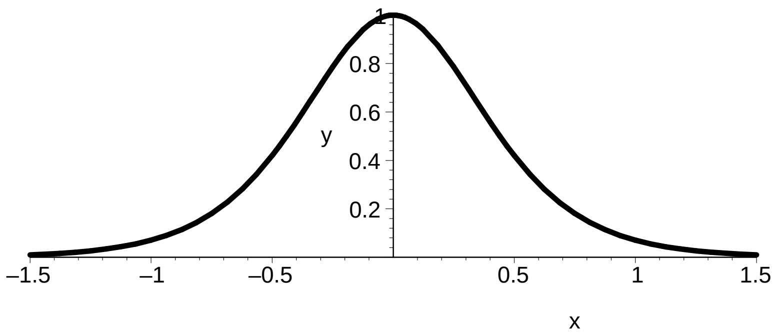


Figure 2.5: The first derivative of the sigmoid function

- its integral has the following form:

$$\int \sigma_d^{(\beta)}(x)\, dx = -\frac{1}{\beta} \ln\left(\sigma_d^{(-\beta)}(x)\right) \tag{2.9}$$

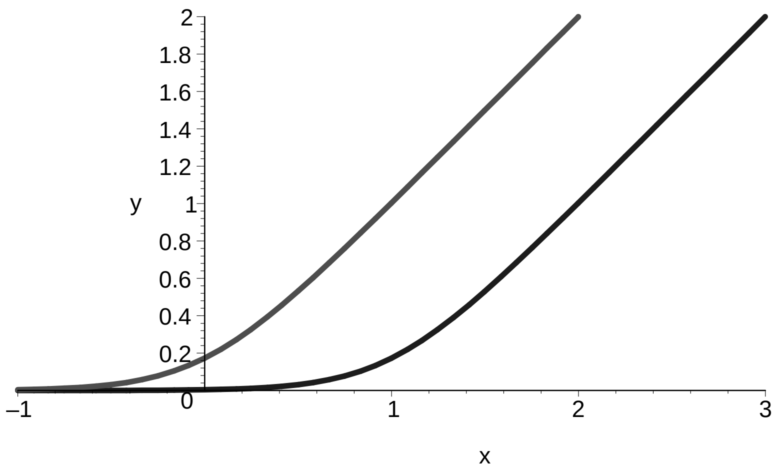


Figure 2.6: The integral of the sigmoid function, one is shifted by 1

Because the sigmoid function is asymptotically 1 as x tends to infinity, the integral of the sigmoid function is asymptotically x (see Fig. 2.6).

2.2 The squashing function on an interval

In order to get an approximation of the generalized cut function, let us integrate the difference of two sigmoid functions, which are translated by a and b ($a < b$), respectively.

$$\begin{aligned}\frac{1}{b-a} \int \left(\sigma_a^{(\beta)}(x) - \sigma_b^{(\beta)}(x)\right)\, dx = \\ &= \frac{1}{b-a}\left(\int \sigma_a^{(\beta)}(x)\, dx - \int \sigma_b^{(\beta)}(x)\, dx\right) = \\ &= \frac{1}{b-a}\left(-\frac{1}{\beta}\ln\left(\sigma_a^{(-\beta)}(x)\right) + \frac{1}{\beta}\ln\left(\sigma_b^{(-\beta)}(x)\right)\right)\end{aligned} \tag{2.10}$$

After simplification we get the interval $[a, b]$ squashing function:

Definition 2.2. *Let the interval* $[a, b]$ *squashing function be*

$$\begin{aligned}S_{a,b}^{(\beta)}(x) &= \frac{1}{b-a}\ln\left(\frac{\sigma_b^{(-\beta)}(x)}{\sigma_a^{(-\beta)}(x)}\right)^{1/\beta} \\ &= \frac{1}{b-a}\ln\left(\frac{1+e^{\beta(x-a)}}{1+e^{\beta(x-b)}}\right)^{1/\beta}.\end{aligned}$$

The parameters a and b affect the placement of the interval squashing function, while the β parameter determines the precision of the approximation. We prove that $S_{a,b}^{(\beta)}(x)$ is indeed an approximation of the generalized cut function.

Theorem 2.3. *Let* $a, b \in \mathbb{R}$, $a < b$ *and* $\beta \in \mathbb{R}^+$. *Then*

$$\lim_{\beta\to\infty} S_{a,b}^{(\beta)}(x) = [x]_{a,b}$$

and $S_{a,b}^{(\beta)}(x)$ *is continuous in* x, a, b *and* β.

Proof. It is easy to see the continuity, because $S_{a,b}^{(\beta)}(x)$ is a simple composition of continuous functions and because the sigmoid function has a range of $[0, 1]$ the quotient is always positive.

In proving the limit we separate three cases, depending on the relation between a, b and x.

- Case 1 ($x < a < b$): Since $\beta(x-a) < 0$, so $e^{\beta(x-a)} \to 0$ and similarly $e^{\beta(x-b)} \to 0$. Hence the quotient converges to 1 if $\beta \to \infty$, and the logarithm of one is zero.

- Case 2 ($a \leq x \leq b$):

$$\begin{aligned}
&\frac{1}{b-a}\ln\left(\lim_{\beta\to\infty}\left(\frac{1+e^{\beta(x-a)}}{1+e^{\beta(x-b)}}\right)^{1/\beta}\right) = \\
&= \frac{1}{b-a}\ln\left(\lim_{\beta\to\infty}\left(\frac{e^{\beta(x-a)}\left(e^{-\beta(x-a)}+1\right)}{\left(1+e^{\beta(x-b)}\right)}\right)^{1/\beta}\right) = \\
&= \frac{1}{b-a}\ln\left(\lim_{\beta\to\infty}\frac{e^{x-a}\left(e^{-\beta(x-a)}+1\right)^{1/\beta}}{\left(1+e^{\beta(x-b)}\right)^{1/\beta}}\right) = \\
&= \frac{1}{b-a}\ln\left(e^{x-a}\lim_{\beta\to\infty}\frac{\left(e^{-\beta(x-a)}+1\right)^{1/\beta}}{\left(1+e^{\beta(x-b)}\right)^{1/\beta}}\right)
\end{aligned}$$

  We transform the nominator so that we can take the e^{x-a} out of the limes. In the nominator $e^{-\beta(x-a)}$ remained which converges to 0 as well as $e^{\beta(x-b)}$ in the denominator so the quotient converges to 1 if $\beta \to \infty$. So as the result, the limit of the interval squashing function is $(x-a)/(b-a)$, which by definition equals to the generalized cut function in this case.

- Case 3 ($a < b < x$):

$$\begin{aligned}
&\frac{1}{b-a}\ln\left(\lim_{\beta\to\infty}\left(\frac{1+e^{\beta(x-a)}}{1+e^{\beta(x-b)}}\right)^{1/\beta}\right) = \\
&= \frac{1}{b-a}\ln\left(\lim_{\beta\to\infty}\left(\frac{e^{\beta(x-a)}\left(e^{-\beta(x-a)}+1\right)}{e^{\beta(x-b)}\left(e^{-\beta(x-b)}+1\right)}\right)^{1/\beta}\right) = \\
&= \frac{1}{b-a}\ln\left(\lim_{\beta\to\infty}\frac{e^{x-a}\left(e^{-\beta(x-a)}+1\right)^{1/\beta}}{e^{x-b}\left(e^{-\beta(x-b)}+1\right)^{1/\beta}}\right) = \\
&= \frac{1}{b-a}\ln\left(\frac{e^{x-a}}{e^{x-b}}\lim_{\beta\to\infty}\frac{\left(e^{-\beta(x-a)}+1\right)^{1/\beta}}{\left(e^{-\beta(x-b)}+1\right)^{1/\beta}}\right)
\end{aligned}$$

  We do the same transformations as in the previous case but we take e^{x-b} from the denominator, too. After these transformations the remaining quotient converges to 1, so

$$\begin{aligned}
\lim_{\beta\to\infty} S_{a,b}^{(\beta)}(x) &= \frac{1}{b-a}\ln\left(\frac{e^{x-a}}{e^{x-b}}\right) \\
&= \frac{1}{b-a}\ln\left(e^{x-a-(x-b)}\right) \\
&= \frac{1}{b-a}\ln\left(e^{b-a}\right) = \frac{b-a}{b-a} = 1.
\end{aligned}$$

$\blacksquare$

On Fig. 2.7 the interval squashing function can be seen with various β parameters. The following proposition states some properties of the interval squashing function.

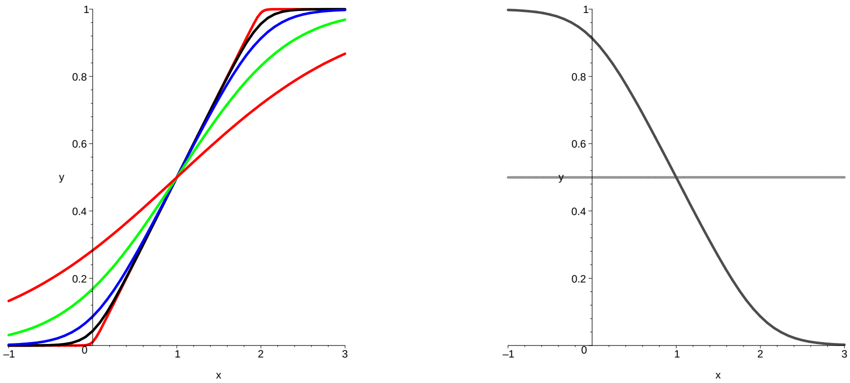


Figure 2.7: On the left: the interval squashing function with an increasing β parameter ($a = 0$ and $b = 2$). On the right: the interval squashing function with a zero and a negative β parameter

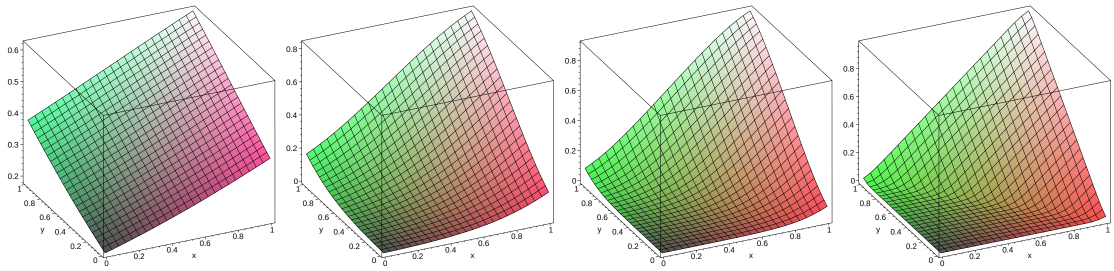


Figure 2.8: The approximation of the nilpotent conjunction with β values 1,4,8 and 32

Proposition 2.4.

$$\lim_{\beta \to 0} S_{a,b}^{(\beta)}(x) = 1/2$$

$$S_{a,b}^{(-\beta)}(x) = 1 - S_{a,b}^{(\beta)}$$

As an another example, the nilpotent conjunction is approximated with the interval squashing function on Fig. 2.8.

For further use, let us introduce an another form of the interval squashing function's formula. Instead of using parameters a and b which were the "bounds" on the x axis, from now on we'll use a and δ, where a gives the center of the squashing function and where δ gives its steepness. Together with the new formula we introduce its pliant notation.

Definition 2.5. *Let the squashing function be*

$$\langle a <_\delta x \rangle_\beta = S_{a,\delta}^{(\beta)}(x) = \frac{1}{2\delta} \ln \left(\frac{\sigma_{a+\delta}^{(-\beta)}(x)}{\sigma_{a-\delta}^{(-\beta)}(x)} \right)^{1/\beta},$$

where $a \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$.

If the a and δ parameters are both $1/2$ we will use the following notation for simplicity:

$$\langle x \rangle_\beta = S_{\frac{1}{2},\frac{1}{2}}^{(\beta)}(x),$$

which is the approximation of the cut function.

The inequality relation in this notation refers to the fact that the squashing function can be interpreted as the truthness of the relation $a < x$ with decision level $1/2$, according to a fuzziness parameter δ and an approximation parameter β (see Fig. 2.9).

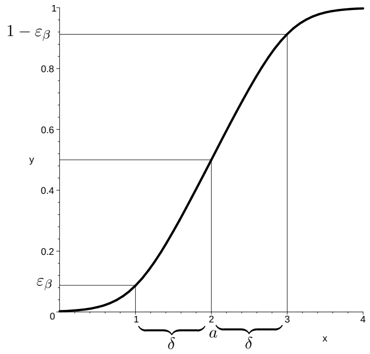


Figure 2.9: The meaning of $\langle a <_\delta x\rangle_\beta$

The derivatives of the squashing function are continuous and can be expressed by itself and sigmoid functions:

$$\frac{\partial S^{(\beta)}_{a,\delta}(x)}{\partial x} = \frac{1}{2\delta}\left(\sigma^{(\beta)}_{a-\delta}(x) - \sigma^{(\beta)}_{a+\delta}(x)\right) \tag{2.11}$$

$$\frac{\partial S^{(\beta)}_{a,\delta}(x)}{\partial a} = \frac{1}{2\delta}\left(\sigma^{(\beta)}_{a+\delta}(x) - \sigma^{(\beta)}_{a-\delta}(x)\right) \tag{2.12}$$

$$\frac{\partial S^{(\beta)}_{a,\delta}(x)}{\partial \delta} = \frac{1}{2\delta}\left(\sigma^{(\beta)}_{a+\delta}(x) + \sigma^{(\beta)}_{a-\delta}(x)\right) - \frac{1}{\delta}S^{(\beta)}_{a,\delta}(x) \tag{2.13}$$

2.3 The error of the approximation

The squashing function approximates the cut function with an error. This error can be defined in many ways. We have chosen the following definition.

Definition 2.6. *Let the approximation error of the squashing function be*

$$\varepsilon_\beta = \langle 0 <_\delta (-\delta)\rangle_\beta = \frac{1}{2\delta}\ln\left(\frac{\sigma^{(-\beta)}_{\delta}(-\delta)}{\sigma^{(-\beta)}_{-\delta}(-\delta)}\right)^{1/\beta}$$

where $\beta > 0$.

Because of the symmetry of the squashing function $\varepsilon_\beta = 1 - \langle 0 <_\delta \delta\rangle_\beta$, see Fig. 2.9.

The purpose of measuring the approximation error is the following inverse problem: we want to get the corresponding β parameter for a desired ε_β error. We state the following lemma on the relationship between ε_β and β.

Lemma 2.7. *Let us fix the value of δ. The following holds for ε_β:*

$$\varepsilon_\beta < c\cdot\frac{1}{\beta},$$

where $c = \frac{\ln 2}{2\delta}$ is constant.

Proof.

$$\varepsilon_\beta = \frac{1}{2\delta\beta} \ln \left(\frac{1 + e^{\beta(-\delta+\delta)}}{1 + e^{\beta(-\delta-\delta)}} \right) = \frac{1}{2\delta\beta} \ln \left(\frac{2}{1 + e^{-2\delta\beta}} \right) =$$
$$= \frac{\ln 2}{2\delta\beta} - \frac{\ln(1 + e^{-2\delta\beta})}{2\delta\beta} < c \cdot \frac{1}{\beta}$$

■

So the error of the approximation can be upper bounded by $c \cdot \frac{1}{\beta}$, which means that by increasing parameter β, the error decreases by the same order of magnitude.

2.4 Approximation of piecewise linear membership functions

In fuzzy theory triangular and trapezoidal membership functions play an important role. For example fuzzy control uses mainly this type of membership functions because of their easy handling. They are piecewise linear, hence they can not be continuously differentiated. The main motivation was to construct an approximation which has the same properties in the limit as the approximated membership function and has a continuous gradient. If we are using approximated piecewise linear membership functions in fuzzy control systems then they can be tuned by a gradient based optimization method and we can get the optimal parameters of the membership functions.

Piecewise linear membership functions can be constructed from generalized cut functions, and thus approximated by using squashing functions with a suitable conjunction operator. We have chosen the Łukasiewicz conjunction. The formula of conjunction also uses the squashing function in place of the cut function. This way, the membership function and the operator are both constructed from the same component.

To describe a trapezoidal membership function using the conjunction operator and two squashing functions four parameters are required, namely a_1, δ_1 and a_2, δ_2, where a_1 and a_2 give the positions of its left and right sides, and δ_1 and δ_2 give its left and right slopes. The two β parameters of the squashing functions have to have opposite signs to form a trapezoid or triangle, and of course the equations $a_1 < a_2$ and $a_1 + \delta_1 \leq a_2 - \delta_2$ must hold.

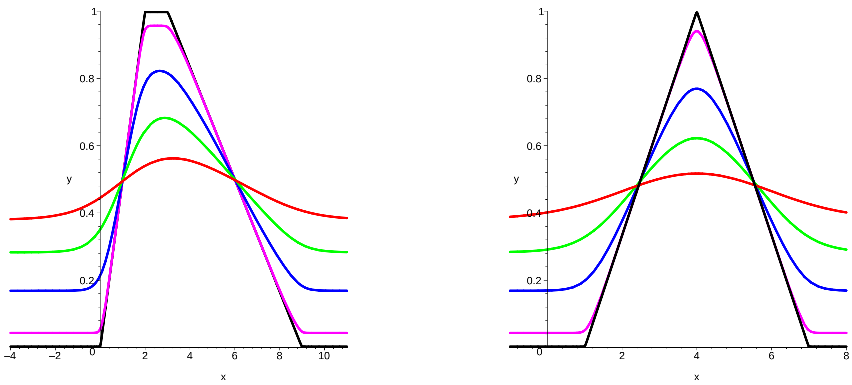


Figure 2.10: The approximation of a trapezoid and a triangular membership function

The approximation of a trapezoid membership function is the following (see Fig. 2.10):

$$S^{(\beta)}_{\frac{1}{2},\frac{1}{2}}\left(S^{(\beta)}_{a_1,\delta_1}(x) + S^{(-\beta)}_{a_2,\delta_2}(x) - 1\right). \tag{2.14}$$

with pliant notation:

$$\langle\langle a_1 <_{\delta_1} x\rangle_\beta + \langle a_2 <_{\delta_2} x\rangle_{(-\beta)} - 1\rangle_\beta. \tag{2.15}$$

As a special case of the trapezoid membership function we get the triangular membership function. To describe one, only two parameters are needed, the center a, and its fuzziness δ (see Fig. 2.10).

Definition 2.8. *Let the approximation of the triangular membership function be defined as (in pliant notation)*

$$\begin{aligned}\langle x \sim_\delta a\rangle_\beta = \langle\langle(a-\delta/2) <_{\delta/2} x\rangle_\beta + \\ + \langle(a+\delta/2) <_{\delta/2} x\rangle_{-\beta} - 1\rangle_\beta.\end{aligned}$$

where a is its center and δ is its fuzziness.

This way an approximation of a trapezoidal or triangular fuzzy number can be represented by a pair of squashing functions. This approximation eliminates piecewise linearity, but can be continuously differentiated, has good analytical properties, for example simple derivatives, fast convergence and low calculation overhead.

Chapter 3

Rule based fuzzy classification using squashing functions

In the past decades neural networks were successfully used in input-output mapping with very good learning abilities. However the comprehensibility of neural networks are low, they lack of logical justification, one does not know why a trained network gives a certain answer. Its knowledge is distributed in its weights and structure and it cannot be directly translated into simple logical formulas. The problem of creating logical rules to describe a set of input-output data or a black box system's internal behavior is still an active area of Computational Intelligence. Many approaches were suggested to explain a neural network's output.

Fuzzy rule extraction/refinement methods based on knowledge based neural networks (KBANN) introduced by Towell and Shavlik [83] are proved to be very popular. KBANN is the most widely known hybrid learning framework with extraction algorithms like Subset and MofN.

Besides KBANN other rule extraction methods were proposed for example the RuleNet technique of McMillan, Mozer and Smolensky [58], the rule-extraction-as-learning technique of Craven and Shavlik [25], the VIA algorithm of Thrun [80], the DEDEC technique of Tickle, Orlowski and Diederich [81], the BRAINNE system of Sestino and Dillon [76], the RULENEG technique of Pop et al. [72] and the RULEX technique of Andrews and Geva [6]. The preceding techniques give crisp rules, thus one does not know the probability of correctness of a classified instance. So fuzzy rule extraction models were developed to overcome this loss of information.

Huang and Xing [52] represent the continuous valued input parameters of the network by linguistic terms and extract rules by dominance. Pedrycz and Reformat [71] at first apply a genetic algorithm to evolve a network with weights fixed to one, and then optimize it using standard backpropagation relaxing the weights to the interval $[0, 1]$. Although integer weights are lost during optimization (which are necessary for logical rules), they are corrected by rounding them to zero or one.

In this chapter a hybrid method is proposed to construct concise and comprehensible fuzzy rules from given training data.

3.1 Problem definition and solution outline

The main task is to learn fuzzy rules describing a set of training data. Comprehensibility and classification performance are the most important attributes of a rule set. The first one is determined by the size of the rule set, and the number of antecedents per rule. To avoid complex formulas, we are only concerned with disjunctions of conjunctions i.e. formulas in disjunctive normal forms. Despite the wide class of t-norms and t-conorms we will use a squashing function-based approximation of the well-known Łukasiewicz connectives in formulas.

The training data is supposed to be a set of n-dimensional real-valued vectors $\mathbf{x}_i$ ($i = 1 \dots n_d$). A target class c_i ($i = 1 \dots n_c$) is assigned to every training data. We suppose the input values are in the interval $[0, 1]$. This normalization does not constrain applicability. The target class labels are transformed into binary valued vectors.

In short, the three-stage rule construction algorithm is the following.

1. The training data is fuzzified using approximated trapezoidal membership functions for each input dimension.
2. The structures of the logic rules are evolved by a genetic algorithm.
3. A gradient based local optimization is applied to fine-tune the membership functions.

The third step of the rule construction algorithm requires that both the membership functions and the logical connectives have a continuous gradient. The Łukasiewicz operators and the widely used trapezoidal memberships do not fulfill this requirement. As a solution an approximation of them is needed. The continuous squashing function-based approximation is used.

3.2 The structure and representation of the rules

The first step of rule learning is a discretization procedure, the fuzzification of the training data. Every input interval is equally divided into k fuzzy sets, where each fuzzy set is a soft triangular or trapezoidal one. Each element of the input vector is fuzzified by these membership functions, so that an n-dimensional data is represented by kn fuzzy membership values. From now on we will denote the fuzzified input data as x_{ij}, ($i = 1 \dots n$, $j = 1 \dots k$). The advantage of the initial fuzzification is that the output will not only provide crisp yes/no answers but classification reliabilities, too.

A set of rules is represented by a constrained neural network in the following way. The activation functions of the neurons are squashing functions with fixed $a = 1/2$ and $\lambda = 1$, and all weights of the network are zero or one. The network is further restricted to one hidden layer with any number of neurons. There are two kinds of neurons in the network: one functioning as a Łukasiewicz t-norm and one as a Łukasiewicz t-conorm, both approximated by the squashing function. Since the activation function of a neuron is given, *its type is determined by the neuron's bias*. A neuron is conjunctive if it has a bias of $n - 1$ (where n is the number of its input synapses), and disjunctive if it has a zero bias. With a given network structure these biases are constant, but for every new network with a different structure these biases must be recalculated to preserve the types of the neurons. The network is additionally constrained so that the hidden layer contains only conjunctive neurons and the output layer contains only disjunctive neurons. These restrictions affect the shape of the decision surface, too. The representable decision borders are parallel to the axises, and the decision surface is a union of local ridges.

Every output neuron corresponds to one rule. Because of the special structure of the network every rule is in disjunctive normal form (DNF). For multi class problems several networks (with one output node) can be trained, one network per class. The output class is decided by taking the maximum of the activations of the networks' output.

These restrictions on the activation function, the weights and the structure of the network are advantageous for the following reasons. First, a fuzzy logical formula (rule) can be directly assessed from the network. Second, the complexity of the represented logical formulas are greatly reduced. See e.g. [18] for the high complexity of directly extracted formulas from neural networks caused by real valued weights. The third advantage of this special network structure is its high comprehensibility, which means that the learned rules are easily human-readable.

The model has three global parameters.

- The number of conjunctive neurons in the hidden layer. Because a hidden neuron corresponds to one local decision region, this is mainly determined by the complexity of the problem.
- The technical parameter denoted by β controls the power of the approximation. A small β gives smooth membership and activation functions, while a large β gives a better approximation of triangular and trapezoidal membership functions. So the value of β directly affects the smoothness of the decision surface.
- The number of fuzzy sets each input range is divided. It can be modified as necessary to get an adequately fine resolution of the feature space.

3.3 The optimization process

The model defined in the previous section is able to arbitrarily approximate a function with sufficiently many fuzzy sets and hidden neurons. Our aim is to give a good approximation of the input-output relation by a modest number of parameters.

We use a similar approach to Pedrycz and Reformat [71] and Huang and Xing [52] for the description of the rule set but the optimization process is different. The main differences are the fixed network weights and the gradient based fine tuning of the memberships.

The proposed hybrid learning method consists of three separate steps. After the initial fuzzification, first we fix the fuzzy sets of the input and by using a genetic algorithm the synapses of the network are optimized. This optimization gives rules that roughly describe the relation between the input/output training data, so it has to be further refined. In the third step a gradient based local optimization method does the fine-tuning by optimizing the parameters of the fuzzy sets. The latter two steps are discussed in more detail.

3.3.1 Rule optimization by a genetic algorithm

The network is defined so that its weights can be only zero or one. In other words it means that either there is a synapse between two neurons in successive layers or not. In the first step this structure is optimized by a genetic algorithm to give the best possible result. It is obvious to represent the network structure by a bit string, where a bit corresponds to a connection between the neurons. A simple fitness function of the genetic algorithm is the negative of the sum of squared errors between the network output and the target value.

$$F(\mathbf{x}) = -\sum_{i=1}^{n}(z_i - t_i)^2, \tag{3.1}$$

where $\mathbf{z}$ denotes the output of the network and $\mathbf{t}$ denotes the desired output or target value and n is the number of training data. Of course, other fitness functions are reasonable too, for example by subtracting from F a value proportional to the number of synapses. This way rewarding structures with less synapses.

The network optimized by the genetic algorithm will contain the necessary synapses to roughly describe the connection between the input and output data with the initial fuzzy sets. This rule set is coarse because the initial fuzzy sets most likely do not suit the problem well.

3.3.2 Gradient based local optimization of memberships

The refinement of the initial fuzzy sets is achieved by fine tuning the parameters of the soft trapezoidal membership functions. Our purpose of using soft membership functions was to have the opportunity to use a simple gradient based local optimization algorithm. The optimization is the following: modify the parameters of the fuzzy sets so that the overall error of the network decreases. By applying this optimization the resulting set of rules will possibly have a better description of the underlying system. We note that only those fuzzy sets are optimized which have (an indirect) connection to the output neuron. It is because the gradient of the not connected ones is zero, thus the optimization algorithm does not change their value.

In order to examine the gradient of the error of the network we must introduce some notations. Let W_h denote the matrix of weights between the input and the hidden layer, W_o the vector of weights between the hidden and the output layer (since there is only one output neuron), and $\mathbf{b}$ the biases of the hidden layer. Let $\mathbf{x}_i$ denote the input of the network, where $i = 1 \dots n_d$. Let $\mathbf{y}_i$ denote the activation of the hidden neurons. Like above z_i denotes the network output.

The error of the network is the following.

$$E = \frac{1}{2}\sum_{i=1}^{n}(z_i - t_i)^2 \tag{3.2}$$

Let us denote the set of parameters by $\mathbf{p}$. These are the parameters of the trapezoidal fuzzy sets, four for each one: the center and width of its left and right sides. The partial derivative of E by $\mathbf{p}$ is

$$\frac{\partial E}{\partial \mathbf{p}} = \sum_{i=1}^{n}\left(\frac{\partial z_i}{\partial \mathbf{p}}\right)^T (z_i - t_i). \tag{3.3}$$

In the network z_i is calculated by the following formula

$$z_i = S^{(\beta)}_{1/2,1}\left(W_o \mathbf{y}_i\right) \tag{3.4}$$

because all the biases of the output layer are zero. Its partial derivative according to $\mathbf{p}$ is

$$\frac{\partial z_i}{\partial \mathbf{p}} = \frac{\partial S^{(\beta)}_{1/2,1}\left(W_o \mathbf{y}_i\right)}{\partial\left(W_o \mathbf{y}_i\right)} W_o \frac{\partial \mathbf{y}_i}{\partial \mathbf{p}}, \tag{3.5}$$

The partial derivatives of the hidden neurons' activation can be calculated similarly because

$$\mathbf{y}_i = S^{(\beta)}_{1/2,1}\left(W_h \mathbf{x}_i - \mathbf{b}\right), \tag{3.6}$$

so

$$\frac{\partial \mathbf{y}_i}{\partial \mathbf{p}} = Diag\left(\frac{\partial S^{(\beta)}_{1/2,1}\left(W_h \mathbf{x}_i - \mathbf{b}\right)}{\partial\left(W_h \mathbf{x}_i - \mathbf{b}\right)}\right) W_h \frac{\partial \mathbf{x}_i}{\partial \mathbf{p}}, \tag{3.7}$$

where $Diag(\xi)$ denotes a diagonal matrix constructed from the vector ξ.

Because the network's inputs are fuzzified values according to given trapezoidal fuzzy memberships, its partial derivatives $\frac{\partial \mathbf{x}_i}{\partial \mathbf{p}}$ can be easily calculated.

The role of the parameter β is very important in the learning process. If its value is too low, there is no real distinction between the different fuzzy sets on the same input interval. If its value is too high (i.e. the squashing function approximates the generalized cut function very well), the optimization is not effective since the gradient of it is either zero or a non-zero constant. For these reasons this optimization step is realized as an iterative process with increasing β values. As a result the final approximation is negligible and the fuzzy sets are represented by piecewise-linear trapezoidal or triangular memberships.

After the two optimization steps the set of rules can be easily extracted from the network. There is a one-to-one correspondence between a network structure and a set of rules. The advantage of this rule learning method is twofold. First, the rules are easily interpretable fuzzy rules (because of the disjunctive normal form) with expressed confidence in the result. However, interpretability could be further increased by constraining the possible settings of membership functions, or by assigning linguistic variables to final membership functions. Second, there are no real valued weights in the network during the optimization which would have to be rounded (and thus losing information) to get a logic interpretation of the input/output relation of the training data.

3.4 Applications of the classification method

In this section we show some examples of the above defined rule construction method. The example problem sets are the Iris, Wine, Ionosphere and the Thyroid datasets from the UCI machine learning repository. In all four experiments the genetic algorithm was run with the following setting:

population:	100
max. generations:	100
crossover method:	scattered
mutation prob.:	2%

The following shorthand notation will be used for the description of membership functions.

Notation 3.1. *Let us denote a trapezoidal membership function by*

$$[a_1 <_{\lambda_1} x <_{\lambda_2} a_2]\,, \tag{3.8}$$

where a_i denote the centers and λ_i denote the widths of the left and right slopes. If one side of the trapezoid is outside of the corresponding input interval then it is omitted.

The Iris dataset is the following. The input feature space is four dimensional, which comprises of the sepal length, the sepal width, the petal length and the petal width of an Iris flower. One has to decide the class of the Iris flower i.e. whether it is an Iris Setosa, an Iris Virginica or an Iris Versicolor. There are 150 entries in the dataset. Three networks were trained, one for each class. Each input interval was divided by three trapezoidal fuzzy sets, the number of hidden neurons was one in each case. The learned rules for the Iris problem are:

- Iris Setosa: $[x_3 <_{1.7} 3.8]$
- Iris Virginica: $[1.5 <_{0.5} x_4]$
- Iris Versicolor: $[0.35 <_{3.76} x_3 <_{1.55} 6.6]$
  AND $[0.27 <_{1.28} x_4 <_{0.32} 1.9]$

These rules give 96% accuracy with 5 misclassified samples. Only two features are used, and the average certainty factors are [98% 92% 96%] for the classes.

The Wine dataset contains 178 instances of 13 dimensional real-valued input vectors. There are three types of wines to classify, so three separate networks were trained. The following rule set has been learned with three fuzzy sets for each input:

- Wine 1: $[435 <_{683} x_{13}]$
- Wine 2: $[x_{10} <_{3.36} 5.9]$
- Wine 3: $[x_7 <_{1.26} 1.74]$

These rules give 95% accuracy with 6 misclassified and 3 undecided samples. Note that only three features are used (x_7, x_{10}, x_{13}) in the rules. The average certainty factors are [88% 85% 85%].

The Ionosphere dataset is a binary classification problem which contains 351 instances of 34 dimensional radar data. One has to decide whether there is evidence of some type of structure in the ionosphere. The following rule has been learned with only one hidden neuron:

$$[0.69 <_{0.5} x_1] \text{ AND } [-0.19 <_{0.013} x_5]$$

With 1/2 threshold, this simple rule gives 88% accuracy.

The Thyroid gland dataset contains 215 instances of 5 dimensional real-valued input vectors. There are three classes (normal, hypo and hyper), each class was classified with only one hidden neuron:

- Normal: $[4.92 <_{2.46} x_2 <_{6.95} 14.57]$

- Hypo: $[x_2 <_{3.75} 6.2]$
- Hyper: $[10.95 <_{4.31} x_2 <_{12.77} 36.8]$

These rules give 94.8% accuracy with 11 misclassifications. Note that only x_2 is used. The average certainty factors are [95% 88% 94%].

3.5 Summary

In this chapter a hybrid method with a genetic algorithm and a gradient based local optimization is introduced for fuzzy logical rule learning. The genetic algorithm is used to find those features of the input with which the separation of classes is optimal. The second step of the method refines the initial fuzzy membership functions in order to give better accuracy. The model is novel in the sense that logical information is directly available and that the fuzzy membership functions are optimized instead of the network weights, so that there is no need to round the weights to integers and thus lose information. The rules are concise and easily understandable because of their disjunctive normal form which is guaranteed by the special network structure. Tests show that the method is very useful in revealing hidden input/output relations.

Chapter 4

Reasoning using approximated fuzzy intervals

The concept of Approximate Reasoning appeared very early after the introduction of fuzzy sets. In fact, one of the main motivations behind fuzzy sets was the need to deal with vague propositions and to make inferences with them. The first model of inference which handled fuzzy rules used the compositional rule of inference (CRI), proposed by Zadeh [95]. The CRI is based on the cylindrical extension and projection of fuzzy relations, and is one of the most fundamental inference mechanisms. Many papers dealt with the generalizations of CRI and the proper selection of operations satisfying various requirements. All authors agree that the CRI should generalize the classical modus ponens principle. This case is well characterized by now, see e.g. [17].

In 1979 Baldwin [10] and Tsukamoto [87] proposed another inference mechanism called fuzzy truth-value (FTV) based reasoning. It is based on the ideas of Bellman and Zadeh, who introduced the concept of local fuzzy truth-values. Many authors generalized the original ideas of Baldwin and Tsukamoto, and revealed that the two inference mechanisms are equivalent.

Later, another subfield of Approximate Reasoning emerged, which handled inference based on the similarity of the input the antecedents of the rules, see Ruspini [74]. This methodology declined some axioms of the logic based view of rules, and defined new ones, which much more emphasize the similarity view.

In this chapter we examine the properties of indetermination in generalized CRI. We then introduce a new reasoning method in which the inference does not produce indetermination. This method, called Membership Driven Inference (MDI), is applicable when rules and premises may be expressed by combinations of sigmoid-like membership functions. This family of membership functions includes approximations of trapezoidal, S-, and Z-shaped membership functions. It is shown that MDI can be efficiently calculated when the rules and premises are so-called squashing functions.

4.1 The compositional rule of inference and fuzzy truth qualification

Let $A, A^* \in \mathscr{F}(X)$ and $B \in \mathscr{F}(Y)$ be fuzzy sets on the universes of discourse X and Y, where $\mathscr{F}$ is the set of all fuzzy sets. The compositional rule of inference (CRI) introduced by Zadeh [93] states that knowing A^* and the rule "IF A THEN B", the conclusion $B^* \in \mathscr{F}(Y)$ is calculated by means of the combination/projection principle of the form

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \wedge R(A(x), B(y))\}, \tag{4.1}$$

where $R \subset \mathcal{I}^X \times \mathcal{I}^Y$ is a fuzzy relation. The behavior of the compositional rule of inference has been intensively studied. Many authors investigated and compared various conjunctions and implications in the CRI or generalized the original inference mechanism [42, 66, 91, 34, 8, 53, 67, 17]. In this chapter the following modification is considered: substituting a t-norm for the min, and constraining the fuzzy relation to the t-norm's residual implication, i.e.

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \bigtriangleup (A(x) \rhd B(y))\}. \tag{4.2}$$

This setting is the maximal solution regarding t-norms fulfilling the very natural property called generalized modus ponens (i.e. if $A^* \equiv A$ then $B^* \equiv B$). In the following this reasoning scheme will be called generalized CRI.

Another approach to fuzzy reasoning has its roots in the theory of truth qualification, introduced by Bellman and Zadeh [14]. In their model they generalized the notion of boolean truth values to fuzzy truth values (FTV). Every statement is relatively (or locally) true to another statement, according to the following equivalence:

$$\text{"}x \text{ is } A \text{ is } f\text{-true"} \Leftrightarrow \; \exists B, \text{"}x \text{ is } B\text{" is true and } \mu_B = \varphi(\mu_f, \mu_A).$$

The membership function μ_f, i.e. the degree of truth of "x is A" assuming that "x is B" is true, can be calculated by (according to the extension principle)

$$\mu_{B|A}(u) = \begin{cases} \bigvee\limits_{\mu_A(x)=u} \mu_B(x) & \text{if } \mu_A^{-1}(u) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

FTV's are fuzzy sets of the unit interval, or equivalently they can be considered as hedges or unary operators.

Baldwin [10, 12, 11] and Tsukamoto [87] were the first to utilize the theory of fuzzy truth values and proposed a reasoning mechanism based on it. The inference with fuzzy truth values is performed by the following steps:

- The fuzzy truth value $f(u) = \mu_{A^*|A}(u)$ is calculated.
- The inference is done in truth-value space taking into account the t-norm $\bigtriangleup$ and its residual implication $\rhd$
$$g(v) = \bigvee_u \{f(u) \bigtriangleup (u \rhd v)\}$$
- Finally the conclusion is computed as
$$\mu_{B^*}(y) = g(\mu_B(y)).$$

The second step of this algorithm can be interpreted [86, 48] as a mapping $M_{\triangleright} : \mathcal{F} \to \mathcal{F}$, where $\mathcal{F}$ is the set of fuzzy truth values:

$$M_{\triangleright}(f)(v) = g(v) = \bigvee_{u} \{f(u) \triangle (u \triangleright v)\} .$$

Note that in this formulation $\triangleright$ is considered to be given instead of $\triangle$, and $\triangle$ is called the modus ponens generating function associated with $\triangleright$ (i.e. the t-norm whose residual is $\triangleright$). The following properties hold for $M_{\triangleright}$, where f and g are fuzzy truth values, and inclusion means pointwise order on the set of functions:

FTV1: $M_{\triangleright}(f) \supseteq f$ for all $f \in \mathcal{F}$

FTV2: if $f \subseteq g$ then $M_{\triangleright}(f) \subseteq M_{\triangleright}(g)$

FTV3: $M_{\triangleright} \circ M_{\triangleright} = M_{\triangleright}$

In fact, from the definition of $M_{\triangleright}$ it also follows that

- for all t-norms $\triangle$ and $f \subseteq id$ s.t. $f(0) = 0$ and $f(1) = 1$, $M_{\triangleright}(f) \equiv id$
- for all order-reversing f s.t. $f(0) = 0$ and $f(1) = 1$, $M_{\triangleright}(f) \equiv 1$

The above elementary reasoning schemes, the generalized CRI and fuzzy truth qualification based inference are proved to be equivalent [82].

4.2 The indetermination part of the conclusion

An elemental feature of CRI based reasoning is the so-called indetermination part of the conclusion. It means that the membership values of the conclusion (i.e. B^*) are no less than a non-zero constant. This phenomenon of CRI based reasoning appeared in the literature under various names: level of indetermination [8], residual uncertainty [38], level of uncertainty [88] or tail-effect [20].

Suppose the reasoning scheme is the generalized CRI. Mantaras and Godo [26] showed that in this setting a non-zero level of indetermination is due to the incompatibility between the membership functions of the antecedent A and the input A^*. The incompatibility occurs when a significant part of A^* (where $A^*(x) > 0$) is not included in A, i.e. there exists $x \in X$ such that $A^*(x) > 0$ and $A^*(x) > A(x)$. De Baets and Kerre [8] did an extensive study on CRI-based reasoning with triangular fuzzy sets. One of their results was that for nearly all combinations of fundamental t-norms and implications (not only their residuals) the conclusions show a non-zero level of indetermination. They also remark that inference rules with a low level of indetermination are preferred.

Under some special conditions the indetermination of the conclusion may appear even if $A^* = A$, as it shown by Turksen and Yao [88]. Moreover, Dubois and Prade [38] proved that by using min-based CRI with Kleene-Dienes implication it may occur even if $A^* \subseteq A$. See Figure 4.1 for the appearance of the indetermination part in CRI for various implication operators as fuzzy relations.

Chang et al. [20] studied min-based CRI reasoning with Gaines-Rescher implication. They proposed a simple solution to avoid the indetermination of the conclusion in the discrete case: change the zero membership values of A and A^* to small positive values. This

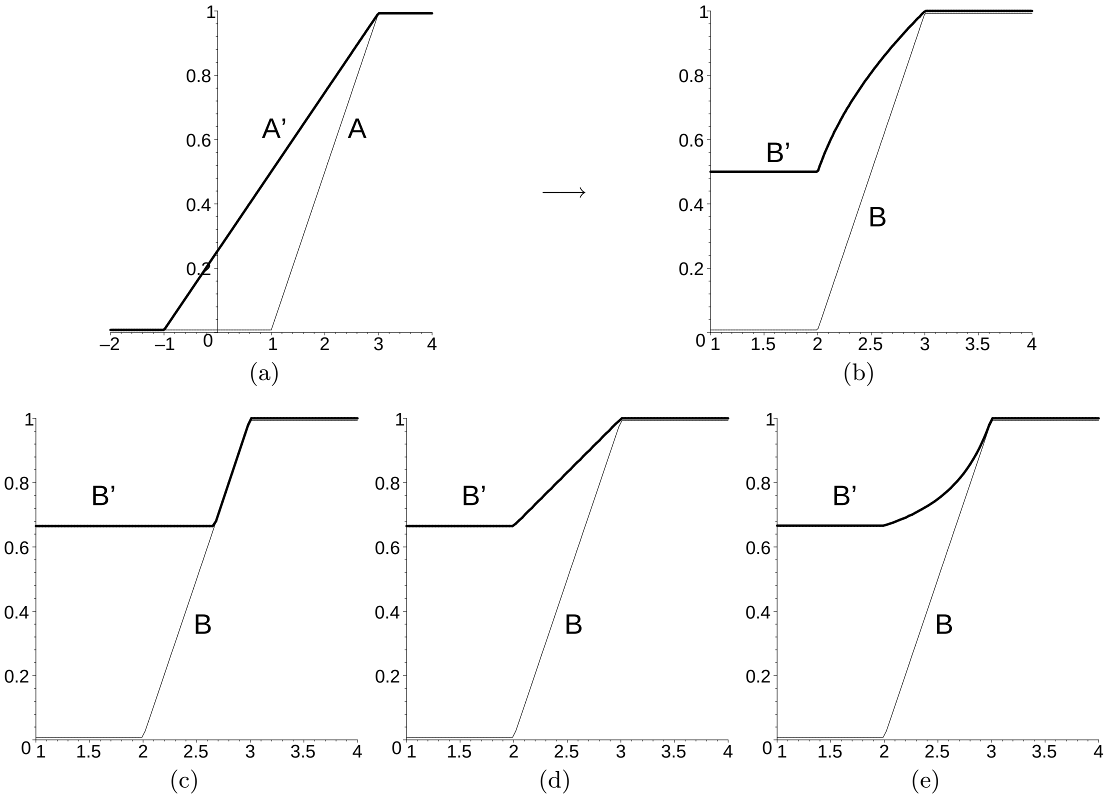


Figure 4.1: (a) The fuzzy sets A and A^* (thick), (b)-(e) The fuzzy set B and the conclusions B^* (thick) using the sup-min composition and Goguen, Kleene-Dienes, Łukasiewicz and Reichenbach implications

way the meaning of the fuzzy sets does not change considerably, but the indetermination vanishes. At this point it is important to point out that it is not enough to change all zero membership values to a sufficiently small ε. In this case it remains, see Figure 4.2. To avoid it, the membership functions have to be strictly monotone.

Since the generalized CRI is equivalent to fuzzy truth value based reasoning, the indetermination part can also be observed in the latter case. A non-zero level of indetermination of B^* emerges if the consequent fuzzy truth value $f(0) > 0$. For example, from the two fuzzy truth values on Figure 4.3 the left one causes a non-zero level of indetermination.

4.3 Logic vs. interpolative reasoning

There are many different axiom systems defining minimum requirements for an inference mechanism. In the following two of the most well-known are revisited. It is assumed that a rule with antecedent A and consequent B is given. Inference is denoted by $\rightarrow$, a premise is indicated by A^* and resulting conclusion by B^* (with or without subscripts).

- Baldwin and Pilsworth [12]

  B1: $B \subseteq B^*$

  B2: $A'_1 \subseteq A'_2 \Rightarrow B'_1 \subseteq B'_2$ (monotonicity)

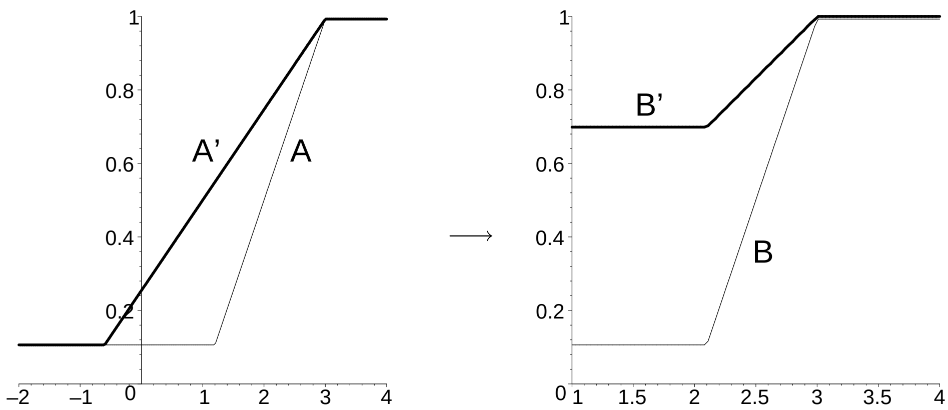


Figure 4.2: The indetermination level remains non-zero even if the fuzzy sets are ’raised’ by a constant ε

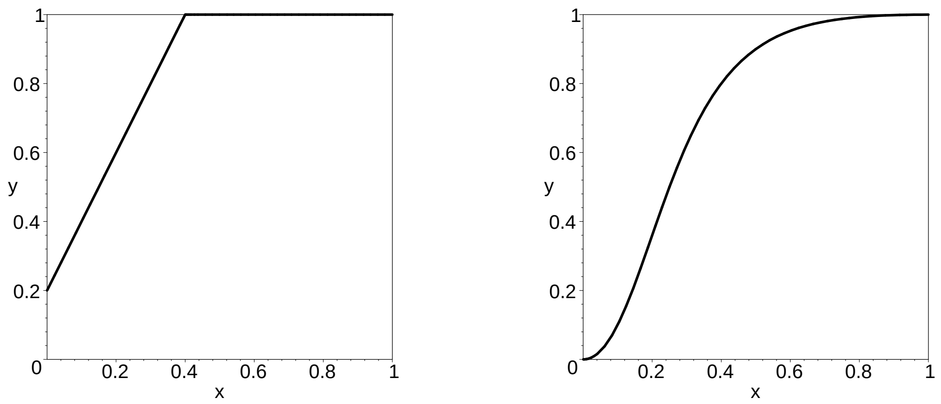


Figure 4.3: Two fuzzy truth values, if applied to the conclusion the first one is causing a non-zero level of indetermination

B3: $n(A) \to V$ (from ’*not A*’ nothing can be concluded)

B4: $A^* \to B^* \equiv n(B^*) \to n(A^*)$ (the equivalence of modus ponens and modus tollens)

- Fukami, Mizumoto and Tanaka [42]

  F1: $A \to B$ (classical modus ponens property)

  F2: *very* $A \to$ *very* B

  F3: *more or less* $A \to$ *more or less* B

  F4: $n(A) \to n(B)$ for a negation n

As it can be instantly seen some axioms are contradictory and the different axioms systems can not be totally fulfilled simultaneously, e.g. B1 and F2. This is due to the fact the two axiom systems represent two very different views on reasoning.

The axiom system of Baldwin and Pilsworth represents the logical view of reasoning. In fact, in axiom B1 it states that the inferred value should never be more restrictive than the rule’s consequent, i.e. from the premise $'very\ A^*$ the inferred conclusion is ”B” and not ”*very* B”. Such a rule is asymmetric in the sense that the conclusion is never more precise than the rule’s consequent and if a premise does not fit at all the rule’s antecedent (axiom B3) then nothing can be concluded. These axioms suggest that the relation between a rule's antecedent and consequent is implicative. This is the way the CRI and the FTV reasoning mechanisms work. In fact, there is a close correspondence between axioms B1-B4 and properties FTV1-FTV3.

On the other hand the axiom system of Fukami et al. represents the interpolative view of reasoning. These axioms express more than an implicative relation. They suggest the use of gradual if-then rules, which relate the antecedent and the consequent in a functional way. Such a rule is symmetric in the sense that no matter how the premise is related to the rule antecedent (i.e. whether A^* is "*more or less A*" or "*very A*") this relationship should also hold for the conclusion and the rule consequent.

There are two main differences between these views. The first is the treatment of the more precise premise compared to a rule's antecedent. The logic view does not allow a more precise conclusion than the rule's consequent. The second is that in the logical setting the appearance of the level of indetermination is natural and necessary (for arguments see [38]). On the contrary, in the interpolative view the indetermination of the conclusion violates the functional relationship i.e. it should be avoided.

On one hand in this chapter we follow the interpolative view of reasoning i.e. consider axioms F1-F4. On the other hand our starting point is the CRI reasoning scheme, which obviously violates these axioms. From our point of view, in order to fulfill F1-F4 it is necessary to avoid the non-zero level of indetermination. Our first aim is to generalize the technique presented in [20] to non-discrete fuzzy sets.

4.4 Inference with sigmoid-like functions

From now on we will only consider fuzzy sets on the real interval.

Definition 4.1. *Let* $\varphi : \mathbb{R} \to [0,1]$ *be a continuous and strictly monotone function. It is called sigmoid-like if*

$$\lim_{x \to -\infty} \varphi(x) = s, \tag{4.3}$$

$$\lim_{x \to +\infty} \varphi(x) = 1 - s, \tag{4.4}$$

where $s \in \{0, 1\}$.

We note that by definition sigmoid-like functions can be increasing or decreasing, and because of the strict monotonicity they take values in $]0,1[$ for any finite x. Also, a sigmoid-like function has a unique inverse function since it is a bijection. Examples for sigmoid-like functions are the logistic function and the squashing function.

As it can be instantly seen, sigmoid-like functions are not fuzzy intervals since they are not normal (i.e. for all finite x: $\varphi(x) < 1$). But, as it will be shown later, any fuzzy interval can be arbitrarily approximated by proper combinations of sigmoid-like functions.

Let us introduce ν-sharpness. A sigmoid-like function A^* is ν-sharper than A if $A(x_\nu) = A^*(x_\nu) = \nu$, $A^*(x) < A(x)$ for all $x < x_\nu$ and $A^*(x) > A(x)$ for all $x > x_\nu$. It is clear that if A^* is ν-sharper than A then it is less fuzzy, so ν-sharpening is closely related to fuzziness measures. We note that ν-sharpening is a generalization of the following definition of *contrast intensification* in hedge theory:

$$\mu_{Int(F)} = \begin{cases} 2\mu_F^2(x) & \text{if } \mu_F(x) \leq 0.5 \\ 1 - 2(1-\mu_F(x))^2 & \text{otherwise} \end{cases}$$

The indetermination part of the conclusion of CRI inference can be avoided by using sigmoid-like membership functions. Next the computation of the generalized CRI reasoning scheme is shown regarding the three representative t-norms.

4.5 Closure properties of the generalized CRI with sigmoid-like functions

In the following we will establish conditions under which the generalized CRI reasoning scheme with a continuous t-norm $\triangle$ and its residual $\triangleright$ is closed under sigmoid-like functions. First we discuss sup-min composition with sigmoid-like functions.

Theorem 4.2. *Let A, A^*, B be sigmoid-like fuzzy sets, and let the reasoning scheme be the original CRI. Then B^* can be calculated as follows.*

If A and A^* have the same type of monotonicity, i.e. both are strictly increasing or decreasing, then

$$B^*(y) = B(y) \vee A^*(A^{-1}(B(y))) \tag{4.5}$$

where A^{-1} is the (unique) inverse function of A, and B^* is also a sigmoid-like function.

If $A^*(x) = A'(x)$ for a negation $'$ i.e. if the functions have different type of monotonicity, then $B^*(y) = 1$, i.e. the conclusion is interpreted as unknown.

Proof. Let $\triangleright_\wedge$ denote the residual implication of the minimum operator, i.e.

$$\triangleright_\wedge(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

The inference rule is

$$\begin{aligned} B^*(y) &= \bigvee_{x\in\mathbb{R}} \{A^*(x) \wedge (A(x) \triangleright_\wedge B(y))\} \\ &= \bigvee_{x:A(x)>B(y)} \{A^*(x) \wedge B(y)\} \vee \bigvee_{x:A(x)\leq B(y)} \{A^*(x) \wedge 1\} \end{aligned} \tag{4.6}$$

Suppose A and A^* have the same type of monotonicity. There are two cases regarding the first supremum. Either there exists x_0 such that $A(x_0) > B(y)$ and $A^*(x_0) = B(y)$, or not. In the latter case for all $x : A(x) > B(y)$ the inequality $A^*(x) > B(y)$ must also hold due to the limit properties of A^*. In both cases

$$\bigvee_{x:A(x)>B(y)} \{A^*(x) \wedge B(y)\} = B(y), \tag{4.7}$$

and so

$$B^*(y) = B(y) \vee \bigvee_{x:A(x)\leq B(y)} A^*(x). \tag{4.8}$$

Since $A^*(x)$ takes its maximum at $x = A^{-1}(B(y))$ under the restriction $A(x) \leq B(y)$,

$$B^*(y) = B(y) \vee A^*(A^{-1}(B(y))). \tag{4.9}$$

Now suppose $A^*(x) = A'(x)$. Then

$$\bigvee_{x:A(x)\leq B(y)} A^*(x) = 1$$

so due to the maximum in (4.6), $B(y) = 1$ for all y. ■

In the above Theorem both A and A^* are sigmoid-like functions, and so let D denote the union of open intervals where $A^*(x) > A(x)$. If $x \in D$ then the conclusion is $B^*(y) = A^*(A^{-1}(B(y)))$, otherwise $B^*(y) = B(y)$.

Corollary 4.3. *Three special cases of Theorem 4.2 are:*

- *If $A^*(x) > A(x)$ for all x then $A^*(A^{-1}(x)) > x$ and so $B^*(y) = A^*(A^{-1}(B(y)))$ for all y.*
- *If $A^*(x) \leq A(x)$ for all x then $A^*(A^{-1}(x)) \leq x$ and so $B^*(y) = B(y)$ for all y.*
- *If A^* is ν-sharper than A (or vice versa) then $B^*(y)$ has two parts divided by $A^{-1}(\nu)$ and can be calculated according to the previous two cases.*

The case $A^*(x) > A(x)$ for all x stands for the weakening of A, i.e. here A^* has the meaning "*more or less A*". Similarly, the case $A^*(x) < A(x)$ for all x stands for the strengthening of A, i.e. "*very A*". The ν-sharpening of A can be thought of as a precisiation of A. See Fig. 4.4 for the illustration of the inference mechanism for three typical settings.

A similar closure theorem holds regarding the product t-norm and its residual in the generalized CRI reasoning scheme with sigmoid-like functions, however its calculation can not be simplified as in the previous case.

Theorem 4.4. *Let A, A^*, B be sigmoid-like fuzzy sets, and let the reasoning scheme be the generalized CRI with the product t-norm and its residual, the Goguen implication. Then B^* can be calculated as follows.*

If A and A^* have the same type of monotonicity then

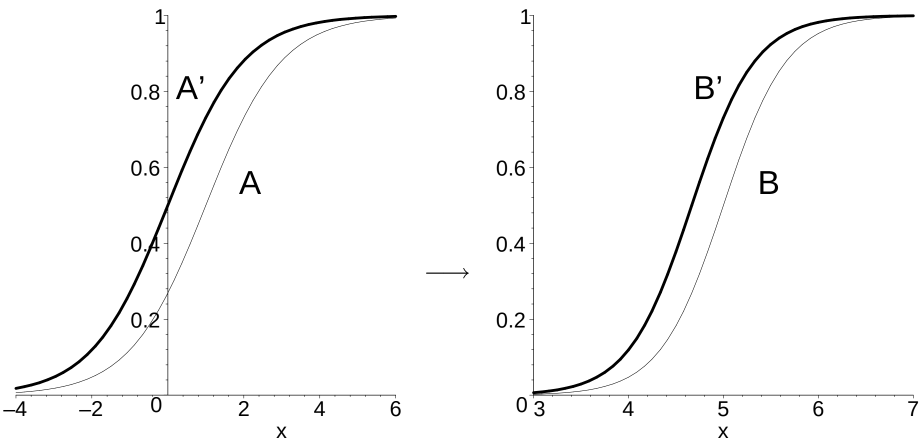
$$B^*(y) = B(y) \cdot \bigvee\nolimits_{x:A(x)\geq B(y)} \{A^*(x)/A(x)\}, \tag{4.10}$$

where B^* is also a sigmoid-like function.

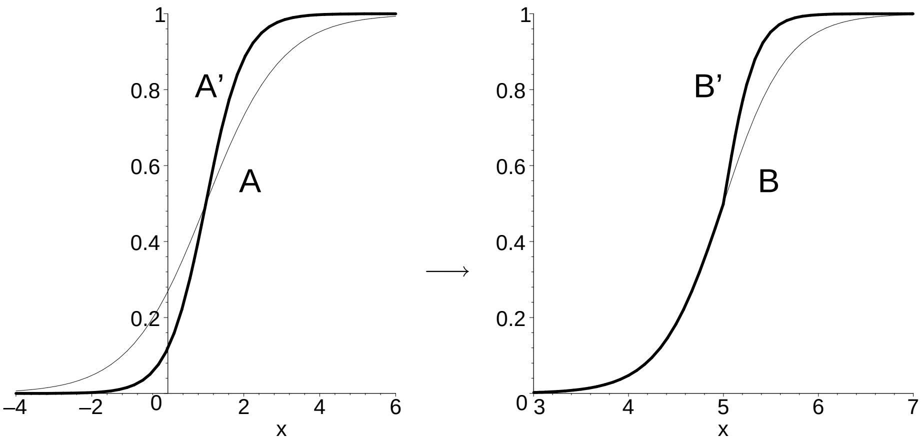
If $A^*(x) = A'(x)$ for a negation $'$, then $B^*(y) = 1$.

Proof. From the CRI we have

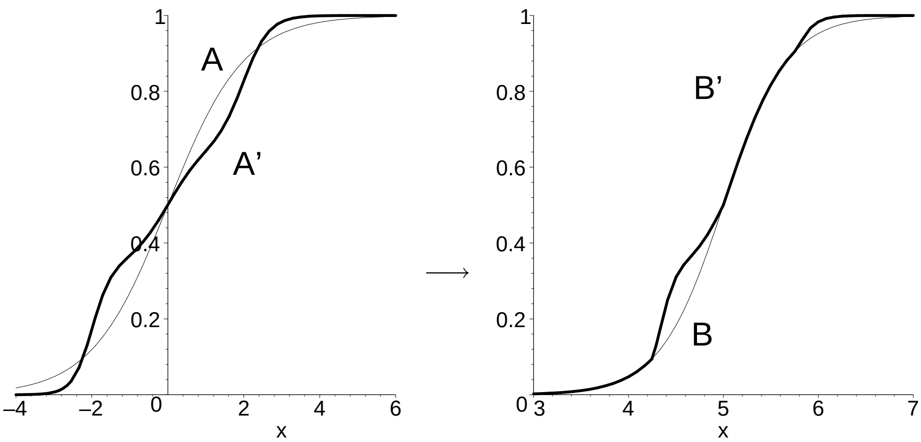
$$\begin{aligned} B^*(y) &= \bigvee_{x:A(x)\leq B(y)} A^*(x) \vee \bigvee_{x:A(x)>B(y)} \{A^*(x) \cdot B(y)/A(x)\} && (4.11)\\ &= A^*(A^{-1}(B(y))) \vee \bigvee_{x:A(x)>B(y)} \{A^*(x) \cdot B(y)/A(x)\} && (4.12) \end{aligned}$$



(a) A^* is *more or less* A



(b) A^* is the ν-sharpening of A



(c) General case, no special relationship between A and A^*

Figure 4.4: The min-based CRI reasoning scheme with sigmoid-like functions. On the left the antecedent A and the input A^* (thick). On the right the conclusion B and the output B^* (thick)

Suppose A and A^* have the same type of monotonicity, then A^* takes its maximum value at x_A where $A(x_A) = B(y)$. So the two terms in the maximum can be unified to

$$B^*(y) = \bigvee_{x:A(x)\geq B(y)} \{A^*(x)/A(x) \cdot B(y)\}. \tag{4.13}$$

We show that B^* is indeed sigmoid-like. Its continuity is trivial. In the limit case $B(y) \to 1$, the supremum of the quotient tends to 1, and in case $B(y) \to 0$, the supremum of the quotient tends to a constant greater than one. To show strict monotonicity the following are needed. Let

$$F(w) = \bigvee_{x:A(x)\geq w} \{A^*(x)/A(x)\}, \quad w \in (0,1) \tag{4.14}$$

so $B^*(y) = B(y) \cdot F(B(y))$. The function F is non-increasing and has a lower bound 1 (since in the limit $A^* = A$), and has a strict upper bound $1/w$. If $F(B(y)) = 1$ then it can not decrease further, and from the strict monotonicity of B, follows of B^*.

In case $F(B(y)) > 1$, suppose $B(y_0)$ is increased to $B(y_1) = c \cdot B(y_0)$, with $c > 1$. First we show that

$$1/c \cdot F(B(y_0)) < F(B(y_1)). \tag{4.15}$$

Let us denote

$$x_0 = \arg_x F(B(y_0)) \quad \text{and} \quad x_1 = \arg_x F(B(y_1)).$$

If $x_0 = x_1$ then $F(B(y_0)) = F(B(y_1))$ and (4.15) holds. Otherwise, there are two similar cases, i.e. when both A and A^* are increasing or decreasing. We consider only the first case, then $x_0 < x_1$. From the strict monotonicity of A^*, $A^*(x_0) < A^*(x_1)$. Because $B(y)$ is increased by the factor c, and by the constraint in the supremum in F, $A(x_1) \geq c \cdot A(x_0)$. So

$$\begin{aligned} 1/c \cdot F(B(y_0)) &= 1/c \cdot A^*(x_0)/A(x_0) \\ &< 1/c \cdot A^*(x_1)/A(x_0) \leq A^*(x_1)/A(x_1) = F(B(y_1)) \end{aligned} \tag{4.16}$$

And so the strict monotonicity of B^* follows because

$$B(y_1) \cdot F(B(y_1)) > B(y_1) \cdot 1/c \cdot F(B(y_0)) \tag{4.17}$$

$$B^*(y_1) > cB(y_0) \cdot 1/c \cdot F(B(y_0)) = B^*(y_0). \tag{4.18}$$

The proof of the second case is similar to that of Theorem 4.2. ■

Remark 4.5. *Due to the lower bound on F, B^* is no less than B, i.e. $B \subseteq B^*$.*

The case of Łukasiewicz t-norm based CRI is different from the previous two, because even in case of A and A^* have same type of monotonicity, the sigmoid-like form of B^* is not guaranteed.

Theorem 4.6. *Let A, A^*, B be sigmoid-like fuzzy sets, and let the reasoning scheme be the generalized CRI with the Łukasiewicz t-norm and its residual. Then B^* can be calculated as follows.*

If A and A^* have the same type of monotonicity then

$$B^*(y) = B(y) + \bigvee_{x:A(x)\geq B(y)} \{A^*(x) - A(x)\}. \tag{4.19}$$

Here B^* is a sigmoid-like function if and only if $A^* \subseteq A$.

If $A^*(x) = A'(x)$ for a strict negation then $B^*(y) = 1$.

Proof. The conclusion B^* is

$$\begin{aligned} B^*(y) &= \bigvee_x \{0 \vee (A^*(x) + (1 \wedge (1 - A(x) + B(y))) - 1)\} \\ &= \bigvee_{x:A(x)\leq B(y)} A^*(x) \vee \bigvee_{x:A(x)>B(y)} \{0 \vee (A^*(x) - A(x) + B(y))\} \\ &= A^*(A^{-1}(B(y))) \vee \left(B(y) + \bigvee_{x:A(x)>B(y)} \{A^*(x) - A(x)\} \right). \end{aligned}$$

Similarly to the proof of Theorem 4.4 the two terms can be unified, so

$$B^*(y) = B(y) + \bigvee_{x:A(x)\geq B(y)} \{A^*(x) - A(x)\}.$$

In case $A^* \nsubseteq A$ i.e. if there exists x_0 s.t. $A^*(x_0) > A(x_0)$ then in case $B(y) \to 0$

$$\lim_{B(y)\to 0} \bigvee_{x:A(x)\geq B(y)} \{A^*(x) - A(x)\} = \bigvee_{x:A(x)\geq 0} \{A^*(x) - A(x)\} > 0 \tag{4.20}$$

and so B^* is non-zero, i.e. it is not sigmoid-like. If $A^* \subseteq A$ then

$$\bigvee_{x:A(x)\geq B(y)} \{A^*(x) - A(x)\} = 0, \tag{4.21}$$

since in the limit the values of A and A^* are equal, and so B^* is indeed sigmoid-like.

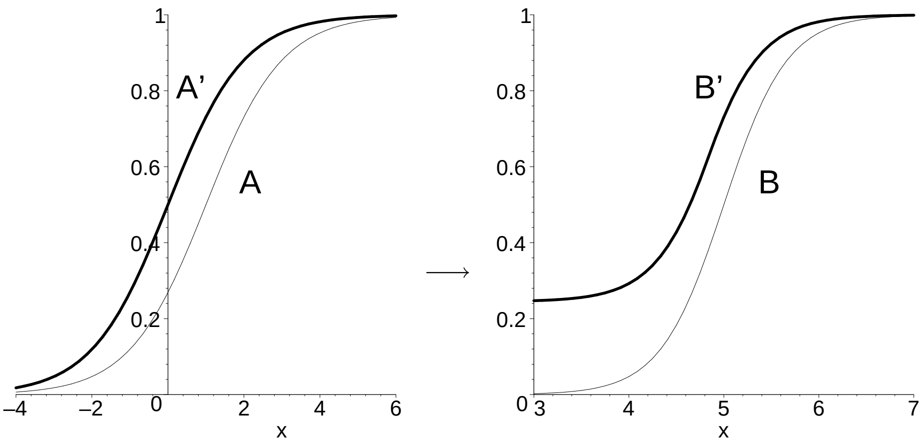
The proof of the second case is similar to that of Theorem 4.2. ∎

Equation (4.20) shows, that the use of the Łukasiewicz t-norm introduced a non-zero indetermination level of the conclusion B^* if $A^* \nsubseteq A$. In this case

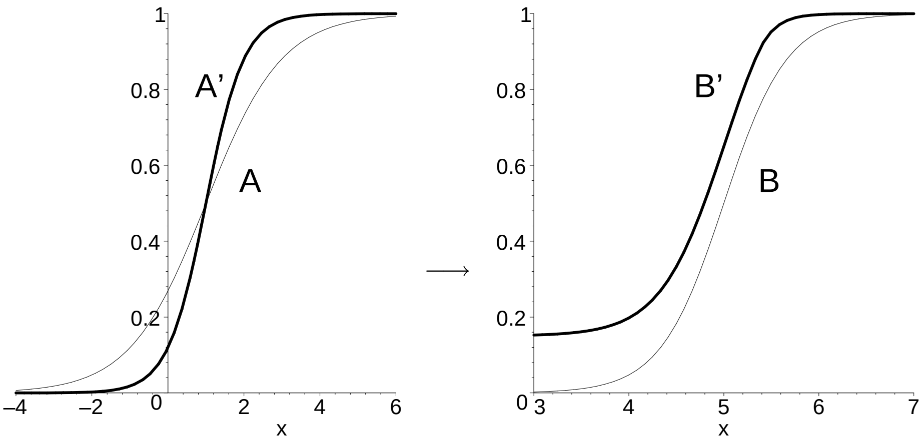
$$B^*(y) \geq \bigvee_x \{A^*(x) - A(x)\} \text{ for all } y \in Y. \tag{4.22}$$

Clearly, the previous Theorems are also valid regarding respective isomorphic t-norms to the minimum, the product and the Łukasiewicz t-norm. In case of the minimum, it is easy to see, since it is only isomorph to itself. In the other cases, i.e. regarding Archimedean t-norms, a strictly increasing bijection transformation of the generator function does not change the assertions of the proofs. Regarding ordinal sums, the previous Theorems can be applied for each t-norm summand separately. Based on these considerations, the following Theorem holds.

Theorem 4.7. *Let $\triangle$ be an arbitrary ordinal sum of a family of continuous t-norms, $\triangleright$*



(a) A^* is *more or less* A



(b) A^* is the ν-sharpening of A

Figure 4.5: The Łukasiewicz t-norm based CRI reasoning scheme with sigmoid-like functions. On the left the antecedent A and the input A^* (thick). On the right the conclusion B and the output B^* (thick)

its residual. Let A,A^* and B be sigmoid-like functions. Let

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \triangle (A(x) \triangleright B(y))\}.$$

If A and A^* are both increasing or decreasing, then B^* is sigmoid-like if all summands of $\triangle$ are either the minimum or strict. If additionally $A^* \subseteq A$ then B^* is sigmoid like for all continuous ordinal sums $\triangle$. If A and A^* have different type of monotonicity then $B^* \equiv 1$.

To sum up, the Łukasiewicz t-norm based generalized CRI is closed under sigmoid-like functions only if $A^* \subseteq A$, since in this case $B^* \equiv B$. In the general case, it introduces a non-zero level of indetermination of the conclusion. Using the product t-norm, the conclusion is sigmoid-like, but to perform the inference the calculation of a pointwise supremum is required, which makes it inefficient in the general case. The min-based CRI is also closed under sigmoid-like functions and moreover its calculation is simple, too. Regarding these properties, from the above three representative variants of logic based reasoning, the min-based CRI fits best to sigmoid-like functions.

4.6 The Membership Driven Inference

The above discussed generalized CRI reasoning schemes on sigmoid-like functions represent the logic view of reasoning and fulfill axioms B1-B4. From the three representative t-norms, the minimum proved to be the most useful. In order to get a reasoning scheme which satisfies axioms F1-F4 (which can serve as the basis for the interpolative view) instead of B1-B4, only a slight modification of Eq. 4.5 is required. The reasoning scheme of Theorem 4.2 is asymmetric due to the maximum operation and the term $B(y)$. By leaving them we get a new reasoning mechanism.

Definition 4.8. *Let the Membership Driven Inference (MDI) reasoning scheme be*

$$B^* = A^* \circ A^{-1} \circ B, \tag{4.23}$$

where A and B are the antecedent and the consequent of a rule, A^* is the input and B^* is the output of the rule.

This reasoning scheme is simple, and like Eq. 4.5 it is interesting from the point of view that it depends only on the sigmoid-like membership functions of A, A^* and B. It does not contain explicitly any conjunctive, implicative or other operation, nor any similarity or distance measure. Although, MDI originates from the min-based generalized CRI it can also be regarded as a modified FTV reasoning where the mapping M_I responsible for the inference in truth value space is the identity. We remark here that there exists no t-norm for which $M_I \equiv id$, hence the MDI is not a special case of the FTV reasoning scheme.

As the following Theorem shows the MDI is simple yet powerful and represents the similarity view of reasoning.

Theorem 4.9. *Let the reasoning mechanism be the Membership Driven Inference $B^* = A^* \circ A^{-1} \circ B$, where A, A^* and B are sigmoid-like functions. It fulfills the following properties:*

i) If $A^* = A$ then $B^* = B$ (generalized modus ponens)

ii) If $B^* = {}' \circ B$ then $A^* = {}' \circ A$ for any negation $'$ (generalized modus tollens)

iii) If $C^* = B^* \circ B^{-1} \circ C$ then $C^* = A^* \circ A^{-1} \circ C$ (generalized chain rule)

A more general rule is valid, covering the first two cases:

iv) For any unary operator f, $A^* = f \circ A$ if and only if $B^* = f \circ B$. Note that with the proper f function this case involves the ν-sharpening of A, too.

Moreover, let alone sigmoid-like functions, for any A and B, $A^* \equiv 0$ i.e. undefined if and only if $B^* \equiv 0$, and $A^* \equiv 1$ i.e. unknown if and only if $B^* \equiv 1$.

Proof. Immediate by substitutions. ■

We remark that the problem of fuzzy abduction [59] is also fulfilled by the MDI reasoning scheme: in case B^* is given, and A^* is unknown, then it is easy to see that $A^* = B^* \circ B^{-1} \circ A$.

The properties in the above Theorem are in correspondence to the axioms F1-F4. Property (i) refers to F1 and property (iv) with different f unary operators refers to F2-F4:

- $f \subset id \sim$ F2
- $f \supset id \sim$ F3
- for any f negation $\sim$ F4

Strictly speaking, this inference mechanism is not equivalent to the generalized CRI reasoning scheme of Zadeh (due to different fulfilled axioms), nor to similarity based reasoning (due to the lack of similarity measure). And since $A^*(A^{-1}(x))$ can be treated as an unary operator (a truth-function) or as some kind of (non-commutative) similitude between A and A^* (a "linguistically represented similarity" as it is remarked in [89]), this reasoning scheme can be positioned in between the generalized CRI and the similarity based reasoning schemes.

4.7 Efficient computation of the MDI reasoning scheme

The concept of sigmoid-like functions were fundamental in previous sections. However, its definition is too general for practical use, because it even allows the use of non-analytical functions, of which the calculation of the inverse can be inefficient. This wide range of applicable functions could be narrowed in many ways. We argue that the squashing function is a good choice for representing membership functions, for the following reasons. The squashing function is a strictly monotone approximation of piecewise-linear S-, or Z-shaped functions, which appear frequently in literature. For example Hellendoorn [51] refers to them as "increasing and decreasing fuzzy numbers". Its advantages are the analytical form and the continuous derivative. By proper constructions, any piecewise-linear fuzzy set can be approximated by a summation of squashing functions.

The following theorem shows that calculations of the MDI reasoning scheme is simple with squashing membership functions and the calculations can be performed on the parameters.

Theorem 4.10. *Let the inference mechanism be the Membership Driven Inference*

$$B^*(y) = A^*(A^{-1}(B(y))).$$

If all membership functions are squashing functions, i.e. if

$$\begin{aligned} A(x) &= S(x; \beta, a, \delta_a) \\ A^*(x) &= S(x; \beta, a^*, \delta_{a^*}) \\ B(x) &= S(x; \beta, b, \delta_b) \end{aligned}$$

then $B^*(x) = S(x; \beta, b^*, \delta_{b^*})$, *where*

$$b^* = b + \frac{\delta_b}{\delta_a}(a^* - a) \qquad \delta_{b^*} = \frac{\delta_b \delta_{a^*}}{\delta_a} \tag{4.24}$$

Proof. The inverse of A is $\delta_a S^{-1}(x;\beta) + a$, so

$$
\begin{aligned}
B^*(x) = A^*(A^{-1}(B(x))) &= S((\delta_a/\delta_b(x-b)+a-a^*)/\delta_{a^*};\beta) \\
&= S\left(\frac{\delta_a}{\delta_{a^*}\delta_b}\left(x-\left(b+\frac{\delta_b}{\delta_a}(a^*-a)\right)\right);\beta\right) \\
&= S\left(x;\beta, b+\frac{\delta_b}{\delta_a}(a^*-a), \frac{\delta_{a^*}\delta_b}{\delta_a}\right)
\end{aligned}
$$

■

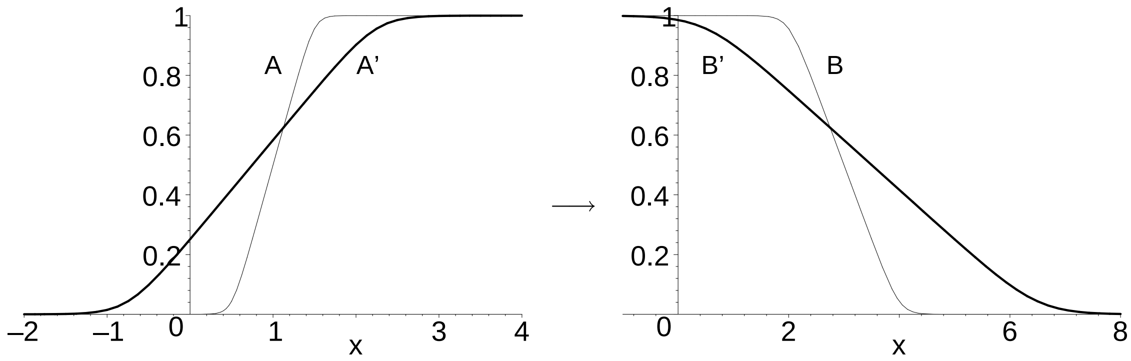


Figure 4.6: An example for MDI reasoning with squashing functions. On the left are the antecedent $A(x) = S(x;16,1,1)$ and the input $A^*(x) = S(x;16,3/4,3)$ (thick), on the right the conclusion $B(x) = S(x;-16,3,2)$ and the output $B^*(x) = S(x;-16,7/2,6)$ (thick)

Trapezoidal fuzzy intervals are special piecewise-linear fuzzy sets on the real line. Alternatively to their common, case-based definitions, they can be defined in the following way, too:

$$\Pi(x;a_L,\delta^L,a_R,\delta^R) = [(x-a_L)/\delta^L] - [(x-a_R)/\delta^R],$$

where a_L and a_R are the centers and δ^L, δ^R are the widths of the left and right spreads. In order to preserve the trapezoidal or triangular shape the following inequality must hold: $a_L + \delta^L/2 \leq a_R - \delta^R$. These Π-functions can be approximated by the following construction of squashing functions

$$A\Pi(x;a_L,\delta^L,a_R,\delta^R,\beta) = S(x;\beta,a_L,\delta^L) - S(x;\beta,a_R,\delta^R). \tag{4.25}$$

Reasoning with approximated trapezoidal fuzzy sets can also be done symbolically. In this case, similarly to LR fuzzy intervals, the calculations must be done independently for the left and right hand sides, and the result is a recombination of them.

Theorem 4.11. *Let the inference mechanism be the Membership Driven Inference*

$$B^*(y) = A^*(A^{-1}(B(y))).$$

Suppose $\beta > 0$ *and finite. If all fuzzy sets are trapezoidal fuzzy intervals approximated by squashing functions, i.e. if*

$$A(x) = A\Pi(x;\beta,a_L,\delta_a^L,a_R,\delta_a^R)$$

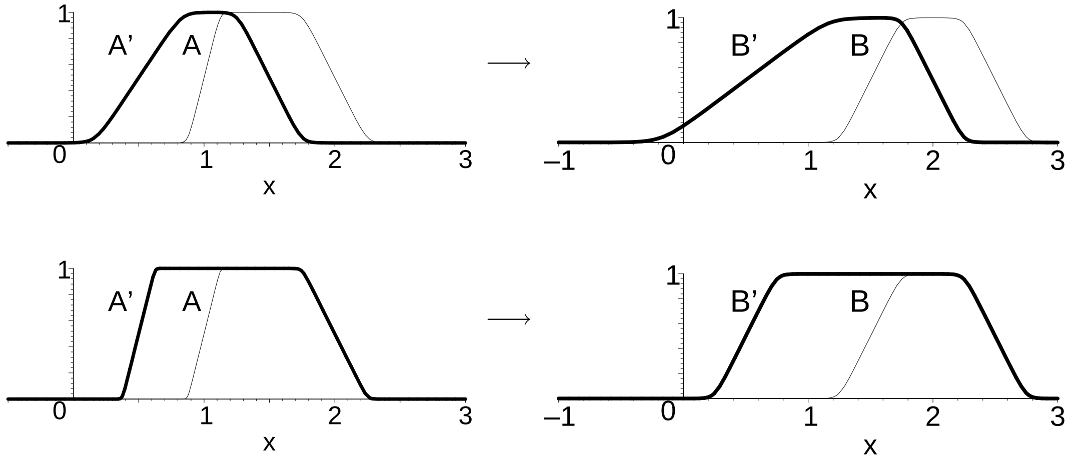


Figure 4.7: An example for MDI reasoning with squashing trapezoidals.

$$A^*(x) = A\Pi(x; \beta, a_L^*, \delta_{a^*}^L, a_R^*, \delta_{a^*}^R)$$
$$B(x) = A\Pi(x; \beta, b_L, \delta_b^L, b_R, \delta_b^R)$$

then $B^*(x) = A\Pi(x; \beta, b_L^*, \delta_{b^*}^L, b_R^*, \delta_{b^*}^R)$, *where*

$$b_L^* = b_L + \frac{\delta_b^L}{\delta_a^L}(a_L^* - a_L) \qquad \delta_{b^*}^L = \frac{\delta_b^L \delta_{a^*}^L}{\delta_a^L} \tag{4.26}$$

$$b_R^* = b_R + \frac{\delta_b^R}{\delta_a^R}(a_R^* - a_R) \qquad \delta_{b^*}^R = \frac{\delta_b^R \delta_{a^*}^R}{\delta_a^R} \tag{4.27}$$

Remark 4.12. *In certain settings, the conclusion may not be an approximated trapezoidal or triangular fuzzy interval, i.e. the inequality* $b_L^* + \delta_{b^*}^L/2 \leq b_R^* - \delta_{b^*}^R/2$ *may not hold. In such cases the conclusion may be considered undefined.*

4.8 Summary

In the first part of this chapter we have investigated two fundamental approaches to approximate reasoning, the compositional rule of inference and the fuzzy truth values based reasoning. They are common in the sense that the so-called indetermination of the conclusion appears in both models. We have argued that this phenomenon is acceptable only in the logic view of reasoning, and so it should be avoided regarding the similarity based view. Sigmoid-like membership functions were introduced to avoid the indetermination of the conclusion. The generalized CRI reasoning scheme was investigated for all three representative t-norms. Only the Łukasiewicz t-norm based CRI scheme is not closed under sigmoid-like functions and from all three, the minimum is the best regarding complexity. The Membership Driven Inference was introduced by modifying the min-based CRI on sigmoid-like functions in order to get a simple yet powerful reasoning scheme. It was shown that it has a series of good properties, it fulfills the generalized modus ponens, the generalized modus tollens, the generalized chain rule, and more.

In the second part we have focused on the efficient computation of the MDI reasoning scheme. The class of squashing functions is revisited. This class of membership functions can arbitrarily approximate piecewise-linear fuzzy sets. Moreover, we have shown that by using squashing membership functions, the MDI reasoning scheme can be calculated on the parameters of the memberships instead of a pointwise computation. This efficient calculation of a rule's output can be applied to approximated trapezoidal and triangular fuzzy intervals, too, by an LR decomposition of them.

Chapter 5

Calculations of operations on fuzzy truth values

The concept of fuzzy truth values (FTVs) was introduced by Bellman and Zadeh [14]. According to their interpretation, every statement (A) can be regarded as relatively (locally) true to another statement (B). The degree of truth f of "x is A" assuming that "x is B" is true can be calculated by applying the extension principle. The theory of fuzzy truth values had many applications, mainly in the field of approximate reasoning [37, 10, 12, 11, 87, 48]. Fuzzy truth values are the next level of generalization of truth values following classical two-valued, type-1 and interval-valued fuzzy logic. Although every level allowed a more subtle representation of truth, and induced various interpretations, they required more and more complex computations.

In recent years, the popularity of type-2 fuzzy sets has been rapidly increasing. Type-2 fuzzy systems show promising results in outperforming type-1 fuzzy systems. Nowadays only interval-valued fuzzy systems are used instead of fully type-2 ones, mainly because of efficiency reasons. For example, the calculation of the conjunction of interval-valued fuzzy sets is far less complex that of general type-2 fuzzy sets. The main bottleneck of type-2 fuzzy systems is the computational complexity of set-operations, like logical operations, type-reduction or defuzzification. A consequence of the strong connection between type-2 fuzzy sets and fuzzy truth values is that the results on the latter can be interpreted in both.

In this chapter we show formulas with low computational complexity for logical operations on specific classes of non interval-valued fuzzy truth values. Our goal is not to approximate the resultant fuzzy truth value (e.g. a conjunction of two) with simple functions, but to give explicit, pointwise formulas for efficient calculations. The results can be directly applied to type-2 fuzzy systems and to reasoning systems based on fuzzy truth values.

5.1 Preliminaries

Definition 5.1. *Fuzzy truth values are mappings of $\mathcal{I}$ onto itself. The set of fuzzy truth values is denoted by $\mathcal{F}$.*

Fuzzy truth values have many interpretations. They can be used as fuzzy truth-qualifications, modifier functions, fuzzy quantifiers and many more. The classical example of the use of fuzzy truth values interpreted as truth-qualifications is the following. Suppose two truth-qualified facts

$$\begin{aligned}&(x\ is\ A)\ \text{is}\, f,\\&(y\ is\ B)\ \text{is}\, g,\end{aligned}$$

where A, B are fuzzy sets, and $f, g \in \mathcal{F}$ are truth-qualifications. The conjunction and the disjunction of these facts are the truth-qualified compound statements

$$\begin{aligned}&((x\ is\ A)\,\text{and}\,(y\ is\ B))\,\text{is}\, f \blacktriangle g,\\&((x\ is\ A)\ \text{ or }\ (y\ is\ B))\,\text{is}\, f \blacktriangledown g.\end{aligned}$$

where $f \blacktriangle g$ and $f \blacktriangledown g$ are compound truth-qualifications, and $\blacktriangle$ and $\blacktriangledown$ are operations on truth-qualifications, meaning the conjunction and disjunction of them. For example, suppose the facts *'John is tall' is fairly true*, and *'John is strong' is very true*. The compound statement *'John is tall and strong'* is then qualified by the conjunction of *'fairly true'* and *'very true'*.

Mizumoto and Tanaka [64, 65] and Baldwin and Guild [10, 11] were the first who tackled the problem of calculating compound fuzzy truth values. They considered the following formulas – derived directly from Zadeh's extension principle – for the conjunction and the disjunction of fuzzy truth values

$$(f \sqcap g)(z) = \bigvee_{z=x\wedge y} \left(f(x) \wedge g(y)\right), \tag{5.1}$$

$$(f \sqcup g)(z) = \bigvee_{z=x\vee y} \left(f(x) \wedge g(y)\right). \tag{5.2}$$

In fact, these are the extended minimum and maximum operations on the set of fuzzy truth values. In recent type-2 fuzzy logic literature they are referred as meet and join. Baldwin and Guild proposed pointwise formulas for their calculations:

$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g), \tag{5.3}$$

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g), \tag{5.4}$$

where the unary operations R and L have the following definitions.

Definition 5.2. *For all fuzzy truth values f let*

$$f^R(x) = \bigvee_{y\geq x} f(y) \quad \textit{and} \quad f^L(x) = \bigvee_{y\leq x} f(y). \tag{5.5}$$

Note, that f^L and f^R are monotonic functions (see fig. 5.1), and that $f^{LR} = \left(f^L\right)^R = \left(f^R\right)^L$ is a constant function which takes the supremum of f.

The algebraic properties of the operations $\sqcap$ and $\sqcup$ (especially on normal and convex fuzzy truth values) are thoroughly investigated by Walker and Walker [90].

The above operations, i.e. (5.1) and (5.2) inherently assume the non-interactivity of their arguments. This non-interactivity is represented by the t-norm $\wedge$ inside the supremum between $f(x)$ and $g(y)$. Non-interactivity is a similar notion to independence in probability theory, it means that the fuzzy truth values in question have no effect on each other. Interactivity is usually modeled by a t-norm.

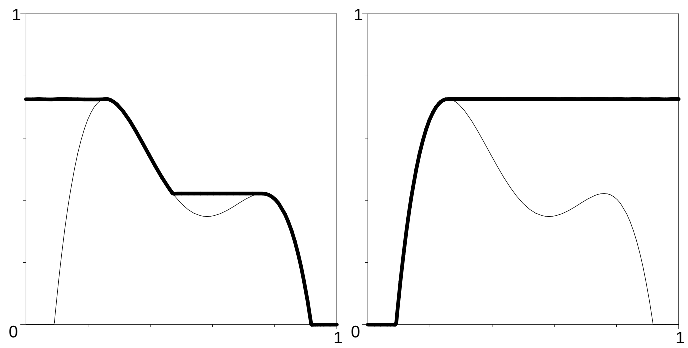


Figure 5.1: The fuzzy truth values f^R and f^L (thick lines)

Godo et al. [48] were the first who considered operations on interactive fuzzy truth values, and generalized the formulas of conjunction and disjunction. Instead of extended *min* and *max* they extended an arbitrary t-norm $\triangle$ and a t-conorm $\bigtriangledown$, which also served to realize interactivity in their setting:

$$(f \blacktriangle g)(z) = \bigvee_{z=x\triangle y} (f(x) \triangle g(y)) \,, \tag{5.6}$$

$$(f \blacktriangledown g)(z) = \bigvee_{z=x\bigtriangledown y} (f(x) \triangle g(y)) \,. \tag{5.7}$$

There can be two other definitions for the disjunction. Instead of the extension principle, these are based on the De Morgan identity between $\triangle$ and $\bigtriangledown$ assuming different negation operations.

$$(f \blacktriangledown_2 g)(z) = \bigwedge_{z=x\bigtriangledown y} (f(x) \bigtriangledown g(y)) \,, \tag{5.8}$$

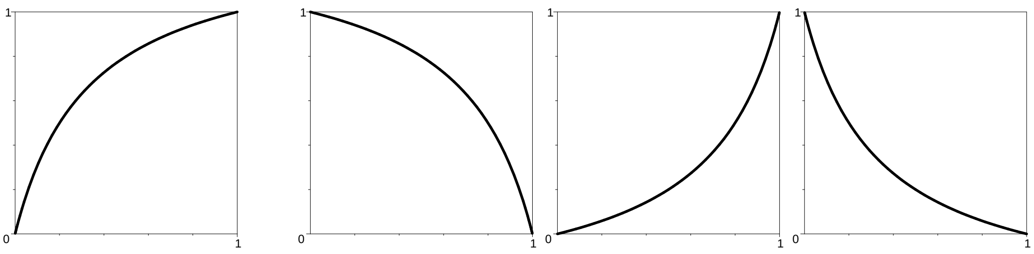
$$(f \blacktriangledown_3 g)(z) = \bigwedge_{z=x\triangle y} (f(x) \bigtriangledown g(y)) \,. \tag{5.9}$$

The three disjunctions (eqs. (5.7)–(5.9)) are implied by the following three possible definitions of a negated fuzzy truth value:

- $f_1^*(x) = f(x')$ (e.g. fairly true – fairly false)
- $f_2^*(x) = (f(x'))'$ (e.g. fairly true – very true)
- $f_3^*(x) = (f(x))'$ (e.g. fairly true – very false)

where $'$ denotes a strong negation on $\mathcal{I}$ (see fig. 5.2).

In this chapter, according to the extension principle and analogously to Walker and Walker [90] and Godo et al. [48], but generalizing their definitions, we will consider the conjunction, the disjunction, and the negation of fuzzy truth values as convolutions of the t-norm $\triangle_1$, t-conorm $\bigtriangledown$ and negation $'$ with respect to the t-norms $\triangle$, $\triangle_2$ and $\vee$. The next definition is fundamental, we will refer to it often.



(a) f *'fairly true'* (b) f_1^* *'fairly false'* (c) f_2^* *'very true'* (d) f_3^* *'very false'*

Figure 5.2: Results of different negations on *fairly true*.

Definition 5.3. *Let $f, g \in \mathcal{F}$ be fuzzy truth values, $\triangle$, $\triangle_1$ and $\triangle_2$ t-norms, $\bigtriangledown$ a t-conorm, and $'$ a strong negation. The conjunction and disjunction of fuzzy truth values are functions $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$:*

$$\begin{aligned}(f \blacktriangle g)(z) &= \bigvee_{z = x \,\triangle_1\, y} (f(x) \triangle_2 g(y)) \,, \\ (f \blacktriangledown g)(z) &= \bigvee_{z = x \,\bigtriangledown\, y} (f(x) \triangle g(y)) \,.\end{aligned}$$

The negation of a fuzzy truth value is a function $\mathcal{F} \to \mathcal{F}$:

$$f^*(z) = \bigvee_{z = y'} f(y) = f(z'). \tag{5.10}$$

For specific operations we will use indexes on $\blacktriangle$ and $\blacktriangledown$. For example, the extended product operation between non-interactive fuzzy truth values will be denoted by $\blacktriangle_P^{\wedge}$, i.e.

$$(f \blacktriangle_P^{\wedge} g)(z) = \bigvee_{z = xy} (f(x) \wedge g(y)) \,.$$

Analogously to P, W will denote the Łukasiewicz operations. Clearly, the operations $\sqcap$ (meet) and $\sqcup$ (join) could alternatively be denoted by $\blacktriangle_\wedge^\wedge$ and $\blacktriangledown_\vee^\wedge$.

Note, that the above definitions are generalizations of (5.6) and (5.7), and that the negation of a fuzzy truth value f defined as the unary convolution of $'$ w.r.t. $\vee$ coincides with f_1^*. We will use the terms *strict conjunction (disjunction)* and *nilpotent conjunction (disjunction)* for the convolutions with strict/nilpotent t-norm $\triangle_1$ and t-conorm $\bigtriangledown$, independently of $\triangle_2$, i.e. the interactivity between them.

Godo et al. have shown properties of the conjunction $\blacktriangle$, in the special case $\triangle_1 = \triangle_2$, for fuzzy truth values of the form

$$h_\mu(x) = \begin{cases} (x/\mu) \wedge 1, & \text{if } \mu \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 5.4 (Godo et al. [48])**.** *If $\triangle = \triangle_1 = \triangle_2$, the following hold for the conjunction $\blacktriangle$:*

- *it is commutative and associative,*
- $h_\mu \blacktriangle h_\nu \geq h_\mu \vee h_\nu$*,*

- $h_\mu \sqcap h_\nu = h_\mu \vee h_\nu = h_{\mu\wedge\nu}$.

It is easy to see that by duality, the operation $\blacktriangledown$ also satisfies

- commutativity and associativity,
- $h_\mu \blacktriangledown h_\nu \leq h_\mu \wedge h_\nu$,
- $h_\mu \sqcup h_\nu = h_\mu \wedge h_\nu = h_{\mu\vee\nu}$.

Definition 5.5. *Fuzzy truth values are mappings of $\mathcal{I}$ onto itself. The set of fuzzy truth values is denoted by $\mathcal{F}$.*

We remark, that this definition of fuzzy truth values can be generalized in many ways (for example by considering a lattice instead of $\mathcal{I}$), but we will use the above.

For a fuzzy truth value f let

$$f^R(x) = \bigvee_{y\geq x} f(y) \quad \text{and} \quad f^L(x) = \bigvee_{y\leq x} f(y).$$

Note, that $\left(f^L\right)^R = \left(f^R\right)^L = f^{LR}$ is a constant function which takes the supremum of f everywhere. The following hold for all $f, g \in \mathcal{F}$ ($\leq$ is a pointwise relation).

1. $f \leq f^R; \quad f \leq f^L$.
2. $(f \wedge g)^R \leq f^R \wedge g^R; \quad (f \wedge g)^L \leq f^L \wedge g^L$.
3. $(f \vee g)^R = f^R \vee g^R; \quad (f \vee g)^L = f^L \vee g^L$.

Definition 5.6. *A fuzzy truth value $f \in \mathcal{F}$ is*

1. *left-maximal (resp. right-maximal) if $f^L = f^{LR}$ (resp. $f^R = f^{LR}$).*
2. *normal if f^{LR} is the constant function 1. The set of normal fuzzy truth values is denoted by $\mathcal{F}_N$.*
3. *convex if for all $x \leq y \leq z$, $f(x) \wedge f(z) \leq f(y)$, or equivalently if $f = f^L \wedge f^R$ (see [90]). The set of convex fuzzy truth values is denoted by $\mathcal{F}_C$.*
4. *an interval if it is the characteristic function of a closed subinterval of $\mathcal{I}$. The set of interval fuzzy truth values is denoted by $\mathcal{F}_I$.*
5. *monotone increasing if and only if $f^L = f$, and monotone decreasing if and only if $f^R = f$. These sets of fuzzy truth values will be denoted by $\mathcal{F}^+$ and $\mathcal{F}^-$, respectively.*

Note, that there are further conditions of normality which are equivalent to the definition. For example, $f \in \mathcal{F}_N$ if and only if $f^R(0) = 1$ (or $f^L(1) = 1$), because $f^R(0)$ is the supremum of f over the real unit interval.

Two special fuzzy truth values are the following.

$$\mathbf{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{1}(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

According to Zadeh's extension principle, a two-place function $\circ : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ can be extended to $\bullet : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ by the convolution of $\circ$ with respect to $\wedge$ and $\vee$. Let $f, g \in \mathcal{F}$, then

$$(f \bullet g)(z) = \bigvee_{z = x \circ y} \left(f(x) \wedge g(y) \right).$$

We suppose non-interactivity, i.e. $\wedge$ is not generalized to a t-norm.

A unary function, such as a strong negation $'$ on $\mathcal{I}$ extended on the set of fuzzy truth values has the following definition:

$$f^*(x) = \bigvee_{x = y'} f(y) = f(x').$$

In particular, if $\circ$ is a t-norm $\triangle$ or a t-conorm $\triangledown$, its extension is called a type-2 t-norm or t-conorm. We have the following definitions for type-2 t-norms and t-conorms.

Definition 5.7. *Let $\triangle$ and $\triangledown$ be a t-norm and a t-conorm, then their extensions $\blacktriangle$ and $\blacktriangledown$ are defined as follows.*

$$(f \blacktriangle g)(z) = \bigvee_{z = x \triangle y} \left(f(x) \wedge g(y) \right),$$
$$(f \blacktriangledown g)(z) = \bigvee_{z = x \triangledown y} \left(f(x) \wedge g(y) \right).$$

The following properties hold for these operations (for a more comprehensive list see [90]):

1. both are commutative and associative.
2. $(f \blacktriangle g)^L = f^L \blacktriangle g^L$; $(f \blacktriangledown g)^L = f^L \blacktriangledown g^L$.
3. $f \blacktriangle \mathbf{1} = f$; $f \blacktriangledown \mathbf{0} = f$ for all $f \in \mathcal{F}$.
4. $\mathbf{1} \blacktriangledown \mathbf{1} = \mathbf{1}$; $\mathbf{0} \blacktriangle \mathbf{0} = \mathbf{0}$.

Here we prove only the first equality of item 4. Let $z = 1$, then

$$(\mathbf{1} \blacktriangledown \mathbf{1})(1) = \bigvee_{1 = x \triangledown y} \left(\mathbf{1}(x) \wedge \mathbf{1}(y) \right) = \left(\mathbf{1}(1) \wedge \mathbf{1}(1) \right) \vee \bigvee_{1 = x \triangledown y} \left(\mathbf{1}(x) \wedge \mathbf{1}(y) \right),$$

which clearly equals to 1, since $\mathbf{1}(1) \wedge \mathbf{1}(1) = 1$. For all $z < 1$,

$$(\mathbf{1} \blacktriangledown \mathbf{1})(z) = \bigvee_{1 > z = x \triangledown y} \left(\mathbf{1}(x) \wedge \mathbf{1}(y) \right).$$

This subset of (x, y) pairs clearly does not contain $(1, 1)$ (because it would imply $z = 1$), so x or y is strictly less than 1, i.e. $\mathbf{1}(x)$ or $\mathbf{1}(y)$ is zero, which implies that their minimum is always 0.

The extended minimum and maximum (usually referred as meet and join) are fundamental operations on fuzzy truth values. These operations and their pointwise expressions are

$$\begin{aligned}(f \sqcap g)(z) &= \bigvee_{z=x\wedge y} (f(x) \wedge g(y)) &= \left(\left(f \wedge g^R\right) \vee \left(f^R \wedge g\right)\right)(z),\\(f \sqcup g)(z) &= \bigvee_{z=x\vee y} (f(x) \wedge g(y)) &= \left(\left(f \wedge g^L\right) \vee \left(f^L \wedge g\right)\right)(z).\end{aligned}$$

These operations define two partial orders $\sqsubseteq$ and $\preccurlyeq$ on $\mathcal{F}$. In particular, let

$$\begin{aligned}f \sqsubseteq g &\quad \text{if and only if} \quad f \sqcap g = f,\\f \preccurlyeq g &\quad \text{if and only if} \quad f \sqcup g = g.\end{aligned}$$

On the basis of the above, we will consider the algebra $\mathbf{F} = (\mathcal{F}, \sqcap, \sqcup, \blacktriangle, \blacktriangledown, ^*, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preccurlyeq)$ equipped with an implicative operation. This algebra is very general, it has many subalgebras with interesting and important properties. The subalgebra $\mathbf{F}_{CN}$ of $\mathbf{F}$ (where $\mathbf{F}_{CN} = (\mathcal{F}_{CN}, \sqcap, \sqcup, \blacktriangle, \blacktriangledown, ^*, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preccurlyeq)$ i.e. the algebra of convex normal fuzzy truth values) is of special importance. In $\mathbf{F}_{CN}$, the distributive and absorption laws hold for $\sqcap$ and $\sqcup$ (see [65]), i.e. for all $f, g, h \in \mathbf{F}_{CN}$,

$$f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h); \quad f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h),$$

$$f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) = f.$$

As a consequence, in $\mathbf{F}_{CN}$ the partial orders $\sqsubseteq$ and $\preccurlyeq$ coincide, it is a bounded maximal lattice in $\mathbf{F}$ (maximal among lattices containing an isomorph subalgebra to $\mathcal{I}$), and a De Morgan algebra. It is also complete, i.e. the operations $\sqcap$ and $\sqcup$ can be naturally extended to infinite operands since they are associative.

5.2 Extended operations on continuous fuzzy truth values

In [90] the authors show a pointwise expression similar to (5.3) and (5.4) for the extended minimum and maximum in case of product-interactive fuzzy truth values.

Theorem 5.8 (Walker [90]). *If $\triangle_1 = \wedge$, $\triangledown = \vee$, and $\triangle_2$ is the product, then the following hold for all $f, g \in \mathcal{F}$:*

$$\begin{aligned}\left(f \blacktriangle_{\wedge}^{P} g\right)(z) &= \bigvee_{z=x\wedge y} (f(x)g(y)) &= \left(\left(f^R g\right) \vee \left(f g^R\right)\right)(z),\\\left(f \blacktriangledown_{\vee}^{P} g\right)(z) &= \bigvee_{z=x\vee y} (f(x)g(y)) &= \left(\left(f^L g\right) \vee \left(f g^L\right)\right)(z).\end{aligned}$$

Since any strict t-norm is isomorphic to the product, the theorem applies to all strict t-norms as well. It is easy to see, that theorem 5.8 also holds for the Łukasiewicz t-norm instead of the product. So the next theorem generalizes the results of [10, 11, 90].

Theorem 5.9. *If $\triangle_1 = \wedge$, $\triangledown = \vee$, and $\triangle_2 = \triangle$ is an arbitrary continuous and*

Archimedean t-norm, then the following hold for all $f, g \in \mathcal{F}$:

$$\begin{aligned}(f \blacktriangle_\wedge g)(z) &= \bigvee_{z=x\wedge y} (f(x) \triangle g(y)) &= \left(\left(f^R \triangle g\right) \vee \left(f \triangle g^R\right)\right)(z),\\ (f \blacktriangledown_\vee g)(z) &= \bigvee_{z=x\vee y} (f(x) \triangle g(y)) &= \left(\left(f^L \triangle g\right) \vee \left(f \triangle g^L\right)\right)(z).\end{aligned} \tag{5.11}$$

Proof. Let $f, g \in \mathcal{F}$.

$$\begin{aligned}(f \blacktriangle_\wedge g)(z) &= \bigvee_{z=x\wedge y} (f(x) \triangle g(y))\\ &= \left(\bigvee_{\substack{z=x\\ y\geq z}} (f(x) \triangle g(y))\right) \vee \left(\bigvee_{\substack{z=y\\ x\geq z}} (f(x) \triangle g(y))\right)\\ &= \left(f(z) \triangle \left(\bigvee_{y\geq z} g(y)\right)\right) \vee \left(\left(\bigvee_{x\geq z} f(x)\right) \triangle g(z)\right)\\ &= \left(f(z) \triangle g^R(z)\right) \vee \left(f^R(z) \triangle g(z)\right).\end{aligned}$$

The disjunction can be proved analogously. ■

This theorem covers the extended minimum and maximum operators in case of $\triangle$-interactivity, where $\triangle$ is an arbitrary continuous and Archimedean t-norm. From now on, we restrict our investigations to extensions of continuous and Archimedean t-norms and t-conorms. The following theorems show transformations of the convolutions of Definition 5.3 and serve as a basis for computational simplifications.

Theorem 5.10. *If $\triangle_1$ and $\triangle_2$ are t-norms, s.t. $\triangle_1$ is continuous and Archimedean, then the following hold for all $f, g \in \mathcal{F}$. For $z > 0$:*

$$(f \blacktriangle g)(z) = \bigvee_{x\geq z} (f(x) \triangle_2 g(x \rhd_1 z)) = \bigvee_{y\geq z} (f(y \rhd_1 z) \triangle_2 g(y)). \tag{5.12}$$

If $\triangle_1$ is strict then for $z = 0$:

$$(f \blacktriangle g)(0) = \left(f(0) \triangle_2 g^R(0)\right) \vee \left(f^R(0) \triangle_2 g(0)\right), \tag{5.13}$$

and if $\triangle_1$ is nilpotent then for $z = 0$:

$$(f \blacktriangle g)(0) = \bigvee_x \left(f(x) \triangle_2 g^L(x')\right) = \bigvee_y \left(f^L(y') \triangle_2 g(y)\right), \tag{5.14}$$

where $\rhd_1$ denotes the residual implication of $\triangle_1$, and $x' = (x \rhd_1 0)$ is the strong negation corresponding to $\rhd_1$.

Proof. Continuous and Archimedean t-norms have a generator functional form, so let

$$x \triangle_1 y = \varphi^{-1} \{\varphi(0) \wedge (\varphi(x) + \varphi(y))\},$$

where $\varphi : [0, 1] \to [0, \infty]$ is a strictly decreasing and continuous function with $\varphi(1) = 0$.

So, by definition

$$(f \blacktriangle g)(z) = \bigvee_{\varphi(z)=(\varphi(x)+\varphi(y))\wedge\varphi(0)} (f(x) \triangle_2 g(y)) .$$

By supposing $z > 0$, i.e. $\varphi(z) < \varphi(0)$, the constraint is $\varphi(z) = \varphi(x) + \varphi(y)$. This implies $\varphi(x) \leq \varphi(z)$, i.e. $x \geq z$, since φ is non-negative and strictly decreasing. Furthermore, $y = \varphi^{-1}(\varphi(z) - \varphi(x)) = x \rhd_1 z$, where $\rhd_1$ is the residual implication of $\triangle_1$. Analogously, $y \geq z$, and $x = \varphi^{-1}(\varphi(z) - \varphi(y)) = y \rhd_1 z$. So in case $z > 0$ we have

$$(f \blacktriangle g)(z) = \bigvee_{x \geq z} (f(x) \triangle_2 g(x \rhd_1 z)) = \bigvee_{y \geq z} (f(y \rhd_1 z) \triangle_2 g(y)) .$$

Suppose $z = 0$ and $\triangle_1$ is strict. Since $x \triangle_1 y = 0$ if and only if $x \wedge y = 0$, Theorem 5.9 can be applied with $z = 0$.

Now, suppose $z = 0$ and $\triangle_1$ is nilpotent. In this case $x \triangle_1 y = 0$ if and only if $y \leq (x \rhd_1 0) = x'$, thus we have

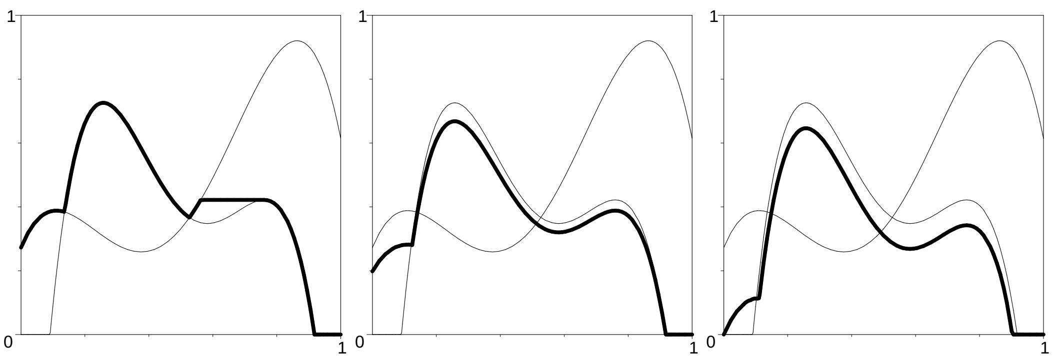
$$\begin{aligned}(f \blacktriangle g)(0) &= \bigvee_{\substack{x\in[0,1]\\ y\leq x'}} (f(x) \triangle_2 g(y)) = \bigvee_{x} \left(f(x) \triangle_2 \bigvee_{y \leq x'} g(y) \right) \\ &= \bigvee_{x} \left(f(x) \triangle_2 g^L(x') \right)\end{aligned}$$

An equivalent condition to $y \leq x'$ is $y' \geq x$, and so by similar transformations

$$(f \blacktriangle g)(0) = \bigvee_{y} \left(f^L(y') \triangle_2 g(y) \right) .$$

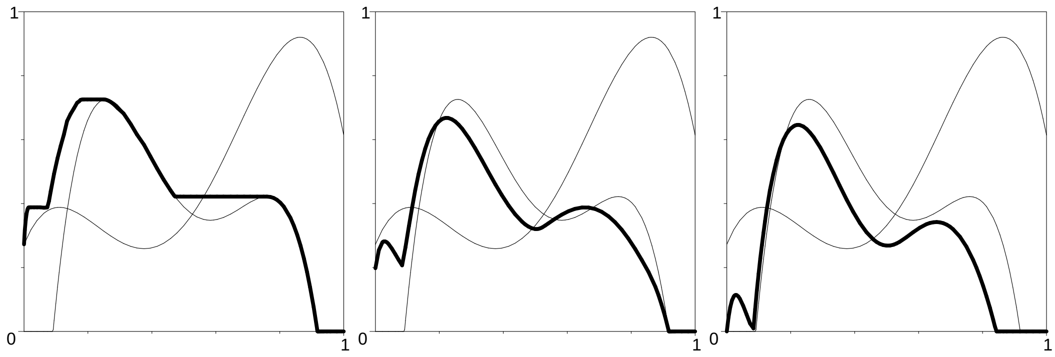
■

Note that the symmetry in (5.12) stems from the commutativity of $\triangle_1$. See figures 5.3, 5.4 and 5.5 for examples of extended t-norms in a general setting. A similar theorem holds for extended Archimedean t-conorms, which can be proved in an analogous matter. Figures 5.6, 5.7 and 5.8 show examples of extended t-conorms.



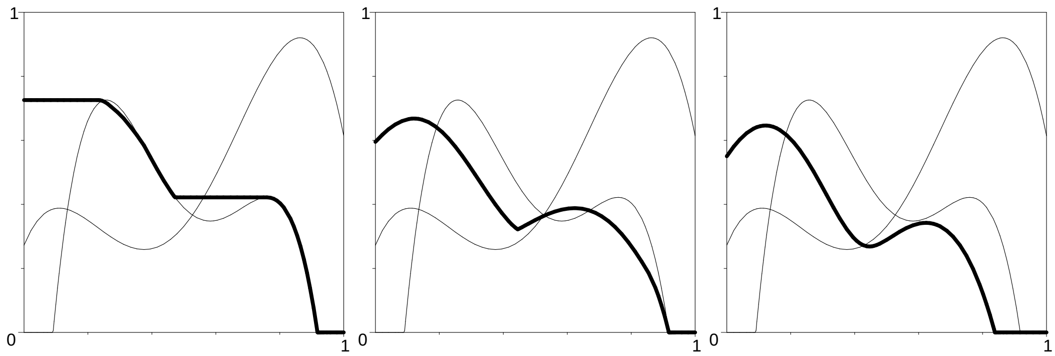
(a) $f \sqcap g$ (b) $f \blacktriangle^P_\wedge g$ (c) $f \blacktriangle^W_\wedge g$

Figure 5.3: The extended minimum of fuzzy truth values with different interactivity



(a) $f \blacktriangle_P^{\wedge} g$ (b) $f \blacktriangle_P^{P} g$ (c) $f \blacktriangle_P^{W} g$

Figure 5.4: The extended product of fuzzy truth values with different interactivity



(a) $f \blacktriangle_W^{\wedge} g$ (b) $f \blacktriangle_W^{P} g$ (c) $f \blacktriangle_W^{W} g$

Figure 5.5: The extended Łukasiewicz t-norm of fuzzy truth values with different interactivity

Theorem 5.11. *If $\triangle$ is a t-norm, $\bigtriangledown$ is a continuous and Archimedean t-conorm, then the following hold for all $f, g \in \mathcal{F}$. For $z < 1$:*

$$(f \blacktriangledown g)(z) = \bigvee_{x \leq z} (f(x) \triangle g(x \lhd z)) = \bigvee_{y \leq z} (f(y \lhd z) \triangle g(y)). \tag{5.15}$$

where $\lhd$ denotes the residual coimplication of $\bigtriangledown$. If $\bigtriangledown$ is strict then for $z = 1$:

$$(f \blacktriangledown g)(1) = \left(f^L(1) \triangle g(1)\right) \vee \left(f(1) \triangle g^L(1)\right), \tag{5.16}$$

and if $\bigtriangledown$ is nilpotent then for $z = 1$:

$$(f \blacktriangledown g)(1) = \bigvee_{x} \left(f(x) \triangle g^R(x')\right) = \bigvee_{y} \left(f^R(y') \triangle g(y)\right), \tag{5.17}$$

where $\lhd$ denotes the residual coimplication of $\bigtriangledown$, and $x' = (x \lhd 1)$ is a strong negation.

5.2.1 Left- and right-maximal and monotonic fuzzy truth values

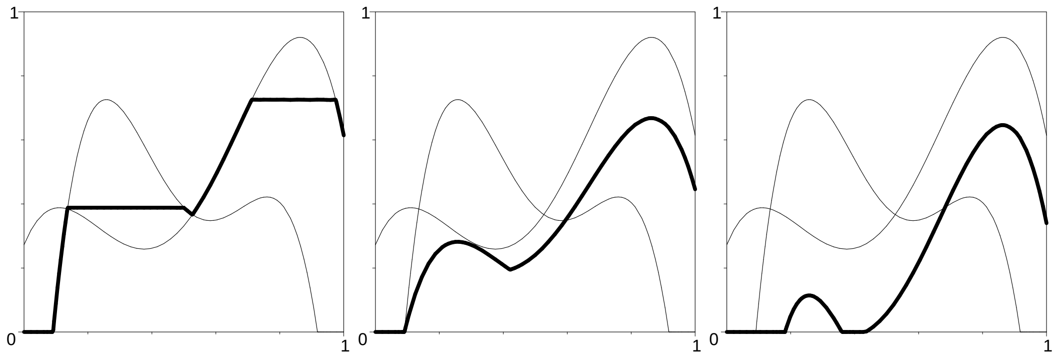
Theorems 5.10 and 5.11 in general do not considerably decrease computational complexity of the extended operations. In this subsection, we restrict our investigations to special classes of fuzzy truth values. With these restrictions, corollaries of the above theorems are shown with practical results.

Definition 5.12 (Nieminen [69], Walker [90]). *A fuzzy truth value f is endmaximal if $f^L = f^R$, left-maximal if $f^L = f^{LR}$, right-maximal if $f^R = f^{LR}$ and normal if $f^{LR} = 1$.*

It is easy to see the following.

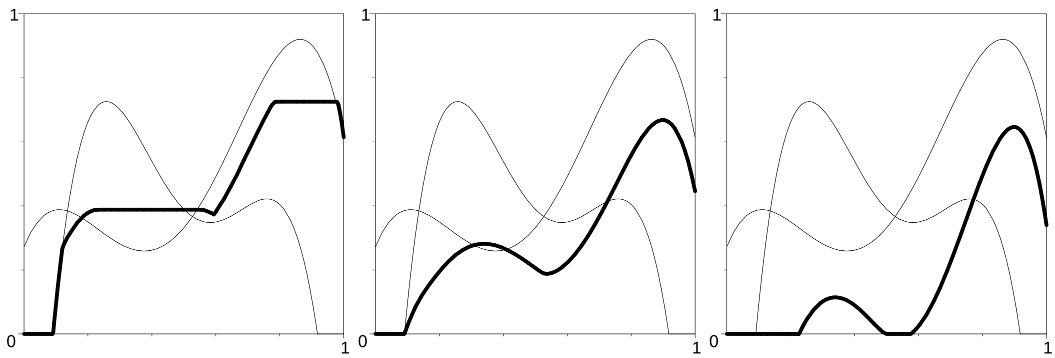
Proposition 5.13. *For all $f \in \mathcal{F}$,*

1. *f is right-maximal iff $f(1) = f^R(0)$.*
2. *f is left-maximal iff $f(0) = f^L(1)$.*



(a) $f \sqcup g$ (b) $f \blacktriangledown^P_\vee g$ (c) $f \blacktriangledown^W_\vee g$

Figure 5.6: The extended maximum of fuzzy truth values with different interactivity

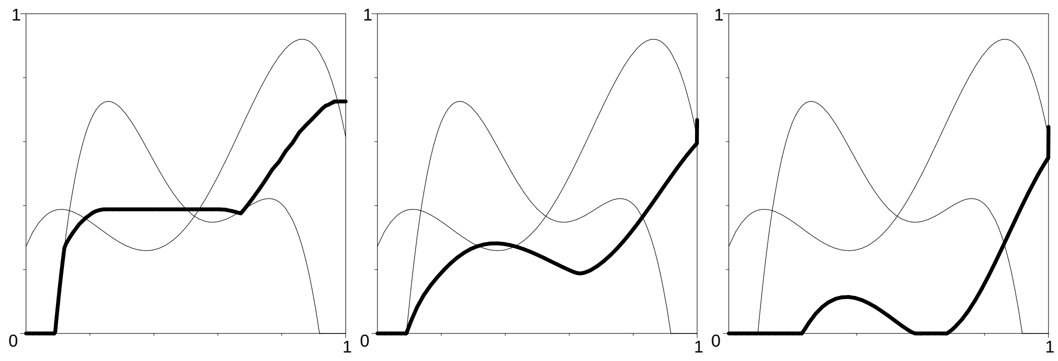


(a) $f \blacktriangledown^\wedge_P g$ (b) $f \blacktriangledown^P_P g$ (c) $f \blacktriangledown^W_P g$

Figure 5.7: The extended algebraic sum of fuzzy truth values with different interactivity

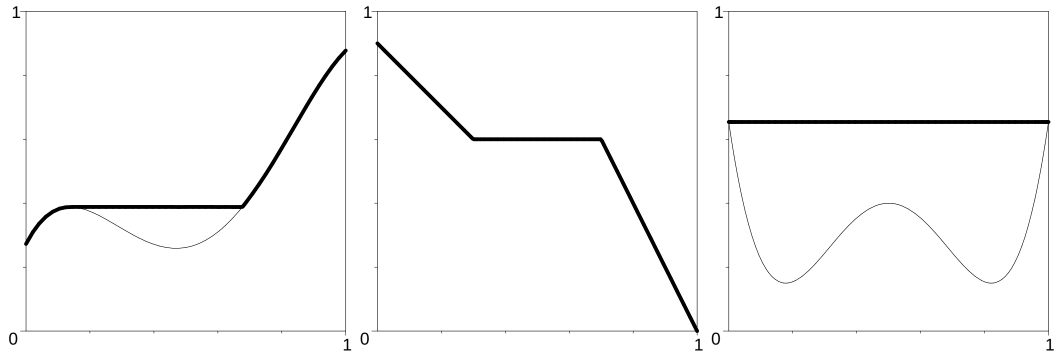
From now on, let $\mathcal{F}^+$ and $\mathcal{F}^-$ denote the set of non-decreasing and non-increasing continuous fuzzy truth values. Note, that if $f \in \mathcal{F}^+$ (resp., $\mathcal{F}^-$) then it is also right-maximal (left-maximal). Monotonic fuzzy truth values are widespread in modeling linguistic modifiers such as *'true', 'very true', 'more or less true', 'false', 'very false', 'more or less false'*, etc..

Next, we give pointwise expressions for the operations $\blacktriangle$ and $\blacktriangledown$ in case of left- and rightmaximal and monotonic fuzzy truth values. Note, that if $f \in \mathcal{F}^+$ then f^R is the constant function which takes the value $f(1)$ everywhere, and $f^L = f$. Analogously, if $f \in \mathcal{F}^-$ then $f^L(x) = f(1)$ for all $x \in \mathcal{I}$, and $f^R = f$. An immediate consequence of Theorem 5.9 is the following.



(a) $f \blacktriangledown_W^{\wedge} g$ (b) $f \blacktriangledown_W^{P} g$ (c) $f \blacktriangledown_W^{W} g$

Figure 5.8: The extended Łukasiewicz t-conorm of fuzzy truth values with different interactivity



(a) (b) (c)

Figure 5.9: (a) a right-maximal f and f^L, (b) a left-maximal and monotonic f and f^R, (c) endmaximal f and f^{LR}

Corollary 5.14. *For all right-maximal $f, g \in \mathcal{F}$*

$$f \blacktriangle_{\wedge} g = \left(f^{LR} \vartriangle g\right) \vee \left(f \vartriangle g^{LR}\right), \tag{5.18}$$

moreover, if $f, g \in \mathcal{F}^+$

$$\left(f \blacktriangledown_{\vee} g\right)(x) = f(x) \vartriangle g(x). \tag{5.19}$$

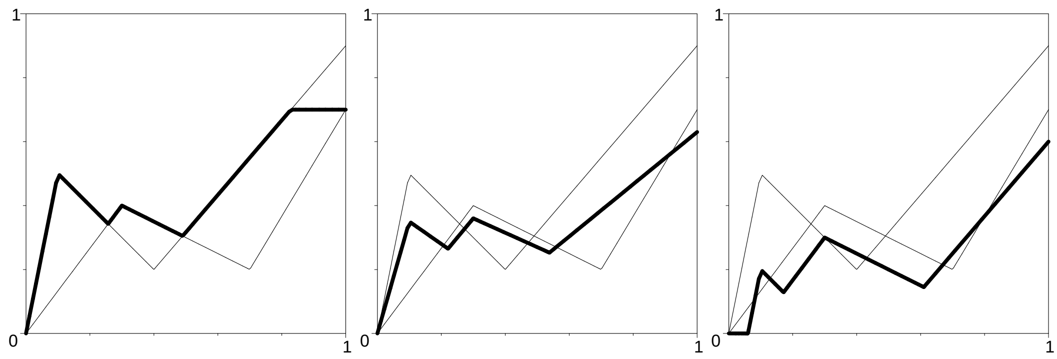
Analogously, for all left-maximal $f, g \in \mathcal{F}$

$$f \blacktriangledown_{\vee} g = \left(f^{LR} \vartriangle g\right) \vee \left(f \vartriangle g^{LR}\right), \tag{5.20}$$

moreover if $f, g \in \mathcal{F}^-$

$$\left(f \blacktriangle_{\wedge} g\right)(x) = f(x) \vartriangle g(x). \tag{5.21}$$

Note, that according to (5.19) and (5.21) the maximum and minimum of two monotone fuzzy truth values can be calculated pointwise with the t-norm representing their



(a) $f \blacktriangle_{\wedge}^{\wedge} g$ (b) $f \blacktriangle_{\wedge}^{P} g$ (c) $f \blacktriangle_{\wedge}^{W} g$

Figure 5.10: The extended minimum operation on right-maximal fuzzy truth values (corollary 5.14)

interactivity.

Corollary 5.15. *For all left-maximal $f, g \in \mathcal{F}$*

$$(f \blacktriangle g)(0) = f(0) \bigtriangleup_2 g(0). \tag{5.22}$$

For all right-maximal f, g:

$$(f \blacktriangledown g)(1) = f(1) \bigtriangleup_2 g(1). \tag{5.23}$$

Important corollaries of Theorem 5.10 and 5.11 are the following.

Corollary 5.16. *If f is right-maximal and $g \in \mathcal{F}^-$, then*

$$(f \blacktriangle g)(x) = f^{LR}(x) \bigtriangleup_2 g(x), \tag{5.24}$$

and $f \blacktriangle g \in \mathcal{F}^-$. Furthermore, if f is also normal, then $f \blacktriangle g = g$, i.e. f acts as a unit element.

Proof. If $z > 0$ then

$$(f \blacktriangle g)(z) = \bigvee_{x \geq z} \left(f(x) \bigtriangleup_2 g(x \bigtriangleup_1 z)\right).$$

It always has a supremum at $x = 1$, so

$$(f \blacktriangle g)(z) = f(1) \bigtriangleup_2 g(1 \bigtriangleup_1 z) = f(1) \bigtriangleup_2 g(z).$$

In case $\bigtriangleup_1$ is strict and $z = 0$,

$$\begin{aligned}(f \blacktriangle g)(0) &= \big(f(0) \bigtriangleup_2 g^R(0)\big) \vee \big(f^R(0) \bigtriangleup_2 g(0)\big) \\ &= \left(f(0) \bigtriangleup_2 g(0)\right) \vee \left(f(1) \bigtriangleup_2 g(0)\right) = f(1) \bigtriangleup_2 g(0).\end{aligned}$$

In case $\bigtriangleup_1$ is nilpotent and $z = 0$,

$$(f \blacktriangle g)(0) = \bigvee_{x} \big(f(x) \bigtriangleup_2 g^L(x \rhd_1 0)\big) = \bigvee_{x} \left(f(x) \bigtriangleup_2 g(0)\right)$$

$$= f(1) \bigtriangleup_2 g(0).$$

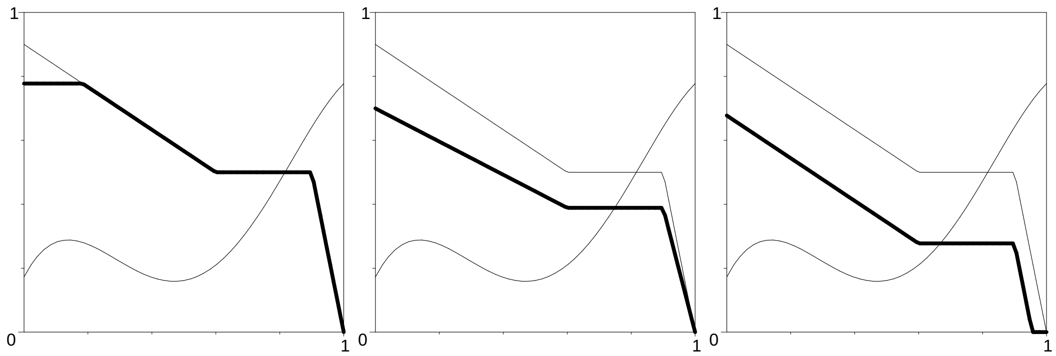
■

We have a similar result for extended disjunctions, which can be proved analogously.

Corollary 5.17. *If f is left-maximal and $g \in \mathcal{F}^+$, then*

$$\left(f \blacktriangledown g\right)(x) = f^{LR}(x) \bigtriangledown g(x), \tag{5.25}$$

and $f \blacktriangledown g \in \mathcal{F}^+$. Furthermore, if f is also normal, then $f \blacktriangledown g = g$.



(a) $f \blacktriangle^{\wedge} g$ (b) $f \blacktriangle^{P} g$ (c) $f \blacktriangle^{W} g$

Figure 5.11: Conjunctions of a right-maximal and a non-increasing fuzzy truth value (corollary 5.16) with various levels of interactivity

Note, that since all non-decreasing fuzzy truth values are right-maximal (and all non-increasing fuzzy truth values are left-maximal), the above results apply to monotonic ones, too. These two corollaries express the natural intuition that the conjunction (resp. disjunction) of a positive, 'true-like' and a negative, 'false-like' fuzzy truth value is the 'weaker' (resp. 'stronger') one. This is in accordance with Boolean logic and type-1 fuzzy logic, too.

5.2.2 Continuity of Operations on Fuzzy Truth Values

In this section we give sufficient conditions for the continuity of the compound fuzzy truth values $f \blacktriangle g$ and $f \blacktriangledown g$. Let $\mathcal{F}_c$ denote the set of continuous fuzzy truth values.

Proposition 5.18. *The strict conjunction $f \blacktriangle g$ of $f, g \in \mathcal{F}_c$ is continuous if f or g is left- or right-maximal.*

Proof. To prove the continuity of $f \blacktriangle g$, it suffices to show

$$\lim_{z \to 0} \left(f \blacktriangle g\right)(z) = \left(f \blacktriangle g\right)(0),$$

since according to (5.12) for all $z > 0$, $\left(f \blacktriangle g\right)(z)$ is continuous. Recall, that for any strict conjunction

$$\left(f \blacktriangle g\right)(0) = \left(f(0) \bigtriangleup_2 g^R(0)\right) \vee \left(f^R(0) \bigtriangleup_2 g(0)\right). \tag{5.26}$$

Now,

$$\begin{aligned}
\lim_{z\to 0} (f \blacktriangle g)(z) &= \lim_{z\to 0} \bigvee_{x\geq z} (f(x) \bigtriangleup_2 g(x \rhd_1 z)) \\
&= \bigvee_{x} (f(x) \bigtriangleup_2 g(x \rhd_1 0)) \\
&= (f(0) \bigtriangleup_2 g(1)) \vee \bigvee_{x>0} (f(x) \bigtriangleup_2 g(0)) \\
&= (f(0) \bigtriangleup_2 g(1)) \vee \left(f^R(0) \bigtriangleup_2 g(0)\right), \qquad (5.27)
\end{aligned}$$

since $\rhd_1$ is a strict residual implication, i.e.

$$x \rhd_1 0 = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If g is right-maximal i.e. $g(1) = g^R(0)$, then clearly (5.27) equals to $(f \blacktriangle g)(0)$.

Furthermore, if g is left-maximal i.e. $g(0) = g^R(0)$, then in both (5.26) and (5.27) the second term dominates the first one, i.e

$$f^R(0) \bigtriangleup_2 g(0) \geq f(0) \bigtriangleup_2 g^R(0)$$

and

$$f^R(0) \bigtriangleup_2 g(0) \geq f(0) \bigtriangleup_2 g(1)$$

and so (5.26) and (5.27) are equal.

It can be shown analogously, that the left- or right-maximality of f is also sufficient for the continuity of $f \blacktriangle g$. ∎

Proposition 5.19. *The nilpotent conjunction $f \blacktriangle g$ of $f, g \in \mathcal{F}_c$ is continuous if $f \in \mathcal{F}_c^+$ or $g \in \mathcal{F}_c^+$, i.e. if f or g is monotone increasing.*

Proof. It suffices to show that

$$\lim_{z\to 0} (f \blacktriangle g)(z) = (f \blacktriangle g)(0).$$

Recall, that for any nilpotent conjunction

$$(f \blacktriangle g)(0) = \bigvee_{x} \left(f(x) \bigtriangleup_2 g^L(x \rhd_1 0)\right). \qquad (5.28)$$

Now,

$$\begin{aligned}
\lim_{z\to 0} (f \blacktriangle g)(z) &= \lim_{z\to 0} \bigvee_{x\geq z} (f(x) \bigtriangleup_2 g(x \rhd_1 z)) \\
&= \bigvee_{x} (f(x) \bigtriangleup_2 g(x \rhd_1 0)) \\
&= \bigvee_{x} \left(f(x) \bigtriangleup_2 g(x')\right). \qquad (5.29)
\end{aligned}$$

It is easy to see, that (5.28) and (5.29) are equal if $g \in \mathcal{F}^+$, i.e. $g = g^L$.

It can be shown analogously, that the monotonicity of f is also sufficient for the continuity of $f \blacktriangle g$. ∎

The next two propositions can be proved analogously.

Proposition 5.20. *The strict disjunction $f \blacktriangledown g$ of $f, g \in \mathcal{F}_c$ is continuous if f or g is left- or right-maximal.*

Proposition 5.21. *The nilpotent disjunction $f \blacktriangledown g$ of $f, g \in \mathcal{F}_c$ is continuous if $f \in \mathcal{F}_c^-$ or $g \in \mathcal{F}_c^-$.*

5.3 Extended Łukasiewicz operations on linear fuzzy truth values

The Łukasiewicz t-norm $0 \vee (x + y - 1)$ will be denoted by $\triangle_W$. Its residual implication $1 \wedge (1 - x + y)$ and coimplication $0 \vee (y - x)$ will be denoted by $\rhd_W$ and $\lhd_W$.

Definition 5.22. *Let the Łukasiewicz conjunction and disjunction of fuzzy truth values be*

$$(f \blacktriangle_W g)(z) = \bigvee_{z=(x+y-1)\vee 0} ((f(x) + g(y) - 1) \vee 0) \tag{5.30}$$

$$(f \blacktriangledown_W g)(z) = \bigvee_{z=(x+y)\wedge 1} ((f(x) + g(y) - 1) \vee 0) \tag{5.31}$$

Note, that this is a special case of definition 5.3. In this setting the arguments are interactive, and this interactivity is represented by the Łukasiewicz t-norm. Here we do not discuss the non-interactive case, for references, see Nguyen's theorem in [68]. Fullér and Keresztfalvi [43] generalized the Nguyen theorem to non-interactive arguments, but did not provide explicit formulas for computations. Here, we provide pointwise, easy-to-compute formulas for the above operations on linear fuzzy truth values.

Definition 5.23. *Let $\mathcal{L} \subset \mathcal{F}_c$ be the set of linear fuzzy truth values characterized by*

$$f_{a,b} \in \mathcal{L} \quad \Longleftrightarrow \quad f_{a,b}(x) = \left\{ \frac{x - a}{b - a} \right\}_0^1, \tag{5.32}$$

where $a \neq b$, $x \in [0, 1]$ and $\{t\}_a^b = a \vee t \wedge b$.

Let $\mathcal{L}^+ \subset \mathcal{F}_c^+$ denote the set of non-decreasing, and $\mathcal{L}^- \subset \mathcal{F}_c^-$ the set of non-increasing linear fuzzy truth values. A linear $f_{a,b}$ is non-decreasing iff $a < b$ and non-increasing iff $a > b$.

The set of normal, non-decreasing (non-increasing) linear fuzzy truth values is $\mathcal{L}_1^+$ (resp. $\mathcal{L}_1^-$) and characterized by $b \leq 1$ (resp. $b \geq 0$).

The next theorem states that the Łukasiewicz conjunction of non-decreasing linear fuzzy truth values can be calculated by a computationally simple pointwise formula instead of a convolution.

Theorem 5.24. *The following hold for all $f_i = f_{a_i,b_i} \in \mathcal{L}^+$ $(i = 1, 2)$.*

$$\begin{aligned}(f_1 \blacktriangle_W f_2)(z) = \quad & (f_1(1) \vartriangle_W f_2(\{b_1\}_z \rhd_W z)) \\ & \vee (f_2(1) \vartriangle_W f_1(\{b_2\}_z \rhd_W z)) \,,\end{aligned} \tag{5.33}$$

where $\{x\}_z = z \vee x \wedge 1$.

The following Lemma is required for the proof.

Lemma 5.25. *Let $f_1 = f_{a_1,b_1} \in \mathcal{L}^+$ and $f_2 = f_{a_2,b_2} \in \mathcal{L}^-$ be two linear fuzzy truth values, let S denote their sum, $S(x) = f_{a_1,b_1}(x) + f_{a_2,b_2}(x)$. Then,*

$$S^R(z) = \bigvee_{x \geq z} S(x) = S(\{b_1\}_z) \vee S(\{b_2\}_z) \quad \forall z \in [0, 1].$$

where $\{x\}_z = z \vee x \wedge 1$.

Proof. Based on the relationship between b_1,b_2 and z, there are four cases:

1. If $b_1 \vee b_2 \leq z$, then $S(x) = 1 + f_2(x)$ for all $x \geq z$. Thus, $S^R(z) = S(z)$.
2. If $b_1 \leq z < b_2$, then similarly to the previous case $S^R(z) = S(z) = S(b_2 \wedge 1)$.
3. If $b_2 \leq z < b_1$, then $S^R(z) = S(z) \vee S(b_1 \wedge 1)$.
4. If $z < b_1 \wedge b_2$, then $S^R(z) = S(b_1 \wedge 1) \vee S(b_2 \wedge 1)$.

$\blacksquare$

Now, we are ready to prove theorem 5.24.

Proof. Since $\mathcal{L}^+ \subset \mathcal{F}_c^+$, according to proposition 5.19 $f_1 \blacktriangle_W f_2$ is continuous, and so the case $z = 0$ of theorem 5.10 need not be considered separately. Then,

$$\begin{aligned}(f_1 \blacktriangle_W f_2)(z) &= \bigvee_{z=(x+y-1)\vee 0} (f_1(x) + f_2(y) - 1) \vee 0 \\ &= \bigvee_{x \geq z} (f_1(x) + f_2(1 - x + z) - 1) \vee 0 \\ &= 0 \vee \left(-1 + \bigvee_{x \geq z} \left(f_1(x) + f_2'(x) \right) \right)\end{aligned}$$

where $f_2'(x) = f_2(1 - x + z)$. This way $f_2' \in \mathcal{L}^-$ is also linear with parameters

$$a_2' = 1 - a_2 + z \quad \text{and} \quad b_2' = 1 - b_2 + z.$$

By lemma 5.25, the supremum is either at $x = \{b_1\}_z$ or $x = \{b_2'\}_z$, and

$$(f_1 \blacktriangle_W f_2)(z) = 0 \vee \left(-1 + \left(f_1(\{b_1\}_z) + f_2'(\{b_1\}_z)\right) \vee \left(f_1(\{b_2'\}_z) + f_2'(\{b_2'\}_z)\right)\right).$$

Consider the following equalities.

$$\begin{aligned}
f_2'(\{b_1\}_z) &= f_2(\{b_1\}_z \rhd_W z), \\
f_1(\{b_2'\}_z) &= f_1(\{b_2\}_z \rhd_W z), \\
f_2'(\{b_2'\}_z) &= f_2(\{b_2\}_z), \\
f_i(\{b_i\}_z) &= f_i(1)
\end{aligned}$$

To prove the third one, for example,

$$\begin{aligned}
f_2'(\{b_2'\}_z) &= f_2(1 - (z \vee (1 - b_2 + z) \wedge 1) + z) \\
&= f_2((1 - z) \wedge (b_2 - z) \vee 0 + z) = f_2(z \vee b_2 \wedge 1) = f_2(\{b_2\}_z).
\end{aligned}$$

It follows that

$$\begin{aligned}
(f_1 \blacktriangle_W f_2)\,(z) &= 0 \vee (-1 + (f_1(1) + f_2(\{b_1\}_z \rhd_W z)) \vee (f_1(\{b_2\}_z \rhd_W z) + f_2(1))) \\
&= (f_1(1) \bigtriangleup_W f_2(\{b_1\}_z \rhd_W z)) \vee (f_1(\{b_2\}_z \rhd_W z) \bigtriangleup_W f_2(1))\,.
\end{aligned}$$

■

Example 5.26. *To clarify theorem 5.24 consider the following example. Let* $f_1 = f_{a_1,b_1} = f_{.6,.7}$ *and* $f_2 = f_{a_2,b_2} = f_{.7,1}$ *be linear fuzzy truth values. The pointwise calculation of* $(f_1 \blacktriangle_W f_2)\,(.5)$ *is as follows (see fig. 5.12(a)). According to theorem 5.24,*

$$\begin{aligned}
(f_1 \blacktriangle_W f_2)(.5) = \quad & (f_1(1) \bigtriangleup_W f_2(\{.7\}_{.5} \rhd_W .5)) \\
& \vee (f_2(1) \bigtriangleup_W f_1(\{1\}_{.5} \rhd_W .5))\,,
\end{aligned}$$

where $\{x\}_y = y \vee x \wedge 1$. *Furthermore, we have*

$$\begin{aligned}
(f_1 \blacktriangle_W f_2)(.5) &= (1 \bigtriangleup_W f_2(.7 \rhd_W .5)) \vee (1 \bigtriangleup_W f_1(1 \rhd_W .5)) \\
&= (0 \vee (1 + f_2(1 \wedge (1 - .7 + .5)) - 1)) \\
&\quad \vee (0 \vee (1 + f_1(1 \wedge (1 - 1 + .5)) - 1)) \\
&= (0 \vee (1 + f_2(.8)) - 1)) \vee (0 \vee (1 + f_1(.5) - 1)) \\
&= (0 \vee (1 + .33 - 1)) \vee (0 \vee (1 + 0 - 1)) = .33
\end{aligned}$$

According to proposition 5.19, the Łukasiewicz conjunction of non-decreasing linear fuzzy truth values is continuous in any case. Although, it is not always linear (see for example fig. 5.13), linearity is preserved for normal fuzzy truth values.

Corollary 5.27. *For all* $f_i = f_{a_i,b_i} \in \mathcal{L}_1^+$ *(*$i = 1, 2$*)*

$$(f_1 \blacktriangle_W f_2)\,(z) = f_1(b_2 \rhd_W z) \vee f_2(b_1 \rhd_W z). \tag{5.34}$$

Furthermore, $f_1 \blacktriangle_W f_2$ *is also linear with parameters*

$$\begin{aligned}
a_{\blacktriangle_W} &= (a_1 + b_2 - 1) \wedge (a_2 + b_1 - 1), \\
b_{\blacktriangle_W} &= b_1 + b_2 - 1.
\end{aligned}$$

Proof. If f_i are normal then $f_i(1) = 1$ and since $b_i \leq 1$, $(\{b_i\}_z \rhd_W z) = (b_i \rhd_W z)$, hence (5.34) holds.

Furthermore, note that for example $f_1(b_2 \rhd_W z) = f_1'(z)$, where

$$a_1' = a_1 + b_2 - 1 \quad \text{and} \quad b_1' = b_1 + b_2 - 1.$$

Indeed,

$$\begin{aligned} f_1(b_2 \rhd_W z) &= \left\{\frac{1 \wedge (1 - b_2 + z) - a_1}{b_1 - a_1}\right\} = \left\{\frac{b_2 \wedge z - (a_1 + b_2 - 1)}{b_1 + b_2 - 1 - (a_1 + b_2 - 1)}\right\} \\ &= f_1'(b_2 \wedge z) = f_1'(z), \end{aligned}$$

since for all $b_2 \leq z$, $f_1'(z) = 1$. Similarly, f_2' is such that

$$a_2' = a_2 + b_1 - 1 \quad \text{and} \quad b_2' = b_1 + b_2 - 1.$$

Thus f_i' are linear fuzzy truth values with equal upper endpoints. Their maximum is the one with the minimal lower endpoint. ∎

Theorem 5.24 can only be applied to fuzzy truth values in $\mathcal{L}^+$. For the Łukasiewicz conjunction on $\mathcal{L}^-$ two cases need to be considered, since its continuity is not guaranteed. The next theorem can be easily proved analogously to theorem 5.24 considering theorem 5.10 and corollary 5.15.

Theorem 5.28. *The following hold for all $f_i = f_{a_i,b_i} \in \mathcal{L}^-$ $(i = 1, 2)$. For $z > 0$,*

$$\begin{aligned} (f_1 \blacktriangle_W f_2)(z) = \; & (f_1(z) \triangle_W f_2(\{b_1\}_z \rhd_W z)) \\ & \vee (f_2(z) \triangle_W f_1(\{b_2\}_z \rhd_W z)), \end{aligned} \tag{5.35}$$

and if $z = 0$, then

$$(f_1 \blacktriangle_W f_2)(0) = f_1(0) \triangle_W f_2(0). \tag{5.36}$$

The next corollary gives necessary and sufficient conditions on the continuity of the Łukasiewicz conjunction between the elements of $\mathcal{L}^-$.

Corollary 5.29. *For $f_i = f_{a_i,b_i} \in \mathcal{L}^-$, the Łukasiewicz conjunction $f_1 \blacktriangle_W f_2$ is continuous if and only if $b_1 + b_2 \geq 1$.*

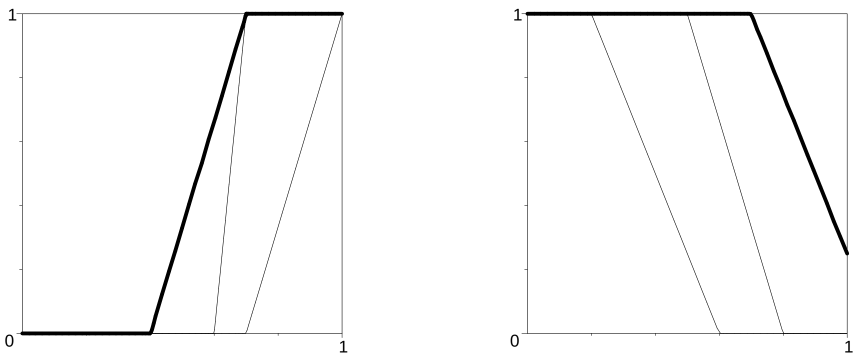
Proof. $(f_1 \blacktriangle_W f_2)(z)$ may be discontinuous only at $z = 0$. Thus, $f_1 \blacktriangle_W f_2$ is continuous if and only if the equality

$$(f_1(0) \triangle_W f_2(1 - \{b_1\})) \vee (f_2(0) \triangle_W f_1(1 - \{b_2\})) = f_1(0) \triangle_W f_2(0)$$

holds. It is equivalent to

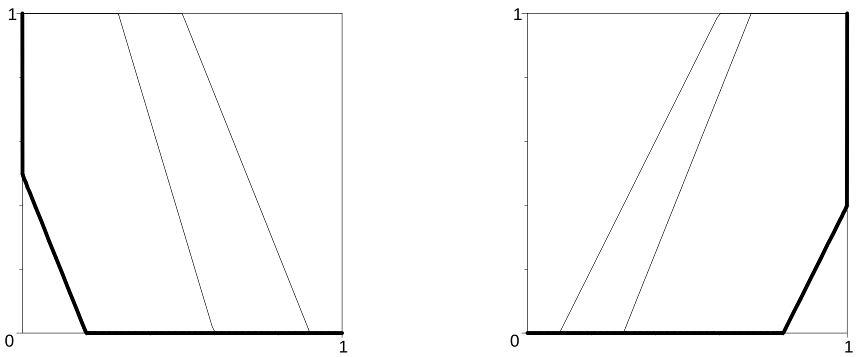
$$f_2(1 - \{b_1\}) = f_2(0) = f_2(\{b_2\}) \quad \text{and} \quad f_1(1 - \{b_2\}) = f_1(0) = f_1(\{b_1\})$$

These equations hold iff $\{b_1\} + \{b_2\} \geq 1$, which is equivalent to the condition $b_1 + b_2 \geq 1$. ∎



(a) $f_{.6,.7} \blacktriangle_W f_{.7,1} = f_{.4,.7}$ (b) $f_{.8,.5} \blacktriangledown_W f_{.6,.2} = f_{1.1,.7}$

Figure 5.12: Examples for continuous Łukasiewicz conjunction and disjunction of linear fuzzy truth values



(a) $f_{.9,.5} \blacktriangle_W f_{.6,.3}$ (b) $f_{.1,.6} \blacktriangledown_W f_{.3,.7}$

Figure 5.13: Examples for discontinuous Łukasiewicz conjunction and disjunction of linear fuzzy truth values

The Łukasiewicz disjunction of linear fuzzy truth values can be handled in a totally analogous manner. We provide the following results without proofs.

Lemma 5.30. *Let $f_1 = f_{a_1,b_1} \in \mathcal{L}^+$ and $f_2 = f_{a_2,b_2} \in \mathcal{L}^-$ be two linear fuzzy truth values, let S denote their sum, $S(x) = f_{a_1,b_1}(x) + f_{a_2,b_2}(x)$. Then,*

$$S^L(z) = \bigvee_{x \le z} S(x) = S(\{b_1\}^z) \vee S(\{b_2\}^z) \quad \forall z \in [0,1].$$

where $\{x\}^z = 0 \vee x \wedge z$.

Theorem 5.31. *For all $f_i = f_{a_i,b_i} \in \mathcal{L}^-$ $(i = 1, 2)$*

$$\begin{aligned}(f_1 \blacktriangledown_W f_2)(z) = \quad & (f_1(0) \bigtriangleup_W f_2(\{b_1\}^z \lhd_W z)) \\ & \vee (f_2(0) \bigtriangleup_W f_1(\{b_2\}^z \lhd_W z))\,, \end{aligned} \tag{5.37}$$

where $\{x\}^z = 0 \vee x \wedge z$.

Example 5.32. *Let $f_1 = f_{a_1,b_1} = f_{.8,.5}$ and $f_2 = f_{a_2,b_2} = f_{.6,.2}$ be linear fuzzy truth values. The pointwise calculation of $(f_1 \blacktriangledown_W f_2)\,(.7)$ is as follows (see fig. 5.12(b)). According to* *theorem 5.31,*

$$\begin{aligned}(f_1 \blacktriangledown_W f_2)(.7) = \quad & \left(f_1(0) \bigtriangleup_W f_2(\{.5\}^{.7} \lhd_W .7)\right) \\ & \vee \left(f_2(0) \bigtriangleup_W f_1(\{.2\}^{.7} \lhd_W .7)\right),\end{aligned}$$

where $\{x\}^y = 0 \vee x \wedge y$*. Furthermore, we have*

$$\begin{aligned}(f_1 \blacktriangledown_W f_2)(.7) &= (1 \bigtriangleup_W f_2(.5 \lhd_W .7)) \vee (1 \bigtriangleup_W f_1(1 \rhd_W .5)) \\ &= (0 \vee (1 + f_2(1 \wedge (1 - .7 + .5)) - 1)) \\ &\quad \vee (0 \vee (1 + f_1(1 \wedge (1 - 1 + .5)) - 1)) \\ &= (0 \vee (1 + f_2(.8)) - 1)) \vee (0 \vee (1 + f_1(.5) - 1)) \\ &= (0 \vee (1 + .33 - 1)) \vee (0 \vee (1 + 0 - 1)) = .33\end{aligned}$$

Corollary 5.33. *For all* $f_i = f_{a_i,b_i} \in \mathcal{L}_1^-$ $(i = 1, 2)$

$$(f_1 \blacktriangledown_W f_2)(z) = f_1(b_2 \lhd_W z) \vee f_2(b_1 \lhd_W z). \tag{5.38}$$

Furthermore, $f_1 \blacktriangledown_W f_2$ *is also linear with parameters*

$$\begin{aligned}a_{\blacktriangledown_W} &= (a_1 + b_2) \vee (a_2 + b_1), \\ b_{\blacktriangledown_W} &= b_1 + b_2.\end{aligned}$$

Theorem 5.34. *The following hold for all* $f_i = f_{a_i,b_i} \in \mathcal{L}^+$ $(i = 1, 2)$*. For* $z < 1$*,*

$$\begin{aligned}(f_1 \blacktriangledown_W f_2)(z) = \quad & (f_1(z) \bigtriangleup_W f_2(\{b_1\}^z \lhd_W z)) \\ & \vee (f_2(z) \bigtriangleup_W f_1(\{b_2\}^z \lhd_W z)),\end{aligned} \tag{5.39}$$

and if $z = 1$*, then*

$$(f_1 \blacktriangledown_W f_2)(1) = f_1(1) \bigtriangleup_W f_2(1). \tag{5.40}$$

The Łukasiewicz disjunction is always continuous on the elements of $\mathcal{L}^-$. On $\mathcal{L}^+$ the following holds.

Corollary 5.35. *For* $f_i = f_{a_i,b_i} \in \mathcal{L}^+$*, the Łukasiewicz disjunction* $f_1 \blacktriangledown_W f_2$ *is continuous if and only if* $b_1 + b_2 \leq 1$*.*

Next, we provide some further properties of the operation $\blacktriangle_W$ on linear fuzzy truth values. Similar ones can be easily obtained for $\blacktriangledown_W$. The proofs are straightforward considering the above results. The constant function 1 is denoted by $\mathbf{1}$.

Proposition 5.36. *The following hold for the Łukasiewicz conjunctions of non-decreasing linear fuzzy truth values.*

1. *idempotency*

$$f_{a,b} \blacktriangle_W f_{a,b} = f_{a,b} \quad \Longleftrightarrow \quad b = 1$$

2. *unit elements*

$$f_{a_1,b_1} \blacktriangle_W f_{a_2,b_2} = f_{a_1,b_1} \quad \Longleftrightarrow \quad \begin{array}{c} b_2 = 1 \text{ and } 1 - a_2 \leq b_1 - a_1, \\ \text{i.e. } f_{a_2,b_2} \text{ is steeper than } f_{a_1,b_1} \end{array}$$

3. *nilpotency*

$$f_{a_1,b_1} \blacktriangle_W f_{a_2,b_2} = \mathbf{1} \quad \Longleftrightarrow \quad b_1 + b_2 \leq 1$$

4. *zero element*

$$\mathbf{1} \blacktriangle_W f = \mathbf{1}$$

5. *monotonicity*

$$\begin{array}{c} f_{a_1,b_1} \geq f_{a_2,b_2} \\ \text{i.e. } a_1 \leq a_2 \text{ and } b_1 \leq b_2 \end{array} \quad \Longrightarrow \quad f_{a_1,b_1} \blacktriangle_W g \geq f_{a_2,b_2} \blacktriangle_W g$$

5.4 Summary

In this chapter we have discussed extended t-norms and t-conorms on continuous and interactive fuzzy truth values. Sufficient conditions were given on the continuity of the resultant functions. We have shown easy-to-implement pointwise formulas for the conjunction and disjunction of fuzzy truth values with different monotonicity.

As an important special case, we have considered the extended Łukasiewicz operations on interactive linear fuzzy truth values. It was shown that the complex convolutions of the extended Łukasiewicz operations are equivalent to simple operations on the parameters on the linear fuzzy truth values. We have given necessary and sufficient conditions when these operations preserve continuity and linearity.

These results can be directly applied to type-2 fuzzy systems and to reasoning systems based on fuzzy truth values.

Chapter 6

Type-2 implications on fuzzy truth values

Fuzzy logic in narrow sense is a generalization of classical two-valued logic, it considers a range of truth values, usually the unit interval. Although fuzzy logic became the "language" of vague propositions, its $[0, 1]$-valued truth values are still precise. In recent years, research related to type-2 fuzzy logic has become even more active than ever as they seem to provide a better framework for the "computing with words" paradigm than classical fuzzy sets [60]. Type-2 fuzzy logic takes the generalization a step further by considering truth values that are themselves fuzzy. This means that every truth value (i.e. every element of $[0, 1]$) has a fuzzy membership degree (which is again an element of $[0, 1]$). This mapping from the unit interval to itself is the truth value, hence its name fuzzy truth value.

The recent popularity of type-2 fuzzy logic mainly stem from the works of Mendel, Karnik and John [55, 61, 62]. Besides the numerous papers with applications of type-2 fuzzy sets, there are many contributions from the theoretical line of research, too. The theory of type-2 fuzzy sets was established by Zadeh [94], Mizomoto and Tanaka [64, 65], Dubois and Prade [36, 37], and Nieminen [69]. Recent publications by Walker and Walker [90] and Starczewski [78, 79] unfold the rich algebraic structure of fuzzy truth values. These papers consider type-2 t-norms and t-conorms on fuzzy truth values, either in general or by restricting the set of fuzzy truth values to – for example – normal, convex, triangular, trapezoidal or bell-shaped functions.

As the basic building blocks of any inference process, fuzzy implications have always been in the mainstream research. The study of (type-1) fuzzy implications [39, 40, 41, 77, 85, 86] provided the basis for the theory and practice of approximate reasoning. Although implicative operations (such as implications and coimplications) were investigated on interval-valued fuzzy sets (which are special type-2 fuzzy sets with intervals as truth values), a general discussion of type-2 implicative operations did not exist. In this paper we study type-2 implicative operations on non-interactive fuzzy truth values. The use of type-2 implicative operations is not straightforward, for example not all properties of a type-1 fuzzy implication apply to its extension.

Fuzzy implications are defined on the algebra $\mathbf{I} = (\mathcal{I}, \wedge, \vee, \leq, 0, 1)$. We define type-2 fuzzy implications analogously to their type-1 counterparts. The underlying set of truth values is generalized from $\mathcal{I}$ to a subset of $\mathcal{F}$, and since it may not be a lattice, the two partial orders defined by $\sqcap$ and $\sqcup$ are considered instead of $\leq$.

Definition 6.1. *Let $\mathbf{A} = (\mathcal{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preccurlyeq)$, where $\mathcal{A} \subseteq \mathcal{F}$. A function $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a type-2 fuzzy implication over $\mathbf{A}$ if and only if it satisfies the boundary conditions*

$$\mathbf{0} \bullet \mathbf{0} = \mathbf{0} \bullet \mathbf{1} = \mathbf{1} \bullet \mathbf{1} = \mathbf{1};\ \mathbf{1} \bullet \mathbf{0} = \mathbf{0},$$

and it is antitone in the first and monotone in the second argument w.r.t. at least one of the partial orders $\sqsubseteq$ or $\preccurlyeq$.

We make a careful distinction between extended fuzzy implications and type-2 fuzzy implications. For example, a fuzzy implication extended to $\mathcal{F}$ is obviously an extended fuzzy implication, but it is a type-2 fuzzy implication (on a subset of $\mathcal{F}$) only if it satisfies the above conditions.

6.1 Extended S-implications and S-coimplications

S-implications are formed by a t-conorm $\bigtriangledown$ and a strong negation $'$ according to the formula $x' \bigtriangledown y$. S-coimplications are dual to S-implications, and are defined as $x' \bigtriangleup y$. The extensions of these operations are as follows.

$$\begin{aligned}(f \blacktriangleright g)(z) &= \bigvee_{z = x' \bigtriangledown y} (f(x) \wedge g(y)) &= (f^* \blacktriangledown g)(z),\\ (f \blacktriangleleft g)(z) &= \bigvee_{z = x' \bigtriangleup y} (f(x) \wedge g(y)) &= (f^* \blacktriangle g)(z).\end{aligned}$$

We assume that the underlying negation $'$, t-conorm $\bigtriangledown$ and t-norm $\bigtriangleup$ of the operations $\blacktriangleright$ and $\blacktriangleleft$ are continuous. Since the operation $\blacktriangleleft$ is dual to $\blacktriangleright$, i.e. $f \blacktriangleleft g = (f^* \blacktriangleright g^*)^*$ for all $f, g \in \mathcal{F}$, thus any statement involving $\blacktriangleright$ has its dual with $\blacktriangleleft$. From now on, we omit the proofs of dual statements, these easily follow by duality.

Proposition 6.2. *The operations $\blacktriangleright$ and $\blacktriangleleft$ are closed on $\mathcal{F}_C$.*

Proof. The preservation of the convexity of fuzzy intervals over the real line was proved by e.g. [15] for any extended continuous function. Any fuzzy truth value can be naturally extended to the real line for example by

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,1],\\ 0, & \text{otherwise.}\end{cases}$$

Since this extension does not affect the convexity of f, the statement follows. ∎

Proposition 6.3. *The following hold for all $g, h \in \mathcal{F}$ if and only if f is convex.*

1. $(g \sqcap h) \blacktriangleright f = (g \blacktriangleright f) \sqcup (h \blacktriangleright f), \quad f \blacktriangleright (g \sqcap h) = (f \blacktriangleright g) \sqcap (f \blacktriangleright h),$
2. $(g \sqcup h) \blacktriangleright f = (g \blacktriangleright f) \sqcap (h \blacktriangleright f), \quad f \blacktriangleright (g \sqcup h) = (f \blacktriangleright g) \sqcup (f \blacktriangleright h).$

Proof. Straightforward from the distributivity of $\blacktriangledown$ over $\sqcap$ and $\sqcup$ (see [90]) and the De Morgan law between $\sqcap$, $\sqcup$ and *. ∎

Note, that similar laws apply to type-2 S-coimplications, too, because $\blacktriangle$ also distributes over $\sqcap$ and $\sqcup$ by the same conditions.

Proposition 6.4. *The operations $\blacktriangleright$ and $\blacktriangleleft$ are closed on $\mathcal{F}_N$. Moreover, $f \blacktriangleright g$ and $f \blacktriangleleft g$ are normal if and only if $f, g \in \mathcal{F}_N$.*

Proof. We prove that $\blacktriangledown$ is closed on $\mathcal{F}_N$, then the first statement follows from $f \blacktriangleright g = f^* \blacktriangledown g$, since * preserves normality. Suppose $f, g \in \mathcal{F}_N$, i.e. $f^L(1) = g^L(1) = 1$. Then,

$$\begin{aligned}(f \blacktriangledown g)^L(1) &= \left(f^L \blacktriangledown g^L\right)(1) = \bigvee_{1 = x \,\triangledown\, y} \left(f^L(x) \wedge g^L(y)\right) \\ &= \left(f^L(1) \wedge g^L(1)\right) \vee \bigvee_{\substack{1 = x \,\triangledown\, y \\ x<1 \text{ or } y<1}} \left(f^L(x) \wedge g^L(y)\right) = 1,\end{aligned}$$

thus $f \blacktriangledown g$ is normal.

Now suppose $f \blacktriangledown g \in \mathcal{F}_N$. We have

$$1 = (f \blacktriangledown g)^R(0) = \left(f^R \blacktriangledown g^R\right)(0) = \bigvee_{0 = x \,\triangledown\, y} \left(f^R(x) \wedge g^R(y)\right) = f^R(0) \wedge g^R(0).$$

Note, that in the last step the following property of t-conorms was used: $x \triangledown y = 0$ if and only if $x = 0$ and $y = 0$. The above equality implies that $f^R(0) = g^R(0) = 1$, thus $f, g \in \mathcal{F}_N$.

Since the extended negation does not affect normality, the statement for $\blacktriangleleft$ follow by duality. ■

Two important properties follow from the definition of type-1 fuzzy implications. For all fuzzy implications $\rhd$,

$$0 \rhd x = 1 \qquad \text{and} \qquad x \rhd 1 = 1 \quad \forall x \in [0, 1].$$

These properties can be naturally "extended" to fuzzy truth values:

$$\mathbf{0} \blacktriangleright f = \mathbf{1} \qquad \text{and} \qquad f \blacktriangleright \mathbf{1} = \mathbf{1} \quad \forall f \in \mathbf{A},$$

for a subalgebra $\mathbf{A}$ of $\mathbf{F}$. The following proposition shows that these properties hold (and so it is reasonable to use $\blacktriangleright$ on a subalgebra of $\mathbf{A}$) if and only if $\mathbf{A} \subseteq \mathbf{F}_N$.

Proposition 6.5. *The equations*

$$\mathbf{0} \blacktriangleright f = \mathbf{1} \qquad \text{and} \qquad f \blacktriangleright \mathbf{1} = \mathbf{1},$$

hold if and only if $f \in \mathcal{F}_N$.

Proof. We provide a proof only for the first equality, the other one can be proved similarly due to the duality between the utilized properties of (type-1) t-norms and t-conorms.

Let $f \in \mathcal{F}$, we have $\mathbf{0} \blacktriangleright f = \mathbf{1} \blacktriangledown f$, and by definition

$$(\mathbf{1} \blacktriangledown f)(z) = \bigvee_{z = x \,\triangledown\, y} \left(\mathbf{1}(x) \wedge f(y)\right).$$

For a given $z < 1$, the minimum inside the sup is 0 whenever $x < 1$, since then $\mathbf{1}(x) = 0$. Since $x = 1$ would imply $z = 1$, the minimum is zero for every possible (x, y) pairs, hence $(\mathbf{0} \blacktriangleright f)(z) = 0$ for all $z < 1$.

In case $z = 1$, our claim is that

$$(\mathbf{1} \blacktriangledown f)(1) = \bigvee_{1 = x \triangledown y} (\mathbf{1}(x) \wedge f(y)) = 1.$$

Note, that $x = 1$ is necessary, because otherwise $\mathbf{1}(x) = 0$, and the sup would never reach 1. So we have

$$\bigvee_{1 = 1 \triangledown y} (\mathbf{1}(1) \wedge f(y)) = \bigvee_{y \in [0,1]} f(y),$$

which equals to 1 if and only if f is normal. ■

Clearly, by duality

$$\mathbf{1} \blacktriangleleft f = \mathbf{0} \quad \text{and} \quad f \blacktriangleleft \mathbf{0} = \mathbf{0} \quad \text{if and only if} \quad f \in \mathcal{F}_N.$$

The main result of this section shows that extended S-implications are type-2 fuzzy S-implications only on $\mathcal{F}_{CN}$.

Theorem 6.6. *The operation* $\blacktriangleright$ *is a type-2 fuzzy implication over* $\mathbf{A} \subseteq \mathbf{F}$ *if and only if* $\mathbf{A}$ *is a subalgebra of the algebra of convex normal functions* $\mathbf{F}_{CN}$*.*

Proof. Let $f, g, h \in \mathbf{A}$. By definition $f \blacktriangleright g = f^* \blacktriangledown g$. The boundary conditions hold since $f \blacktriangledown \mathbf{0} = f$, for all $f \in \mathcal{F}$, and $\mathbf{1} \blacktriangledown \mathbf{1} = \mathbf{1}$.

The sufficiency of the condition can be easily proved by proposition 6.3 and considering that for convex normal fuzzy truth values the two partial orders coincide. Necessity can be proved as follows.

The operation $\blacktriangleright$ is monotone in the second argument if $g \sqsubseteq h$ implies $f \blacktriangleright g \sqsubseteq f \blacktriangleright h$ (a similar argument follows from the use of $\preccurlyeq$) which is equivalent to

$$(f \blacktriangleright g) \sqcap (f \blacktriangleright h) = f \blacktriangleright g.$$

Now, $g \sqsubseteq h$, i.e. $g \sqcap h = g$, thus we have

$$(f \blacktriangleright g) \sqcap (f \blacktriangleright h) = f \blacktriangleright (g \sqcap h).$$

By proposition 6.3 this holds if and only if f is convex, thus $\mathbf{A} \subseteq \mathbf{F}_C$.

The operation $\blacktriangleright$ is antitone in the first argument if $g \sqsubseteq f$ implies $f \blacktriangleright h \sqsubseteq g \blacktriangleright h$ (again, a similar argument follows from the use of $\preccurlyeq$) which is equivalent to

$$(f \blacktriangleright h) \sqcap (g \blacktriangleright h) = f \blacktriangleright h.$$

Now, $g \sqsubseteq f$, i.e. $g \sqcap f = g$, thus we have

$$(f \blacktriangleright h) \sqcap ((f \sqcap g) \blacktriangleright h) = f \blacktriangleright h.$$

We may assume convexity, thus by proposition 6.3, we have

$$(f \blacktriangleright h) \sqcap ((f \blacktriangleright h) \sqcup (g \blacktriangleright h)) = f \blacktriangleright h.$$

This is the absorption law, which holds if and only if $f \blacktriangleright h$ is convex and $g \blacktriangleright h$ is normal [90] . So, by proposition 6.4, $g, h \in \mathcal{F}_N$ and thus $\mathbf{A} \subseteq \mathbf{F}_{CN}$. ■

Clearly, by duality the operation $\blacktriangleleft$ is a type-2 fuzzy coimplication over $\mathbf{A} \subseteq \mathbf{F}$ if and only if $\mathbf{A}$ is a subalgebra of the algebra of convex normal functions $\mathbf{F}_{CN}$.

Further properties of extended S-implications are summarized as follows.

Proposition 6.7. *The following hold for all $f, g, h \in \mathbf{F}$,*

1. $\mathbf{1} \blacktriangleright f = f$.
2. $f \blacktriangleright (g \blacktriangleright h) = g \blacktriangleright (f \blacktriangleright h)$.
3. $f \blacktriangleright g = g^* \blacktriangleright f^*$.
4. $g \sqsubseteq f \blacktriangleright g$ *where* $f, g \in \mathbf{F}_{CN}$.

Proof. Let $f, g, h \in \mathbf{F}$.

1. $\mathbf{1} \blacktriangleright f = (\mathbf{1}^*) \blacktriangledown f = \mathbf{0} \blacktriangledown f = f$,
2. $f \blacktriangleright (g \blacktriangleright h) = f^* \blacktriangledown (g^* \blacktriangledown h) = f^* \blacktriangledown g^* \blacktriangledown h = g^* \blacktriangledown (f^* \blacktriangledown h) = g \blacktriangleright (f \blacktriangleright h)$,
3. $g^* \blacktriangleright f^* = (g^*)^* \blacktriangledown f^* = f^* \blacktriangledown g = f \blacktriangleright g$.
4. To prove $g \sqsubseteq f \blacktriangleright g$ on $\mathbf{F}_{CN}$ consider that

$$g \sqsubseteq f \blacktriangleright g \quad \text{iff} \quad g \sqcap (f \blacktriangleright g) = g,$$

and

$$g \sqcap (f \blacktriangleright g) = (\mathbf{1} \blacktriangleright g) \sqcap (f \blacktriangleright g) = (\mathbf{1} \sqcup f) \blacktriangleright g = \mathbf{1} \blacktriangleright g = g.$$

■

Clearly, by duality we have

1. $\mathbf{0} \blacktriangleleft f = f$.
2. $f \blacktriangleleft (g \blacktriangleleft h) = g \blacktriangleleft (f \blacktriangleleft h)$.
3. $f \blacktriangleleft g = g^* \blacktriangleleft f^*$.
4. $f \blacktriangleleft g \sqsubseteq g$ where $f, g \in \mathbf{F}_{CN}$.

To sum up the above results, the operation $\blacktriangleright$ is a type-2 fuzzy implication on the lattice of convex normal fuzzy truth values $\mathbf{F}_{CN}$, and possesses similar properties that a type-1 fuzzy S-implication has.

Since interval fuzzy truth values are normal convex, and type-2 t-norms, t-conorms and negations are closed on this set, the following result is immediate. (See also [28, 7])

Corollary 6.8. *The operation $\blacktriangleright$ is a type-2 fuzzy implication on the subalgebra of interval fuzzy truth values $\mathbf{F}_I$.*

6.1.1 The extended S-implications of fundamental t-norms

The three basic t-norms/t-conorms are the min/max, the product/algebraic sum and the Łukasiewicz operators. In this section we discuss the properties of extended S-implications formed by these operators. Certainly, such implications have all the previously proved properties.

First, we discuss the meet ($\sqcap$) and join ($\sqcup$), which are the extensions of $\wedge$ and $\vee$. Let us introduce the notations $\blacktriangleright_\vee$ and $\blacktriangleleft_\wedge$ for the extended S-implication and coimplication formed by join and meet and an extended strong negation. Thus

$$f \blacktriangleright_\vee g = f^* \sqcup g = \left(f^* \wedge g^L\right) \vee \left(\left(f^*\right)^L \wedge g\right),$$
$$f \blacktriangleleft_\wedge g = f^* \sqcap g = \left(f^* \wedge g^R\right) \vee \left(\left(f^*\right)^R \wedge g\right).$$

Meet and join are special in the sense that they form a distributive lattice on the set of normal convex fuzzy truth values. From the absorption laws in $\mathbf{F}_{CN}$ we have the following.

Corollary 6.9. *For all $f, g \in \mathcal{F}_{CN}$,*

$$f \blacktriangleright_\vee (f \blacktriangleleft_\wedge g) = f \blacktriangleleft_\wedge (f \blacktriangleright_\vee g) = f^*.$$

Because of the idempotency of $\sqcap$ and $\sqcup$ we have the following.

Proposition 6.10. *The following distributive laws hold for all $f, g, h \in \mathcal{F}$.*

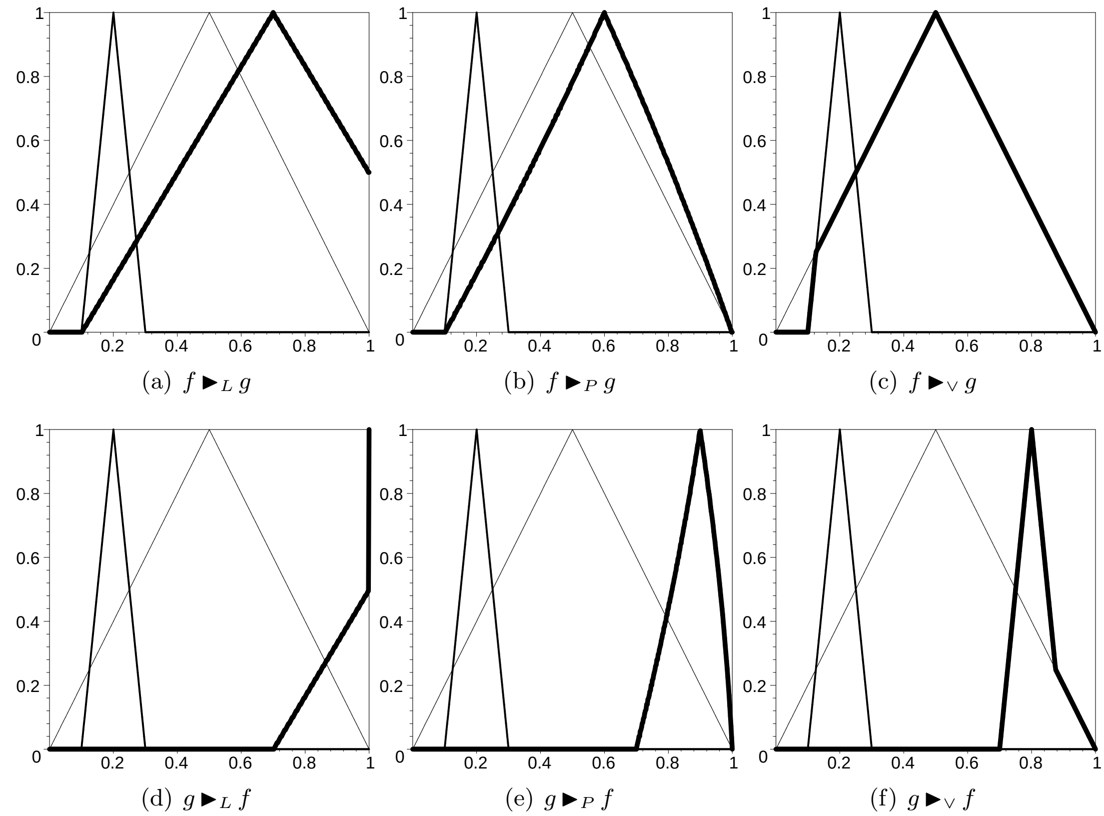
1. $f \blacktriangleright_\vee (g \sqcup h) = (f \blacktriangleright_\vee g) \sqcup (f \blacktriangleright_\vee h)$; $\quad f \blacktriangleleft_\wedge (g \sqcap h) = (f \blacktriangleleft_\wedge g) \sqcap (f \blacktriangleleft_\wedge h)$.
2. $(f \sqcap g) \blacktriangleright_\vee h = (f \blacktriangleright_\vee h) \sqcup (g \blacktriangleright_\vee h)$; $\quad (f \sqcup g) \blacktriangleleft_\wedge h = (f \blacktriangleleft_\wedge h) \sqcap (g \blacktriangleleft_\wedge h)$.

According to Theorems 5.10 and 5.11, the extended Łukasiewicz S-implication and S-coimplication are

$$(f \blacktriangleright_L g)(z) = (f^* \blacktriangledown_L g)(z) = \begin{cases} \bigvee\limits_{y \le z} (f(1-z+y) \wedge g(y)), & \text{if } z < 1, \\ \left(f \wedge g^R\right)^{LR}(1), & \text{otherwise,} \end{cases}$$
$$(f \blacktriangleleft_L g)(z) = (f^* \blacktriangle_L g)(z) = \begin{cases} \bigvee\limits_{y \ge z} (f(y-z) \wedge g(y)), & \text{if } z > 0, \\ \left(f \wedge g^L\right)^{LR}(0), & \text{otherwise.} \end{cases}$$

Similarly, the extended S-implication and S-coimplication of product/algebraic sum are

$$(f \blacktriangleright_P g)(z) = (f^* \blacktriangledown_P g)(z) = \begin{cases} \bigvee\limits_{y \le z} (f((1-z)/(1-y)) \wedge g(y)), & \text{if } z < 1, \\ \left(\left(f^* \wedge g^{LR}\right) \vee \left(f^{LR} \wedge g\right)\right)(1), & \text{otherwise,} \end{cases}$$
$$(f \blacktriangleleft_P g)(z) = (f^* \blacktriangle_P g)(z) = \begin{cases} \bigvee\limits_{y \ge z} (f((y-z)/y) \wedge g(y)), & \text{if } z > 0, \\ \left(\left(f^* \wedge g^{LR}\right) \vee \left(f^{LR} \wedge g\right)\right)(0), & \text{otherwise.} \end{cases}$$



(a) $f \blacktriangleright_L g$ (b) $f \blacktriangleright_P g$ (c) $f \blacktriangleright_\vee g$

(d) $g \blacktriangleright_L f$ (e) $g \blacktriangleright_P f$ (f) $g \blacktriangleright_\vee f$

Figure 6.1: Extended S-implications of the three basic t-norms on normal convex fuzzy truth values (thin line – f, normal line – g, thick line – result).

Specific calculations with these formulas can be done efficiently with a discretization of the unit interval. Figure 6.1 shows the differences between these extended implications on triangular fuzzy truth values.

6.2 Extended residual implications and coimplications

In this section, first we introduce extensions of residual implications in general. Then, mainly because of the special role of meet and join, we examine their extended residuals in detail.

In this section let $\triangleright$ (resp. $\triangleleft$) denote the residual implication (coimplication) of a t-norm $\triangle$ (resp. t-conorm $\triangledown$). Their extensions to fuzzy truth values are defined as

$$(f \blacktriangleright g)(z) = \bigvee_{z = x \triangleright y} (f(x) \wedge g(y)),$$
$$(f \blacktriangleleft g)(z) = \bigvee_{z = x \triangleleft y} (f(x) \wedge g(y)).$$

Similarly to the case of S-implications, there is a duality between residual implications and coimplications, i.e. we have

$$f \blacktriangleleft g = (f^* \blacktriangleright g^*)^*,$$

for an extended strong negation *. Thus, the forthcoming results are proved only for implications, and can be applied to coimplications as well.

A fundamental property of residual implications is that $x \rhd y = 1$ if and only if $x \leq y$. Due to duality, $x \lhd y = 0$ whenever $x \geq y$. Based on these we can establish the following.

Proposition 6.11. *For any extended residual implication* $\blacktriangleright$*, coimplication* $\blacktriangleleft$*, and* $f, g \in \mathcal{F}$*,*

$$(f \blacktriangleright g)(1) = \left(f \wedge g^R\right)^{LR}(1),$$
$$(f \blacktriangleleft g)(0) = \left(f \wedge g^L\right)^{LR}(0).$$

Proof. Let $f, g \in \mathcal{F}$.

$$\begin{aligned}(f \blacktriangleright g)(1) &= \bigvee_{1 = x \rhd y} (f(x) \wedge g(y)) = \bigvee_{x \leq y} (f(x) \wedge g(y)) = \bigvee_{x} \bigvee_{x \leq y} (f(x) \wedge g(y)) \\ &= \bigvee_{x} \left(f(x) \wedge \bigvee_{x \leq y} g(y) \right) = \bigvee_{x} \left(f(x) \wedge g^R(x)\right) = \left(f \wedge g^R\right)^{LR}(1).\end{aligned}$$

Proof is analogous for $\blacktriangleleft$. ■

Note, that

$$\bigvee_{x \leq y} (f(x) \wedge g(y)) = \bigvee_{z} \bigvee_{x \leq z \leq y} (f(x) \wedge g(y)) = \bigvee_{z} \left(\bigvee_{x \leq z} f(x) \right) \wedge \left(\bigvee_{z \leq y} g(y) \right),$$

thus we have

$$\left(f \wedge g^R\right)^{LR} = \left(f^L \wedge g\right)^{LR} = \left(f^L \wedge g^R\right)^{LR},$$

and also by similar reasoning we have

$$\left(f \wedge g^L\right)^{LR} = \left(f^R \wedge g\right)^{LR} = \left(f^R \wedge g^L\right)^{LR}.$$

Lemma 6.12. *The following hold for all* $f \in \mathcal{F}$*.*

1. $\mathbf{1} \blacktriangleright f = f$*;* $\mathbf{0} \blacktriangleleft f = f$*.*
2. $f \blacktriangleright \mathbf{1} = \mathbf{0} \blacktriangleright f = f^{LR} \wedge \mathbf{1}$*;* $f \blacktriangleleft \mathbf{0} = \mathbf{1} \blacktriangleleft f = f^{LR} \wedge \mathbf{0}$*.*

Proof. Let $f \in \mathcal{F}$.

$$(\mathbf{1} \blacktriangleright f)(1) = \left(\mathbf{1}^L \wedge f^R\right)^{LR}(1) = \left(\mathbf{1} \wedge f^R\right)^{LR}(1) = f(1).$$

For all $z < 1$,

$$(\mathbf{1} \blacktriangleright f)(z) = \bigvee_{z = x \rhd y} (\mathbf{1}(x) \wedge f(y)) = \bigvee_{z = 1 \rhd y} (\mathbf{1}(1) \wedge f(y)) = \bigvee_{z = y} f(y) = f(z).$$

Furthermore,

$$(f \blacktriangleright \mathbf{1})(1) = \left(f^L \wedge \mathbf{1}^R\right)^{LR}(1) = \left(f^L\right)^{LR}(1) = f^{LR}(1).$$

For all $z < 1$,

$$(f \blacktriangleright \mathbf{1})(z) = \bigvee_{z = x \rhd y} (f(x) \wedge \mathbf{1}(y)) .$$

According to the properties of residual implications, $z < 1$ implies $y < 1$, thus $\mathbf{1}(y) = 0$, and $(f \blacktriangleright \mathbf{1})(z) = 0$. $\mathbf{0} \blacktriangleright f$ can be proved similarly, the others follow by duality. ∎

Lemma 6.12 imply that all extended residual implications and coimplications fulfill the necessary boundary conditions of implicative operators, i.e.

$$\mathbf{0} \blacktriangleright \mathbf{0} = \mathbf{0} \blacktriangleright \mathbf{1} = \mathbf{1} \blacktriangleright \mathbf{1} = \mathbf{1}; \quad \mathbf{1} \blacktriangleright \mathbf{0} = \mathbf{0},$$
$$\mathbf{0} \blacktriangleleft \mathbf{0} = \mathbf{1} \blacktriangleleft \mathbf{0} = \mathbf{1} \blacktriangleleft \mathbf{1} = \mathbf{0}; \quad \mathbf{0} \blacktriangleleft \mathbf{1} = \mathbf{1}.$$

Proposition 6.13. *For all $f, g, h \in \mathcal{F}$,*

$$f \blacktriangleright (g \blacktriangleright h) = g \blacktriangleright (f \blacktriangleright h) .$$

Proof. Let $f, g, h \in \mathcal{F}$,

$$\begin{aligned}(f \blacktriangleright (g \blacktriangleright h))(z) &= \bigvee_{z = x \rhd y} \left(f(x) \wedge \bigvee_{y = u \rhd v} (g(u) \wedge h(v)) \right) \\ &= \bigvee_{z = x \rhd (u \rhd v)} (f(x) \wedge g(u) \wedge h(v)) .\end{aligned}$$

A similar argument can be given for $g \blacktriangleright (f \blacktriangleright h)$, and so the statement follows from the identity $x \rhd (u \rhd v) = u \rhd (x \rhd v)$, which holds for all residual fuzzy implications. ∎

Figure 6.2 shows the difference between the extended residual implications of the three basic t-norms, min, product and Łukasiewicz. The formulas for the extended residual of the product and the Łukasiewicz t-norms are

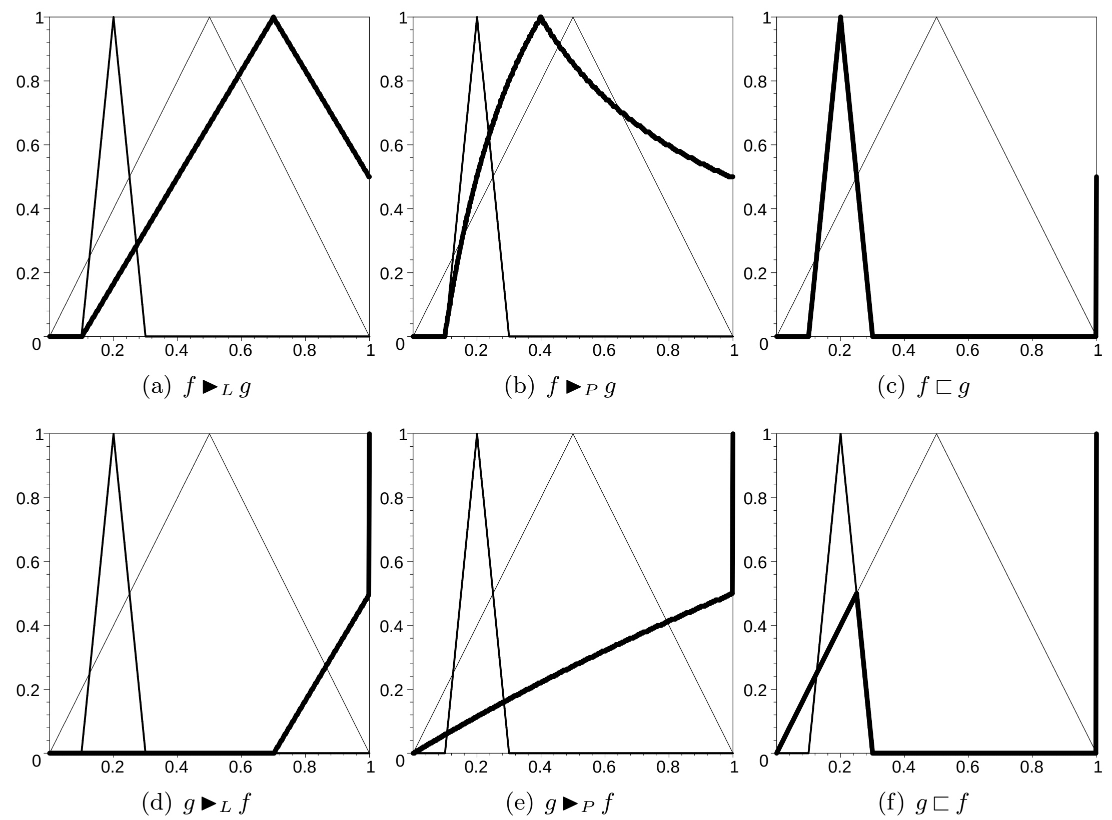
$$(f \blacktriangleright_P g)(z) = \begin{cases} \left(f \wedge g^R\right)^{LR}(1), & \text{if } z = 1, \\ \bigvee\limits_{x > 0} (f(x) \wedge g(zx)) , & \text{otherwise,} \end{cases}$$

$$(f \blacktriangleright_L g)(z) = \begin{cases} \left(f \wedge g^R\right)^{LR}(1), & \text{if } z = 1, \\ \bigvee\limits_{x} (f(x) \wedge g(x + z - 1)) , & \text{otherwise.} \end{cases}$$

For a pointwise formula of the extended residual implication of minimum (denoted by $\sqsubset$) see theorem 6.14 in the next subsection.

6.2.1 The extended residuals of $\wedge$ and $\vee$

As well as the minimum ($\wedge$) and maximum ($\vee$) operators, their type-2 extensions, meet ($\sqcap$) and join ($\sqcup$) are widely used in many applications. These operations are fundamental in type-2 fuzzy logic systems, see [90] for a thorough discussion on their algebraic properties.



(a) $f \blacktriangleright_L g$ (b) $f \blacktriangleright_P g$ (c) $f \sqsubset g$

(d) $g \blacktriangleright_L f$ (e) $g \blacktriangleright_P f$ (f) $g \sqsubset f$

Figure 6.2: Extended residual implications of the three basic t-norms on normal convex fuzzy truth values (thin line – f, normal line – g, thick line – result).

The residuals of $\wedge$ and $\vee$ have the well-known formulas

$$x \rhd_\wedge y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases} \qquad \text{and} \qquad x \lhd_\vee y = \begin{cases} 0 & \text{if } y \leq x, \\ y & \text{otherwise.} \end{cases}$$

In this subsection we consider the extensions of $\rhd_\wedge$ and $\rhd_\vee$. We will use the unique notation $\sqsubset$ and $\sqsupset$ for these operators, i.e.

$$(f \sqsubset g)(z) = \bigvee_{z = x \rhd_\wedge y} (f(x) \wedge g(y)),$$
$$(f \sqsupset g)(z) = \bigvee_{z = x \lhd_\vee y} (f(x) \wedge g(y)).$$

To simplify the forthcoming formulas, we introduce the following notations, which are strict counterparts of the operators ^{R} and ^{L}. For all $f \in \mathcal{F}$ let

$$f^r(x) = \begin{cases} \bigvee_{y>x} f(y), & \text{if } x < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad f^l(x) = \begin{cases} \bigvee_{y<x} f(y), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Note, that $f^R = f^r \vee f$ and also $f^L = f^l \vee f$. As a consequence,

$$f^r \vee f^L = f^r \vee f^L \vee f = f^R \vee f^L = f^{LR},$$

and on similar considerations $f^l \vee f^R = f^{LR}$.

The operations $\sqsubset$ and $\sqsupset$ can be expressed in terms of pointwise operations. This theorem will be used extensively throughout this section.

Theorem 6.14. *For all $f, g \in \mathcal{F}$,*

$$(f \sqsubset g)(z) = \begin{cases} \left(f \wedge g^R\right)^{LR}(1) & \text{if } z = 1, \\ \left(f^r \wedge g\right)(z) & \text{otherwise.} \end{cases}$$

$$(f \sqsupset g)(z) = \begin{cases} \left(f \wedge g^L\right)^{LR}(1) & \text{if } z = 0, \\ \left(f^l \wedge g\right)(z) & \text{otherwise.} \end{cases}$$

Proof. The case $z = 1$ follows from proposition 6.11. For all $z < 1$, $(x \rhd_\wedge y) = y$ and so

$$(f \sqsubset g)(z) = \bigvee_{x>z, y=z} (f(x) \wedge g(y)) = \left(\bigvee_{x>z} f(x) \right) \wedge g(z) = \left(f^r \wedge g\right)(z).$$

The formula for $\sqsupset$ can be proved similarly. ■

It is known, that $x \rhd y = 1$ iff $x \leq y$ holds for any residual fuzzy implication. The extended counterpart of this equivalence is

$$f \blacktriangleright g = \mathbf{1} \quad \text{iff} \quad f \preceq g$$

for a binary relation $\preceq$ over $\mathcal{F}$. Here, we give necessary and sufficient conditions in the special case of $\sqsubset$.

Theorem 6.15. *For all $f, g \in \mathcal{F}$, $f \sqsubset g = \mathbf{1}$ if and only if*

1. *$f, g \in \mathcal{F}_N$, and*
2. *$g^l(x_0) = 0$, where $x_0 = \sup\{x \mid f(x) > 0\}$.*

Proof. Suppose $f \sqsubset g = \mathbf{1}$. On the one hand it implies $(f \sqsubset g)(1) = 1$, i.e. by definition $\left(f \wedge g^R\right)^{LR}(1) = 1$ or equivalently $f \wedge g^R \in \mathcal{F}_N$. It is easy to see, that the normality of f and g^R (and thus the normality of g) is necessary.

On the other hand, for all $z < 1$, $(f \sqsubset g)(z) = 0$ if and only if $\left(f^r \wedge g\right)(z) = 0$, by definition. Let

$$x_0 = \sup\{x \mid f(x) > 0\},$$

i.e. the least upper bound of the (not necessarily convex) support of f. It always exists, since as we have seen before, f is necessarily normal. Note, that $f(x_0)$ is zero if f is right-continuous at x_0, and non-zero otherwise. Also note that $f^r(x) = 0$ for all $x \geq x_0$. There are three cases:

- if $x_0 = 1$, i.e. $f^r(x) > 0$ for all $x < 1$, then it is necessary that $g(x) = 0$ for all $x < 1$, i.e. $g^l(1) = 0$. Note, that because of normality this means that $g = \mathbf{1}$.
- if $x_0 = 0$, i.e. $f = \mathbf{0}$ (since f must be normal), then there is no constraint on the value of $g(x)$ for $x < 1$ (due to the definition of ^{l} we can write $g^l(0) = 0$).
- if $x_0 \in (0, 1)$, then it is necessary that $g(x) = 0$ for all $x < x_0$, i.e. $g^l(x_0) = 0$.

Summarized, $(f \sqsubset g)(z) = 0$ for all $z < 1$ implies $g^l(x_0) = 0$.

Now, suppose $f, g \in \mathcal{F}_N$, and $g^l(x_0) = 0$. For all $z < 1$, it is clear that $(f^r \wedge g)(z) = 0$, by considering the following cases:

- for all $z < x_0$, $g^l(x_0) = 0$ implies $g(z) = 0$,
- for all $z \geq x_0$, $f^r(z) = 0$, since x_0 is the least upper bound of the support of f.

In case $z = 1$, by definition $(f \sqsubset g)(z) = 1$ if and only if $f \wedge g^R$ is normal. By the definition of x_0 and the assumptions $g^l(x_0) = 0$ and $g \in \mathcal{F}_N$ follows, that $g^R(x_0) = g^R(0) = 1$, i.e. g is constantly 1 on the interval $[0, x_0]$. Note, that since f is normal and x_0 is the least upper bound of its support, the restriction of f on $[0, x_0]$ is also normal. Now, this implies that $f \wedge g^R$ is normal on $[0, x_0]$, and thus on the real unit interval, too. ■

6.2.2 Distributive properties of $\sqsubset$ and $\sqsupset$

Proposition 6.16. *The following distributive laws hold for all $f, g, h \in \mathcal{F}$,*

1. $f \sqsubset (g \vee h) = (f \sqsubset g) \vee (f \sqsubset h)$; $\quad f \sqsupset (g \vee h) = (f \sqsupset g) \vee (f \sqsupset h)$.
2. $(f \vee g) \sqsubset h = (f \sqsubset h) \vee (g \sqsubset h)$; $\quad (f \vee g) \sqsupset h = (f \sqsupset h) \vee (g \sqsupset h)$.

Proof. We prove only the first equality of item 1. Item 2 can be shown analogously, and the formulas with $\sqsupset$ follow from duality. Let $f, g, h \in \mathcal{F}$.

$$\begin{aligned}
(f \sqsubset (g \vee h))(1) &= \left(f \wedge (g \vee h)^R\right)^{LR} = \left(f \wedge \left(g^R \vee h^R\right)\right)^{LR} \\
&= \left(\left(f \wedge g^R\right) \vee \left(f \wedge h^R\right)\right)^{LR} = \left(f \wedge g^R\right)^{LR} \vee \left(f \wedge h^R\right)^{LR} \\
&= (f \sqsubset g)(1) \vee (f \sqsubset h)(1).
\end{aligned}$$

For any $x < 1$,

$$\begin{aligned}
(f \sqsubset (g \vee h))(x) &= (f^r \wedge (g \vee h))(x) = ((f^r \wedge g) \vee (f^r \wedge h))(x) \\
&= ((f \sqsubset g) \vee (f \sqsubset h))(x).
\end{aligned}$$

■

The operations $\sqsubset$ and $\sqsupset$ do not distribute over $\wedge$ in general, only the following inequalities hold.

Proposition 6.17. *For all $f, g, h \in \mathcal{F}$,*

1. $f \sqsubset (g \wedge h) \leq (f \sqsubset g) \wedge (f \sqsubset h)$; $\quad f \sqsupset (g \wedge h) \leq (f \sqsupset g) \wedge (f \sqsupset h)$.
2. $(f \wedge g) \sqsubset h \leq (f \sqsubset h) \wedge (g \sqsubset h)$; $\quad (f \wedge g) \sqsupset h \leq (f \sqsupset h) \wedge (g \sqsupset h)$.

Proof. Let $f, g, h \in \mathcal{F}$.

$$\begin{aligned}
(f \sqsubset (g \wedge h))(1) &= \left(f^L \wedge g \wedge h\right)^{LR} = \left(\left(f^L \wedge g\right) \wedge \left(f^L \wedge h\right)\right)^{LR} \\
&\leq \left(f^L \wedge g\right)^{LR} \wedge \left(f^L \wedge h\right)^{LR} = (f \sqsubset g)(1) \wedge (f \sqsubset h)(1).
\end{aligned}$$

For any $x < 1$,

$$\begin{aligned}(f \sqsubset (g \wedge h))(x) &= (f^r \wedge g \wedge h)(x) = (f^r \wedge g \wedge f^r \wedge h)(x) \\ &= (f \sqsubset g)(x) \wedge (f \sqsubset h)(x).\end{aligned}$$

Item 2 can be proved similarly, and the formulas with $\sqsupset$ follow by duality. ■

In general, $\sqsubset$ does not distribute over $\sqcap$ and $\sqcup$, only the following inequalities hold.

Theorem 6.18. *For all $f, g, h \in \mathcal{F}$,*

$$f \sqsubset (g \sqcap h) \leq (f \sqsubset g) \sqcap (f \sqsubset h); \quad f \sqsubset (g \sqcup h) \leq (f \sqsubset g) \sqcup (f \sqsubset h).$$

To prove this theorem we need the following two lemmas.

Lemma 6.19. *For all $f, g \in \mathcal{F}$,*

$$(f \sqsubset g)^R = \left(f^R \wedge g\right)^R \vee \left(f \wedge g^R\right)^{LR}, \tag{6.2}$$

$$(f \sqsubset g)^L(z) = \begin{cases} \left(f^{LR} \wedge g^{LR}\right)(1) & \textit{if } z = 1, \\ (f^r \wedge g)^L(z) & \textit{otherwise.} \end{cases} \tag{6.3}$$

Proof. Let $f, g \in \mathcal{F}$.

$$(f \sqsubset g)^R(1) = (f \sqsubset g)(1) = \left(f \wedge g^R\right)^{LR}(1).$$

For all $x < 1$,

$$(f \sqsubset g)^R(x) = \bigvee_{y \geq x} (f \sqsubset g)(y) = (f \sqsubset g)(1) \vee \bigvee_{1 > y \geq x} (f \sqsubset g)(y).$$

But since $(f^r \wedge g)(1) = 0$,

$$\bigvee_{1 > y \geq x} (f \sqsubset g)(y) = \bigvee_{y \geq x} (f^r \wedge g)(y) = (f^r \wedge g)^R(x),$$

thus

$$(f \sqsubset g)^R = (f^r \wedge g)^R \vee \left(f \wedge g^R\right)^{LR}.$$

Since $(f \wedge g)^R \leq \left(f \wedge g^R\right)^{LR}$, we have

$$\begin{aligned}(f^r \wedge g)^R \vee \left(f \wedge g^R\right)^{LR} &= (f^r \wedge g)^R \vee (f \wedge g)^R \vee \left(f \wedge g^R\right)^{LR} \\ &= ((f^r \wedge g) \vee (f \wedge g))^R \vee \left(f \wedge g^R\right)^{LR} \\ &= ((f^r \vee f) \wedge g)^R \vee \left(f \wedge g^R\right)^{LR} \\ &= \left(f^R \wedge g\right)^R \vee \left(f \wedge g^R\right)^{LR}.\end{aligned}$$

In case $x < 1$, the formula for $(f \sqsubset g)^L(x)$ is straightforward from the definition of $\sqsubset$.

Also by definition,

$$(f \sqsubset g)^L(1) = (f \sqsubset g)^{LR}(1) = \left((f^r \wedge g)^{LR} \vee \left(f^L \wedge g\right)^{LR}\right)(1).$$

Furthermore,

$$\begin{aligned}(f^r \wedge g)^{LR} \vee \left(f^L \wedge g\right)^{LR} &= \left((f^r \wedge g) \vee \left(f^L \wedge g\right)\right)^{LR} = \left(\left(f^r \vee f^L\right) \wedge g\right)^{LR} \\ &= \left(f^{LR} \wedge g\right)^{LR} = f^{LR} \wedge g^{LR}.\end{aligned}$$

■

Lemma 6.20. *For all $f, g \in \mathcal{F}$,*

$$f \sqsubset g \leq f \sqsubset g^R \leq (f \sqsubset g)^R, \tag{6.4}$$

$$f \sqsubset g \leq f \sqsubset g^L \leq (f \sqsubset g)^L. \tag{6.5}$$

Proof. Recall theorem 6.14, the pointwise formulas for $\sqsubset$. The first inequality of (6.4) holds, because $(f \sqsubset g)(1) = \left(f \sqsubset g^R\right)(1)$ by definition, and $f^r \wedge g \leq f^r \wedge g^R$. As for the second inequality, $\left(f \sqsubset g^R\right)(1) = (f \sqsubset g)^R(1)$ also by definition. For all $x < 1$, the inequality

$$\left(f^r \wedge g^R\right)(x) \leq \left(\left(f^R \wedge g\right)^R \vee \left(f \wedge g^R\right)^{LR}\right)(x)$$

holds, as shown by the following reasoning:

$$\begin{aligned}f^r \wedge g^R \leq f^R \wedge g^R = f^R \sqcap g^R = (f \sqcap g)^R &= \left((f^R \wedge g) \vee (f \wedge g^R)\right)^R \\ = (f^R \wedge g)^R \vee (f \wedge g^R)^R &\leq (f^R \wedge g)^R \vee (f \wedge g^R)^{LR}.\end{aligned}$$

The first inequality of (6.5) can be proved analogously as above. Furthermore,

$$\left(f \sqsubset g^L\right)(1) = (f \sqsubset g)^L(1),$$

since $\left(f \wedge g^{LR}\right)^{LR} = f^{LR} \wedge g^{LR}$. So, to prove the second inequality of (6.5) we need to show that

$$f^r \wedge g^L \leq (f^r \wedge g)^L. \tag{6.6}$$

Suppose (6.6) does not hold, i.e.

$$\exists x : (f^r \wedge g)^L(x) < \left(f^r \wedge g^L\right)(x), \text{ i.e.}$$

$$\exists x : (f^r \wedge g)^L(x) < f^r(x) \quad \text{and} \quad (f^r \wedge g)^L(x) < g^L(x). \tag{6.7}$$

By the definition of the operator ^{L}, the first inequality of (6.7) is equivalent to

$$\forall y \leq x : f^r(y) < f^r(x) \quad \text{or} \quad g(y) < f^r(x).$$

Since $f^r \in \mathcal{F}^-$, $f^r(y) < f^r(x)$ can not hold for any $y \leq x$, and so $g(y) < f^r(x)$ must hold for all $y \leq x$, i.e. the first inequality of (6.7) implies

$$g^L(x) < f^r(x). \tag{6.8}$$

Analogously, the second inequality of (6.7) is equivalent to

$$\forall y \leq x : f^r(y) < g^L(x) \quad \text{or} \quad g(y) < g^L(x).$$

Now, since $g(y) < g^L(x)$ can not hold for all $y \leq x$, it implies that

$$\exists y \leq x : f^r(y) < g^L(x). \tag{6.9}$$

So, by combining (6.8) and (6.9), (6.7) implies

$$\exists x : g^L(x) < f^r(x) \text{ and } \exists y \leq x : f^r(y) < g^L(x).$$

This is a contradiction, because $f^r \in \mathcal{F}^-$, i.e. (6.6) and so (6.5) holds. ■

Now we can prove theorem 6.18.

Proof. Let $f, g, h \in \mathcal{F}$.

$$\begin{aligned}
f \sqsubset (g \sqcap h) &= f \sqsubset \left(\left(g \wedge h^R\right) \vee \left(g^R \wedge h\right)\right) = \left(f \sqsubset \left(g \wedge h^R\right)\right) \vee \left(f \sqsubset \left(g^R \wedge h\right)\right) \\
&\leq \left(\left(f \sqsubset g\right) \wedge \left(f \sqsubset h^R\right)\right) \vee \left(\left(f \sqsubset g^R\right) \wedge \left(f \sqsubset h\right)\right) \\
&\leq \Big(\left(f \sqsubset g\right) \wedge \left(f \sqsubset h\right)^R\Big) \vee \Big(\left(f \sqsubset g\right)^R \wedge \left(f \sqsubset h\right)\Big) \\
&= \left(f \sqsubset g\right) \sqcap \left(f \sqsubset h\right).
\end{aligned}$$

The other equality follows analogously. ■

In the next subsections we investigate the operation $\sqsubset$ on the main subalgebras of $\mathbf{F}$.

6.2.3 Convex and normal fuzzy truth values

The most elementary subset of $\mathcal{F}$ is the set of singleton fuzzy truth values $\mathcal{F}_S$. A fuzzy truth value f_x is a singleton if there exists exactly one $x \in \mathcal{I}$ such that $f_x(x) = 1$, and for all $y \neq x$, $f_x(y) = 0$. It is proved that $(\mathcal{F}_S, \sqcap, \sqcup, {}^*, \mathbf{0}, \mathbf{1})$ is isomorphic to the algebra of (type-1) truth values $(\mathcal{I}, \wedge, \vee, {}', 0, 1)$ by the bijection $x \mapsto f_x$ from $\mathcal{I}$ to $\mathcal{F}_S$.

Is is easy to see, that the elements of $\mathcal{F}_S$ are normal convex, it is closed w.r.t $\sqcap$, $\sqcup$, and the partial orders $\sqsubseteq$ and $\preccurlyeq$ coincide (in fact, $(\mathcal{F}_S, \sqsubseteq)$ is a chain).

Proposition 6.21. *For all $f_x, g_y \in \mathcal{F}_S$,*

$$f_x \sqsubseteq g_y \quad \textit{if and only if} \quad x \leq y, \tag{6.10}$$

$$f_x \sqsubset g_y = \begin{cases} \mathbf{1}, & \textit{if } f_x \sqsubseteq g_y, \\ g_y, & \textit{otherwise}. \end{cases} \tag{6.11}$$

Proof is straightforward from the formulas of $\sqsubseteq$ and $\sqsubset$. Having $f_x \sqsubset g_y$, the following theorem can be established stating that the algebras $\mathbf{I}$ and $\mathbf{F}$ equipped with the residual implication $\rhd_\wedge$ and its extension are also isomorphic.

Theorem 6.22. *The algebra $\mathbf{F}_S = (\mathcal{F}_S, \sqcap, \sqsubset, {}^*, \mathbf{0}, \mathbf{1})$ is isomorph to $\mathbf{I} = (\mathcal{I}, \wedge, \rhd_\wedge, {}', 0, 1)$, where $\rhd_\wedge$ denotes the residual implication of $\wedge$.*

It is easy to see that $\sqsubset$ is a type-2 fuzzy implication on $\mathbf{F}_S$, i.e. besides the boundary conditions, it is antitone/monotone. However, it is an open question whether it is the largest such subalgebra of $\mathbf{F}$ containing $\mathbf{F}_S$.

An undoubtedly important and recently most popular subalgebra of $\mathbf{F}$ is the algebra of interval fuzzy truth values $\mathbf{F}_I$. It is proved to be isomorphic to the algebra $(I^{[2]}, \wedge, \vee, ', 0, 1)$, where $I^{[2]}$ denotes the set of closed intervals in I. Thus, the elements of $\mathbf{F}_I$ can be represented by a pair $(a, b) \in I^{[2]}$. It is important to remark that $I^{[2]}$ contains only closed intervals, and so the elements of $\mathbf{F}_I$ are closed interval fuzzy truth values. In fact, the proof of the next theorem is based on this observation.

Theorem 6.23. *The algebra $\mathbf{F}_I$ of interval fuzzy truth values is not closed w.r.t. $\sqsubset$ and $\sqsupset$.*

Proof. We prove by example. Let $f, g \in \mathbf{F}_I$ be represented by the intervals $(1/3, 2/3)$ and $(1/2, 3/4)$, respectively. Then $f \sqsubset g$ is not a closed interval. Moreover, it is not even an interval fuzzy truth value. Indeed, for all $x < 1$, $(f \sqsubset g)(x) = (f^r \wedge g)(x)$, and $(f \sqsubset g)(2/3) = f^r(2/3) \wedge g(2/3) = 0$ while for all $1/2 \leq y < 2/3$, $(f \sqsubset g)(y) = 1$. To see that it is not even an interval, note that $(f \sqsubset g)(1) = 1$. A similar example can be given for $\sqsupset$. ■

By this negative result on $\mathbf{F}_I$ it is natural to ask the following. What is the largest subalgebra $\mathbf{A}$ of $\mathbf{F}$ containing $\mathbf{F}_S$ such that for all $f, g \in \mathbf{A}$, $f \sqsubset g$ reduces to

$$f \sqsubset g = \begin{cases} \mathbf{1}, & \text{if } f \sqsubseteq g, \\ g, & \text{otherwise.} \end{cases}$$

Since interval fuzzy truth values are convex and normal, it is straightforward to investigate the latter. In fact, the following corollary is immediate from the proof of theorem 6.23 since the result there is not even convex.

Corollary 6.24. *The set $\mathcal{F}_C$ of convex fuzzy truth values is not closed w.r.t. $\sqsubset$ and $\sqsupset$.*

We have a positive result on normal fuzzy truth values.

Theorem 6.25. *The set $\mathcal{F}_N$ of normal fuzzy truth values is closed w.r.t. $\sqsubset$ and $\sqsupset$. Moreover, $f \sqsubset g \in \mathcal{F}_N$ (resp. $f \sqsupset g \in \mathcal{F}_N$) if and only if $f, g \in \mathcal{F}_N$.*

Proof. By lemma 6.19 we have

$$(f \sqsubset g)^L(1) = \left(f^{LR} \wedge g^{LR}\right)(1) = f^{LR}(1) \wedge g^{LR}(1) = f^L(1) \wedge g^L(1),$$

so $(f \sqsubset g)^L(1) = 1$ if and only if $f^L(1) = 1$ and $g^L(1) = 1$. The operation $\sqsupset$ preserves normality by duality. ■

It is proved in [90], that the lattice of normal convex fuzzy truth values is a maximal lattice in $\mathbf{F}$. Since the operation $\sqsubset$ is not closed on $\mathcal{F}_C$, the next theorem is straightforward.

Theorem 6.26. *The operations $\sqcap$ and $\sqsubset$ do not form an adjoint pair on the lattice*

$$(F_{CN}, \sqcap, \sqcup, \sqsubset, \mathbf{0}, \mathbf{1}).$$

Naturally arises the following. What is the largest sublattice of F_{CN} for which $\sqcap$ and $\sqsubset$ are an adjoint pair? The question is correct, since for example, on the algebra $\mathbf{F}_S$ of singleton fuzzy truth values $\sqsubset$ is the residual of $\sqcap$.

6.2.4 Left- and right-maximal fuzzy truth values

In the following let $\mathcal{F}_{LM}$ and $\mathcal{F}_{RM}$ denote the sets of left-maximal and right-maximal fuzzy truth values.

Proposition 6.27. *If $f \in \mathcal{F}_{RM}$, then*

$$(f \sqsubset g)(x) = \begin{cases} \left(f \wedge g^R\right)^{LR}(1) & \text{if } x = 1, \\ \left(f^{LR} \wedge g\right)(x) & \text{otherwise.} \end{cases} \tag{6.12}$$

Furthermore, if also $g \in \mathcal{F}_{RM}$, then it simplifies to

$$f \sqsubset g = f^{LR} \wedge g, \tag{6.13}$$

and so $f \sqsubset g \in \mathcal{F}_{RM}$. If $g \in \mathcal{F}_{LM}$, then $f \sqsubset g \in \mathcal{F}_{LM}$.

Proof. The right-maximality of f implies $f^r(x) = f^{LR}$ for all $x < 1$, hence the formula $\left(f^{LR} \wedge g\right)(x)$ in case $x < 1$. If in addition g is also right-maximal, then

$$\left(f \wedge g^R\right)^{LR}(1) = \left(f \wedge g^{LR}\right)^{LR}(1) = \left(f^{LR} \wedge g^{LR}\right)(1) = \left(f^{LR} \wedge g\right)(1).$$

It is easy to see that in this case $f^{LR} \wedge g$ is right-maximal, too.

The left-maximality of $f \sqsubset g$ assuming g is left-maximal is as follows. According to (6.12), $f \sqsubset g$ is left-maximal if $(f \sqsubset g)(1) \leq (f \sqsubset g)(0)$, i.e.

$$\left(f \wedge g^R\right)^{LR} \leq \left(f^{LR} \wedge g\right)(0) = f^{LR} \wedge g^{LR}.$$

It holds for all f, g, since $(f \wedge g)^{LR} \leq f^{LR} \wedge g^{LR}$. ∎

By (6.13), it is easy to check the following.

Corollary 6.28. *If $f, g \in \mathcal{F}_+$, then $f \sqsubset g \in \mathcal{F}_+$, i.e. $\sqsubset$ is closed on $\mathcal{F}_+$.*

Proposition 6.29. *If $f \in \mathcal{F}_{LM}$, then*

$$(f \sqsubset g)(z) = \begin{cases} f^{LR} \wedge g^{LR} & \text{if } z = 1, \\ \left(f^r \wedge g\right)(z) & \text{otherwise,} \end{cases} \tag{6.14}$$

moreover, it is right-maximal for all $g \in \mathcal{F}$.

Proof. Consider the following inequalities. For all $x < 1$,

$$\left(f^r \wedge g\right)(x) \leq \left(f^R \wedge g^R\right)(x) \leq \left(f^R \wedge g^R\right)(0) = f^R(0) \wedge g^R(0) = (f \sqsubset g)(1).$$

∎

Summarizing the above results, the following theorem holds.

Theorem 6.30. *The algebra* $\mathbf{F}_M = (\mathcal{F}_{LM} \cup \mathcal{F}_{RM}, \sqcap, \sqcup, ^*, \sqsubset, \mathbf{0}, \mathbf{1})$ *i.e. the algebra of left- or right-maximal fuzzy truth values is a subalgebra of* $(\mathcal{F}, \sqcap, \sqcup, ^*, \sqsubset, \mathbf{0}, \mathbf{1})$.

Proof. According to the previous propositions, the operation $\sqsubset$ is closed on the union set of left- or right-maximal functions. **0** and **1** are clearly elements of it, and it is also easy to check that $\sqcap$ and $\sqcup$ are also closed on $\mathbf{F}_M$. ∎

6.3 Summary

Current research is just starting to discover the rich structure of fuzzy truth values. In this chapter we have discussed extended fuzzy implications from two distinct views: extended fuzzy S-implications and S-coimplications and the extended residual implications and coimplications (especially the extended residual of $\wedge$).

First, we have discussed type-2 S-implication operators which have a well established background due to recent literature on type-2 t-norms and t-conorms, in particular to meet and join. Our investigations show that type-2 S-implications have similar properties on the lattice of convex normal fuzzy truth values like type-1 S-implications on the unit interval.

Residual implications (i.e. residuals of conjunctive operations) are fundamental in fuzzy logic. A (type-1) residual implication can be extended to the set of fuzzy truth values. In section 6.2 we have discussed such extended operations, in particular the operation $\sqsubset$, the extended residual of $\wedge$.

The following questions are left open for further research:

1. What is the adjoint operation of $\sqsubset$ (the extended residual implication of $\wedge$) on **F**?

2. What is the largest subalgebra of **F**, where $\sqsubset$ is a type-2 fuzzy implication (definition 6.1)?

3. What is the largest subalgebra of **F**, where $\sqsubset$ and the residual of $\sqcap$ coincide?

Bibliography

[1] N.H. Abel. Untersuchung der functionen zweier unabhängig veränderlichen grössen x und y wie $f(x, y)$, welche die eigenschaft haben, dass $f(z, f(x, y))$ eine symmetrische function von x, y und z ist. *J. Reine Angew. Math.*, 1:11–15, 1826.

[2] R. Ackermann. *An introduction to many-valued logics.* Dover, New York, 1967.

[3] J. Aczél. Über eine klasse von funktionalgleichungen. *Comment. Math. Helv.*, 54:247–256, 1948.

[4] J. Aczél. Sur les opérations définies pour des numbres réels. *Bull. Soc. Math. de France*, 76:59–64, 1949.

[5] J. Aczél. *Lectures on functional equations and their applications.* Academic Press, New York, 1966.

[6] R. Andrews and S. Geva. Inserting and extracting knowledge from constrained error back propagation networks. In *Proc. 6th Australian Conference on Neural Networks, Sydney*, 1995.

[7] M. Baczyński. On some properties of intuitionistic fuzzy implications. In M. Wagenknecht and R. Hampel, editors, *Proc. 3rd International Conference in Fuzzy Logic and Technology*, pages 168–171, 2003.

[8] B. De Baets and E. E. Kerre. The generalized modus ponens and the triangular fuzzy data model. *Fuzzy Sets and Systems*, 59:305–318, 1993.

[9] A. Bagis. Determining fuzzy membership functions with tabu search an application to control. *Fuzzy Sets and Systems*, 139:209–225, 2003.

[10] J. F. Baldwin. A new approach to approximate reasoning using a fuzzy logic. *Fuzzy Sets and Systems*, 2:309–325, 1979.

[11] J. F. Baldwin and N. C. F. Guild. Feasible algorithms for approximate reasoning using fuzzy logic. *Fuzzy Sets and Systems*, 3:225–251, 1980.

[12] J. F. Baldwin and B. W. Pilsworth. Axiomatic approach to implication for approximate reasoning with fuzzy logic. *Fuzzy Sets and Systems*, 3:193–219, 1980.

[13] W. Bandler and L. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:13–30, 1980.

[14] R. E. Bellman and L. A. Zadeh. Local and fuzzy logics. In G. Epstein, editor, *Modern Uses of Multiple-Valued Logic*, pages 103–165. Reidel, Dordrecht, 1977.

[15] L. Biacino and A. Lettieri. Equations with fuzzy numbers. *Inf. Sci.*, 47(1):63–76, 1989.

[16] S. Bodjanova. A generalized α-cut. *Fuzzy Sets and Systems*, 126:157–176, 2002.

[17] B. Bouchon-Meunier, R. Mesiar, C. Marsala, and M. Rifqi. Compositional rule of inference as an analogical scheme. *Fuzzy Sets and Systems*, 138:53–65, 2003.

[18] J. L. Castro and E. Trillas. The logic of neural networks. *Mathware & Soft Computing*, 5:23–37, 1998.

[19] P. Chandra and Y. Singh. An activation function adapting training algorithm for sigmoidal feedforward networks. *Neurocomputing*, 61:429–437, 1994.

[20] T.C. Chang, K. Hasegawa, and C.W. Ibbs. The effects of membership function on fuzzy reasoning. *Fuzzy Sets and Systems*, 44:169–186, 1991.

[21] C.-L. Chen and W.-C. Chen. Fuzzy controller design by using neural network techniques. *Fuzzy Systems, IEEE Transactions on*, 2(3):235–244, 1994.

[22] C. L. Chen, S. N. Wang, C. T. Hsieh, and F. Y. Chang. Theoretical analysis of a fuzzy-logic controller with unequally spaced triangular membership functions. *Fuzzy Sets and Systems*, 101:87–108, 1999.

[23] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning.* Springer, 2000.

[24] M. R. Civanlar and H. J. Trussel. Constructing membership functions using statistical data. *Fuzzy Sets and Systems*, 17:1–14, 1986.

[25] M. W. Craven and J. Shavlik. Using sampling and queries to extract rules from trained neural networks. In *Machine Learning: Proc. of the 11th International Conferenceand San Francisco, CA, USA*, 1994.

[26] R.L. de Mantaras and L. Godo. From fuzzy logic to fuzzy truth-valued logic for expert systems: a survey. In *Fuzzy Systems, 1993., Second IEEE International Conference on*, pages 750–755vol.2, 28 March-1 April 1993.

[27] M. Denna, , G. Mauriand, and A.M. Zanaboni. Learning fuzzy rules with tabu search-an application to control. *IEEE Trans. Fuzzy Systems*, 7:295–318, 1999.

[28] G. Deschrijver and E. E. Kerre. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems*, 133(2):227–235, 2003.

[29] J. Dombi. Basic concepts for a theory of evaluation: the aggregative operator. *European Journal of Operations Research*, 10:282–293, 1982.

[30] J. Dombi. Membership function as an evaluation. *Fuzzy Sets and Systems*, 35:1–21, 1990.

[31] J. Dombi and Zs. Gera. The approximation of piecewise linear membership functions and łukasiewicz operators. *Fuzzy Sets and Systems*, 154:275–286, 2005.

[32] J. Dombi and Zs. Gera. Approximation of the continuous nilpotent operator class. *Acta Polytechnica Hungarica*, 2:45–58, 2005.

[33] J. Dombi and Zs. Gera. Fuzzy rule based classifier construction using squashing functions. *Journal of Intelligent and Fuzzy Systems*, 19(1):3–8, 2008.

[34] D. Driankov. Inference with a single fuzzy conditional proposition. *Fuzzy Sets and Systems*, 24:51–63, 1987.

[35] D. Dubois and H. Prade (Eds.). *Fundamentals of fuzzy sets*. Kluwer, 2000.

[36] D. Dubois and H. Prade. Operations on fuzzy numbers. *Int. J. Syst. Sci.*, 9:613–626, 1978.

[37] D. Dubois and H. Prade. Operations in a fuzzy-valued logic. *Information & Control*, 43(2):224–240, 1979.

[38] D. Dubois and H. Prade. Fuzzy sets in approximate reasoning, part i and ii. *Fuzzy Sets and Systems*, 40:143–244, 1991.

[39] D. Dubois, H. Prade, and J. C. Bezdek, editors. *Fuzzy Sets in Approximate Reasoning and Information Systems*. Kluwer Academic Publishers, Norwell, MA, USA, 1999.

[40] J. Fodor. On fuzzy implication operators. *Fuzzy Sets and Systems*, 42:293–300, 1991.

[41] J. Fodor and M. Roubens. *Fuzzy preference modelling and multicriteria decision support*. Kluwer, 1994.

[42] S. Fukami, M. Mizumoto, and K. Tanaka. Some considerations on fuzzy conditional inference. *Fuzzy Sets and Systems*, 4:243–273, 1980.

[43] R. Fullér and T. Kereszfalvi. On generalization of nguyen's theorem. *Fuzzy Sets and Systems*, 41:371–374, 1991.

[44] B. Gaines. Foundations of fuzzy reasoning. *International Journal of Man-Machine Studies*, 8(6):623–668, 1976.

[45] Zs. Gera. Computationally efficient reasoning using approximated fuzzy intervals. *Fuzzy Sets and Systems*, 158(7):689–703, 2007.

[46] Zs. Gera and J. Dombi. Exact calculations of extended logical operations on fuzzy truth values. *Fuzzy Sets and Systems*, 159:1309–1326, 2008.

[47] Zs. Gera and J. Dombi. Type-2 implications on non-interactive fuzzy truth values. *Fuzzy Sets and Systems*, 159:3014–3032, 2008.

[48] L. Godo, J. Jacas, and L. Valverde. Fuzzy values in fuzzy logic. *International Journal of Intelligent Systems*, 6:199–212, 1991.

[49] A. Grauel and L. A. Ludwig. Construction of differentiable membership functions. *Fuzzy Sets and Systems*, 101:219–225, 1999.

[50] P. Hájek. *Metamathematics of fuzzy logic*. Kluwer, 1998.

[51] J. Hellendoorn. *Reasoning with fuzzy logic*. PhD thesis, Delft University of Technology, Delft, 1990.

[52] S. Huang and H. Xing. Extract intelligible and concise fuzzy rules from neural networks. *Fuzzy Sets and Systems*, 132:233–243, 2002.

[53] S. Jenei. Continuity in zadeh's compositional rule of inference. *Fuzzy Sets and Systems*, 104:333–339, 1999.

[54] D. Karaboga. Design of fuzzy logic controllers using tabu search algorithm. In *Fuzzy Information Processing Society, 1996. NAFIPS. 1996 Biennial Conference of the North American*, pages 489–491, 1996.

[55] N. Karnik and J. M. Mendel. Operations on type-2 fuzzy sets. *Fuzzy Sets and Systems*, 122:327–348, 2001.

[56] M. Kodaka. Requirements for generating sigmoidal timecourse aggregation in nucleation-dependent polymerization model. *Biophysical Chemistry*, 107:243–253, 2004.

[57] C.H. Ling. Representation of associative functions. *Publicationes Mathematicae, Debrecen*, 12:189–212, 1965.

[58] C. McMillan, M. C. Mozer, and P. Smolensky. The connectionist scientist game: rule extraction and refinement in a neural network. In *Proc. 13th Annual Conference of the Cognitive Science Society. Hillsdale, NJ, USA*, 1991.

[59] N. Mellouli and B. Bouchon-Meunier. Abductive reasoning and measures of similitude in the presence of fuzzy rules. *Fuzzy Sets and Systems*, 137:177–188, 2003.

[60] J. M. Mendel. Fuzzy sets for words: a new beginning. In *IEEE International Conference on Fuzzy Systems*, page 3742, St. Louis, MO, 2003.

[61] J. M. Mendel. Advances in type-2 fuzzy sets and systems. *Inf. Sci.*, 177(1):84–110, 2007.

[62] J. M. Mendel and R. I. John. Type-2 fuzzy sets made simple. *IEEE Trans. Fuzzy Syst.*, 10:117–127, 2002.

[63] K. Menger. Statistical metric spaces. *Proceedings of the National Academy of Science of the USA*, 28:535–537, 1942.

[64] M. Mizumoto and K. Tanaka. Some properties of fuzzy sets of type 2. *Information and Control*, 31:312–340, 1976.

[65] M. Mizumoto and K. Tanaka. Fuzzy sets of type 2 under algebraic product and algebraic sum. *Fuzzy Sets and Systems*, 5:277–290, 1981.

[66] M. Mizumoto and H.-J. Zimmermann. Comparison of fuzzy reasoning methods. *Fuzzy Sets and Systems*, 8:253–283, 1982.

[67] N. N. Morsi and A. A. Fahmy. On generalized modus ponens with multiple rules and a residuated implication. *Fuzzy Sets and Systems*, 129:267–274, 2002.

[68] H. T. Nguyen. A note on the extension principle for fuzzy sets. *J. Math. Anal. Appl.*, 64:369–380, 1978.

[69] J. Nieminen. On the algebraic structure of fuzzy sets of type-2. *Kybernetika*, 13:261–273, 1977.

[70] S. Ovchinnikov and M. Roubens. On strict preference relations. *Fuzzy Sets and Systems*, 43:319–326, 1991.

[71] W. Pedrycz and M. Reformat. Genetically optimized logic models. *Fuzzy Sets and Systems*, 150:351–371, 2005.

[72] E. Pop, R. Hayward, and J. Diederich. Ruleneg: extracting rules from a trained ann by stepwise negation. Technical Report NRC QUT December 1994, Queensland University, 1994.

[73] T. A. Runkler and J. T. Bezdek. Function approximation with polynomial membership functions and alternating cluster information. *Fuzzy Sets and Systems*, 101:207–218, 1999.

[74] E. H. Ruspini. On the semantics of fuzzy logic. *International Journal of Approximate Reasoning*, 5:45–88, 1991.

[75] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* North Holland, Amsterdam, 1983.

[76] S. Sestino and T. Dillon. Automated knowledge acquisition of rules with continuously valued attributes. In *Proc. 12th Int. Conf. on Expert Systems and their Applications, Avignon, France*, pages 645–656, 1992.

[77] P. Smets and P. Magrez. Implication in fuzzy logic. *Int J. of Approximate Reasoning*, 1:327–347, 1987.

[78] J. T. Starczewski. A triangular type-2 fuzzy logic system. In *Proc. FUZZ-IEEE 2006*, pages 7231–7239, Vancouver, Canada, 2006.

[79] J.T. Starczewski. Extended triangular norms on gaussian fuzzy sets. In *Proceedings of EUSFLAT-LFA, Barcelona, Spain*, pages 872–877, 2005.

[80] S. B. Thrun. Extracting provably correct rules from artificial neural networks. Technical Report IAI-TR-93-5, Institut for Informatik III Universitat Bonn, Germany, 1994.

[81] A. B. Tickle, , M. Orlowski, , and J. Diederich. Dedec: decision detection by rule extraction from neural networks. Technical Report NRC QUT September 1994, Queensland University, 1994.

[82] R. M. Tong and J. Efstathiou. A critical assessment of truth functional modification and its use in approximate reasoning. *Fuzzy Sets and Systems*, 7:103–108, 1982.

[83] G. Towell and J. Shavlik. The extraction of refined rules from knowledge based neural networks. *Machine Learning*, 131:71–101, 1993.

[84] E. Trillas. Sobre funciones de negación en la teoría de conjuntos difusos. *Stochastica*, III:47–60, 1979.

[85] E. Trillas and L. Valverde. On some functionally expressable implications for fuzzy set theory. In *3rd Intern. Seminar on Fuzzy Set Theory*, pages 173–190. Johannes Kepler Universitat, 1981.

[86] E. Trillas and L. Valverde. On inference in fuzzy logic. In *Proc. Second Inter. Fuzzy Systems Association (IFSA'87) Congress*, pages 294–297, 1987.

[87] Y. Tsukamoto. An approach to fuzzy reasoning method. In M.M. Gupta, R.K. Ragade, and R.R. Yager, editors, *Advances in Fuzzy Set Theory and Applications*, pages 137–149. North-Holland, Amsterdam, 1979.

[88] I.B. Turksen and D.D.W. Yao. Representation of connectives in fuzzy reasoning: the view through normal forms. *IEEE Trans. Systems Man. Cybernet.*, 14:146–151, 1984.

[89] K. Uehara and M. Fujise. Multistage fuzzy inference formulated as linguistic-truth-value propagation and its learning algorithm based on back-propagating error information. *IEEE Transactions on Fuzzy Systems*, 1(3):205–221, 1993.

[90] C. L. Walker and E. A. Walker. The algebra of fuzzy truth values. *Fuzzy Sets and Systems*, 149:309–347, 2005.

[91] W. Wangming. Fuzzy reasoning and fuzzy relational equations. *Fuzzy Sets and Systems*, 20:67–78, 1986.

[92] R. R. Yager. An approach to inference in approximate reasoning. *International Journal of Man-Machine Studies*, 13:323–338, 1980.

[93] L. A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Trans. Syst. Man. Cybernet.*, 3:28–44, 1973.

[94] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning - 1. *Information Sciences*, 8:199–249, 1975.

[95] L. A. Zadeh. A theory of approximate reasoning. In J.E. Hayes, D. Michie, and L.I. Mikulich, editors, *Machine Intelligence 9*, pages 149–194. Wileyand New York, 1979.

[96] L.A. Zadeh. Fuzzy sets. *Information Control*, 8:338–353, 1965.

Summary of the results of the thesis

Chapter 1 reviewed some basic fuzzy concepts. In Chapter 2, the so called "squashing" function was introduced. It approximates the piecewise linear cut function in such a way that its derivatives are continuous, too. This Chapter also investigated the error of the approximation as well as other properties. Furthermore an application of the squashing function is shown: the approximation of piecewise linear membership functions such as trapezoidal or triangular ones.

Chapter 3 introduced a hybrid, fuzzy rule learning model, which is partly based on the approximation of piecewise linear fuzzy membership functions. The purpose of the model is to determine a rule set which is concise and easily understandable and well describes the connection between the input-output data. In short, it operates as follows. After the fuzzification of the input data, a genetic algorithm evolves the structure of the rules. Each rule must be in disjunctive normal form, i.e. disjunctions of conjunctions of features, hence the rules remain comprehensible. Then the fine-tuning of fuzzy membership functions is done by gradient-based optimization. It is made possible by the continuous derivatives of the squashing function, which is used to approximate the piecewise linear membership functions. The Chapter ends with the presentation of the results on well known datasets.

The main results presented in Chapter 4 are as follows. I studied the closure properties of the classical Compositional Rule of Inference (CRI) on sigmoid-like membership functions, such as the squashing function. I have shown in which setting regarding the input, and the rule premise and conclusion does the output preserve its shape, and how the calculations can be done. I investigated the properties of CRI and interpolation based reasoning, and the complexity of the calculations of their output. Taken into consideration the former, I introduced a new fuzzy inference method, the Membership Driver Inference (MDI). It does not utilize any implication operator or any measure of similitude, the output is only determined by the membership functions of the input and the rule premise and consequence. I proved that MDI fulfills the fuzzy modus ponens, modus tollens and chain rule properties. Fast calculations of MDI are shown on trapezoid membership functions approximated by squashing functions.

In the MDI model a simple fuzzy rule can be regarded as a fuzzy truth value. To represent composite rules e.g. where the premise is a conjunction of two fuzzy sets, logical operations are required on the set of fuzzy truth values. Chapter 5 shows results on reducing the computational complexity of conjunctions and disjunctions of certain subsets of fuzzy truth values. These subsets are the sets of right-, and leftmaximal, the monotone and the linear fuzzy truth values.

In the last Chapter I studied type-2 fuzzy implications, which are implication operators interpreted on fuzzy truth values. Type-2 fuzzy implications are extended from classical fuzzy implication operators (i.e. on $[0, 1]$). In case of extended S-implications I proved that these operators fulfill the basic requirements of a fuzzy implication only regarding convex and normal fuzzy truth values. By investigating the algebraic properties of type-2 residual implications and specifically the residuals of the type-2 min/max operators, I also proved several important properties (e.g. distributivity, or closure properties) including their necessary and sufficient conditions. I have shown, that the algebra of continuous left- or rightmaximal fuzzy truth values equipped with the type-2 min/max operators and their residuals is a subalgebra of the algebra containing all continuous fuzzy truth values.

Magyar nyelvű összefoglaló

Az els fejezet, azaz a szksges fuzzy alapfogalmak ttekintse utn a második fejezetben az ún. "squashing" függvényt vezettem be, amely a szakaszonként lineáris, $[0, 1]$ értékkészlettel rendelkező vágófüggvény egy olyan approximációja, amelynek deriváltjai is folytonosak. E fejezet vizsgálja továbbá a squashing függvény közelítési hibáját a paraméterei függvényében, illetve egyéb tulajdonságait is. A fejezet végül a squashing függvény egy alkalmazását mutatja be: a trapéz alakú, illetve a trianguáris fuzzy halmazhoztartozási függvények approximációját.

A harmadik fejezet egy olyan hibrid, fuzzy szabálytanuló modellt mutatott be, amelynek egyik alapját a squashing függvényekkel közelített fuzzy tagsági függvények adják. A modell célja felállítani egy könnyen értelmezhető fuzzy szabályhalmazt, amely jól leírja az input és output adatok kapcsolatát. Működése röviden a következő: az input fuzzifikálása (azaz fuzzy tagsági függvények felállítása) után a szabályok struktúrájának kialakítása történik egy genetikus algoritmus segítségével. Minden szabálynak diszjunktív normálformában kell lennie, ezáltal azok könnyen érthetőek maradnak. Ezután a squashing függvénnyel közelített fuzzy tagsági függvények optimalizációja következik: mivel azok deriváltjai folytonosak, egy gradiens alapú lokális optimalizáló eljárás finomhangolja a fuzzy tagsági függvényeket. A fejezet a tesztelt adathalmazokon elért eredményekkel zárul.

A negyedik fejezet fő eredményei az alábbiak. A klasszikus Compositional Rule of Inference (CRI) következtetés zártsági tulajdonságait vizsgáltam ún. szigmoidszerű függvények (ilyen függvény a squashing is) mellett. Tételesen bizonyítottam, hogy mely esetekben (tekintve az inputot, illetve a szably elzmnyt s kvetkezmnyt) marad szigmoidszerű a következmény, illetve az hogyan számolható ki. A CRI és az interpolációs következtetés tulajdonságai és az eredményfggvnyek kiszámítása nehézségeinek vizsgálata után, a fentiek figyelembevételével vezettem be egy új fuzzy következtetési eljárást, a tagsági függvény alapú következtetést (Membership Driven Inference, MDI). Az MDI nem használ sem implikációs operátort, sem hasonlósági mértéket, a következtetés eredményét csak az input és a szabály tagsági függvényei határozzák meg. Bizonyítottam, hogy az MDI teljesíti a fuzzy modus ponens, modus tollens és a láncszabály tulajdonságokat. A fejezet azzal zárult, hogy squashing függvényekkel közelített trapezoid tagsági függvényeken mutattam be az MDI gyors számítását.

Az MDI következtetési modellben egy nem összetett szabály felfogható egy fuzzy igazságértéknek (fuzzy truth value). Összetett szabályokhoz (például, ha egy szabály előzménye több fuzzy halmaz konjunkciója) ezért szükség van fuzzy igazságértékeken végzett logikai műveletekre. Az ötödik fejezet fő eredményei, hogy a folytonos fuzzy igazságértékek főbb részhalmazaira bizonyítottam olyan tételeket, amelyekkel a fuzzy igazságértékeken végzett konjunkció és diszjunkció számításigénye lényegesen csökkenthető. Ezen részhalmazok a jobb- és balmaximális, a monoton, illetve a lineáris fuzzy igazságértékek.

Az utolsó, hatodik fejezetben a fuzzy igazságértékeken értelmezett ún. "type-2" implikációs műveleteket vizsgáltam. Ezek megfeleltethetőek egy-egy, a klasszikus fuzzy logikában (azaz a $[0, 1]$ intervallumon) értelmezett implikációnak. A type-2 S-implikációk tekintetében bizonyítottam, hogy azok csak a konvex és normál fuzzy igazságértékek halmazán teljesítik a fuzzy logikában elvárt alapvető implikációs operátor tulajdonságokat. A type-2 reziduális implikációk és különösképp a min/max operátorok type-2 reziduálisai algebrai tulajdonságait vizsgálva beláttam több fontos tulajdonságot (például disztributivitás, zártsági tételek), illetve azok teljesüléséhez elégséges és szükséges feltételeket. Bizonyítottam, hogy a type-2 min/max operátorokkal és reziduálisukkal rendelkező folytonos jobb- vagy balmaximális fuzzy igazságértékeket tartalmazó algebra részalgebrája az összes folytonos fuzzy igazságértéket magában foglaló implikcis algebrának.