Asymptotic inference for linear stochastic differential equations with time delay

Outline of Ph.D. Thesis

by

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1 Introduction

In the thesis a statistical model of linear stochastic differential equation with time delay is considered. The aim of the investigation is to prove local asymptotic properties of the likelihood function. The list of the papers dealing with investigation of the local asymptotic properties of delay models is not so extensive. The majority of the literature is the works by Gushchin, Küchler and their co-authors. The references can be found in Section 1.3 of the thesis. We have to emphasize the paper was written in 1999 by Gushchin and Küchler [4]. They study the local asymptotic properties of the equation

$$\mathrm{d}X(t) = (aX(t) + bX(t-1))\,\mathrm{d}t + \mathrm{d}W(t).$$

It is shown, that eleven different cases – consisting LAN, LAMN, PLAMN and LAQ (the definitions can be found in this introduction or in Section 2.1 of the thesis) – are possible if the value of the parameter $\boldsymbol{\theta} = (a, b)$ runs through $\mathbb{R}^2$. This study was one of the most motivational and useful paper for the birth of the thesis.

The introduction of the thesis contains some other motivational examples for time delayed models (Section 1.1) and a heuristic description of the basic concept of asymptotic statistics (Section 1.2).

In this outline we recall the basic definitions of asymptotic statistics. In Section 2.1 and 2.2 of the thesis one can find a more detailed introduction of asymptotic statistics.

1.1 Definition. (LAQ) *Let* $\Theta \subset \mathbb{R}^p$ *be an open set. A family* $(X_T, \mathcal{X}_T, \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have locally asymptotically quadratic (LAQ) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if there exist (scaling) matrices* $\boldsymbol{r}_{\boldsymbol{\theta},T} \in \mathbb{R}^{p\times p}$, $T \in \mathbb{R}_{++}$, *measurable functions (statistics)* $\boldsymbol{\Delta}_{\boldsymbol{\theta},T} : X_T \to \mathbb{R}^p$, $T \in \mathbb{R}_{++}$, *and* $\boldsymbol{J}_{\boldsymbol{\theta},T} : X_T \to \mathbb{R}^{p\times p}$, $T \in \mathbb{R}_{++}$, *such that*

$$\log \frac{\mathrm{d}\mathbb{P}_{\boldsymbol{\theta}+\boldsymbol{r}_{\boldsymbol{\theta},T}\boldsymbol{h}_T,T}}{\mathrm{d}\mathbb{P}_{\boldsymbol{\theta},T}} = \boldsymbol{h}_T^\top \boldsymbol{\Delta}_{\boldsymbol{\theta},T} - \frac{1}{2}\boldsymbol{h}_T^\top \boldsymbol{J}_{\boldsymbol{\theta},T}\boldsymbol{h}_T + \mathrm{o}_{\mathbb{P}_{\boldsymbol{\theta},T}}(1) \qquad \text{as } T \to \infty$$

whenever $\boldsymbol{h}_T \in \mathbb{R}^p$, $T \in \mathbb{R}_{++}$, *is a bounded family satisfying* $\boldsymbol{\theta} + \boldsymbol{r}_{\boldsymbol{\theta},T}\boldsymbol{h}_T \in \Theta$ *for all* $T \in \mathbb{R}_{++}$,

$$(\boldsymbol{\Delta}_{\boldsymbol{\theta},T}, \boldsymbol{J}_{\boldsymbol{\theta},T}) = \mathrm{O}_{\mathbb{P}_{\boldsymbol{\theta},T}}(1), \qquad T \in \mathbb{R}_{++},$$

and for each accumulation point $\mu_{\boldsymbol{\theta}}$ *of the family*

$$(\mathcal{L}((\boldsymbol{\Delta}_{\boldsymbol{\theta},T}, \boldsymbol{J}_{\boldsymbol{\theta},T}) \,|\, \mathbb{P}_{\boldsymbol{\theta},T}))_{T\in\mathbb{R}_{++}}$$

as $T \to \infty$, *which is a probability measure on* $(\mathbb{R}^p \times \mathbb{R}^{p\times p}, \mathcal{B}(\mathbb{R}^p \times \mathbb{R}^{p\times p}))$, *we have*

$$(1.1)\quad \mu_{\boldsymbol{\theta}}(\{(\boldsymbol{\Delta}, \boldsymbol{J}) \in \mathbb{R}^p \times \mathbb{R}^{p\times p} : \boldsymbol{J} \text{ is symmetric and strictly positive definite}\}) = 1$$

and

$$(1.2)\qquad \int_{\mathbb{R}^p\times\mathbb{R}^{p\times p}} \exp\left\{\boldsymbol{h}^\top \boldsymbol{\Delta} - \frac{1}{2}\boldsymbol{h}^\top \boldsymbol{J}\boldsymbol{h}\right\} \mu_{\boldsymbol{\theta}}(\mathrm{d}\boldsymbol{\Delta}, \mathrm{d}\boldsymbol{J}) = 1$$

whenever $\boldsymbol{h} \in \mathbb{R}^p$ *such that there exist* $T_k \in \mathbb{R}_{++}$, $k \in \mathbb{N}$, *and* $\boldsymbol{h}_{T_k} \in \mathbb{R}^p$, $k \in \mathbb{N}$, *with* $\boldsymbol{h}_{T_k} \to \boldsymbol{h}$ *as* $k \to \infty$, $\boldsymbol{\theta} + \boldsymbol{r}_{\boldsymbol{\theta},T_k}\boldsymbol{h}_{T_k} \in \Theta$ *for all* $k \in \mathbb{N}$.

1.2 Definition. (LAMN) *Let* $\Theta \subset \mathbb{R}^p$ *be an open set. A family* $(X_T, \mathcal{X}_T, \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have locally asymptotically mixed normal (LAMN) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if it is LAQ at* $\boldsymbol{\theta} \in \Theta$, *and for each accumulation point* $\mu_{\boldsymbol{\theta}}$ *of the family* $(\mathcal{L}((\boldsymbol{\Delta}_{\boldsymbol{\theta},T}, \boldsymbol{J}_{\boldsymbol{\theta},T}) \,|\, \mathbb{P}_{\boldsymbol{\theta},T}))_{T\in\mathbb{R}_{++}}$ *as* $T \to \infty$, *we have*

$$\int_{\mathbb{R}^p\times B} \mathrm{e}^{\mathrm{i}\,\boldsymbol{h}^\top\boldsymbol{\Delta}}\, \mu_{\boldsymbol{\theta}}(\mathrm{d}\boldsymbol{\Delta}, \mathrm{d}\boldsymbol{J}) = \int_{\mathbb{R}^p\times B} \mathrm{e}^{-\boldsymbol{h}^\top\boldsymbol{J}\boldsymbol{h}/2}\, \mu_{\boldsymbol{\theta}}(\mathrm{d}\boldsymbol{\Delta}, \mathrm{d}\boldsymbol{J}),$$

for all $B \in \mathcal{B}(\mathbb{R}^{p\times p})$ *and* $\boldsymbol{h} \in \mathbb{R}^p$ *, i.e., the conditional distribution of* $\boldsymbol{\Delta}$ *given* $\boldsymbol{J}$ *under* $\mu_{\boldsymbol{\theta}}$ *is* $\mathcal{N}_p(\boldsymbol{0}, \boldsymbol{J})$, *or, equivalently,* $\mu_{\boldsymbol{\theta}} = \mathcal{L}((\eta_{\boldsymbol{\theta}}\mathcal{Z}, \eta_{\boldsymbol{\theta}}\eta_{\boldsymbol{\theta}}^\top) \,|\, \mathbb{P})$, *where* $\mathcal{Z} : \Omega \to \mathbb{R}^p$ *and* $\eta_{\boldsymbol{\theta}} : \Omega \to \mathbb{R}^{p\times p}$ *are independent random elements on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\mathcal{L}(\mathcal{Z} \,|\, \mathbb{P}) = \mathcal{N}_p(\boldsymbol{0}, \boldsymbol{I}_p)$.

1.3 Definition. (LAN) *Let* $\Theta \subset \mathbb{R}^p$ *be an open set. A family* $(X_T, \mathcal{X}_T, \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have locally asymptotically normal (LAN) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if it is LAMN at* $\boldsymbol{\theta} \in \Theta$, *and for each accumulation point* $\mu_{\boldsymbol{\theta}}$ *of the family* $(\mathcal{L}((\boldsymbol{\Delta}_{\boldsymbol{\theta},T}, \boldsymbol{J}_{\boldsymbol{\theta},T}) \,|\, \mathbb{P}_{\boldsymbol{\theta},T}))_{T\in\mathbb{R}_{++}}$ *as* $T \to \infty$, *we have*

$$\mu_{\boldsymbol{\theta}} = \mathcal{N}_p(\boldsymbol{0}, \boldsymbol{J}_{\boldsymbol{\theta}}) \times \delta_{\boldsymbol{J}_{\boldsymbol{\theta}}}$$

with some symmetric, strictly positive definite matrix $\boldsymbol{J}_{\boldsymbol{\theta}} \in \mathbb{R}^{p\times p}$, *where* $\delta_{\boldsymbol{J}_{\boldsymbol{\theta}}}$ *denotes the Dirac measure on* $(\mathbb{R}^{p\times p}, \mathcal{B}(\mathbb{R}^{p\times p}))$, *concentrated in* $\boldsymbol{J}_{\boldsymbol{\theta}}$. *The matrix* $\boldsymbol{J}_{\boldsymbol{\theta}}$ *is called the information matrix.*

1.4 Definition. (PLAMN) *Let* $\Theta \subset \mathbb{R}^p$ *be an open set. A family* $(X_T, \mathcal{X}_T, \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \Theta\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments is said to have periodic locally asymptotically mixed normal (PLAMN) likelihood ratios at* $\boldsymbol{\theta} \in \Theta$ *if it is LAQ at* $\boldsymbol{\theta} \in \Theta$, *and*

$$(\boldsymbol{\Delta}_{\boldsymbol{\theta},kD+d}, \boldsymbol{J}_{\boldsymbol{\theta},kD+d}) \xrightarrow{\mathcal{D}} (\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d), \boldsymbol{J}_{\boldsymbol{\theta}}(d)) \qquad \text{as} \quad k \to \infty$$

for all $d \in [0, D)$, *and for each* $d \in [0, D)$, *the conditional distribution of* $\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d)$ *given* $\boldsymbol{J}_{\boldsymbol{\theta}}(d)$ *is* $\mathcal{N}_p(\boldsymbol{0}, \boldsymbol{J}_{\boldsymbol{\theta}}(d))$, *or, equivalently, there exist a random vector* $\mathcal{Z} : \Omega \to \mathbb{R}^p$ *and a random matrix* $\eta_{\boldsymbol{\theta}}(d) : \Omega \to \mathbb{R}^{p\times p}$ *such that they are independent,* $\mathcal{Z} \stackrel{\mathcal{D}}{=} \mathcal{N}_p(\boldsymbol{0}, \boldsymbol{I}_p)$, *and* $\boldsymbol{\Delta}_{\boldsymbol{\theta}}(d) = \eta_{\boldsymbol{\theta}}(d)\mathcal{Z}$, $\boldsymbol{J}_{\boldsymbol{\theta}}(d) = \eta_{\boldsymbol{\theta}}(d)\eta_{\boldsymbol{\theta}}^\top(d)$.

2 Heston model

For illustrating the local asymptotic properties of the likelihood function a well-known financial model, the Heston model has been investigated in Section 2.3 of thesis. The results have been published in Benke and Pap [3].

Let us consider a Heston model

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma_1\sqrt{Y_t}\,\mathrm{d}W_t, \\ \mathrm{d}X_t = (\alpha - \beta Y_t)\,\mathrm{d}t + \sigma_2\sqrt{Y_t}\big(\varrho\,\mathrm{d}W_t + \sqrt{1-\varrho^2}\,\mathrm{d}B_t\big), \end{cases} \qquad t \geqslant 0,$$

where $a > 0$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\varrho \in (-1, 1)$ and $(W_t, B_t)_{t\geqslant 0}$ is a 2-dimensional standard Wiener process. Here one can interpret X_t as the log-price of an asset, and Y_t as the stochastic volatility of the asset price at time $t \geqslant 0$. One can distinguish three cases: subcritical if $b > 0$, critical if $b = 0$ and supercritical if $b < 0$. We study local asymptotic properties of the likelihood ratios of this model concerning the drift parameter (a, α, b, β).

Let $\mathbb{P}_{\boldsymbol{\theta},T}$ denote the probability measure induced by $(Y_t, X_t)_{t\in[0,T]}$ on the measurable space $(C([0, T], \mathbb{R}^2), \mathcal{B}(C([0, T], \mathbb{R}^2))$. Introduce the family

$$(2.1) \qquad (\mathcal{E}_T) := \big(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}^2)), \{\mathbb{P}_{\boldsymbol{\theta},T} : \boldsymbol{\theta} \in \mathbb{R}_{++} \times \mathbb{R}^3\}\big)$$

of statistical experiments, where $T \in \mathbb{R}_{++}$, and the notation

$$(2.2) \qquad \boldsymbol{S} := \begin{bmatrix} \sigma_1^2 & \varrho\sigma_1\sigma_2 \\ \varrho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

2.1 Theorem. (Subcritical case) *If* $a \in \big(\frac{\sigma_1^2}{2}, \infty\big)$, $b \in \mathbb{R}_{++}$, *and* $\alpha, \beta \in \mathbb{R}$, *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments, given in* (2.1), *is LAN at* $\boldsymbol{\theta} := (a, \alpha, b, \beta)$ *with scaling matrices* $\boldsymbol{r}_{\boldsymbol{\theta},T} := \frac{1}{\sqrt{T}}\boldsymbol{I}_4$, $T \in \mathbb{R}_{++}$, *and with information matrix*

$$\boldsymbol{J}_{\boldsymbol{\theta}} := \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix} \otimes \boldsymbol{S}^{-1}.$$

Consequently, the family $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}^2)), \{\mathbb{P}_{\boldsymbol{\theta}+\boldsymbol{h}/\sqrt{T},T} : \boldsymbol{h} \in \mathbb{R}^4\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments converges to the statistical experiment* $(\mathbb{R}^4 \times \mathbb{R}^{4\times 4}, \mathcal{B}(\mathbb{R}^4 \times \mathbb{R}^{4\times 4}), \{\mathcal{N}_4(\boldsymbol{J}_{\boldsymbol{\theta}}\boldsymbol{h}, \boldsymbol{J}_{\boldsymbol{\theta}}) : \boldsymbol{h} \in \mathbb{R}^4\})$ *as* $T \to \infty$.

2.2 Theorem. (Critical case) *If* $a \in \big(\frac{\sigma_1^2}{2}, \infty\big)$, $b = 0$, *and* $\alpha, \beta \in \mathbb{R}$, *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments, given in* (2.1), *is LAQ at* $\boldsymbol{\theta} := (a, \alpha, b, \beta)$ *with scaling matrices*

$$\boldsymbol{r}_{\boldsymbol{\theta},T} := \begin{bmatrix} \frac{1}{\sqrt{\log T}} & 0 \\ 0 & \frac{1}{T} \end{bmatrix} \otimes \boldsymbol{I}_2, \quad T \in \mathbb{R}_{++},$$

and with

$$\big(\boldsymbol{\Delta}_{\boldsymbol{\theta},T}(Y, X), \boldsymbol{J}_{\boldsymbol{\theta},T}(Y, X)\big) \stackrel{\mathcal{D}}{\longrightarrow} (\boldsymbol{\Delta}_{\boldsymbol{\theta}}, \boldsymbol{J}_{\boldsymbol{\theta}}) \qquad \text{as } T \to \infty,$$

where

$$\boldsymbol{\Delta}_{\boldsymbol{\theta}} := \begin{bmatrix} \left(a - \frac{\sigma_1^2}{2}\right)^{-1/2} \begin{bmatrix} \sigma_1 & \sigma_2\varrho \\ 0 & \sigma_2\sqrt{1-\varrho^2} \end{bmatrix}^{-1} \boldsymbol{Z}_2 \\ \boldsymbol{S}^{-1} \begin{bmatrix} a - \mathcal{Y}_1 \\ \alpha - \mathcal{X}_1 \end{bmatrix} \end{bmatrix},$$

$$\boldsymbol{J}_{\boldsymbol{\theta}} := \begin{bmatrix} \left(a - \frac{\sigma_1^2}{2}\right)^{-1} & 0 \\ 0 & \int_0^1 \mathcal{Y}_s \, \mathrm{d}s \end{bmatrix} \otimes \boldsymbol{S}^{-1},$$

where $(\mathcal{Y}_t, \mathcal{X}_t)_{t\in\mathbb{R}_+}$ *is the unique strong solution of the SDE*

$$\begin{cases} \mathrm{d}\mathcal{Y}_t = a\,\mathrm{d}t + \sigma_1\sqrt{\mathcal{Y}_t}\,\mathrm{d}\mathcal{W}_t, \\ \mathrm{d}\mathcal{X}_t = \alpha\,\mathrm{d}t + \sigma_2\sqrt{\mathcal{Y}_t}\big(\varrho\,\mathrm{d}\mathcal{W}_t + \sqrt{1-\varrho^2}\,\mathrm{d}\mathcal{B}_t\big), \end{cases} \qquad t \in \mathbb{R}_+,$$

with initial value $(\mathcal{Y}_0, \mathcal{X}_0) = (0, 0)$, *where* $(\mathcal{W}_t, \mathcal{B}_t)_{t\in\mathbb{R}_+}$ *is a* 2*-dimensional standard Wiener process,* $\boldsymbol{Z}_2$ *is a* 2*-dimensional standard* *normally distributed random vector independent of* $\big(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_t \,\mathrm{d}t, \mathcal{X}_1\big)$, *and* $\boldsymbol{S}$ *is defined in* (2.2).

Consequently, the family $(C(\mathbb{R}_+,\mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+,\mathbb{R}^2)), \{\mathbb{P}_{\boldsymbol{\theta}+\boldsymbol{r}_{\boldsymbol{\theta},T}\boldsymbol{h},T} : \boldsymbol{h} \in \mathbb{R}^4\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments converges to the statistical experiment* $(\mathbb{R}^4 \times \mathbb{R}^{4\times 4}, \mathcal{B}(\mathbb{R}^4 \times \mathbb{R}^{4\times4}), \{\mathbb{Q}_{\boldsymbol{\theta},\boldsymbol{h}} : \boldsymbol{h} \in \mathbb{R}^4\})$ *as* $T \to \infty$, *where*

$$\mathbb{Q}_{\boldsymbol{\theta},\boldsymbol{h}}(B) := \mathbb{E}\left(\exp\left\{\boldsymbol{h}^\top \boldsymbol{\Delta}_{\boldsymbol{\theta}} - \frac{1}{2}\boldsymbol{h}^\top \boldsymbol{J}_{\boldsymbol{\theta}}\boldsymbol{h}\right\} \mathbb{1}_B(\boldsymbol{\Delta}_{\boldsymbol{\theta}}, \boldsymbol{J}_{\boldsymbol{\theta}})\right),$$

whenever $B \in \mathcal{B}(\mathbb{R}^4 \times \mathbb{R}^{4\times4})$ *and* $\boldsymbol{h} \in \mathbb{R}^4$.

If $b = 0$ *and* $\beta \in \mathbb{R}$ *are fixed, then the subfamily*

$$\left(C(\mathbb{R}_+,\mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+,\mathbb{R}^2)), \left\{\mathbb{P}_{\boldsymbol{\theta},T} : a \in \left(\frac{\sigma_1^2}{2}, \infty\right),\ \alpha \in \mathbb{R}\right\}\right)_{T\in\mathbb{R}_{++}}$$

of statistical experiments is LAN at (a, α) *with scaling matrices* $\boldsymbol{r}^{(1)}_{\boldsymbol{\theta},T} := \frac{1}{\sqrt{\log T}}\boldsymbol{I}_2$, $T \in \mathbb{R}_{++}$, *and with information matrix* $\boldsymbol{J}^{(1)}_{\boldsymbol{\theta}} := \left(a - \frac{\sigma_1^2}{2}\right)^{-1} \boldsymbol{S}^{-1}$.

Consequently, the family $(C(\mathbb{R}_+,\mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+,\mathbb{R}^2)), \{\mathbb{P}_{\boldsymbol{\theta}+\boldsymbol{h}/\sqrt{\log T},T} : \boldsymbol{h}_1 \in \mathbb{R}^2\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments converges to the statistical experiment* $(\mathbb{R}^2 \times \mathbb{R}^{2\times2}, \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^{2\times2}), \{\mathcal{N}_2(\boldsymbol{J}^{(1)}_{\boldsymbol{\theta}}\boldsymbol{h}_1, \boldsymbol{J}^{(1)}_{\boldsymbol{\theta}}) : \boldsymbol{h}_1 \in \mathbb{R}^2\})$ *as* $T \to \infty$, *where* $\boldsymbol{h} := (\boldsymbol{h}_1, \boldsymbol{0})^\top \in \mathbb{R}^4$.

2.3 Theorem. (Supercritical case) *If* $a \in \big[\frac{\sigma_1^2}{2}, \infty\big)$, $b \in \mathbb{R}_{--}$, *and* $\alpha, \beta \in \mathbb{R}$, *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments, given in* (2.1), *is not LAQ at* $\boldsymbol{\theta} := (a, \alpha, b, \beta)$ *with scaling matrices*

$$\boldsymbol{r}_{\boldsymbol{\theta},T} := \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{bT/2} \end{bmatrix} \otimes \boldsymbol{I}_2, \quad T \in \mathbb{R}_{++},$$

although

$$\big(\boldsymbol{\Delta}_{\boldsymbol{\theta},T}(Y,X), \boldsymbol{J}_{\boldsymbol{\theta},T}(Y,X)\big) \xrightarrow{\mathcal{D}} (\boldsymbol{\Delta}_{\boldsymbol{\theta}}, \boldsymbol{J}_{\boldsymbol{\theta}}) \qquad \text{as } T \to \infty,$$

with

$$\boldsymbol{\Delta}_{\boldsymbol{\theta}} := \left(\boldsymbol{I}_2 \otimes \begin{bmatrix} \sigma_1 & \sigma_2\varrho \\ 0 & \sigma_2\sqrt{1-\varrho^2} \end{bmatrix}^{-1}\right) \begin{bmatrix} \sigma_1^{-1}\widetilde{\mathcal{V}} \\ Z_1 \\ \left(-\frac{\widetilde{\mathcal{Y}}_{-1/b}}{b}\right)^{1/2} \boldsymbol{Z}_2 \end{bmatrix},$$

$$\boldsymbol{J}_{\boldsymbol{\theta}} := \begin{bmatrix} \int_0^{-1/b} \widetilde{\mathcal{Y}}_u \, \mathrm{d}u & 0 \\ 0 & -\frac{\widetilde{\mathcal{Y}}_{-1/b}}{b} \end{bmatrix} \otimes \boldsymbol{S}^{-1},$$

where $(\widetilde{\mathcal{Y}}_t)_{t\in\mathbb{R}_+}$ *is a CIR process given by the SDE*

$$\mathrm{d}\widetilde{\mathcal{Y}}_t = a\mathrm{d}t + \sigma_1\sqrt{\widetilde{\mathcal{Y}}_t}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\widetilde{\mathcal{Y}}_0 = y_0$, *where* $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process,*

$$\widetilde{\mathcal{V}} := \log \widetilde{\mathcal{Y}}_{-1/b} - \log y_0 - \left(a - \frac{\sigma_1^2}{2}\right) \int_0^{-1/b} \widetilde{\mathcal{Y}}_u \, \mathrm{d}u,$$

Z_1 *is a* 1*-dimensional standard normally distributed random variable,* $\boldsymbol{Z}_2$ *is a* 2*-dimensional standard normally distributed random vector such that* $(\widetilde{\mathcal{Y}}_{-1/b}, \int_0^{-1/b} \widetilde{\mathcal{Y}}_u \, \mathrm{d}u)$, Z_1 *and* $\boldsymbol{Z}_2$ *are independent, and* $\boldsymbol{S}$ *is defined in* (2.2)*. Moreover,* (1.1) *also holds, but* (1.2) *is not valid.*

If $a \in \big(\frac{\sigma_1^2}{2}, \infty\big)$ *and* $\alpha \in \mathbb{R}$ *are fixed, then the subfamily*

$$\big(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}^2)), \big\{\mathbb{P}_{\boldsymbol{\theta},T} : b \in \mathbb{R}_{--},\, \beta \in \mathbb{R}\big\}\big)_{T\in\mathbb{R}_{++}}$$

of statistical experiments is LAMN at (b, β) *with scaling matrices* $\boldsymbol{r}^{(2)}_{\boldsymbol{\theta},T} := \mathrm{e}^{bT/2}\boldsymbol{I}_2$, $T \in \mathbb{R}_{++}$, *and with*

$$\boldsymbol{\Delta}^{(2)}_{\boldsymbol{\theta}} := \left(-\frac{\widetilde{\mathcal{Y}}_{-1/b}}{b}\right)^{1/2} \begin{bmatrix} \sigma_1 & \sigma_2\varrho \\ 0 & \sigma_2\sqrt{1-\varrho^2} \end{bmatrix}^{-1} \boldsymbol{Z}_2,$$

$$\boldsymbol{J}^{(2)}_{\boldsymbol{\theta}} := \left(-\frac{\widetilde{\mathcal{Y}}_{-1/b}}{b}\right) \boldsymbol{S}^{-1}.$$

Consequently, the family $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}^2)), \{\mathbb{P}_{\boldsymbol{\theta}+\mathrm{e}^{bT/2}\boldsymbol{h},T} : \boldsymbol{h}_2 \in \mathbb{R}^2\})_{T\in\mathbb{R}_{++}}$ *of statistical experiments converges to the statistical experiment* $(\mathbb{R}^2 \times \mathbb{R}^{2\times2}, \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^{2\times2}), \{\mathcal{L}((\boldsymbol{\Delta}^{(2)}_{\boldsymbol{\theta}} + \boldsymbol{J}^{(2)}_{\boldsymbol{\theta}}\boldsymbol{h}_2, \boldsymbol{J}^{(2)}_{\boldsymbol{\theta}}) \,|\, \mathbb{P}) : \boldsymbol{h}_2 \in \mathbb{R}^2\})$ *as* $T \to \infty$, *where* $\boldsymbol{h} := (\boldsymbol{0}, \boldsymbol{h}_2)^\top \in \mathbb{R}^4$.

3 Uniformly distributed delay case

In Chapter 3 of the thesis the main results are presented. Assume that we have a stochastic process $(X(t))_{t\in[-r,T]}$, which satisfies the linear stochastic delay differential equation

$$\mathrm{d}X(t) = \vartheta \int_{[-r,0]} X(t+u)\, a(\mathrm{d}u)\, \mathrm{d}t + \mathrm{d}W(t), \qquad t \geqslant 0,$$

where a is a finite signed measure on $[-r,0]$ and W is a standard Wiener process. The real parameter ϑ is unknown, hence a good estimation from the continuous sample $(X(t))_{t\in[-r,T]}$ is needed. For this, the local asymptotic properties of the likelihood function are studied.

After a preliminaries in Section 3.2 a special case have studied, when the delay is uniform. The results for this case have been published in Benke and Pap [1]. Namely, assume $(X^{(\vartheta)}(t))_{t\in\mathbb{R}_+}$ is the solution of the SDDE

$$\begin{cases} \mathrm{d}X(t) = \vartheta \int_{-1}^{0} X(t+u)\, \mathrm{d}u\, \mathrm{d}t + \mathrm{d}W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-1,0], \end{cases}$$

where $(X_0(t))_{t\in[-1,0]}$ is a fixed continuous initial process. Further, for all $T \in \mathbb{R}_{++}$, let $\mathbb{P}_{\vartheta,T}$ be the probability measure induced by $(X^{(\vartheta)}(t))_{t\in[-1,T]}$ on $(C([-1,T],\mathbb{R}), \mathcal{B}(C([-1,T],\mathbb{R})))$. Introduce the family

$$(3.1) \qquad (\mathcal{E}_T) := \big(C(\mathbb{R}_+,\mathbb{R}), \mathcal{B}(C(\mathbb{R}_+,\mathbb{R})), \{\mathbb{P}_{\vartheta,T} : \vartheta \in \mathbb{R}\}\big)$$

of statistical experiments, where $T \in \mathbb{R}_{++}$.

3.1 Theorem. *If $\vartheta \in \left(-\frac{\pi^2}{2}, 0\right)$ then the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments given in (3.1) is LAN at ϑ with scaling $r_{\vartheta,T} = \frac{1}{\sqrt{T}}$, $T \in \mathbb{R}_{++}$, and with*

$$J_\vartheta = \int_0^\infty \left(\int_{-1}^0 x_{0,\vartheta}(t+u)\, \mathrm{d}u \right)^2 \mathrm{d}t.$$

3.2 Theorem. *The family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments given in (3.1) is LAQ at 0 with scaling $r_{0,T} = \frac{1}{T}$, $T \in \mathbb{R}_{++}$, and with*

$$\Delta_0 = \int_0^1 \mathcal{W}(t)\, \mathrm{d}\mathcal{W}(t), \qquad J_0 = \int_0^1 \mathcal{W}(t)^2\, \mathrm{d}t,$$

where $(\mathcal{W}(t))_{t\in[0,1]}$ *is a standard Wiener process.*

3.3 Theorem. *The family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is LAQ at* $-\frac{\pi^2}{2}$ *with scaling* $r_{-\frac{\pi^2}{2},T} = \frac{1}{T}$, $T\in\mathbb{R}_{++}$, *and with*

$$\begin{aligned}\Delta_{-\frac{\pi^2}{2}} = \frac{1}{\pi(\pi^2+16)}\bigg(16\int_0^1 (\mathcal{W}_1(s)\,\mathrm{d}\mathcal{W}_2(s) - \mathcal{W}_2(s)\,\mathrm{d}\mathcal{W}_1(s))\\ -4\pi\int_0^1 (\mathcal{W}_1(s)\,\mathrm{d}\mathcal{W}_1(s) + \mathcal{W}_2(s)\,\mathrm{d}\mathcal{W}_2(s))\bigg),\end{aligned}$$

$$J_{-\frac{\pi^2}{2}} = \frac{16}{\pi^2(\pi^2+16)}\int_0^1 (\mathcal{W}_1(t)^2 + \mathcal{W}_2(t)^2)\,\mathrm{d}t,$$

where $(\mathcal{W}_1(t),\mathcal{W}_2(t))_{t\in[0,1]}$ *is a 2-dimensional standard Wiener process.*

3.4 Theorem. *If* $\vartheta\in(0,\infty)$ *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is LAMN at* ϑ *with scaling* $r_{\vartheta,T} = \mathrm{e}^{-v_0(\vartheta)T}$, $T\in\mathbb{R}_{++}$, *and with*

$$\Delta_\vartheta = Z\sqrt{J_\vartheta}, \qquad J_\vartheta = \frac{(1-\mathrm{e}^{-v_0(\vartheta)})^2}{2v_0(\vartheta)(v_0(\vartheta)^2+2v_0(\vartheta)-\vartheta)^2}(U^{(\vartheta)})^2,$$

with

$$U^{(\vartheta)} = X_0(0)+\vartheta\int_{-1}^0\int_u^0 \mathrm{e}^{-v_0(\vartheta)(s-u)}X_0(s)\,\mathrm{d}s\,\mathrm{d}u+\int_0^\infty \mathrm{e}^{-v_0(\vartheta)s}\,\mathrm{d}W(s),$$

and Z *is a standard normally distributed random variable independent of* J_ϑ.

3.5 Theorem. *If* $\vartheta\in\left(-\infty,-\frac{\pi^2}{2}\right)$ *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (3.1) *is PLAMN at* ϑ *with period* $D=\frac{\pi}{\kappa_0(\vartheta)}$, *with scaling* $r_{\vartheta,T}=\mathrm{e}^{-v_0(\vartheta)T}$, $T\in\mathbb{R}_{++}$, *and with*

$$\Delta_\vartheta(d) = Z\sqrt{J_\vartheta(d)}, \qquad J_\vartheta(d) = \int_0^\infty \mathrm{e}^{-2v_0(\vartheta)s}(V^{(\vartheta)}(d-s))^2\,\mathrm{d}s,$$

whenever $d \in \left[0, \frac{\pi}{\kappa_0(\vartheta)}\right)$, *where*

$$V^{(\vartheta)}(t) = X_0(0)\varphi_\vartheta(t) + \vartheta \int_{-1}^{0} \int_{u}^{0} \varphi_\vartheta(t+u-s) \mathrm{e}^{-v_0(\vartheta)(s-u)} X_0(s) \, \mathrm{d}s \, \mathrm{d}u$$
$$+ \int_0^\infty \varphi_\vartheta(t-s) \mathrm{e}^{-v_0(\vartheta)s} \, \mathrm{d}W(s), \qquad t \in \mathbb{R}_+,$$

with

$$\varphi_\vartheta(t) := A_0(\vartheta) \cos(\kappa_0(\vartheta)t) + B_0(\vartheta) \sin(\kappa_0(\vartheta)t), \qquad t \in \mathbb{R},$$

and Z *is a standard normally distributed random variable independent of* $J_\vartheta(d)$.

4 General delay case

In Section 3.3 of the thesis the general case are presented. The results have been published in Benke and Pap [2]. Consider now the SDDE

$$\begin{cases} \mathrm{d}X(t) = \vartheta \int_{[-r,0]} X(t+u)\, a(\mathrm{d}u)\,\mathrm{d}t + \mathrm{d}W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-r, 0], \end{cases}$$

where now the delay measure a is an arbitrary finite signed measure on $[-r, 0]$.

The asymptotic behaviour is connected with the so-called characteristic function $h_\vartheta : \mathbb{C} \to \mathbb{C}$, given by

$$h_\vartheta(\lambda) := \lambda - \vartheta \int_{[-r,0]} \mathrm{e}^{\lambda u}\, a(\mathrm{d}u), \qquad \lambda \in \mathbb{C},$$

and the set Λ_ϑ of the (complex) solutions of the so-called characteristic equation

$$\lambda - \vartheta \int_{[-r,0]} \mathrm{e}^{\lambda u}\, a(\mathrm{d}u) = 0.$$

A crucial quantity is the real part of the rightmost characteristic root, which is defined by

$$v_0(\vartheta) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \Lambda_\vartheta\} < \infty.$$

For $\lambda \in \Lambda_\vartheta$, denote by $m_\vartheta(\lambda)$ the multiplicity of the characteristic root λ.

In the uniform delay case and in each of the earlier results, which concern local asymptotic properties of delayed models, LAN has been proved in case of $v_0(\vartheta) < 0$. Accordingly one can imagine, that this condition would be the sufficient for LAN. But it turns out, that is not the truth. We give an example, where $v_0(\vartheta) = 0$, and LAN holds (see, Example 3.3.7 in the thesis). A modification is needed to give the proper condition. For this, for each $\lambda \in \Lambda_\vartheta$, denote by $\widetilde{m}_\vartheta(\lambda)$ the degree of the complex-valued polynomial

$$P_{\vartheta,\lambda}(t) := \sum_{\ell=0}^{m_\vartheta(\lambda)-1} c_{\vartheta,\lambda,\ell}\, t^\ell$$

with

$$c_{\vartheta,\lambda,\ell} := \frac{1}{\ell!} \sum_{j=0}^{m_\vartheta(\lambda)-1-\ell} \frac{A_{\vartheta,-j-1-\ell}(\lambda)}{j!} \int_{[-r,0]} u^j \mathrm{e}^{\lambda u}\, a(\mathrm{d}u),$$

where $A_{\vartheta,k}(\lambda)$, $k \in \{-m_\vartheta(\lambda), -m_\vartheta(\lambda)+1, \ldots\}$ denotes the coefficients of the Laurent's series of $1/h_\vartheta(z)$ at $z = \lambda$, and the degree of the zero polynomial is defined to be $-\infty$. Put

$$v_\vartheta^* := \sup\{\operatorname{Re}(\lambda) : \lambda \in \Lambda_\vartheta,\ \widetilde{m}_\vartheta(\lambda) \geqslant 0\},$$
$$m_\vartheta^* := \max\{\widetilde{m}_\vartheta(\lambda) : \lambda \in \Lambda_\vartheta,\ \operatorname{Re}(\lambda) = v_\vartheta^*\},$$

where $\sup\emptyset := -\infty$ and $\max\emptyset := -\infty$.

For all $T \in \mathbb{R}_{++}$, let $\mathbb{P}_{\vartheta,T}$ be the probability measure induced by $(X^{(\vartheta)}(t))_{t\in[-r,T]}$ on $(C([-r,T],\mathbb{R}), \mathcal{B}(C([-r,T],\mathbb{R})))$. Introduce the family

$$(\mathcal{E}_T) := \big(C(\mathbb{R}_+,\mathbb{R}), \mathcal{B}(C(\mathbb{R}_+,\mathbb{R})), \{\mathbb{P}_{\vartheta,T} : \vartheta \in \mathbb{R}\}\big) \tag{4.1}$$

of statistical experiments, where $T \in \mathbb{R}_{++}$.

4.1 Theorem. *If $\vartheta \in \mathbb{R}$ with $v_\vartheta^* < 0$, then the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments given in* (4.1) *is LAN at ϑ with scaling $r_{\vartheta,T} = T^{-1/2}$, $T \in \mathbb{R}_{++}$, and with*

$$J_\vartheta = \int_0^\infty \left(\int_{[-r,0]} x_{0,\vartheta}(t+u)\, a(\mathrm{d}u) \right)^2 \mathrm{d}t.$$

Particularly, if $a([-r,0]) = 0$, then $v_0^* = -\infty$, $m_0^* = -\infty$, and the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments given in (4.1) *is LAN at 0 with scaling $r_{0,T} = T^{-1/2}$, $T \in \mathbb{R}_{++}$, and with*

$$J_0 = \int_0^r a([-t,0])^2\, \mathrm{d}t.$$

4.2 Theorem. *If $\vartheta \in \mathbb{R}$ with $v_\vartheta^* = 0$, then the family $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ of statistical experiments given in* (4.1) *is LAQ at ϑ with scaling $r_{\vartheta,T} = T^{-m_\vartheta^*-1}$ and with*

$$\Delta_\vartheta = \sum_{\substack{\lambda\in\Lambda_\vartheta\cap(\mathrm{i}\mathbb{R})\\ \widetilde{m}_\vartheta(\lambda)=m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} \int_0^1 \mathcal{Z}_{\operatorname{Im}(\lambda),m_\vartheta^*}(s)\, \mathrm{d}\overline{\mathcal{Z}_{\operatorname{Im}(\lambda),0}(s)},$$

$$J_\vartheta = \sum_{\substack{\lambda\in\Lambda_\vartheta\cap(\mathrm{i}\mathbb{R})\\ \widetilde{m}_\vartheta(\lambda)=m_\vartheta^*}} |c_{\vartheta,\lambda,m_\vartheta^*}|^2 \int_0^1 |\mathcal{Z}_{\operatorname{Im}(\lambda),m_\vartheta^*}(s)|^2\, \mathrm{d}s,$$

with

$$\mathcal{Z}_{\varphi,0} := \begin{cases} \mathcal{W}, & \text{if } \varphi = 0, \\ \frac{1}{\sqrt{2}}(\mathcal{W}_{\varphi,\mathrm{Re}} + \mathrm{i}\mathcal{W}_{\varphi,\mathrm{Im}}), & \text{if } \varphi \in \mathbb{R}_{++}, \\ \overline{\mathcal{Z}_{-\varphi,0}}, & \text{if } \varphi \in \mathbb{R}_{--}, \end{cases}$$

where $(\mathcal{W}(s))_{s\in[0,1]}$, $(\mathcal{W}_{\varphi,\mathrm{Re}}(s))_{s\in[0,1]}$ *and* $(\mathcal{W}_{\varphi,\mathrm{Im}}(s))_{s\in[0,1]}$, $\varphi \in \mathbb{R}_{++}$, *are independent standard Wiener processes, and*

$$\mathcal{Z}_{\varphi,\ell}(s) := \int_0^s (s-u)^\ell \,\mathrm{d}\mathcal{Z}_{\varphi,0}(u), \qquad s \in [0,1], \quad \varphi \in \mathbb{R}, \quad \ell \in \mathbb{N}.$$

Particularly, if $a([-r,0]) \neq 0$, *then* $v_0^* = 0$, $m_0^* = 0$, *and the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (4.1) *is LAQ at* 0 *with scaling* $r_{0,T} = T^{-1}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_0 = a([-r,0]) \int_0^1 \mathcal{W}(s)\,\mathrm{d}\mathcal{W}(s), \qquad J_0 = a([-r,0])^2 \int_0^1 \mathcal{W}(s)^2\,\mathrm{d}s.$$

4.3 Theorem. *Let* $\vartheta \in \mathbb{R}$ *with* $v_\vartheta^* > 0$. *If*

$$H_\vartheta := \{\mathrm{Im}(\lambda) : \lambda \in \Lambda_\vartheta \cap (v_\vartheta^* + \mathrm{i}\mathbb{R}_{++}),\ \widetilde{m}_\vartheta(\lambda) = m_\vartheta^*\} \neq \emptyset,$$

and the numbers in H_ϑ *have a common divisor* D_ϑ *(namely, they are pairwise commensurable, and the quotients of these numbers and* D_ϑ *are integers), then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (4.1) *is PLAMN at* ϑ *with period* $\frac{2\pi}{D_\vartheta}$, *with scaling* $r_{\vartheta,T} = T^{-m_\vartheta^*}\mathrm{e}^{-v_\vartheta^* T}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_\vartheta(d) = Z\sqrt{J_\vartheta(d)},$$

$$J_\vartheta(d) = \int_0^\infty \mathrm{e}^{-2v_\vartheta^* t}\,\mathrm{Re}\Bigg(\sum_{\substack{\lambda\in\Lambda_\vartheta\cap(v_\vartheta^*+\mathrm{i}\mathbb{R})\\ \widetilde{m}_\vartheta(\lambda)=m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} U_\lambda^{(\vartheta)} \mathrm{e}^{\mathrm{i}(d-t)\,\mathrm{Im}(\lambda)}\Bigg)^2 \mathrm{d}t,$$

for $d \in [0, \frac{2\pi}{D_\vartheta})$, *where*

$$U_\lambda^{(\vartheta)} = X_0(0) + v_\vartheta^* \int_{[-r,0]} \int_u^0 \mathrm{e}^{-\lambda(s-u)} X_0(s)\,\mathrm{d}s\,a(\mathrm{d}u) + \int_0^\infty \mathrm{e}^{-\lambda s}\,\mathrm{d}W(s),$$

whenever $\lambda \in \mathbb{C}$ *and* Z *is a standard normally distributed random variable independent of* $(X_0(t))_{t\in[-r,0]}$ *and* $(W(t))_{t\in\mathbb{R}_+}$.

If $H_\vartheta = \emptyset$, *then the family* $(\mathcal{E}_T)_{T\in\mathbb{R}_{++}}$ *of statistical experiments given in* (4.1) *is LAMN at* ϑ *with scaling* $r_{\vartheta,T} = T^{-m_\vartheta^*}\mathrm{e}^{-v_\vartheta^* T}$, $T \in \mathbb{R}_{++}$, *and with*

$$\Delta_\vartheta = Z\sqrt{J_\vartheta}, \qquad J_\vartheta = \frac{c^2_{\vartheta,v_\vartheta^*,m_\vartheta^*}}{2v_\vartheta^*}(U^{(\vartheta)}_{v_\vartheta^*})^2.$$

References

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[3] Benke, J. M. and Pap, G. (2017). Local asymptotic quadraticity of statistical experiments connected with a Heston model. *Acta Scientiarum Mathematicarum* **83(1-2)** 313–344.

[4] Gushchin, A. A. and Küchler, U. (1999). Asymptotic inference for a linear stochastic differential equation with time delay. *Bernoulli* **5(6)** 1059–1098.