Asymptotic Bernstein type inequalities

Thesis

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2005

Bernstein and Markov type inequalities are important tools in approximation theory. Since the original two article [3] and [2], many generalizations have appeared. The aim of this dissertation is to extend Bernstein's inequality onto a wider class of sets and explain the "geometrical" factor with potential theory. It consists of three parts corresponding to the articles [4], [6] and [5]. For an introduction to potential theory, we refer to [8] or [7].

Remark.
The numbering of the theorems and formulas here and in the dissertation coincide for easier reading.

Preliminaries

The well-known (complex) Bernstein's inequality states that

$$\left|P_n'(z_0)\right| \leq n||P_n||_D \ , \tag{1.1}$$

where P_n is arbitrary complex polynomial with degree n and $||P_n||_D$ denotes its supremum norm over the unit disk $D = \{z \in \mathbf{C} : |z| \leq 1\}$. By a simple substitution, one can obtain another inequality which now involves the interval $I = [-1, 1]$

$$\left|P_n'(t)\right| \leq n\frac{1}{\sqrt{1-t^2}}||P_n||_I. \tag{1.2}$$

This is also called as Bernstein's inequality.
The factor $(1-t^2)^{-1/2}$ is closely related to the "geometry" of $[-1, 1]$, and with potential theory, it can be expressed as $(1-t^2)^{-1/2} = \pi\omega_I(t)$ where $\omega_I(t)$ is the density function of the equilibrium measure of I (with respect to the Lebesgue measure). Using this potential theoretical approach, the following generalization has been recently proved in [9] and in [1]

Theorem.

$$\left|P_n'(t)\right| \leq n\pi\omega_K(t)||P_n||_K \tag{1.3}$$

where $K \subset \mathbf{C}$ is a compact set, $\omega_K(t)$ is the density function of the equilibrium measure ν_K, $\omega_K(t)dt = d\nu_K(t)$ and $t \in \operatorname{Int} K$ (so that $\omega_K(t)$ be finite), and $||P_n||_K$ is the supremum norm of P_n over K.

We also use the notion of the Green's function and for $K \subset \mathbf{C}$ compact set, the $g_K(z) = g(K, z)$ denotes the Green's function of the (unbounded) component of $\mathbf{C} \setminus K$ with pole at infinity.

It is worth to mention that the density function $\omega_K(t)$ is closely related to the (outward) normal derivative of Green's function, more precisely:

Theorem. *If $K \subset \mathbf{C}$ is a compact set such that ∂K is a union of finitely many $C^{1+\delta}$ smooth curves ($\delta > 0$), then the equilibrium measure is absolutely continuous with respect to arc length measure, furthermore,*

$$\frac{d\nu_K(z)}{ds} = \frac{1}{2\pi} \frac{\partial}{\partial \mathbf{n}_z} g_K(z)$$

where ds denotes the arc-length measure on ∂K and $\partial / \partial \mathbf{n}_z$ denotes differentation at z in the direction of the outer normal $\mathbf{n}_z$.

Main result

The proof of the inequality (1.3) uses an exhaustion technique which turned out to be quite useful tool in one (complex or real) dimension and also appears in higher dimension in some form. The following notion describes the generality.

A compact set $K \subset \mathbf{C}$ is called Jordan fat, if the boundary of every connected component of its interior $\operatorname{Int}(K)$ is a Jordan curve and K is the closure of its interior: $K = \overline{\operatorname{Int}(K)}$.

This dissertation uses this idea to extend Bernstein's inequality to the following level of generality

Theorem (9). *Let K be a Jordan fat compact set on the plane with connected complement. Let z_0 be a point on the boundary of K and* *let us suppose that this boundary is a twice continuously differentiable Jordan arc in a neighborhood of* z_0*. Then*

$$|P_n'(z_0)| \le n(1+o(1))\frac{\partial g(K,z_0)}{\partial \mathbf{n}}\|P_n\|_K, \qquad (3.3)$$

where the $o(1)$ *tends to* 0 *uniformly in the polynomials* P_n *of degree at most* n *as* $n \to \infty$*.*

The sharpness of this inequality is also discussed.

The proof of this inequality consists of several steps, proving (3.3) in higher and higher generality.

The (3.3) on lemniscates

Let us recall a notion which describes the class of sets for which inequality (3.3) is proved first.

Denote the unit disk by D, $D = \{z \in \mathbf{C} : |z| \le 1\}$.

Definition (1). *The set* $L \subset \mathbf{C}$ *is a* lemniscate *if for some complex polynomial* r*,* $L = r^{-1}[\partial D]$*, that is,* $z \in L \Leftrightarrow |r(z)| = 1$*. The set* $r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| \le 1\}$ *is called the* interior of the lemniscate L*.*

Note that the interior of a lemniscate is not the topological interior of the lemniscate (which is actually $\{z \in \mathbf{C} : |r(z)| < 1\}$).

A lemniscate usually behaves nicely near one of its points. More precisely, a lemniscate is a system of finitely many closed Jordan curves. They are not necessarily simple curves, so we distinguish its points. If $z \in L = r^{-1}[\partial D]$ is a point from the lemniscate L with $r'(z) \ne 0$, then we say z is a simple point (of the lemniscate L). In other words, z is not a critical point of r. It is also equivalent to the fact that L is a simple curve near z_0 (does not cross itself). Moreover, if $r'(z) \ne 0$, then $L = r^{-1}[\partial D]$ is a smooth (actually, analytic) curve near z. First, it is proved on special sets, namely, on lemniscates in [4].

The following lemma shows how much the notion of lemniscates fits in this setting.

Lemma (4). *Let* $K := r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| \le 1\}$. *Denote the Green function of the unbounded component of the complement of* K *by* g_K. *If* $z_0 \in \partial K$ *and* $r'(z_0) \neq 0$, *then*

$$\frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) = \frac{1}{\deg r} |r'(z_0)| \ . \tag{2.2}$$

Another important tool in this step is the following symmetrization trick. For a given $z \in r^{-1}[\partial D]$, we denote by $z^{(j)}$, $j = 0, \ldots, \deg r - 1$ those places which $r(z) = r(z^{(j)})$ for all j. In other words, these points are "associated" to z on the branches of $r^{-1}[\partial D]$ and for some j_0, $z = z^{(j_0)}$.

We consider the "periodic extension" of P_n which is defined as

$$P^*(z) := \sum_{j=0}^{\deg r - 1} P_n(z^{(j)}) \cdot Q(z_0; z^{(j)}). \tag{2.8}$$

Using this symmetrization and Lemma 4, we obtain that (3.3) holds on lemniscates, that is,

Theorem (2). *Let* $K \subset \mathbf{C}$ *be the interior of a lemniscate of some polynomial* r, *that is,* $K = r^{-1}[D]$ *and let* $z_0 \in \partial K$ *be fixed. Assume that* z_0 *is a simple point of* ∂K. *Denote the Green's function of the unbounded component of* $\mathbf{C}_\infty \setminus K$ *by* $g_K(z)$. *Then, for every polynomial* P_n *with* $\deg P_n = n$ *we have*

$$|P_n'(z_0)| \le \big(1 + o(1)\big) \cdot \ n \cdot \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) \cdot ||P_n||_K \ , \tag{1}$$

where the term $o(1)$ *is to be understood as* $n \to \infty$ *and depends only on* K *and* z_0 *and is independent of* P_n.

The result is sharp in the following two senses.

Theorem (3). *i) For a given fixed n, the factor $1 + o(1)$ can be arbitrarily large, if we choose the set K and the polynomial P_n appropriately.*

ii) For every lemniscate K there exists a sequence of nonzero polynomials $\{P_n\}$ with degrees tending to infinity such that

$$\left|P_n'(z_0)\right| = n||P_n||_K \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0)$$

where $\deg P_n = n$*,* $z_0 \in K$ *and* z_0 *is a simple point of* K*.*

In other words, the $1 + o(1)$ factor cannot be left out if we choose the compact set and the polynomial suitably, and the constant (the factor $\frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0)$ on the right hand side) cannot be replaced by anything smaller.

The (3.3) in general

We use the following notion and two theorems.

Let γ^* and Γ^* be twice continuously differentiable in a neighborhood of P and touching each other at P. We say that they $\mathcal{K}$-*touch* each other if their (signed) curvature at P is different (signed curvature is seen from the outside of Γ^*). Equivalently we can say that in a neighborhood of P the two curves are separated by two circles one of them lying in the interior of the other one.

Theorem (7). *Let $\gamma^* = \cup_{j=1}^m \gamma_j$ and $\Gamma^* = \cup_{j=1}^m \Gamma_j$ be as above, and let γ^* $\mathcal{K}$-touch Γ^* in finitely many points $P_1, \ldots, P_k$ in a neighborhood of which both curves are twice continuously differentiable. Then there is a lemniscate σ that separates γ^* and Γ^* and $\mathcal{K}$-touches both γ^* and Γ^* at each P_j.*

Furthermore, σ lies strictly in between γ^* and Γ^* except for the points $P_1, \ldots, P_k$, and has precisely one connected component in between each γ_j and Γ_j, $j = 1, \ldots, m$, and these m components are Jordan curves.

Its proof is fairly technical, the outline is as follows. For a simpler notation, we leave out here the index j and denote K_0 the compact set γ^* encloses.

- First we remove a small part of the closed inner domain K_0 around the point P, the rest will be denoted by K_1.
- The removed part will be replaced by a rotated and shifted copy $T^{\theta,\delta}(S)$ of a lens shaped region S for which the bounding circular arcs have curvature lying in between the curvatures of Γ and γ at the point P.
- The Green line will be for some small τ the τ-level curve of Green's function $g(K_1 \cup T^{\theta,\delta}(S), z)$ of $\mathbf{C}_\infty \setminus (K_1 \cup T^{\theta,\delta}(S))$ with pole at infinity.
- To analyze these τ-level lines close to the boundary of $T^{\theta,\delta}(S)$ we use the reflection principle to continue the Green's functions $g(K_1 \cup T^{\theta,\delta}(S), z)$ over the circular arc $\partial T^{\theta,\delta}(S)$, and complete these continued harmonic functions to analytic functions. This way the τ-level line of $g(K_1 \cup T^{\theta,\delta}(S), z)$ coincides with the image of a line segment under the inverse of these analytic functions, and simple analytic properties can be used for the analysis (Lemma 11).
- We shall use the Brouwer fixed point theorem to prove that for appropriate (and small) rotation (by angle θ) and shift (by δ), the τ-level line will pass through the point P and will have the same tangent line there as Γ (and γ).
- For small τ this τ-level line will lie very close to $K_1 \cup T^{\theta,\delta}(S)$, hence it will separate each γ_j from Γ_j, and along the boundary of $T^{\theta,\delta}(S)$ it will have curvature very close to that of $\partial T^{\theta,\delta}(S)$, which is the same as the curvature of ∂S.
- As a consequence, in the neighborhood of P we are working in, the curvature of the τ-level line will lie in between the curvatures of γ and Γ and at the same time it touches both of these

curves at P. Hence, by a variant of Blaschke's rolling theorem (given in Lemma 13) the level line will lie in between these two curves in a smaller neighborhood.

- Elsewhere the τ-level line follows closely the boundary of $K_1 \cup T^{\theta,\delta}(S)$, hence it lies outside γ^* but inside Γ^*.

Furthermore, we relate the normal derivative at P of the Green's function of the original compact set K and of the exhausting lemniscate σ with the following

Theorem (8). *Let Γ^*, γ^* and $P_1, \ldots, P_k \in \Gamma^*$ be as in the Theorem 7. Then for every $\varepsilon > 0$ there is a lemniscate σ as in the Theorem 7 such that for each P_j we have*

$$\frac{\partial g(L, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(K, P_j)}{\partial \mathbf{n}} + \varepsilon, \tag{3.1}$$

where $\partial(\cdot)/\partial\mathbf{n}$ denotes (outward) normal derivative.

In a similar manner, for every $\varepsilon > 0$ there is a lemniscate σ as in the Theorem 7 such that for each P_j we have

$$\frac{\partial g(K_0, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(L, P_j)}{\partial \mathbf{n}} + \varepsilon. \tag{3.2}$$

Sharpness

We mention that Theorem 9 is sharp in the following two senses. First, in general the inequality

$$|P_n'(z_0)| \leq n \frac{\partial g(K, z_0)}{\partial \mathbf{n}} \|P_n\|_K$$

i.e. (3.3) without the term $1 + o(1)$ is not true.
Second, the "geometrical" constant can not be replaced by smaller, as the following theorem shows:

Theorem (10). *Let K and z_0 be as in Theorem 9. Then for every n there is a polynomial P_n of degree at most n such that*

$$|P_n'(z_0)| > n(1 - o(1)) \frac{\partial g(K, z_0)}{\partial \mathbf{n}} \|P_n\|_K. \tag{3.4}$$

Furthermore, the generalized Hilbert's lemniscate theorem is sharp in the sense that the C^2 smoothness condition can not be dropped. That is, in the proof of Theorem 7 we actually used the fact that the derivates (of the curves up to order 1) coincide and they have derivatives of order 2 which are different.

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