Advances in Bijective Combinatorics

Ph.D. Thesis

by

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CHAPTER 1

Introduction

One of the characteristic feature of mathematics that distinguishes it from other sciences is that it is not a collection of informations about a given topic, but its purpose is the establishing and the comprehension of connections.

Combinatorics has a special role in that effort/pursuit, with its particular formulations of questions. The aim of this discipline is giving insight into the structure of mathematical objects, considering discrete structures. Combinatorics, as its name indicates, is the science of combinations. Given basic rules for assembling simple components, what are the (quantitative or/and structural) properties of the resulting objects?

Determining the size of the set of the given objects respect to parameters is one way to describe objects that are given by definition of various properties. This is the central problem of enumerative combinatorics. R. Stanley introduced the different forms of enumeration in his fundamental book *Enumerative Combinatorics I and II.* [**87**],[**88**]. Various methods of other branches of mathematics are applied, but there is a very specific method, only used in enumerative combinatorics: the bijective proof.

Bijection is a one–to–one correspondence between two sets, that proves that the two sets are equinumerous. The primary importance of a bijection is beyond proving such a fact. A bijection reveals characteristic properties of the integer sequence that arise in connection with the considered sets. By pointing out the common attributes of diverse objects the bijection contributes to the understanding of the underlying structures.

Since mathematicians like ,,nice" proofs in some cases a bijection exists but it is not satisfying. There is no objective measure of the worth of a bijective proof by definition, but the simplicity and a good understandable description of it is expected. Because of this it is often interesting to find new bijections or new way of descriptions of an existing bijection of a known enumeration result.

The goal of this thesis is to show the essence of combinatorial proofs. We present bijective proofs in three different topics. The work is based on the following papers of the author:

- B. Bényi, P. Hajnal. Combinatorics of poly–Bernoulli numbers, accepted for publications in *Studia Scientarium Mathematicarum Hungarica*
- B. Bényi. Bijective proofs of the hook formula for rooted trees, *Ars Combinatoria* **106** (2012), 483–494.
- B. Bényi. A simple bijection between 312–avoiding permutations and triangulations, accepted for publication in *Journal of Combinatorial Mathematics and Combinatorial Computing*

The main part of this thesis consists of the combinatorial investigation of poly–Bernoulli numbers and based on [**6**]. These numbers are natural extensions of the classical Bernoulli numbers and were introduced recently by Kankeko [**47**]. The poly–Bernoulli numbers are natural numbers for negative parameters. Our work [**6**] is the first attack to give a complex combinatorial description of the poly–Bernoulli numbers by collecting the known results that can be found sporadic in the literature. Moreover our summary reveals the connections between the sets arising in different areas, as discrete tomography, graph theory, algebra, etc. We interpret the formula of poly–Bernoulli numbers combinatorially as the number of ordered pairs of partitions of two sets. We show that the permutations that are defined by Callan [**14**] has the same structure and show with a bijection that it is also true for the first combinatorial interpretation of poly–Bernoulli numbers that was known [**13**], for lonesum matrices. Binary matrices can be thought as an assignment of orientations of the edges of a complete bipartite graph. It turns out that a lonesum matrix is essentially the same as a coding of an acyclic orientation of a complete bipartite graph [**15**].

There are two other permutations that are enumerated by poly–Bernoulli numbers. The one, the so called ascending–to–max permutations are dual of Callan permutations. The other one is a permutation class that are characterized by a constraint on the distance between the element and the image of the element in the permutation.

As a main result of the second chapter we present a new class of combinatorial objects that are enumerated by poly–Bernoulli numbers, the Γ–free matrices. A Γ–free matrix is a matrix that is determined by a set of excluded submatrices. The study of matrices that are characterized by excluded matrices is an active research area with many important results and applications ([**32**], [**65**], [**29**]).

Though the interpretations are for itself interesting they can be used to prove known properties of the poly–Bernoulli numbers, that were proved before analytically. Our new interpretation for instance gives a transparent explanation of the recursive formula. The analytical derivation of this formula uses multiple zeta values and is complicated [**48**], [**38**]. The interpretation as the number of acyclic orientations of a complete bipartite graph connects the two explicit formulas of poly–Bernoulli numbers. A bijection on Callan permutations proves a further interesting property of a special sum of poly–Bernoulli numbers.

There are many open questions related to poly–Bernoulli numbers. We close this chapter with a description of some open problems that are of combinatorial interests. We can establish two main directions of future research:

- Each interpretation has some characteristic parameters. Based on these parameters the objects can be generalized. How do the natural generalizations of the different underlying objects relate to each other?
- Researcher studied poly–Bernoulli numbers and defined related sequences purely analytically as for instance poly–Cauchy numbers [**57**] or multi–poly–Bernoulli numbers [**39**]. Are there any interesting combinatorial interpretations of the number sequences that arise through such algebraic manipulations?

In the third chapter we consider hook formula for plane trees. This part is based on the work [**4**]. The hook formula is a surprisingly compact formula that arises in enumerations of linear extensions of particular partial orders as Standard Young Tableaux, Shifted Standard Young Tableaux, and plane trees. In each case the characteristic hook parameter plays the crucial role. Since these formulas are easy to interpret combinatorially it was natural to require a nice bijective proof. However the appearance of the first satisfying result for the case of the Standard Young Tableaux took a relative long time and the simpler case, the case of the plane trees remained neglected. We complete this lack with the description of two bijections: one in the spirit of Novelli, Pak and Stoyanovskii [**68**] and one that is more natural in the case of plane trees. We point out the importance of the case of plane trees with a review of the wealth of identities involving hook length that were derived recently for special classes of trees.

The fourth chapter is devoted to 312–avoiding permutations and based on the paper [**5**]. Pattern avoidance in permutations is a well studied classical problem. The interest was waked up with the fact that the number of permutations that avoid a pattern of length three is the Catalan number for all patterns. The Catalan sequence is itself a ubiquitous number sequence in enumerative combinatorics.

In this work we provide a simple direct bijection between 312–avoiding permutations and triangulations. In how many ways can a convex polygon divided into triangles by non–intersecting diagonals was the first question that was answered by the Catalan numbers. Our crucial observation, on that our bijection is based, that the triangles can be labeled according to their middle vertex, can be extended to the general case of k–triangulations. Our bijection uses the inversion table of 312–avoiding permutations and shows the correspondence with the length of the diagonals in a triangulation.

In this thesis we want to emphasize the role of the inversion tables of 312–avoiding permutations. There are two kinds of inversion tables, vectors with special conditions, both defining 312–avoiding permutations uniquely. The idea of considering inversions in a permutation is central. The lattices that arise by ordering permutations by considering inversions are well known and the lattices that arise in the case of 312–avoiding permutations are the the so called Catalan lattices: Tamari and Dyck lattice. Our unified interpretation allows to better understand the connections between these lattices. Though these are known results we think that our point of view can be important since in many enumeration problems the number of intervals of the Dyck or the Tamari lattice appear. It is an interesting question whether there are nice bijections including pairs of appropriate 312–avoiding permutations.

As a demonstration of the use of our idea we define a simple bijection between a special pattern avoiding matching and pairs of 312–avoiding permutations.

CHAPTER 2

Poly–Bernoulli numbers

,,One picture is worth
thousand words"

1. Introduction

As the name indicates the poly–Bernoulli numbers are generalization of Bernoulli numbers. Since Bernoulli numbers are rational numbers there is no direct combinatorial interpretation of this sequence as the counting any set of combinatorial objects. However through its connections to finite calculus, or the inclusion–exclusion principle, it is a an important sequence also in combinatorics. Surprisingly a generalization of this sequence generate a two parametric sequence of integer numbers that have beautiful, simple and direct combinatorial interpretations. The importance of the notion of poly–Bernoulli numbers is underlined by the fact that there are several drastically different combinatorial descriptions.

However except Brewbakers works [**13, 12**] in the literature we find primarily algebraic proofs of identities involving poly–Bernoulli numbers. The goal of our work is to give an exhaustive presentation of the combinatorics of poly–Bernoulli numbers. Through our work we would like to demonstrate the nature and power of combinatorial proofs and emphasize how this point of view supplement the algebraic ones giving insight in the structure of objects that correspond to sequences coded by the given functions. In order to reach this goal we will rewrite some in the literature given calculations also. In our proofs we try to pick up the optimal candidate to reveal the main point, though as we will see these explanations can be formulated in analogous way for several combinatorial objects.

First of all to see the whole picture we recall some useful and important informations about Bernoulli numbers itself. After this brief survey we introduce poly–Bernoulli numbers according to Kaneko [**47**]. Supplementing this introduction we mention different generalization methods and by using algebraic manipulations from the literature we show that they result the same sequence of numbers. We finish this part with a list of some notable identities that will be proven later by combinatorial methods.

In the main part of this chapter we investigate poly–Bernoulli numbers from the combinatorial point of view. We give a simple, obvious interpretation and show how the combinatorial objects that are known to be counted by poly–Bernoulli numbers are related to this model. We supplement this collection with a new family of permutations that were not connected until now to poly–Bernoulli numbers and even more we define a structurally substantially different new combinatorial family also. Based on these interpretations we prove combinatorially the formulas that were calculated by algebraic methods in the literature.

Finally we collected some related topics and open questions revealing some possible direction of future research. Inspired by the success of the way of generalization of Bernoulli numbers Komatsu [**57**] defined analogously poly–Cauchy numbers. The close relation of poly–Bernoulli numbers to Riemann zeta functions served the idea to introduce multi–poly–Bernoulli numbers. Generalizations of the combinatorial interpretations raised also some new sequences connected to these numbers. These instances serves the demonstration of the fact that the research of this area is just initiated and there are numerous exiting open questions to be answered.

2. Bernoulli numbers

We summarize some basic facts about Bernoulli numbers that guide the reader when considering poly–Bernoulli numbers. We do not presents proofs here. The results are well known and can be found in any book on Bernoulli numbers (for example [**23**]).

The story of Bernoulli numbers starts in the 17th century with the list of the formulas of Johann Faulhaber (1580–1635) [**27**] giving the sum of the m^{th} powers of the first n positive integers. We refer the interested reader to Knuth's article [**55**], discussing the history and background of Faulhaber's paper.

$$S_m(n) = \sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m$$

These formulas are always polynomials in n of degree $m+1$.

The scheme in the coefficients of these polynomials was realized by Jacob Bernoulli (1655–1705). Describing the coefficients he introduced a new sequence of rational numbers that was named after him later, as the Bernoulli numbers.

Let consider the exponential generating function of $\sum_{m\geq 0} S_m(n)\frac{x^m}{m!}$.

$$\sum_{m\geq 0} S_m(n)\frac{x^m}{m!} = \sum_{m\geq 0}\frac{x^m}{m!}\sum_{0\leq k<n} k^m = \sum_{0\leq k<n}\sum_{m\geq 0}\frac{k^m x^m}{m!} \tag{1}$$
$$= \sum_{0\leq k<n} e^{kx} = \frac{e^{nx}-1}{e^x-1} = \frac{e^{nx}-1}{x}\frac{x}{e^x-1}.$$

We introduce now the sequence $\{B_j\}_{j\geq 0}$ of Bernoulli numbers with its exponential generating function

$$\frac{x}{e^x-1} = \sum_{j\geq 0} B_j \frac{x^j}{j!} \tag{2}$$

With the help of Bernoulli numbers we can easily express $S_m(n)$. Rewriting the exponential function of $S_m(n)$ we obtain:

$$(1) = \sum_{k\geq 1}\frac{n^k x^{k-1}}{k!}\sum_{j\geq 0} B_j\frac{x^j}{j!} = \sum_{m\geq 0}\frac{x^m}{m!}\sum_{k+j-1=m}\frac{m!}{k!j!}B_j n^k$$

The symmetry in j and k allows us to write $B_j n^k$ instead of $B_k n^j$. Setting $j = m+1-k$ we have

$$S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

It is interesting that around the same time the Japanese mathematician Seki Kowa (1642–1708) discovered these numbers also, though in that time there weren't any scientific contact between Japan and Europe. This is a nice instance of the mysterious fact that in mathematics new ideas are borne often independently of different people but nearest the same time. Later Leonhard Euler (1707–1783) [**26**] recognized the significance of this sequence (he was the ,,godfather" of the Bernoulli numbers) and described several applications of these numbers. The first few values are: Bernoulli numbers arise today in many

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Table 1. The Bernoulli numbers

branches of mathematics. Bernoulli numbers can be regarded namely from different point of views: for instance as values of the Riemann zeta function, as arithmetical objects or as combinatorial objects.

The famous Riemann zeta function is defined as the analytic continuation of the infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is absolutely convergent for complex s with real part greater than 1. The values of the Riemann zeta function at negative integers are related to Bernoulli numbers as follows:

$$\zeta(1-n) = -\frac{B_n}{n}, \quad n \geq 2.$$

The Riemann hypothesis is one of the most famous conjecture that is desired to be proven by mathematician since David Hilbert listed it in his 23 central unsolved problems in 1900. It is also included in the list of the Millennium Prize Problems stated by Clay Mathematics Institute in 2000. It states that the nontrivial zeros of the Riemann zeta function all have real part $\frac{1}{2}$. The trivial zeros are $-2, -4, -6, \ldots$ as it follows from the above mentioned relation with Bernoulli numbers and that the Bernoulli numbers are zero $B_n = 0$ for odd n greater than 2. The Riemann hypothesis includes numerous propositions which are equivalent to it and some which are implied by it. There is even a reformulation of the Riemann hypothesis using Riesz function, which is defined by Bernoulli numbers. The special values at even integers $s = 2n$, $n \in \mathbb{N}$ were determined by Euler using even indexed Bernoulli numbers:

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

The special case $\zeta(2)$ of this formula is the celebrated solution of the famous Basel problem showed also by Euler:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The generalizations of Riemann zeta functions, as for example the Euler–Zaiger sums or multiple zeta values led to investigations of poly–Bernoulli numbers.

The arithmetical properties of the Bernoulli numbers stand them into the center of many research. We mention here just one of this, that has also a generalized form connecting to poly–Bernoulli numbers [**47**]. The von Staudt Clausen theorem states that for every positive integer n

$$B_{2n} + \sum_{p-1|2n} \frac{1}{p},$$

where the sum extends over all primes p for which $p - 1$ divides $2n$.

From the combinatorial point of view Bernoulli numbers hold also numerous interesting properties. This sequence is the ,,prototype" of the umbral calculus. This powerful technique were developed in the 1970's by Roman and Rota [**79**] and were used extensively for instance by Riordan in his book *Combinatorial Identities* [**78**]. The method is a natural procedure pretending that the indices of the sequences are exponents. Rigorously umbral calculus is an algebra of linear functionals on the vector space of polynomials in a variable.

Simply saying Move n from subscript to a superscript we receive a formal expression for the generating function of Bernoulli numbers (as defined in (2)) that can be used to derive a recursive relation.

$$\sum_{n\geq 0}\frac{B_n x^n}{n!}=\frac{x}{e^x-1}$$

$$\sum_{n\geq 0}\frac{B^n x^n}{n!}=e^{Bx}$$

Multiplying by e^x-1

$$x=e^{Bx}(e^x-1)=e^{(B+1)x}-e^{Bx}=\sum_{n\geq 0}\left[(B+1)^n-B^n\right]\frac{x^n}{n!}$$

Hence by comparing the coefficients

$$0=(B+1)^n-B^n \quad \text{for} \quad n\geq 2$$

and we obtained the recursive relation for Bernoulli numbers:

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k. \tag{3}$$

From this expression the Bernoulli numbers can be computed successively. The correctness of the ad hoc manipulations can be shown by introducing the linear functional L on polynomials in y by $L(y^n)=B_n$.

Bernoulli numbers are intimately related to the theory of finite differences. Several identities are known involving Bernoulli numbers and other number sequences of this area as Stirling numbers of the first and second kind, Euler numbers, Harmonic numbers and the less known Cauchy numbers.

In order to make a strong basis for our investigations on poly–Bernoulli numbers from combinatorial point of view, we recall some basic fact concerning Stirling numbers of the second kind, since this sequence plays a crucial role.

A *partition of a finite set* N is a collection $P=\{B_1,B_2,\ldots,B_r\}$ of subsets of N such that

- $B_i\neq\emptyset$ for each i
- $B_i\cap B_j=\emptyset$ if $i\neq j$
- $B_1\cup B_2\cup\cdots B_r=N$

We call B_i a block of P.

Definition 1. Let n and r be two natural numbers. *Stirling number of the second kind* is the number of partitions of $[n]$ (or any set of n elements) into r classes, and is denoted by $\left\{ {n \atop r} \right\}$.

The special values are $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {0 \atop k} \right\} = 0$ for $k > 0$.
For instance $\left\{ {4 \atop 2} \right\} = 7$ corresponding to the partitions

$$123-4, \quad 124-3, \quad 134-2, \quad 234-1, \quad 12-34, \quad 13-24, \quad 14-23.$$

There is a simple bijection between the equivalence relations $\sim$ on a set N and the partitions of N, viz. the equivalence classes of $\sim$ form a partition of N.

Stirling numbers occur naturally in enumeration of surjective mappings. The number of surjective mappings f from the set N with n elements to the set R with r elements is $r!\left\{ {n \atop r} \right\}$, since the pre–images $f^{-1}(s)$ $s \in R$ form an ordered partition of N

$$f^{-1}(s_1)|f^{-1}(s_2)|\cdots|f^{-1}(s_r).$$

Bernoulli numbers have two representations as a sum of Stirling numbers of the second kind.

$$B_n = \sum_{m=0}^{n} (-1)^m \frac{1}{m+1} m! \left\{ {n \atop m} \right\} \tag{4}$$

$$\begin{aligned}
B_0 &= 1 \\
B_1 &= 0 - \frac{1}{2} = -\frac{1}{2} \\
B_2 &= 0 - \frac{1}{2} + \frac{2}{3} = \frac{1}{6} \\
B_3 &= 0 - \frac{1}{2} + \frac{6}{3} - \frac{6}{4} = 0 \\
B_4 &= 0 - \frac{1}{2} + \frac{14}{3} - \frac{36}{4} + \frac{24}{5} = -\frac{1}{30}
\end{aligned}$$

The other one is also called Worpitzky's representation:

$$B_n = \sum_{m=0}^{n} (-1)^m \frac{1}{m+1} m! \left\{ {n+1 \atop m+1} \right\} \tag{5}$$

$$B_0 = 1$$
$$B_1 = 1 - \frac{1}{2} = \frac{1}{2}$$
$$B_2 = 1 - \frac{3}{2} + \frac{2}{3} = \frac{1}{6}$$
$$B_3 = 1 - \frac{7}{2} + \frac{12}{3} - \frac{6}{4} = 0$$
$$B_4 = 1 - \frac{15}{2} + \frac{50}{3} - \frac{60}{4} + \frac{24}{5} = -\frac{1}{30}$$
$$B_5 = 1 - \frac{31}{2} + \frac{180}{3} - \frac{390}{4} + \frac{360}{5} - \frac{120}{6} = 0$$

3. Introduction of the poly–Bernoulli numbers

As we have seen Bernoulli numbers can be defined in different ways:

- by generating function (2),
- by a recursive relation (3),
- by an inclusion-exclusion type formula (4).

The same can be stated for poly–Bernoulli numbers, though originally the poly–Bernoulli numbers were introduced by a generalization of the generating function of Bernoulli numbers.

Poly–Bernoulli numbers were introduced in 1997 by Kaneko [**48**] as he considered multiple zeta values (or Euler sums). Multiple zeta values are nested generalizations of the Riemann zeta function evaluated at integer values:

$$\zeta(k_1, k_2, \dots k_n) = \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where k_i are positive integers and $k_n \geq 2$. Multiple zeta values have received much attention in recent years. Despite of the wealth of interesting results their precise structure still remains a mystery.

Definition 2. ([**47**]) The $\{B_n^{(k)}\}_{n\in\mathbb{N},k\in Z}$ poly-Bernoulli numbers are defined by the following exponential generating function

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}$$

where

$$Li_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k},$$

i. e. $Li_k(z)$ is the k–th polylogarithm when $k > 0$ and a rational function when $k \leq 0$. For some k we have:

$$Li_1(z) = -\ln(1-z), \quad Li_0(z) = \frac{z}{1-z}, \quad Li_{-1}(z) = \frac{z}{(1-z)^2}.$$

In case $k \geq 0$ the poly–Bernoulli numbers are rational numbers and in case $k < 0$ integers. The values of the first few poly–Bernoulli numbers are given in the table. Obviously $\{B_n^{(1)}\}_n$ are the classical Bernoulli numbers (with $B_1 = \frac{1}{2}$).

The Poly–Bernoulli Numbers

	0	1	2	3	4	5
-5	1	32	454	4718	41506	329462
-4	1	16	146	1066	6902	41506
-3	1	8	46	230	1066	4718
-2	1	4	14	46	146	454
-1	1	2	4	8	16	32
0	1	1	1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0
2	1	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$
3	1	$\frac{1}{8}$	$-\frac{11}{216}$	$-\frac{1}{288}$	$\frac{1243}{54000}$	$-\frac{49}{7200}$

In [**2**] the authors showed that poly–Bernoulli numbers — as expected — can be expressed as special values at negative arguments of certain combinations of multiple zeta values. This recursive relation can be regarded as the appropriate generalization of the well known recurrence (3).

Theorem 1. [**2**]

$$B_n^{(k)} = \frac{1}{n+1}\left(B_n^{(k-1)} - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{(k)}\right)$$

or equivalently

$$B_n^{(k-1)} = B_n^{(k)} + \sum_{m=1}^{n}\binom{n}{m}B_{n-(m-1)}^{(k)}.$$

The generalization of the inclusion–exclusion type formula of the Bernoulli numbers (4) can be derived from the generating function.

Theorem 2. [**47**]

$$B_n^{(k)} = (-1)^n\sum_{m=0}^{n}(-1)^m\frac{m!}{(m+1)^k}\left\{ n \atop m \right\}$$

PROOF.

$$\begin{aligned}
\frac{Li_k(1-e^{-x})}{1-e^{-x}} &= \sum_{m=0}^{\infty} \frac{(1-e^{-x})^m}{(m+1)^k} \\
&= \sum_{m=0}^{\infty} \frac{m!}{(m+1)^k} \frac{(-1)^m (e^{-x}-1)^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{m!}{(m+1)^k} (-1)^m \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{(-x)^n}{n!} \\
&= \sum_{n=0}^{\infty} \left((-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k} \right) \frac{x^n}{n!}
\end{aligned}$$

In the third line we used the generating function of the Stirling numbers of the second kind:

$$\sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!} = \frac{(e^x-1)^m}{m!}$$

□

The multivariate generating function of poly–Bernoulli numbers reveals some unexpected properties of poly–Bernoulli numbers:

- $B_n^{(k)} = B_k^{(n)}$ and
- the poly–Bernoulli numbers $B_n^{(-k)}$ are natural numbers.

Theorem 3. [**47**]

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}$$

PROOF.

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (1-e^{-x})^m (m+1)^k \frac{y^k}{k!} \\
&= \sum_{m=0}^{\infty} (1-e^{-x})^m e^{(m+1)y} \\
&= \frac{e^y}{1-(1-e^{-x})e^y} \\
&= \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.
\end{aligned}$$

□

The symmetry in x and y of this function implies the following theorem:

Theorem 4. [**47**] *For all $n > 0$ and $k > 0$ it holds:*

$$B_n^{(-k)} = B_k^{(-n)}$$

The function shows also the fact that $B_n^{(-k)}$ ($n > 0$ and $k > 0$) are positive integers:

$$\frac{e^{x+y}}{e^x + e^y - e^{x+y}} = \frac{e^{x+y}}{1 - (e^x - 1)(e^y - 1)} =$$
$$= e^{x+y}(1 + (e^x - 1)(e^y - 1) + (e^x - 1)^2(e^y - 1)^2) + \cdots$$

We note that m, n, l, k usually denotes positive integers. We will skip this remark later. $B_n^{(-k)}$ always denote a poly–Bernoulli number with negative upper index.

The next theorem exhibits also the fact that $B_n^{(-k)}$ is a natural number and has initiated the combinatorial investigations of poly–Bernoulli numbers.

Theorem 5. [**1**]

$$B_n^{(-k)} = \sum_{m=0}^{\min(n,k)} m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} m! \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\}.$$

Proof. Several proofs of this formula is known. We give here an elegant proof published by Peregrino using the idea due to Zeilberger [**71**].

$$\begin{aligned}
\frac{e^{x+y}}{e^x + e^y - e^{x+y}} &= e^{x+y} \sum_{m=0}^{\infty} (1 - e^x)^m (1 - e^y)^m \\
&= \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \left[(m+1)(1 - e^x)^m(-e^x)\right] \left[(m+1)(1 - e^y)^m(-e^y)\right] \\
&= \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \mathrm{D}_x[(1 - e^x)^{m+1}] \mathrm{D}_y[(1 - e^y)^{m+1}] \\
&= \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \mathrm{D}_x \left((-1)^{m+1}(m+1)! \sum_{m+1}^{\infty} \left\{ \begin{matrix} n \\ m+1 \end{matrix} \right\} \frac{x^n}{n!} \right) \times \\
&\quad \mathrm{D}_y \left((-1)^{m+1}(m+1)! \sum_{m+1}^{\infty} \left\{ \begin{matrix} k \\ m+1 \end{matrix} \right\} \frac{y^k}{k!} \right) \\
&= \sum_{m=0}^{\infty} (m!)^2 \sum_{n=m}^{\infty} \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \frac{x^n}{n!} \sum_{k=m}^{\infty} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} \frac{y^k}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} (m!)^2 \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} \right) \frac{x^n}{n!} \frac{y^k}{k!}
\end{aligned}$$

□

There are several combinatorially described sequences of sets, such that their size is $B_n^{(-k)}$. We can consider these statements as alternative definitions of poly–Bernoulli numbers as answer to enumeration problems.

4. Obvious interpretation

Seeing the formula of Arakawa and Kaneko one can easily come up with a combinatorial problem such that the answer to it is $B_n^{(-k)}$.

Let N be a set of n elements and K a set of k elements. One can think as $N = [n]$ and $K = [k]$. Extend both sets with a special element: $\widehat{N} = N \dot\cup \{n+1\}$ and $\widehat{K} = K \dot\cup \{k+1\}$. Let $\mathcal{P}_{\widehat{N}}$ resp. $\mathcal{P}_{\widehat{K}}$ denote partitions of $\widehat{N}$ and of $\widehat{K}$. Each partition has a special class: the class of the special element. We call the other classes as *ordinary* classes. Let m denote the number of ordinary classes in $\mathcal{P}_{\widehat{N}}$ (that is the same as the number of ordinary classes in $\mathcal{P}_{\widehat{K}}$). Obviously $m \in \{0, 1, 2, \dots, \min\{n, k\}\}$. Order the ordinary classes arbitrary in both partitions. How many ways can we do this?

For fixed m choosing $\mathcal{P}_{\widehat{N}}$ and ordering its ordinary classes can be done $m!\left\{ {n+1 \atop m+1} \right\}$ ways. Since to each ordered partition of $\mathcal{P}_{\widehat{N}}$ we can choose each of the ordered partition $\mathcal{P}_{\widehat{K}}$ choosing the pair of ordered partitions can be done

$$m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$$

ways. Hence we get the answer to our original question by summing up of these choices for all m:

$$\sum_{m=0}^{\min(n,k)} m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\} = B_n^{(-k)}.$$

5. Binary lonesum matrices

The popular Milton Bradley game Battleship consists of an array of coordinates and a set of different kind of vessels. The simple enumeration problem that concerns the number of the possible configurations of the set of vessels by a given number of squares covered by a ship in each row and column, is in connection with the problems considered for instance in discrete tomography [**12**].

Discrete tomography is a relative young and active studied field dealing with the retrieval of information about objects (lattice sets, binary matrices, digital or labelled images, measurable sets, dominoes etc.) from data about its projections [**43**]. Generally we could say that the problems of DT have to do with determining an unknown function f (whose range is known to be a given discrete set) from weighted sums over subsets of its domain. DT is not simple a special case of CT (computerized tomography), it has its own mathematical theory based mostly on discrete mathematics.

Many problems of DT were first discussed as combinatorial problems. Ryser [**80**] published in the late 1950's, a necessary and sufficient consistency condition for a pair of integral vectors being the row and column sum vector of a binary matrix. (A *binary matrix* is a matrix with entries 0 or 1.) He recognized also the so called *interchange* relation. An interchange operation is one of the following elementary operations on binary matrices:

- replacing the submatrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or
- replacing the submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Given a matrix M, a *submatrix* is a matrix that can be obtained from M by deletion of rows and columns (equivalently by selection of certain rows and columns of M.)

Obviously the application of interchange relation doesn't change the values of row/column sums.

The crucial role of interchange operations are revealed by the next theorem.

Theorem 6. [**80**] *Any binary matrix with a given row and column sum vectors can be transformed into any other binary matrix with the same row and column sum vectors via interchange operations.*

An equivalence class of the set of binary matrices with the same row and column sum vectors is called *Ryser class*. The Ryser classes of size one are also called *lonesum matrices*. An equivalent definition of lonesum matrices is the following:

Definition 3. [**13**] A binary matrix is lonesum iff it can be reconstructed from its row and column sums.

An obvious observation is that lonesum matrices can not contain the submatrices that are involved in any interchange operation. This property is a characterization [**80**]. One direction of this claim is obvious: in the case of the existence of one of the above submatrices we can switch it to the other one without changing the row and column sum vectors, so our matrix is not lonesum.

Given two matrices A and B, we say that A *contains* B, whenever B is equal to a submatrix of A. Otherwise we say that A *avoids* B. With this terminology we can state that the forbidden submatrices in a lonesum matrix are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For another characterization we introduce stair matrices. A *stair matrix* is a matrix whose i–th row is $1^{r_i}0^{n-r_i}$ with $r_1 \geq r_2 \geq \cdots \geq r_n$. The following example is a stair matrix with $(r_1, r_2, r_3, r_4) = (9, 6, 5, 2)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Claim 7. *Every lonesum matrix can be rearranged by row/column order changes into a stair matrix.*

Proof. The statement follows from the next two observations.

(1) In a lonesum matrix if two rows has the same row sum then the positions of 1–entries are exactly the same in these two rows.
(2) For rows R_1 and R_2 with row sums $r_1 < r_2$ it is true that in the positions of the 1–entries in the row R_1, there are also 1 entries in the row R_2.

The different rows of a lonesum matrix form actually a chain in the following sense. Let $\underline{0} < R_1 < \cdots < R_m$ denote rows with different row sums in a lonesum matrix. The row R_{i+1} is created from the row R_i by choosing some 0–positions of R_{i+1} and replacing these 0 entries by 1's. See for an example Figure 1. The same is true for columns. □

	1	2	3	4	5	6	7	8	9
R_2	1	1	1	0	0	1	1	1	0
R_1	1	0	1	0	0	1	0	1	0
$\underline{0}$	0	0	0	0	0	0	0	0	0
R_3	1	1	1	1	0	1	1	1	0
R_1	1	0	1	0	0	1	0	1	0
R_2	1	1	1	0	0	1	1	1	0

$$\underline{0} \xrightarrow{\{1,3,6,8\}} R_1 \xrightarrow{\{2,7,9\}} R_2 \xrightarrow{\{4\}} R_3$$

Figure 1. The chain of different rows of a lonesum matrix

From this characterization we can conclude another property of lonesum matrices. In a stair matrix it is clear that the number of different non–0 row sums is the same as the number of different non–0 column sums. This holds for lonesum matrices also, since the row sums and column sums are the same in the lonesum matrix and the corresponding stair matrix.

Now we are ready to prove combinatorially the following theorem, first presented by Brewbaker in his MSc thesis:

Theorem 8. *(*[**12**],[**13**]*) Let $\mathcal{L}_n^{(k)}$ the set of binary lonesum matrices of size $n \times k$. Then*

$$|\mathcal{L}_n^{(k)}| = \sum_{m=0}^{\min(n,k)} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = B_n^{(-k)}.$$

PROOF. Let M be a binary lonesum matrix of size $n \times k$. Add a special row and column with all 0's. Let $\widehat{M}$ be the extended $(n+1) \times (k+1)$ matrix. 'Having the same row sum' is an equivalence relation. The corresponding partition has a special class, the set of 0 rows. By the extension we ensured that the special class exists/non-empty. Let m be the number of ordinary classes. The ordinary classes are ordered by their corresponding row sums. The same way we obtain an ordered partition of columns. Straightforward to prove that the two ordered partitions give a coding of lonesum matrices. □

6. Acyclic orientations of $K_{n,k}$

It is a fundamental question which graphs in the class of all graphs with given numbers n and m of vertices and edges resp. minimizes or maximizes the value of some graph parameters? The number of acyclic orientations of a graph is an interesting graph parameter. *Acyclic orientation* of a graph is an assignment of direction to each edge of the graph such that there are no directed cycles. There is a unique graph class with minimal number of acyclic orientations [**63**], [**90**], [**16**]. Finding the maximum number of acyclic orientations is considerably more challenging. A conjecture says that if n and m are such that a Turán graph with n vertices and m edges exists then the Turán graph has the maximum number of acyclic orientations. Cameron, Glass and Schumacher computed some values in the case of $m = n^2/4$ and found that these numbers are the poly–Bernoulli numbers $B_n^{(-n)}$. Motivated by this observation they proved the following theorem.

Theorem 9. [**15**] *The number of acyclic orientations of $K_{n,k}$ is $B_n^{(-k)}$.*

$K_{n,k}$ denotes as usual the *complete bipartite graph*, a graph whose vertex set can be partitioned into two classes A and F ($|A| = n$ and $|F| = k$) such that no two vertices in the same class are adjacent and any pair (v, w): $v \in A$ and $w \in F$ are adjacent.

PROOF. Let $\mathcal{O}_n^k$ denote the set of acyclic orientations of $K_{n,k}$. A simple graph theoretical observation gives us that in a complete bipartite graph acyclicity is equivalent to 'not having directed 4–cycle'.

Suppose there are no directed 4–cycles (C_4), but there is a directed cycle in the graph. Let $C_{\min} = (v_1, w_1, v_2, w_2 \ldots, v_l, w_l, v_1)$ be the shortest cycle. Then the edge between v_1 and w_2 must be oriented from v_1 to w_2, since otherwise there would be a C_4: $(v_1, w_1, v_2, w_2, v_1)$. But then $(v_1, w_2, v_3, \ldots, v_l, w_l, v_1)$ is a shorter cycle than $C_{\min}$, a contradiction.

The theorem is immediate from a bijection between $\mathcal{O}_n^k$ and $\mathcal{L}_n^k$ (i.e. the set of lonesum matrices). The bijection is easy and natural. Identify the two parts of nodes in $K_{n,k}$ with the rows and columns of a matrix size $n \times k$. any edge has two possible orientations, hence we can code an actual oriented edge by a bit (0/1) according to its direction between the two colour classes. The oriented graph can be coded by a binary matrix of size $n \times k$. Forbidding directed C_4's is equivalent to forbidding two submatrices of size 2×2. These matrices are the same as the ones in Ryser's characterization of lonesum matrices. So the desired bijection is just the simple coding we have described. □

Using this interpretation of the poly–Bernoulli numbers we give a new proof of the following theorem.

Theorem 10. [**47**]

$$B_n^{(-k)} = (-1)^n \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} (m+1)^k$$

PROOF. It is well known that the number of acyclic orientations of a graph is equal to the absolute value of the chromatic polynomial of the graph evaluated at -1. [**90**]. The chromatic polynomial of $K_{n,k}$ is defined as

$$chr(K_{n,k}, \lambda) = \sum_{m \geq 0} \left\{ {n \atop m} \right\} \lambda(\lambda - 1) \cdots (\lambda - m + 1)(\lambda - m)^k$$

The evaluation of this polynomial at -1 gives exactly the formula in the theorem. □

7. Callan permutations

Callan [**14**] considered the set $[n+k]$. We call the elements $1, 2, \ldots, n$ *left-value elements* (n many of them) and $n+1, n+2, \ldots, n+k$ *right-value elements* (k many of them). We extend our universe with 0, a special left-value element and with $n+k+1$, a special right-value element. Let $N = [n]$, $K = \{n+1, n+2, \ldots, n+k\}$, $\widehat{N} = N \dot\cup \{0\}$ and $\widehat{K} = K \dot\cup \{n+k+1\}$. Consider the permutation of $\widehat{N} \dot\cup \widehat{K}$

$$\pi : 0, \pi_1, \pi_2, \ldots, \pi_{n+k}, n+k+1$$

with the restriction that its first element is 0 and its last element is $n+k+1$.

Consider the following equivalence relation of left-values: two left-values are equivalent iff 'each element in the permutation between them is a left-value'. Similarly one can define an equivalence relation on the right-values: 'each element in the permutation between them is a right-value'. The equivalence classes are just the "blocks" of left- and right-values in permutation π.

The left-right reading of π gives an ordering of the blocks of left-values and right-values. The order starts with a left-value block (the equivalence class of 0, the special class) and ends with a right-value block (the equivalence class of $n+k+1$, the special class). Let m be the common number of ordinary left-value blocks and ordinary right-value blocks.

Callan considered the permutations such that in each block the numbers are in increasing order. Let $\mathcal{C}_n^{(k)}$ the set of these permutations. For example

$$\mathcal{C}_2^{(2)} = \{012\mathbf{345}, 01\mathbf{3}2\mathbf{45}, 014\mathbf{2}3\mathbf{5}, 01\mathbf{34}2\mathbf{5}, 02\mathbf{3}1\mathbf{45}, 024\mathbf{1}3\mathbf{5}, 02\mathbf{34}1\mathbf{5}, 0\mathbf{3}12\mathbf{45},$$

$$0\mathbf{3}1\mathbf{4}2\mathbf{5}, 0\mathbf{3}2\mathbf{4}1\mathbf{5}, 0\mathbf{34}12\mathbf{5}, 0412\mathbf{35}, 041\mathbf{3}2\mathbf{5}, 042\mathbf{3}1\mathbf{5}\}$$

(the right-value elements are written in boldface).

It is easy to see that describing a Callan permutation we need to give the two ordered partitions of the left-value and right-value elements. Indeed, inside the blocks the 'increasing' condition defines the order, and the ordering of the classes let us know how to merge the left-value and right-value blocks. We obtained the following theorem.

Theorem 11 (announced in [**14**] without proof)**.**

$$|\mathcal{C}_n^{(k)}| = \sum_{m=0}^{\min(n,k)} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = B_n^{(-k)}.$$

We reveal the bijection between Callan permutations and lonesum matrices with a simple procedure (for an example see Figure 2.). A Callan permutation in $\mathcal{C}_n^{(k)}$ defines in a $(k+1) \times (n+1)$ rectangle a labelled path from $(0,0)$ to $(k+1, n+1)$. Each left value codes an up step $(x,y) \to (x, y+1)$ and each right value an east step $(x,y) \to (x+1, y)$. The label of a step is the value of the corresponding element of the permutation. The path is the boundary of a stair matrix, the standard form of an extended lonesum matrix. The labellings of the steps show how to receive the unique lonesum matrix from this path: project the labellings to the row and column indices and multiply the stair matrix with the suitable permutation matrices from left and right in order to rearrange the row resp. column indices in increasing order (from top bottom resp. from left to right). The process is reversible.

$\pi = 0 - \mathbf{6} - \mathbf{8} - \mathbf{11} - \mathbf{13} - 2 - 4 - \mathbf{7} - \mathbf{12} - \mathbf{14} - 1 - 5 - \mathbf{9} - 3 - \mathbf{10} - \mathbf{15}$

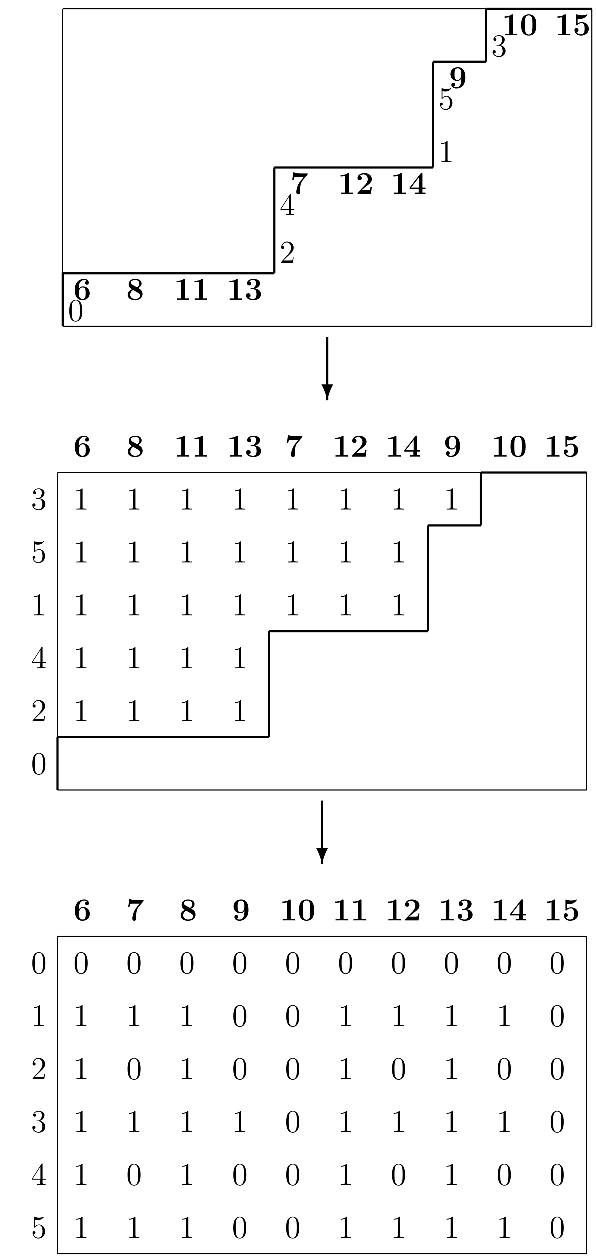


FIGURE 2. The bijection between Callan permutations and lonesum matrices

8. Ascending–to–max permutations

In this section we describe another special type of permutation class, the ascending–to–max permutations that are (in some sense) a "dual" of Callan permutations. We mention that [**85**] does not contain this description of poly-Bernoulli numbers.

He, Munro and Rao [**42**] introduced the notion of ascending–to–max permutations as they formulated a categorization theorem on suffix arrays. Suffix trees and suffix arrays are the data structures designed for pattern searching in a text. They are full text indexes which allow existential and cardinality queries. Suffix tree is more natural, but suffix array is substantially the same as suffix tree since it can be constructed by performing a depth–first traversal of a suffix tree, suffix array have been proposed to reduce space cost.

Suffix array contains the starting positions of the lexicographically sorted suffixes of the string. For instance the next table shows the suffix array of the word ,,bernoulli". The suffix array of a string of length n is a permutation

suffix array	suffixes
1	bernoulli
2	ernoulli
9	i
8	li
7	lli
4	noulli
5	oulli
3	rnoulli
6	ulli

of of $[n]$. It is clear that not every permutation corresponds to a suffix array. There are only k^{n-1} words of length n over an alphabet with k symbols (it is usual to set a terminal symbol as for instance \$ to ensure that no suffix is a prefix of another suffix) and $(n-1)!$ permutations. But is it possible to characterize the permutations that is a feasible suffix array? The answer is yes. He, Munro and Rao indeed gave a characterization of the suitable permutations in the case of binary alphabets by two properties, the one is the better known *non–nesting* property and the other one is defined by He, Muro and Rao, the *Ascending–To–Max* property.

Definition 4. [**42**] Given a permutation $\sigma \in S(n)$ we call it *Ascending–To–Max* iff for any integer i, $1 \leq i \leq n$, we have:

(1) if $\sigma^{-1}(i) < \sigma^{-1}(n)$ and $\sigma^{-1}(i+1) < \sigma^{-1}$ then $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ and
(2) if $\sigma^{-1}(i) > \sigma^{-1}(n)$ and $\sigma^{-1}(i+1) > \sigma^{-1}$ then $\sigma^{-1}(i) > \sigma^{-1}(i+1)$

We illustrate this definition as follows. We write the permutation in word notation and draw an arrow from the value i to the value $(i+1)$. We draw forward arrows above the permutation and backward arrows below the permutation. Then all the arrows that do not enclose the maximum value are in the direction that points towards the maximum value in the permutation.

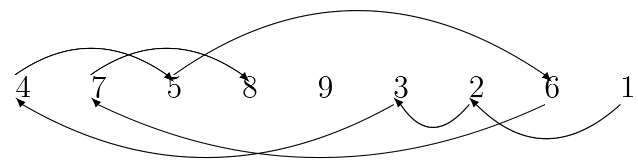


FIGURE 3. The Ascending–To–Max property

He, Munro and Rao's categorization theorem tells which permutations are suffix arrays and which are not [**42**].

suffix array	suffixes
4	aaba$
7	a$
5	aba$
1	abbaaba$
8	$
3	bbaba$
6	ba$
2	bbaaba$

TABLE 2. The suffix array of the text *abbaaba$*

In order to reveal the strong connection with poly–Bernoulli numbers and the ,,duality" to Callan permutations we give a slightly different version of Ascending–To–Max property of permutations than the one presented in [**42**] and call it *ascending–to–max property*

Again we consider

$$\pi : 0, \pi_1, \pi_2, \ldots, \pi_{n+k}, n+k+1$$

permutations of $\widehat{N}\dot{\cup}\widehat{K}$ with the restriction that its first element is 0 and its last element is $n+k+1$. We call the first $k+1$ elements of the permutation *left–position elements* (0 will be referred to as special left–position element) and the remained $n+1$ elements *right–position elements* ($n+k+1$ is the special right–position element). Consider the following equivalence relation/partition of left-positions: two left-positions, say i and j, are equivalent 'iff any integer v between $\pi(i)$ and $\pi(j)$ occupies a left position in π'.

Similarly one can define an equivalence relation on the right-positions: 'each value between the ones, that occupy the positions, is in a right-position'.

Definition 5. A permutation has *the ascending–to–max* property if in a class of positions the values are in increasing order.

For example consider the case when $n = 4$ and $k = 2$. Let π be the permutation 621534. We extend it with a first 0 and last 7:

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \pi : & (0 & 6 & 2 & 1 & 5 & \mathbf{3} & \mathbf{4} & \mathbf{7}). \end{matrix}$$

The top row contains the positions, and the lower row shows the values that are permuted. The numbers in bold face are the values in right positions. The two left positions 1 and 3 are equivalent, since the corresponding values at these positions are 0 and 2; only the value 3 is between then, and that is at a left position. The two left positions 1 and 2 are not equivalent, since the corresponding values at these positions are 0 and 6; the value 3 is between them, but it is in a right position (in the 6^{th}). The equivalence classes of left positions are $\{1, 3, 4\}$ and $\{2, 5\}$. The corresponding values standing in one equivalence classes are $0, 1, 2$, respectively $5, 6$. The equivalence classes of left positions are $\{6, 7\}$ and $\{8\}$. The permutation is not ascending–to–max permutation: since at the positions $1, 3, 4$ (they form an equivalence class) tha values $0, 1, 2$ are not in increasing order. In the case of 512634 the equivalence classes are the same (we permuted the values within positions forming an equivalence class). It is an ascending–to–max permutation.

We give an example

$$\mathcal{A}_2^{(2)} = \{012\mathbf{345}, 013\mathbf{245}, 013\mathbf{425}, 031\mathbf{245}, 031\mathbf{425}, 014\mathbf{235}, 041\mathbf{235}, 023\mathbf{145},$$
$$023\mathbf{415}, 024\mathbf{135}, 024\mathbf{315}, 042\mathbf{135}, 042\mathbf{315}, 034\mathbf{125}\},$$

where boldface denotes the numbers at right-positions.

The definitions of Callan and ascending–to–max permutations are very similar. By exchanging the roles of position/value we transform one of them into the other. Specially if we consider our permutations as a bijection from $\{0, 1, \ldots, n + k, n + k + 1\}$ to itself then "invert permutation" is a bijection from $\mathcal{C}_n^{(k)}$ to $\mathcal{A}_n^{(k)}$.

Theorem 12. *Let $\mathcal{A}_n^{(k)}$ the set of ascending–to–max of $\{0, 1, 2, \ldots, n+k+1\}$. Then*

$$|\mathcal{A}_n^{(k)}| = \sum_{m=0} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = B_n^{(-k)}.$$

9. Vesztergombi permutations

There is another substantially different permutation class which enumeration leads to poly–Bernoulli numbers.

The problem of enumerating the permutations π such that $\pi(i) \in A_i$ for all $i = 1, \ldots, n$, where $A_1, \ldots A_n$ are given subsets of $\{1, \ldots, n\}$ has been subject to many investigations. It is known to be equivalent to determine the number of perfect matchings of a bipartite graph or to determine the permanent of a binary matrix.

Vesztergombi [**92**] investigated permutations with the most natural restriction: with a bound on the distances between the elements and their images.

Let $f(r, n, k)$ denote the number of permutations $\pi \in S(n+k)$ satisfying

$$-(n+r) < i - \pi(i) < k + r.$$

The problem can be formulated as computing the permanent of the following matrix (see [**67**] for the definition of the permanent of a square matrix):

$$a_{ij} = \begin{cases} 1 & \text{if} \quad -(n+r) < i - j < k + r \\ 0 & \text{otherwise} \end{cases}$$

A third possible interpretation of the basic enumeration question comes from graph theory: The counted permutations can be considered as perfect matchings in the following bipartite graph. Its two colour classes are

$$U = \{u_1, \ldots, u_{n+k}\} \quad \text{and} \quad V = \{v_1, \ldots, v_{n+k}\}.$$

u_i and v_j is connected iff

$$-(n+r) < i - j < k + r.$$

Vesztergombi derived a recursion for $f(r, n, k)$ by expanding rules for the permanent. Solving the differential equations which are obtained from these recursions Vesztergombi obtained the following theorem.

Theorem 13. [**92**]

$$f(r, n, k) = r! \sum_{m=0}^{n} (-1)^{n+m} m! \binom{m+r}{m} (m+r)^k \left\{ {n+1 \atop m+1} \right\}$$

or equivalently:

$$f(r, n, k) = \sum_{m=0}^{n} (-1)^{n+m} (m+r)! (m+r)^k \left\{ {n+1 \atop m+1} \right\}$$

Launois [**62**] working on quantum matrices met the same problem. He realized the importance of the work of Vesztergombi and made the connection to the poly–Bernoulli numbers:

$$f(2, n-1, k-1) = \sum_{m=0}^{n-1} (-1)^{n-1+m} (m+2)! (m+2)^{k-1} \left\{ {n \atop m+1} \right\}.$$

After substituting $m+1 \to m$ and observing that $\left\{ {n \atop 0} \right\} = 0$ we obtain

$$f(2, n-1, k-1) = (-1)^n \sum_{m=0}^{n} (-1)^m m! (m+1)^k \left\{ {n \atop m} \right\} = B_n^{(-k)},$$

where $f(2, n-1, k-1)$ counts the permutations with the restriction $-n \le i - \pi(i) \le k$.

Next we prove the following theorem using the method of Lovász ([**64**], Exercise 4.31.) for this more general case. The problem presented there is just a small modification of the following theorem.

Theorem 14. *([**92**],[**62**]) Let $\mathcal{V}_n^{(k)}$ the set of permutations π of $[n+k]$ such that $-n \le i - \pi(i) \le k$ for all i in $[n+k]$.*

$$|\mathcal{V}_n^{(k)}| = \sum_{m=0}^{\min(n,k)} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = B_n^{(-k)}.$$

$|\mathcal{V}_n^{(k)}|$ is the permanent of the $(n+k) \times (n+k)$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if} \quad -n \le i - j \le k, \quad i = 1, \dots, n+k \\ 0 & \text{otherwise.} \end{cases}$$

.

$$A = \begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & \dots & 1 & 1 & 1 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

The permanent of A (denoted by per$A$) counts the number of expansion terms of the matrix A which do not contain a 0 term.

The matrix A is built up of 4 blocks:

$$A = \begin{bmatrix} J_{k,n} & B_k \\ B_n & J_{n,k} \end{bmatrix}$$

where $J_{n,k}$ denotes the $n \times k$ matrix with entries 1, and B_k and B_n are the upper resp. lower triangular matrices with entries 1. and $B_{ij}^n = 1$ iff $i \le j$.

$$B_k = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \quad B_n = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

For a term in the expansion of the permanent we have to select exactly one 1 from each row and each column, so if a term contains m 1's from the upper left block $J_{k,n}$, then it contains $k-m$ 1's from B_k; and then it contains m 1's from the lower right block and finally $n-m$ 1's from B_n. In order to establish the number of ways of selecting $k-m$, resp. $n-m$ 1's from the triangular matrices we turn the problem to the language of graphs and modify the proof of Exercise 4.31. in [**64**] to our problem.

Lemma 15. *Let G_n be the following bipartite graph:*

$$V = \{u_1, \dots, u_n; v_1, \dots, v_n\} \quad \textit{and} \quad E(G) = \{u_i, v_j : i \le j\}.$$

Then the number of $n-m$ element matchings in G is

$$\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}.$$

Proof. Let $\widehat{V} = \{v_0, v_1, \dots, v_n; u_1, \dots, u_n, u_{n+1}\}$ and $\widehat{E} = E$. (So we added two isolated vertices to G). Let $\widehat{G}_n$ be the graph that we obtain this way.

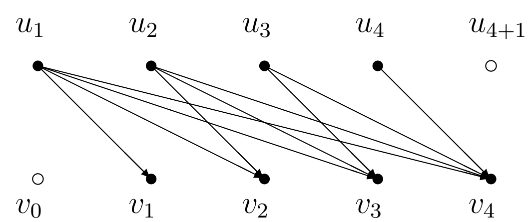


Figure 4. The graph $\widehat{G}_3$

Let M be a matching of size $n-m$ and $e = v_i u_j \in M$. We define a $\tilde{e}$ edge in the T transitive tournament on $\{1, 2, \dots, n\}$:

$$\tilde{e} = \overrightarrow{(i+1)j}.$$

Let $\tilde{M}$ be the $n-m$ edge of T we obtain this way.

It is obvious that $\tilde{M}$ is acyclic and each vertex has out–degree at most 1. If we consider isolated vertices (resp. to $\tilde{M}$) as path of length 0 then we can consider $\tilde{M}$ as a path system in T covering all vertices of T. Since $\tilde{M}$ is an edge set of size $n-m$ on $n+1$ vertices, $\tilde{M}$ is a system of $m+1$ paths. I. e. $\tilde{M}$ defines a partition of $[n+1]$ into $[m+1]$ classes. The correspondence above can be reversed, so it describes a bijection between matchings of size $n-m$ and partitions with $n-m$ classes. This bijection proves the lemma. $\square$

According to the lemma there are $\left\{ {n+1 \atop m+1} \right\}$ ways to select $n-m$ 1's from B_n, there are $\left\{ {k+1 \atop m+1} \right\}$ ways to select $k-m$ 1's from B_k and obviously $(m!)^2$ to select m and m 1's from $J_{k,n}$ resp. $J_{n,k}$. Hence the theorem holds.

10. A new poly-Bernoulli family

Let M be a 0-1 matrix. We say that three 1s in M form a Γ configuration iff two of them are in the same row (one, let us say a, precedes the other) and the third is under a. I.e. the three 1s form the upper left, upper right and lower left elements of a submatrix of size 2×2. So we do not have any condition on the lower right element of the submatrix of size 2×2, containing the Γ. We will consider matrices without Γ configuration. Hence a Γ–free matrix doesn't contain any submatrix from the following set:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

Let $\mathcal{G}_n^{(k)}$ denote the set of all 0-1 matrices of size $n \times k$ without Γ.

The following theorem is our main theorem.

Theorem 16.

$$|\mathcal{G}_n^{(k)}| = B_n^{(-k)}.$$

The rest of the section is devoted to the combinatorial proof of this statement.

The obvious way to prove our claim is to give a bijection to one of the previous sets, where the size is known to be $B_n^{(-k)}$. The obvious candidate is $\mathcal{L}_n^{(k)}$. Γ-free matrices were considered from the point of extremal combinatorics (see [**32**]). It is known that Γ-free matrices of size $n \times k$ contain at most $n+k-1$ many 1s. Among Brewbaker's lonesum matrices (in contrast) there are some with many 1s (for example the all-1 matrix) and there are others with few 1s. We do not know straight, simple bijection between lonesum matrices and matrices with no Γ. Instead, we follow the obvious scheme: we code Γ-free matrices with two partitions and two orders. From this and from the previous bijections one can construct a direct bijection between the two sets of matrices but that is not appealing.

Proof. Let M be a 0-1 matrix of size $n \times k$. We say that a position/element has *height* $n-i$ iff it is in the i^{th} row. The *top*-1 of a column is its 1 element of maximal height. The *height of a column* is the height of its top-1 or 0, whenever it is a 0 column.

Let M be a matrix without Γ configuration. Let $\widehat{M}$ be the extension of it with an all 0s column and row. (We have defined the *height of all*-0 *columns* to be 0. In $\widehat{M}$ non-0 columns have 0 at the bottom, hence their heights are at least 1.) 'Having the same height' is an equivalence relation on the set of columns in $\widehat{M}$. The class of the additional column is the set of 0 columns (that is not empty since we work with the extended matrix). We call the class of the additional column 'the special class'. Its elements are the special columns. So special column means 'all-0 column'. The additional column in $\widehat{M}$ ensures that we have this special class. The other classes are the ordinary classes. Let m be the number of the ordinary classes. These m classes partition the set of non-0 columns. The total number of equivalence classes is $m + 1$.

In order to clarify the details after the formal description we explain the steps on a specific example. $\diamondsuit$ denote the end of example, when we return to the abstract discussion,

Example. M is a Γ-free matrix:

$$M = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \qquad \widehat{M} = \left(\begin{array}{cccccccc|c} 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 1&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ \hline 0&0&0&0&0&0&0&0&0 \end{array}\right)$$

$\widehat{M}$ is "basically the same" as M. It only contains an additional all-0 column (the last one), and an additional all-0 row (the last one). In $\widehat{M}$ each column has a height. The height depends on the position of the top-1 of the considered column (it counts how many positions are under it). The all-0 column has height 0. We used the symbol 1^t for the top-1s and marked the height of the columns at the upper border of our matrix $\widehat{M}$:

$$\begin{array}{c} \begin{matrix} 4&6&4&3&0&3&1&5&0 \end{matrix} \\ \widehat{M} = \begin{pmatrix} 0&1^t&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1^t&0 \\ 1^t&0&1^t&0&0&0&0&0&0 \\ 0&1&0&1^t&0&1^t&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1^t&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, \end{array}$$

'Having the same height' is an equivalence relation among the columns. In our example there are six different heights considering all the columns: $0, 1, 3, 4, 5, 6$. Two columns have height 0 (they are the two all-0 columns). One of them is the additional column of $\widehat{M}$, the other was present in M. They are the special columns, forming the special class of our equivalence relation on columns. The other five heights define five ordinary classes. One of these classes is formed by the first and third column, they are the columns with height 4. ♢

Take $\mathcal{C}$, any non-special class of columns (the columns in $\mathcal{C}$ are ordered as the indices order the whole set of columns). Since our matrix does not contain Γ all columns but the last one has only one 1 (that is necessarily the top-1) of the same height. We say that the last elements/columns of non-special classes are *important columns*. Important columns in $\widehat{M}$ form a submatrix M_0 of size $(n+1) \times m$.

In M_0 the top-1s are called *important elements*. In each row without top-1 the *leading* 1 (the 1 with minimal column index) is also called *important* 1. So in all non-0 rows of M_0 there is exactly one important 1.

Each row has an 'indentation': the position of the important 1, i.e. last top-1 if the row contains a top-1, otherwise the position of the first 1 (or 0 if the row is all 0s). The row indentations determine a partition of the set of rows.

The two partitions have the same number of parts, namely, $m+1$ where m was introduced when describing the column partition.

The last top-1s are in different rows and columns, hence determine an $m \times m$ submatrix which becomes a permutation matrix if all entries except the last top-1s are zeroed out. This permutation matrix determines an identification of the ordinary column classes and ordinary row classes.

Example. M_0 contains the last columns of the ordinary equivalence classes. In our example it has 5 columns (the upper border of our example we see the common height of the column class, and the original index of each row).

There are two rows without top-1. The last row is all-0, the other in not all-0. Its leading 1 is in bold face. We also marked (at the left border of M_0) the indentations of rows. The indentation/label of an all-0 row is 0:

$$M_0 = \begin{array}{c} \\ \\ 1 \\ 5 \\ 2 \\ 3 \\ 2 \\ 4 \\ 0 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} \scriptstyle 6 & \scriptstyle 4 & \scriptstyle 3 & \scriptstyle 1 & \scriptstyle 5 \\ \scriptstyle 2^{\text{nd}} & \scriptstyle 3^{\text{rd}} & \scriptstyle 6^{\text{th}} & \scriptstyle 7^{\text{th}} & \scriptstyle 8^{\text{th}} \end{array} \\ \begin{pmatrix} 1^t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1^t \\ 0 & 1^t & 0 & 0 & 0 \\ 1 & 0 & 1^t & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1^t & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Note that since each column has a top-1 we have $1, 2, 3, 4, 5$ as labels (all-0 columns form the special column class that has no representative in M_0). The last row of M_0 is an all-0 row, hence we have the 0 label too.

The top-1s define an identification of ordinary row and column classes. We use the letters a, b, c, d, e for the identified row/column classes. s marks the special rows (in our case there is only one special row, the last row):

$$M_0 = \begin{array}{c} \\ a \\ e \\ b \\ c \\ b \\ d \\ s \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} \overset{a}{2^{\text{nd}}} & \overset{b}{3^{\text{rd}}} & \overset{c}{6^{\text{th}}} & \overset{d}{7^{\text{th}}} & \overset{e}{8^{\text{th}}} \end{array} \\ \begin{pmatrix} 1^t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1^t \\ 0 & 1^t & 0 & 0 & 0 \\ 1 & 0 & 1^t & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1^t & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

$\diamondsuit$

A partition of columns into $m+1$ classes, and partition of rows into $m+1$ classes, and a bijection between the non-special row- and column-classes — after fixing m — leaves

$$m!\left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\}$$

possibilities. This information (knowing the two partitions and the correspondence) codes a big part of matrix $\widehat{M}$:

We know that the columns and rows of the special classes are all 0s. A non-special column class $\mathcal{C}$ has a corresponding class of rows. The top row of the corresponding row class gives us the common height of the columns in $\mathcal{C}$. So we know each non-important columns (they have only one 1, defining its known height). We narrowed the unknown 1s of M into the non-0 rows of M_0. Easy to check that $\widehat{M}$ contains Γ iff M_0 contains one.

Example. Let us consider our example. $m = 5$ and the row/column partitions are marked at the left and upper border of our matrix. We use a, b, c, d, e, s as names for the classes in both cases (hence the classes of the two partitions are identified). s denotes the two special classes (the class of the last row and last column).

$$\widehat{M} = \begin{array}{c} \\ a \\ e \\ b \\ c \\ b \\ d \\ s \end{array} \!\!\begin{array}{c} \begin{array}{ccccccccc} b & a & b & c & s & c & d & e & s \end{array} \\ \begin{pmatrix} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{pmatrix} \end{array}$$

We can recover a large portion of $\widehat{M}$ from this information. The class of the last column, and the class of the last row are the two special classes (named by s). We must have all 0s in these rows/columns.

The columns of M_0 are the last columns of the ordinary classes. For example the top-1s in the column class a can be decoded from the rows belonging to class a: The highest row with label a marks the common row of top-1s in columns labelled by a. This way we can determine the common heights of the ordinary column classes, hence recover the top-1s. Then we know all elements above a top-1 must have value 0. All columns with label a but the last one contains only its top-1 as non-0 element.

In our example we sum up the information gained so far (the top border contains the recovered heights and labels for the columns from M_0):

$$\widehat{M} = \begin{array}{c} \\ b \\ f \\ a \\ c \\ a \\ e \\ d \end{array} \begin{array}{c} \begin{array}{ccccccccc} 4 & \overset{6}{M_0} & \overset{4}{M_0} & 3 & 0 & \overset{3}{M_0} & \overset{1}{M_0} & \overset{5}{M_0} & 0 \end{array} \\ \left(\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & & 1 & 0 & 0 & 0 & 0 & & 0 \\ 0 & & & 1 & 0 & 1 & 0 & & 0 \\ 0 & & & 0 & 0 & & 0 & & 0 \\ 0 & & & 0 & 0 & & 1 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Knowing the top-1s enable us to recover the indentations (relative to M_0) belonging to the ordinary row classes. If there is a row without a top-1, then from its row label we know the position of its first 1 (hence we know that in that row at previous positions we have 0s):

$$\widehat{M} = \begin{array}{c} \\ 1/M_0 \\ 5/M_0 \\ 2/M_0 \\ 3/M_0 \\ 2/M_0 \\ 4/M_0 \\ 0/M_0 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 4 & \overset{6}{M_0} & \overset{4}{M_0} & 3 & 0 & \overset{3}{M_0} & \overset{1}{M_0} & \overset{5}{M_0} & 0 \end{array} \\ \left(\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & & 1 & 0 & 0 & 0 & 0 & & 0 \\ 0 & & & 1 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & 0 & & 0 & & 0 \\ 0 & & & 0 & 0 & & 1 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$\diamondsuit$

Note that there are many positions where we do not know the elements of our matrix (all are located in M_0). Also when counting the possibilities we have a missing $m!$ factor. The rest of the proof shows that filling in the missing elements (resulting a Γ-free matrix) can be done $m!$ many ways.

Now on we concentrate on M_0 (that is where the unknown elements are). The positions of important ones are known. In each column of M_0 there is a lowest important 1. We call them *crucial* 1s. (Specially crucial 1s are important 1s too.) We have m many crucial 1s, one is in each column of M_0. A 1 in M_0 that is non-important is called *hiding* 1.

Lemma 17. *Consider a hiding* 1 *in* M_0. *Then exactly one of the following two possibilities holds:*

(1) *there is a crucial* 1 *above it and a top-*1 *to the right of it,*
(2) *there is a crucial* 1 *on its left side (and of course a top-*1 *above it).*

Example. The following figure exhibits the two options: $1^{(h1)}$, $1^{(h2)}$ are two hiding 1s, corresponding case (1) and case (2) respectively. The two hiding 1s "share" the crucial 1 in the lemma.

$$\begin{pmatrix} & \vdots & & & & \\ & & \dots & 1^t & & \\ & \vdots & & \uparrow & & \\ \dots & 1^{(c)} & \leftarrow & 1^{(h2)} & & \\ & \uparrow & & \vdots & & \\ & 1^{(h1)} & \rightarrow & & \rightarrow & 1^t \\ & \vdots & & & & \end{pmatrix}$$

$\diamondsuit$

PROOF. Let h be a hiding 1 in M_0.

First, assume that the row of h does not contain a top-1. Then the first 1 in this row (f) is an important 1 (hence it differs from h). Since the matrix is Γ-free, we cannot have a 1 under f, i.e. f is a crucial 1. h is not important, so it is not a top-1. The top-1 in its column must be above it. We obtained that case (2) holds.

Second, assume that the row of h contains a top-1, t. If t is on the left of h then the forbidden Γ ensures that under t there is no other 1. Hence t is crucial and case (2) holds again. If t is on the right of h then the forbidden Γ ensures that under h there is no other 1. Hence the lowest important 1 in the column of h (a crucial 1) is above of it. Case (1) holds.

(1) and (2) cases are exclusive since if both are satisfied then h has a crucial 1 on its left and a top-1 on its right. That is impossible since the 1s in a row of a top-1 are not even important. $\square$

Let h be a hiding 1. There must be a unique crucial 1 corresponding to it: If h satisfies case (1), then it is the crucial one above it. If h satisfies case (2), then it is the crucial one on the left side of it. In this case we say that this crucial c is *responsible* for h.

Take a crucial 1 in M_0, that we call c. For any top-1, t that comes in a later column and it is higher than c the position in the row of c under t we call *questionable*. Also for any top-1, t that comes in a later column and it is lower than c the position in the column of c before t we call *questionable*. In M_0 there are m many crucial 1. If c is in the i^{th} column, then there are $m-i$ column that comes later and each defines one questionable position.

The meaning of the lemma is that each hiding 1 must be in a questionable position.

Example. We continue our previous example (but only M_0 is followed on).

We added an index c to crucial 1s. (All the important 1s are identified so we are able to locate these elements.)

$$\widehat{M} = \begin{pmatrix} 1^{t(c)} & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1^{t(c)} \\ & 1^{t} & 0 & 0 & \\ & & 1^{t(c)} & 0 & \\ 0 & 1^{(c)} & & 0 & \\ & & & 1^{t(c)} & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

According to our argument, for the crucial 1 in the first column there are four top-1s in the later columns and there are four questionable positions corresponding to them. We mark them as $?_1$. Similarly, three questionable positions belongs to the crucial 1 in the second column, marked as $?_2$. (If there is hiding 1 in one of these positions then the crucial 1 of the second column would be responsible to it). We put ? to each questionable position and add an index marking the column of the crucial 1 that is connected to it:

$$M_0 = \begin{pmatrix} 1^{t(c)} & 0 & 0 & 0 & 0 \\ ?_1 & 0 & 0 & 0 & 1^{t(c)} \\ ?_1 & 1^{t} & 0 & 0 & \\ ?_1 & & 1^{t(c)} & 0 & ?_3 \\ 0 & 1^{(c)} & ?_2 & 0 & ?_2 \\ ?_1 & ?_2 & ?_3 & 1^{t(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that there are positions that are not questionable. The lemma says there can not be a hiding 1. Indeed, a 1 at these positions would create a Γ. $\diamondsuit$

First rephrase our lemma:

Corollary 18. *All hiding* 1*s are in questionable positions.*

It is obvious that we have $(m-1)+(m-2)+\ldots+2+1$ many questionable positions (to the crucial 1 in the i^{th} column there are $m-i$ many questionable position is assigned).

Easy to check that if we put the important 1s into M_0 and add a new 1 into a questionable position then we won't create a Γ configuration. The problem is that the different questionable positions are not independent.

Lemma 19. *There are $m!$ ways to fill the questionable positions with 0s and 1s without forming a Γ.*

PROOF. Let c be a crucial 1. We divide the set of questionable positions that corresponds to c, depending their positions relative to c into two parts: Let R_c be the set of questionable positions in the row of c, that is right from c. Let D_c be the set of questionable positions in the column of c, that is down from c.

The following two observation is immediate:

(i) At most one of R_c and D_c contains a 1.
(ii) If D_c contains a 1 (hence R_c is empty), then it contains only one 1.

Indeed, if the two claims are not true then we can easily recognize a Γ.

For each crucial c describe the following 'piece of information': I_1: the position of the first 1 in R_c or I_2: the position of the only one 1 in D_c (this informs us that R_c contains only 0s) or I_3: say "all the positions of $R_c \cup D_c$ contain 0".

If c comes from the first column of M_0, then we have m many outcomes for this piece of information. $m-1$ many of these are such that one position of a 1 is revealed (the first 1 in R_c or the the only one 1 in D_c). Placing a 1 there doesn't harm the Γ-free property of our matrix. One possible outcome of the information is the one that reveals that there is no 1 in $R_c \cup D_c$.

Example. In our example let c be the crucial 1 of the first column, i.e. the first element of the first row:

$$M_0 = \begin{pmatrix} c = 1^{t(c)} & 0 & 0 & 0 & 0 \\ ?_1 & 0 & 0 & 0 & 1^{t(c)} \\ ?_1 & 1^t & 0 & 0 & 0 \\ ?_1 & 0 & 1^{t(c)} & 0 & ?_3 \\ 0 & 1^{(c)} & ?_2 & 0 & ?_2 \\ ?_1 & ?_2 & ?_3 & 1^{t(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

R_c is empty, D_c contains four positions from the first column (2^{nd}, 3^{rd}, 4^{th}, 6^{th}). So when we reveal the above mentioned information about c then we have the following 5 possible outcomes:

$I_2(2)$: "The only 1 under c is in the second row and there is no 1 in the row of c."

$I_2(3)$: "The only 1 under c is in the third row and there is no 1 in the row of c."

$I_2(4)$: "The only 1 under c is in the fourth row and there is no 1 in the row of c."
$I_2(6)$: "The only 1 under c is in the sixth row and there is no 1 in the row of c."
I_3: "There is no 1 under and after c."

Let us assume that we get the the third possibility ($I_2(4)$) as the additional information. Then we can continue filling the missing elements of M_0:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1^{t(c)} \\ 0 & 1^t & 0 & 0 & 0 \\ 1 & 0 & 1^{t(c)} & 0 & ?_3 \\ 0 & c = 1^{(c)} & ?_2 & 0 & ?_2 \\ 0 & ?_2 & ?_3 & 1^{t(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now let c the crucial 1 in the second column. R_c contains two positions (3^{rd} and 5^{th} column), D_c contains one position from the column of the actual c (the one in the 6^{th} row). So when we reveal the above mentioned information about c then we have the following 4 possibilities:

$I_1(3)$: "The first 1 after c is in the third column and there is no 1 in D_c."
$I_1(5)$: "The first 1 after c is in the fifth column and there is no 1 in D_c."
$I_2(6)$: "The only 1 under c is in the sixth row and there is no 1 in the row of c."
I_3: "There is no 1 under and after c."

Let us assume that we get the the first possibility ($I_1(3)$) as an additional information. The hidden 1 that is revealed is under the crucial 1 (c') of its column. The Γ-free property of M (and hence M_0) guarantees that cannot be a hiding 1 after c'. Again we summarize the information gained:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1^{t(c)} \\ 0 & 1^t & 0 & 0 & 0 \\ 1 & 0 & c' = 1^{t(c)} & 0 & 0 \\ 0 & 1^{(c)} & 1 & 0 & ?_2 \\ 0 & 0 & ?_3 & 1^{t(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the third column the old crucial 1 (c') will be replaced by the 1, (c) revealed by the previous information.

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1^{t(c)} \\ 0 & 1^t & 0 & 0 & 0 \\ 1 & 0 & c' = 1^t & 0 & 0 \\ 0 & 1^{(c)} & c = 1^{(c)} & 0 & ?_3 \\ 0 & 0 & ?_3 & 1^{t(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\diamondsuit$If we get I_2 or I_3 then we know all the elements at the questionable positions corresponding to c. In this case we can inductively continue and finish the description of M. If the information, we obtain is I_1 then our knowledge about the 1s at the questionable positions corresponding to c is not complete. But we can deduce many additional information.

Assume that I_1 says that on the right of c the first 1 in questionable position is in the j^{th} column. Let $\widetilde{c}_j$ be the position of this 1. The position c_j is above of it. We know that R_i doesn't contain a 1 (indeed, that would form a Γ with the 1s at c_j and at $\widetilde{c}_j$). For similar reasons also we cannot have a 1 at a questionable position between c_j and $\widetilde{c}_j$.

This knowledge guarantees that we can substitute c_j with $\widetilde{c}_j$ ($\widetilde{c}_j$ will be a crucial 1 substituting c_j). The corresponding questionable positions will be the questionable positions that are down and right from it. We still encounter all the hiding ones (there must be at the questionable positions corresponding to the crucial 1s, we didn't confronted yet). So we can induct.

The above argument proves that any element of

$$\{1, 2, \dots, m\} \times \{1, 2, \dots, m-1\} \times \{1, 2\} \times \{1\}$$

codes the outcome of the information revealing process, hence a Γ-free completion of our previous knowledge. The i^{th} component of the code says that in the i^{th} column of M_0 which information on the actual crucial 1 is true. Our previous argument just describe how to do the first few steps of the decoding and how to recursively continue it.

$\square$

The lemma finishes the enumeration of Γ-free 0-1 matrices of size $n \times k$. Also finishes a description of a constructive bijection from $\mathcal{G}_n^{(k)}$ to the obvious poly-Bernoulli set. Our main theorem is proven a bijective way.

11. Combinatorial proofs

The combinatorial interpretations provide proofs of several properties of the poly–Bernoulli numbers with negative indices. Both the symmetry of the closed formula and the symmetry of the generating function exhibit the symmetry of the poly–Bernoulli numbers, but it is obvious from any of the combinatorial definitions.

Theorem 20 ([**47**]). *For any $n, k \geq 0$ we have*

$$B_n^{(-k)} = B_k^{(-n)}.$$

The recursion was originally proven by Kaneko [**2**],[**38**] and is easily seen using the new combinatorial interpretation, the Γ–free matrices. Our next result is a new, combinatorial proof of the following theorem.

Theorem 21. [**2**]

$$B_n^{(-k)} = B_n^{(-(k-1))} + \sum_{i=1}^{n} \binom{n}{i} B_{n-(i-1)}^{(-(k-1))}.$$

Proof. Our main theorem gives that $B_n^{(-k)}$ counts the Γ-free matrices of size $n \times k$.

Each row of a Γ-free matrix

A. starts with a 0, or
B. starts with a 1, followed only by 0s, or
C. starts with a 1, and contains at least one more 1.

Let j denote the number of rows of type B/C.

If $j = 0$, then the first column is all-0 column, and it has $B_n^{(-(k-1))}$ many extensions as Γ-free matrix.

If $j \geq 1$, then we must choose the j many rows of type B/C. Our decision describes the first column of our matrix. The first $j - 1$ many chosen rows cannot contain any other 1, since a Γ would appear. I.e. they are type B, and completely described.

The further elements (a submatrix of size $(n-j+1) \times (k-1)$) can be filled with an arbitrary Γ-free matrix. The recursion is proven. □

We can state the theorem (without a reference to the main theorem) as a recursion for $|\mathcal{G}_n^{(k)}|$. Since the same recursion is known for $B_n^{(-k)}$, an easy induction proves the main theorem. Our first proof, the main part of this paper is purely combinatorial and explains a previously known recursion without algebraic manipulations of generating functions.

We proof the next result also combinatorially.

Theorem 22.

$$\sum_{\substack{n,k \in \mathbb{N} \\ n+k=N}} (-1)^n B_n^{(-k)} = 0.$$

Proof. In order to proof this theorem we consider Callan's description of poly–Bernoulli numbers. We consider Callan permutations of N objects (the extended base set has size $N+2$). We underline that to speak about Callan permutations we must divide the N objects into left– and right–value category. For this we need to write N as a term sum: $n+k$.

For technical reasons we change the base set of our permutations. The extended left values remain $0, 1, 2, \ldots, n$, the extended right values will be $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}, \mathbf{k+1}$. We note that the calligraphic distinction between the left and right values allow us to use any $n+1$ numbers for the left values and any $k+1$ numbers for the right values.

The combinatorial content of the claim is that if we consider Callan permutations of N objects (with all possible $n+k$ partitions), then those with even many left values have the same number as those with odd many left values.

$B_n^{(-k)}$ is the size of $\mathcal{C}_n^{(k)}$. We divide it into two subsets according to the type of the element following the leading 0. Let $\mathcal{C}_n^k(l)$ be the set of those elements from $\mathcal{C}_n^{(k)}$, where the leading 0 is followed by a left value element. Let $\mathcal{C}_n^k(r)$ be the set of those elements from $\mathcal{C}_n^{(k)}$, where the leading 0 is followed by a right value element.

We will describe a

$$\varphi : \mathcal{C}_n^k(l) \to \mathcal{C}_{n-1}^{(k+1)}(r)$$

bijection. Hence we will have a $\varphi : \mathcal{C}_n^k(r) \to \mathcal{C}_{n+1}^{(k-1)}(l)$ bijection too. Our map reverses the parity of the number of left values and completes the proof.

The bijection goes as follows: Take a permutation from $\mathcal{C}_n^k(l)$. Find '$1'$' in the permutation. It follows the leading 0 or it will be the first element of a block of left values that is preceded by a block of right values, say R. In the first case we substitute 1 by $\mathbf{0}$. In the second case we also substitute 1 by $\mathbf{0}$ but additionally we move the R block right after the leading 0.

We warn the reader that the image permutations have extended left values $0, 2, \ldots, n$ and extended right values $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{k+1}$. This change does not effect the essence.

Next we list a case analysis with examples that proves that φ is a bijection.

A. 1 follows the leading 0.

A1. If the first left–value block of $\pi \in \mathcal{C}_n^{(k)}(l)$ is $\{0, 1\}$, following by a right–value block, then in $\varphi(\pi)$ the left–value block will be $\{0\}$ and $\mathbf{0}$ is added to the next right–value block as a first (least) element.

$$0\,1 - \mathbf{3}\,\mathbf{4} - 4 - \mathbf{1}\,\mathbf{5} - 2\,3 - \mathbf{2}\,\mathbf{6} \longrightarrow 0 - \mathbf{0}\,\mathbf{3}\,\mathbf{4} - 4 - \mathbf{1}\,\mathbf{5} - 2\,3 - \mathbf{2}\,\mathbf{6}$$

A2. If the first left–value block of $\pi \in \mathcal{C}_n^{(k)}(l)$ contains not only $\{0, 1\}$, after replacing 1 by $\mathbf{0}$ the right value $\mathbf{0}$ will cut this first left–value block into two parts by forming a right–value block with a single element.

$$0\,1\,3 - \mathbf{1\,2\,5} - 2 - \mathbf{3} - 4 - \mathbf{4\,6} \longrightarrow 0 - \mathbf{0} - 3 - \mathbf{1\,2\,5} - 2 - \mathbf{3} - 4 - \mathbf{4\,6}$$

B. If the element 1 in π is not in the first left–value block (not in the same block as 0), then there is necessarily a right–value block R before it. In this case the replacing of 1 by **0** break the Callan condition, since two right–value blocks are merged to one block by **0**, but **0**, the least right–value element should have been at the first position of a right–value block. We consider the two different cases in details and see that after our additional moving step we receive in both cases a valid Callan permutation of the set $\mathcal{C}_{n-1}^{k+1}(r)$.

B1. If the left–value block of the element 1 contains only this single element the replacing it by the element **0** create a right–value block with the structure: a sequence of increasing right values followed by **0**, followed again by a sequence of increasing right values. But we moved the elements before **0** to another place in the permutation and the elements followed by **0** are all greater than **0**.

$$0\,3 - \mathbf{2\,4\,5} - 2\,4 - \mathbf{1\,3} - 1 - \mathbf{6} \longrightarrow 0 - \mathbf{1\,3} - 3 - \mathbf{2\,4\,5} - 2\,4 - \mathbf{0\,6}$$

B2. If the block of π that contains the element 1 contains other left–value elements than in $\varphi(\pi)$ the rest of this left–value block is ,,separated" from the left–value block before it by the one–element right–value block $\{\mathbf{0}\}$.

$$0\,3 - \mathbf{1\,5} - 1\,4 - \mathbf{2} - 2 - \mathbf{3\,4\,6} \longrightarrow 0 - \mathbf{1\,5} - 3 - \mathbf{0} - 4 - \mathbf{2} - 2 - \mathbf{3\,4\,6}$$

The inverse of our map can be easily constructed. It must be based on **1**. The details are left to the reader.

□

12. Related sequences

In this section we collected a few results and notes that are generalizations or analogous of poly–Bernoulli numbers and have some combinatorial aspects. The questions, open problems appeared in these line of research are natural directions to extend our results and methods.

12.1. q-ary lonesum matrices. One of the most natural question we can ask after investigating lonesum binary matrices is: are the enumeration results in general matrices determined by their column sums and row sums also so elegant?

Kim, Krotov and Lee [**50**] considered this problem and studied q–ary matrices, matrices with entries from the set $\{0, 1, \ldots, q-1\}$. They defined *strong* *lonesum matrix* as matrices that can be uniquely reconstructed from its row and column sums.

Rather surprisingly strong q–ary lonesum matrices are determined by a set of allowed 2×2 submatrices.

Theorem 23 ([**50**], Theorem 3.2)**.** *A q–ary matrix is a lonesum matrix if and only if each of its 2×2 submatrices is equivalent to one of*

$$\begin{pmatrix} q-1 & q-1 \\ c & d \end{pmatrix}, \quad \begin{pmatrix} q-1 & b \\ q-1 & d \end{pmatrix}, \quad \begin{pmatrix} q-1 & b \\ c & 0 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$

where $\{a, b, c, d\} \subseteq \{0, 1, \ldots, q-2\}$*.*

Using this characterization theorem and generalizing the technique for binary lonesum matrices (considering a stair matrix as an ordered pair of partitions and then permuting rows and columns) Kim, Krotov and Lee established the number of q–ary strong lonesum matrices:

Theorem 24 ([**50**], Theorem 3.4.)**.** *The number of q-ary lonesum $n \times k$ matrices is*

$$1 \quad + \quad \sum_{j=1}^{\min(n,k)} \sum_{\substack{(n_0,n_1,\ldots,n_j)\in S_n^j \\ (k_0,k_1,\ldots,k_j)\in S_k^j}} \binom{n}{n_0,\ldots,n_j}\binom{k}{k_0,\ldots,k_j} \prod_{i=1}^{j} f_q(n_i, k_{j+1-i}),$$

where $f_q(r,s)$ is defined as

$$f_q(r,s) = 1 + (q-2)rs + r((q-1)^s - (q-2)s - 1) + s((q-1)^r - (q-2)r - 1)$$

and the summation runs over the sequences

$$S_l^j = \left\{ (l_0, l_1, \ldots, l_j) \in \mathcal{Z}^{j+1} \,\middle|\, \sum_{i=0}^{j} l_i = l,\ l_0 \geq 0,\ l_i \geq 1 (i \in \{1, 2, \ldots, j\}) \right\}$$

This leads to one of the possible generalization of poly–Bernoulli numbers. Let the $B_n^{(-k)}(q)$ be the sequence of numbers defined by the generating function:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_n^{(-k)}(q)\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{1 - F_q(x,y)}$$

where

$$\begin{aligned} F_q(x,y) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} f_q(r,s)\frac{x^r}{r!}\frac{y^s}{s!} \\ &= 1 - e^x - e^y + (1 - x - y - (q-2)xy + xe^{(q-2)y} + ye^{(q-2)x})e^{x+y}. \end{aligned}$$

Though this generalization is natural in the case of lonesum matrices it is not clear and requires further investigations whether other interpretations of poly–Bernoulli numbers are in connection with q–ary strong lonesum matrices.

The special case, when the matrices are symmetric, is substantially easier. For instance to construct a symmetric binary lonesum matrix, we need only one ordered partition instead of a pair of ordered partitions.

Theorem 25 ([**50**], Theorem 3.5.)**.** *Let $B_n(q)$ the number of q–ary symmetric $n \times n$ lonesum matrices. Then*

$$B_n(q) = 1 + \sum_{(n_0,n_1,\ldots,n_j)\in S_n^j} \binom{n}{n_0, n_1, \ldots, n_j} \left(\prod_{i=1}^{\lfloor \frac{j}{2} \rfloor} f_q(n_{2i-1}, n_{2i}) \right) \left(1 + (q-2)(n - \sum_{i=0}^{2\lfloor \frac{j}{2} \rfloor} n_i) \right).$$

n	$B_n(q)$	$B_n(2)$	$B_n(3)$	$B_n(4)$
1	q	2	3	4
2	$2q^2 + 2q - 6$	6	18	34
3	$9q^3 - 12q^2 + 12q - 22$	26	149	410
4	$16q^4 + 72q^3 - 312q^2 + 392q - 218$	150	1390	5062
5	$25q^5 + 160q^4 + 400q^3 - 3180q^2 + 4920q - 2598$	1082	13377	58362

TABLE 3. The number of symmetric lonesum q–ary matrices

The sequence $B_n(2)$ is well known [**85**], for instance it is the number of necklaces of partitions of $n+2$ labelled beads. But there is no combinatorial explanation for the sequences $B_n(q)$ for $q \geq 3$. See Table 4 for some values of $B_n(3)$ and $B_n(4)$. We hope that our list of interpretations of poly–Bernoulli numbers helps to answer the question the authors asked.

Open Problem 1. *Is there any combinatorial objects that provide a (another) combinatorial meaning of the number sequences $B_n(q)$, $q \geq 3$?*

12.2. Multi–poly–Bernoulli numbers. Hamahata and Masubuchi [**38, 39**] defined a generalized version of the poly–logarithm function $Li_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k}$ and introduced multi–poly–Bernoulli numbers by a generating function. For integers $k_1, k_2, \ldots, k_r$ let

$$Li_{k_1,k_2,\ldots,k_r}(z) = \sum_{\substack{m_1,m_2,\ldots,m_r\in\mathbb{Z} \\ 0<m_1<m_2<\cdots<m_r}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \frac{z^{m_r}}{m_r^{k_r}}.$$

Definition 6. *Multi–poly–Bernoulli numbers* $B_n^{k_1,k_2,\ldots,k_r}$ $(n = 0, 1, 2, \ldots)$ are defined for each integer $k_1, k_2, \ldots, k_r$ by the generating function

$$\frac{Li_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!},$$

Theorem 26 ([**39**], Theorem 7.)**.**

$$B_n^{(k_1,k_2,\ldots,k_r)} = (-1)^n \sum_{m_r=r}^{n+r} \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} (-1)^{m_r-r} \frac{(m_r-r)!}{m_r^{k_r}} \left\{ {n \atop m_r - r} \right\}$$

We recall one of the recursions that Hamahata and Masubuchi derived, since it can be viewed as a generalization of the recurrence we proved combinatorially using our new interpretation of poly–Bernoulli numbers, the Γ–free matrices.

Theorem 27 ([**38**],Theorem 6.)**.** *For $k_r \neq 1$ and $n \geq 1$*

$$B_n^{(k_1,\ldots,k_{r-1},k_r-1)} = (n+r)B_n^{(k_1,\ldots,k_r)} + \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,\ldots,k_r)}$$

This recurrence exhibits the interesting fact that for $k_r \leq 0$ these numbers are always positive integers.

Hamahata and Masubuchi investigated the special case when $k_1 = k_2 = \cdots = k_{r-1} = 0$ named these numbers *special multi–poly–Bernoulli numbers*, $B[r]_n^{(k)}$. The authors derived the generating function and a closed formula for these numbers.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B[r]_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \left(\frac{e^{x+y}}{e^x+e^y-e^{x+y}}\right)^r.$$

Theorem 28 ([**39**], Theorem 8.)**.** *For $n, k \geq 0$, we have*

$$B[r]_n^{(-k)} = \sum_{\substack{n=n_1+\cdots+n_r \\ n_1,\ldots,n_r\geq 0}} \sum_{\substack{k=k_1+\cdots+k_r \\ k_1,\ldots,k_r\geq 0}} \frac{n!k!}{n_1!\cdots n_r!k_1!\cdots k_r!}$$

$$\times \left(\sum_{j_1=0}^{\min(n_1,k_1)} \cdots \sum_{j_r=0}^{\min(n_r,k_r)} (j_1!\cdots j_r!)^2 \left\{ {n_1+1 \atop j_1+1} \right\} \cdots \left\{ {n_r+1 \atop j_r+1} \right\} \left\{ {k_1+1 \atop j_1+1} \right\} \cdots \left\{ {k_r+1 \atop j_r+1} \right\} \right).$$

Or equivalently

$$B[r]_n^{(-k)} = \sum_{\substack{n=n_1+\cdots+n_r \\ n_1,\ldots,n_r\geq 0}} \sum_{\substack{k=k_1+\cdots+k_r \\ k_1,\ldots,k_r\geq 0}} \binom{n}{n_1,\ldots,n_r}\binom{k}{k_1,\ldots,k_r} B_{n_1}^{(-k_1)}\cdots B_{n_r}^{(-k_r)}$$

The symmetry holds:

Theorem 29 ([**39**] Cor. 10.)**.** *For $n, k \geq 0$ we have*

$$B[r]_n^{(-k)} = B[r]_k^{(-n)}.$$

$B_n[2]^{(-1)}$	2	6	18	54	162	486
$B_n[3]^{(-1)}$	3	12	48	192	768	3072

Table 4. Special–multi–poly–Bernoulli Numbers

Though the generalization is purely algebraic, the formula is of combinatorial nature.

Open Problem 2. *Give a combinatorial interpretation of the sequences* $\{B[r]_n^{(-k)}\}_{n\in\mathbb{Z},k\in\mathbb{Z}}$.

12.3. The Arakawa–Kaneko function. Arakawa and Kaneko [**2**] established a connection between the multiple zeta values and the poly–Bernoulli numbers. In order to show that poly–Bernoulli numbers appear as special values of zeta function they introduced a function, that are referred as the Arakawa–Kaneko function in the literature.

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt,$$

where s is a complex value and $k \geq 1$. The special values of this function at non–positive integers are given by

$$\xi_k(-m) = (-1)^m A_m^{(k)},$$

where the generating function of the numbers $\{A_n^{(k)}\}$ are given by

$$\sum_{n=0}^{\infty} A_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1 - e^{-x})}{e^x - 1}.$$

Since the exponential generating functions of the sequences $\{A_n^{(k)}\}$ and $\{B_n^{(k)}\}$ differ only by a factor e^x, the numbers are related by

$$B_n^{(k)} = \sum_{m=0}^{n} \binom{n}{m} A_m^{(k)} \quad \text{and} \quad A_n^{(k)} = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} B_m^{(k)}$$

and hence the relation holds:

$$B_n^{(k)} = A_n^{(k)} + A_{n-1}^{(k-1)}. \tag{6}$$

In [**49**] the authors investigated poly–Bernoulli polynomials and derived some recurrence formulas and interesting identities using umbral calculus. One of his result is

n	1	2	3	4	5	6
0	1	1	1	1	1	1
1	1	3	7	15	31	63
2	1	7	31	115	391	1267
3	1	15	115	675	3451	16275
4	1	31	391	3451	25231	164731

TABLE 5. The Arakawa–Kaneko numbers

Theorem 30. [**49**]

$$\sum_{m=0}^{n-1}(-1)^{n-1-m}\binom{n-1}{m}B_m^{(k-1)} = \sum_{m=0}^{n-1}\binom{n}{m}(-1)^{n-1-m}B_m^{(k)}.$$

We note that this relation is simple the recursion (5).

PROOF. The left hand side is exactly the number $A_{n-1}^{(k-1)}$, the right hand side is

$$\begin{aligned}\sum_{m=0}^{n-1}\binom{n}{m}(-1)^{n-1-m}B_m^{(k)} &= -\sum_{m=0}^{n-1}\binom{n}{m}(-1)^{n-m}B_m^{(k)} - B_m^{(k)} + B_m^{(k)}\\ &= -\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}B_m^{(k)} + B_m^{(k)}\\ &= -A_n^{(k)} + B_n^{(k)}\end{aligned}$$

□

12.4. Poly–Cauchy numbers. The nice properties of poly–Bernoulli numbers motivated some authors to start a study of similarly defined numbers, the poly–Cauchy numbers and related objects, as poly–Cauchy polynomials or hypergeometric Cauchy–numbers.

Cauchy numbers appear first in the book of Comtet Advanced Combinatorics [**23**] ex. 13. pp. 293. Comtet defines two kinds of Cauchy numbers: *Cauchy numbers of the first kind* are defined by the integral

$$C_n = \int_0^1 x^{\underline{n}}dx$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ is the falling factorial and *Cauchy numbers of the second kind* are defined analogous by the integral

$$\widehat{C}_n = \int_0^1 x^{\overline{n}}dx$$

where $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ is the rising factorial. The relation between the two kinds of Cauchy numbers is:

$$C_n = \widehat{C}_n + n\widehat{C}_{n-1}$$

In the explicit formula for Cauchy numbers we see that they are related to the Stirling numbers of the first kind in a similar manner as Bernoulli numbers to the Stirling numbers of the second kind.

$$C_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{n-k}}{k+1} \quad \text{and} \quad \widehat{C}_n = (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{k+1}$$

It would be far reaching to give accurate definitions about this topic, we mention only that one of the most important application of the Cauchy numbers is the so called *Laplace summation formula* which was devised to perform approximate integrals and is an analogous to the Euler–McLaurin formula. It uses instead of the differentiation operator $D = \frac{d}{dt}$ the *difference operator* Δ.

$$\int = \Delta^{-1} \sum_{k=0}^{\infty} \frac{C_k}{k!} \Delta^k,$$

where $\int$ is the *integration operator* and Δ^{-1} is the *indefinite summation operator.*

We refer the interested reader to [**66**], where we find a description of Cauchy numbers. Analogous results and connections suggested the introduction of poly–Cauchy numbers after poly–Bernoulli numbers and was accomplished by Komatsu [**57**]. For k and n positive integers Poly–Cauchy numbers of the second kind can be defined by the generating function

$$\sum_{n=0}^{\infty} \widehat{C}_n^{(k)} \frac{x^n}{n!} = \mathrm{Lif}_k(-\ln(1+x)),$$

where

$$\mathrm{Lif}_k(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!(m+1)^j}$$

is the k–th polylogarithm factorial function. The explicit formula for $\widehat{C}_n^{(-k)}$, $n \geq 0, k \geq 1$ is given by the Stirling number of the first kind:

$$\widehat{C}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{1}{(m+1)^k} \begin{bmatrix} n \\ m \end{bmatrix} \tag{7}$$

As we see this expression is ,,almost the same" as of that for poly–Bernoulli numbers, just that instead of $m!\left\{ {n \atop m} \right\}$ here we have $(-1)^m \left[{n \atop m} \right]$. Komatsu asked

n,k	1	2	3	4	5
1	2	4	8	16	32
2	5	13	35	97	275
3	17	51	161	531	1817
4	74	244	854	3148	12134
5	394	1392	5248	20940	87784

TABLE 6. Poly–Cauchy numbers

for combinatorial objects that are counted by poly–Cauchy numbers with negative indices k. The generating function exhibits that the duality theorem does not hold for poly–Cauchy numbers of negative indices [**57**].

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\widehat{C}_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^y}{(1+x)^{e^y}}.$$

The poly–Bernoulli numbers can be expressed by poly–Cauchy numbers.

Theorem 31 ([**57**]).

$$B_n^{(k)}=(-1)^n\sum_{l=1}^{n}\sum_{m=1}^{n}m!\left\{{n \atop m}\right\}\left\{{m \atop l}\right\}\widehat{C}_l^{(k)},$$

and conversely

$$\widehat{C}_n^{(k)}=(-1)^n\sum_{l=1}^{n}\sum_{m=1}^{n}\frac{1}{m!}\left[{n \atop m}\right]\left[{m \atop l}\right]B_l^{(k)}.$$

Open Problem 3. *Is there any combinatorial interpretation of poly–Cauchy numbers with negative indices?*

12.5. Poly–Bernoulli numbers and the Pascal triangle. Consider the infinite lower triangular matrix $M=(m_{ij})$ $i=0,1,\dots$ and $j=0,1,\dots$.

$$m_{ij}=\begin{cases}0, & \text{when} \quad j=0 \quad \text{or} \quad i<j\\ \binom{i}{j-1}, & \text{when} \quad 1\le j<i\\ i+1, & \text{when} \quad j=i\end{cases}$$

$$\begin{pmatrix}1 &&&&&&\\ 0 & 2 &&&&&\\ 0 & 1 & 3 &&&&\\ 0 & 1 & 3 & 4 &&&\\ 0 & 1 & 4 & 6 & 5 &&\\ 0 & 1 & 5 & 10 & 10 & 6 &\\ 0 & 1 & 6 & 15 & 20 & 15 & 7\end{pmatrix}$$

The matrix M has interesting connections to Bernoulli resp. poly–Bernoulli numbers as mentioned in [**85**] sequence A210381.

(1) The n–th power of the matrix M M^n has generate the poly–Bernoulli numbers. The sums of the entries in the k-th row of the matrix M^n are the poly–Bernoulli numbers $B_n^{(-k)}$.
(2) $M\underline{b} = [1, 1, \dots]^t$, where $\underline{b}$ is the column vector with entries of the Bernoulli numbers (with $B_1 = \frac{1}{2}$)
(3) We can express the recurrence relation that we proved combinatorially with the matrix M:
$$B_n^{(-k)} = \sum_{j=1}^{n} m_{nj} B_j^{(-k)}.$$

CHAPTER 3

The hook formula

,,It's a thing that non–mathematician don't realize. Mathematics is actually an esthetic subject almost entirely." (John Conway)

1. Introduction

The original hook formula concerns partitions of an integer. The notion of partition, rooted in number theory has found many combinatorial links. In this chapter we devote the first section to the introduction of partitions and classical results.

Partitions can be described by their Young diagrams. The diagrams occupies some squares of the plane. There is a natural partial order among the positions. The number of linear extensions of this partial order is given by a nice formula, the hook formula. Similar formulas exist for enumeration of related objects and if such a formula exists we say that the poset has hooklength.

Classically there are three posets having hooklength:

- standard Young tableaux
- shifted standard Young tableaux
- plane trees.

Frame, Robinson and Thrall [**30**] discovered first (1954) the hook–length formula for standard Young tableaux. There are different proofs of this formula, for instance Greene, Nijenhuis, and Wilf [**37**] presented a probabilistic proof of this result in that hook length appeared in a very natural way.

But in view of such a nice combinatorial formula one expects a nice bijective proof that explains the form of the formula. The problem of finding a bijective proof for these surprisingly compact formulas that explains the form of them has a long history for the first case, the case of standard tableaux of an (ordinary) Ferrers shape.

In [**31**] a bijection is described as an algorithm. The description of the algorithm is simple, but difficult to show that it really works. Remmel's [**77**] proof is complicated, it is actually a ,,bijectivization" of recurrence relations and finally Zeilberger's [**94**] proof is a translation of the probabilistic proof of Greene, Nijenhuis and Wilf into a bijection. The most celebrated, regarded as really satisfactory was the bijection given by Novelli, Pak and Stoyanovskii in [**68**] (1997, more than 40 years after the first proof of the formula).

Less attention had been paid to the problem of finding a bijective proof of the shifted hook formula (cf. [**59**]). However, recently Fischer [**28**] succeeded in finding such a bijective proof in the spirit of Novelli, Pak and Stoyanovskii.

The aim of our work is to complete this program for the case of plane trees as well. In fact, we do not only provide a bijective proof of the hook formula for plane trees in the spirit of Novelli, Pak and Stoyanovskii (see Section 3.), we also provide a second, conceptually different, bijective proof (see Section 4.). The plan of this chapter is as follows: in Section 2. we recall some necessary definitions and notations and the main idea of the bijective proofs of hook formulas. In Section 3. and 4. we present bijective proofs for plane trees. In the last section (Section 5.) we mention related topic, a generalization of the hook formula, and some hook formulas for special classes of plane trees from recent research.

2. Hook length formulas

A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ and $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0$ for some r. The *Ferrers diagram* of λ is an array of cells with r left–justified rows and λ_i cells in row i. Given a partition λ of n a *standard Young tableau of shape* λ SYT is a filling of the cells of the Ferrers diagram of λ with the integers $\{1, 2, \dots, n\}$ such that the entries along rows and columns are increasing.

An equivalent description can be done the following way. The squares of the Young diagram of λ have a natural partial order: for two s, s' two squares $s < s'$ iff s is in N, W or NW position from s'. A standard Young tableau of shape λ is a linear extension of this order.

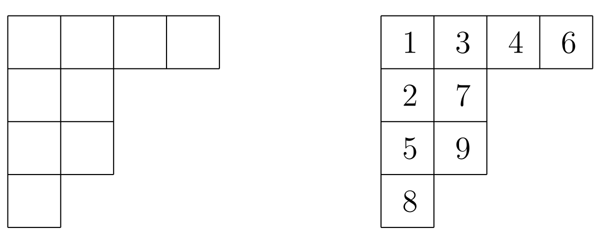


FIGURE 1. Ferrers diagram and a standard Young tableau of shape $(4, 2, 2, 1)$

The *hook* of the cell ρ of the Ferrers diagram of λ is the set of cells that are either in the same row as ρ and to the right of ρ or in the same column as

ρ and below ρ, ρ included. The hook of a cell $\rho = (i, j)$ consists of three parts, the cell itself, the *arm*, which is the set of cells to the right of ρ in the same row as ρ and the *leg*, which is the set of cells below ρ in the same column as ρ. The *hook length* $h_{i,j}$ of cell (i, j) is the number of cells in the hook of $\rho = (i, j)$. In Figure $h_{2,1} = 4$.

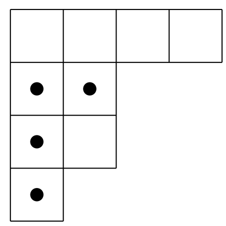


FIGURE 2. The hook of the cell (2,1)

Theorem 32 ([**30**], Hook length formula)**.** *Let* f_λ *denote the number of standard Young tableaux of shape* λ*. Then*

$$f_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$$

If λ is a partition with distinct components, then the *shifted Ferrers diagram of shape* λ is an array of cells with r rows each row intended by one cell to the right with respect to the previous row, and with λ_i cells in row i. Given a partition λ of n the *shifted standard Young tableaux* is a filling of the cells such that the entries along rows and columns are increasing. The *hook* of a

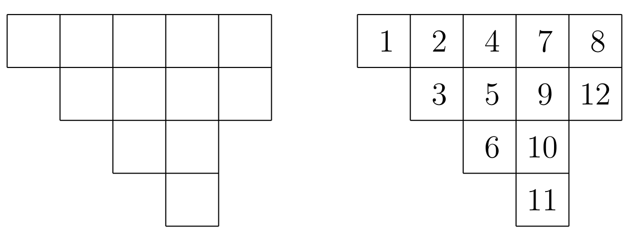


FIGURE 3. Shifted Ferrers diagram and a shifted standard Young tableau of shape $(5, 4, 2, 1)$

cell c of the shifted Ferrers shape of λ includes all cells that are

- either in the same row as c and to the right of c, or
- in the same column as c and below c, or
- if this set contains a cell on the main diagonal, cell (j, j) then also all the cells of the $(j + 1)$–st row belong to the hook of λ.

The *hooklength* $h^*_{i,j}$ of a cell $c = (i, j)$ in the shifted tableaux is the number of the cells in the hook of c. In Figure $h_{1,2} = 7$.

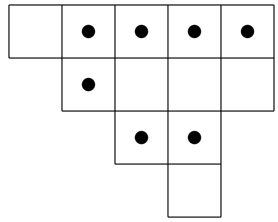


FIGURE 4. The hook of the cell $(1, 2)$

Theorem 33 ([**81**]). *Let f^*_λ denote the number of shifted standard Young tableaux of shape λ. Then*

$$f^*_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h^*_{i,j}}.$$

Trees counts today to the classical combinatorial objects as the fundamental structures in computer science. This fact is underlined in the basic book series The Art of Computer programming of Donald Knuth [**54**], a whole book is dedicated to the study of simple algorithms by which trees of various species can be exhaustively explored.

The significance of this simple case is shown also in the recent research, as we see, that new hook formulas are continuously discovered for special classes of trees. We hope that our bijections contributes the understanding of the hook formula and that it can be used to give bijective explanations of the formulas in the special cases, too. A *plane tree* is a rooted tree for which an ordering is specified for the successors of each vertex. A poset $\mathcal{T} = (V, \leq)$ is a *rooted tree* if it has a unique minimal element. The Hasse diagram of $\mathcal{T}$ is a tree T in the graphic-theoretic sense of the term. The set of the nodes of the tree T is $V = V(T)$. If n is the number of elements in the poset then a bijection $S : V \to [n] = \{1, 2, \ldots, n\}$ is a *labelling* on $\mathcal{T}$. An order preserving bijection $S : V \to [n] = \{1, 2, \ldots, n\}$ is a *standard labelling* on $\mathcal{T}$. If v is a

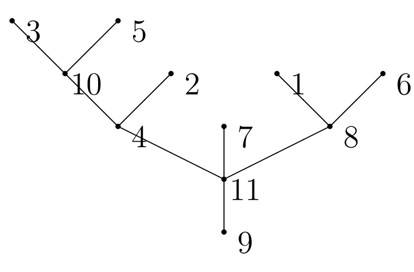


FIGURE 5. Standard labelling of a plane tree

node of T, then the *hook of* v is

$$H_v = \{w \in \mathcal{T} \mid w \geq v\},$$

with corresponding *hook length* $h_v = |H_v|$. In fact the hook length is the number of successors of the node including the node itself. In Figure the hooklength of the node v is 5.

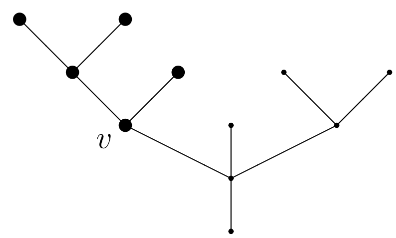


FIGURE 6. The hooklength of v

Theorem 34 ([**53**], Hook length formula for trees)**.** *Let f_T denote the number of standard labellings on the plane tree T. Then*

$$f_T = \frac{n!}{\prod_{v \in V(T)} h_v}.$$

The main result of this chapter is reproving this theorem. Our presentation will be two simple bijective proofs. First we recall Sagan's proof [**83**] to demonstrate the interesting fact that in proofs of hook formulas interestingly algorithmic and probabilistic approaches seemed to be natural.

Let T be a fixed plane tree with n nodes. The following algorithm can be used to choose a labelling of T.

GNW1. Pick a node $v \in T$ uniformly at random. (i.e. with probability $\frac{1}{n}$)
GNW2. If v is maximal (a leaf), then let $L(v) = n$ (label the node v by n) and return to GNW1. with $T \leftarrow T - \{v\}$ and $n \leftarrow (n-1)$, (unless there is no nodes left, in which case the algorithm terminates.)
GNW3. If v is not maximal, then choose a different node $w \in H_v$ uniformly at random, (i.e. with probability $\frac{1}{h_v - 1}$), and return to GNW2 with $v \leftarrow w$.

Theorem 35 ([**83**], Theorem 1.)**.** *If T is a fixed plane tree with n nodes, then GNW1. –GNW3. produce all labellings of T uniformly at random. The probability of a given labelling is*

$$\prod_{v \in T} \frac{h_v}{n!}$$

PROOF. We proof the theorem by induction. Let w be any maximal element. First we compute the probability that w receives the label n denoted by $P(w \mapsto n)$. There are two possibilities:

- w is chosen as an initial node in GNW1. This happens with probability $\frac{1}{n}$.
- w is an endnode of a *trial*, a loop between GNW2. and GNW3. In this case the initial node v is on the unique path W from w to the root of T (excluding itself). The endnode can be chosen with 1 intermediate node, 2 intermediate nodes, ..., h_v intermediate nodes so with the probability

$$1+\sum_{v\in W}\frac{1}{h_v-1}+\sum_{v_1,v_2\in W}\frac{1}{h_{v_1}-1}\frac{1}{h_{v_2}-1}+\cdots+\sum_{v_{i_1},\dots,v_{i_{h_v}}\in W}\prod_{j=1}^{h_v}\frac{1}{h_{v_{i_j}}-1}$$

Hence

$$P(w\mapsto n)=\frac{1}{n}\prod_{v\in W}\left(1+\frac{1}{h_v-1}\right)=\frac{1}{n}\prod_{v\in W}\frac{h_v}{h_v-1}.$$

Assume now that the claim holds for the tree without the leaf w: $T\setminus\{w\}$.

$$P(T\setminus w\mapsto L)=\prod_{v\in T\setminus w}=\frac{h_v}{(n-1)!},$$

where $P(T\setminus w\mapsto L)$ denotes the probability of any given labelling L.

Then by induction

$$P(T\mapsto L)=\frac{1}{(n-1)!}\prod_{v\in T\setminus\{W\}}h_v\prod_{v\in W}(h_v-1)\frac{1}{n}\prod_{v\in W}\frac{h_v}{h_v-1}=\frac{1}{n!}\prod_{v\in T}h_v.$$

Note that the hook lengths of the nodes on the path W is one less in the tree without the element w. □

3. Our first bijection

As we mentioned before as one of the main result of this chapter we present a bijection for the case of plane trees in the spirit of Novelli, Pak and Stoyanovskii.

The first step in this direction is to bring the formula in a form that can be easily interpreted combinatorially, we multiply both sides by the denominator $\prod_{v\in V(T)}h_v$:

$$f_T\times\prod_{v\in V(T)}h_v=n!. \tag{8}$$

The combinatorial interpretation of this formula is obvious: the right side of this equation can be interpreted as the number of arbitrary labellings of the nodes (permutations of $[n]$), and the left side as the number of pairs of a standard labelling and a map $H:V\to\mathbb{Z}$, such that $H(v)\in\{1,2,\dots,h_v\}$ (we call such a map *hook function*).

Next we need to define a total order on the tree with the property that the order of a node is always greater than the order of its successors.

We describe this total order by a map $V \longrightarrow [n]$. First we define *the left most leaf of the tree.* This is the endnode of the unique path $P = \{t_1, t_2, \ldots, t_l\}$, where t_1 is the root and t_{i+1} is the first node (moving from left to the right) among the successors of t_i for all $1 \leq i \leq l-1$.

We construct our map which gives the total order the following way: Consider the left most leaf of the tree and assign the least number to it. Delete this node from the set of the nodes and delete this number from the set of the numbers. A node is denoted by v_j if the number j has been associated to it. See Figure 8. for an example.

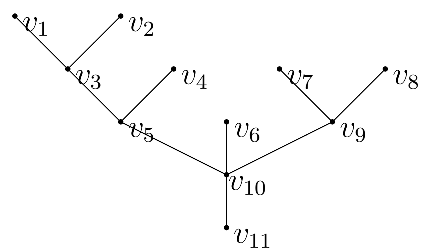


FIGURE 7. The total order

We define first a map from the set of labellings (L) of the tree T to the set of pairs (S, H) where S is a standard labelling and H is a hook function of $\mathcal{T}$. We will see that this is a bijective map.

The map I. transforms a labelling to a pair (S, H) using a sequence of pairs (S_j, H_j), $1 \leq j \leq (n+1)$:

$$L \longrightarrow (S_1, H_1) \xrightarrow{Move(1)} (S_2, H_2) \xrightarrow{Move(2)} \cdots \xrightarrow{Move(n)} (S_{n+1}, H_{n+1})$$

The sequence starts with (S_1, H_1), where S_1 is the labelling L and H_1 is the hook function with $h(v) = 1$ for all $v \in V(T)$.

Move(j) transforms the pair (S_j, H_j) into a pair (S_{j+1}, H_{j+1}) the following way: Start the process with considering the node v_j. We denote the label of a node v_j in S_j by l.

Step 0 Let v_i be the actual node with label l. Consider the set of the direct successors $(D(v_i))$ in S_j. Let v_{min} be the node with the minimal label (l_{min}) in $D(v_i)$.

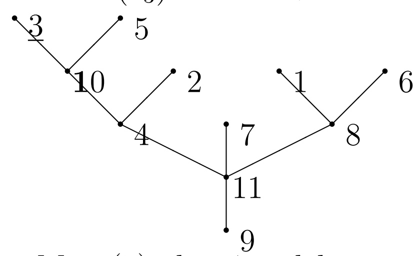
Step 1
- if $l_{min} < l$ then interchange the label of v_{min} and v_i. The actual node with l is v_{min}. Go to Step 0.
- if $l_{min} > l$ Go to Step 2.

Step 2 Let v_k be the node with the label l in S_{j+1}. We point to this node with the hook number which we associate to the node v_j and set $h(v_j) = j - k + 1$.

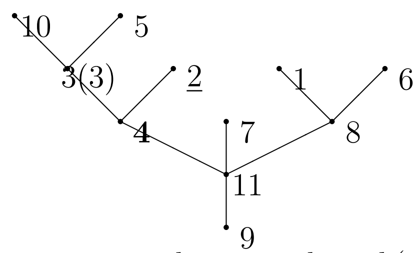
The label l slides actually from v_j to another node v_k along a unique path. The labels of the nodes of this path slide one node down and the endnode of the path receive the label l.

Lets consider an example. We describe only moves when the labelling changes.

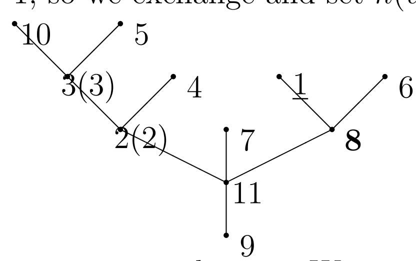
The first exchange will be necessary at Move(3) concerning the node v_3: $l = 10$ $l_{min} = 3$. And we set: $h(v_3) = 3 - 1 + 1 = 3$.



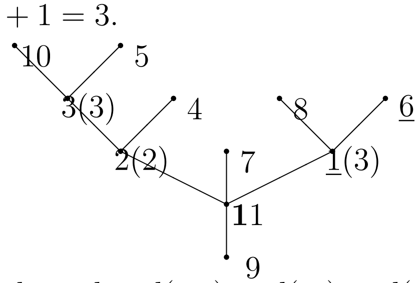
The next exchange is at Move(5): $l = 4$ and $l_{min} = 2$. After the exchange we set: $h(v_5) = 5 - 4 + 1 = 2$.



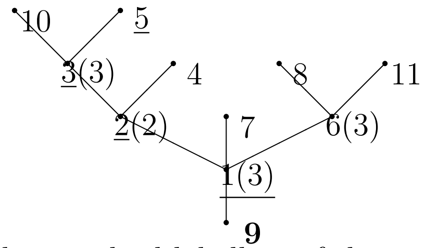
Move(9): $l = 8$, $l_{min} = 1$, so we exchange and set $h(v_9) = 9 - 7 + 1 = 3$.



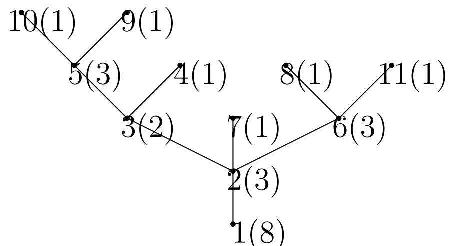
Move(10):$l = 11$, $l_{min} = 1$, we exchange. We consider now the successors of v_9: $l_{min} = 6 < 11$ so we do another exchange and Move(10) ends with setting$h(v_{10}) = 10 - 8 + 1 = 3$.



Move(11): $l = 9$. We have $l > l(v_{10}) > l(v_5) > l(v_3) > l(v_2)$, so Move(11) ends after 4 exchanges and $l = 9$ slides up to the node v_2. We have to set $h(v_{11}) = 11 - 2 + 1 = 8$



The pair (S_{12}, H_{12}). The standard labelling of the tree and the hook function



Next we define map II. that transforms a pair of a standard labelling and a hook function (S, H) into a labelling L of the tree T using a sequence of pairs (S'_j, H'_j), $1 \leq j \leq n$:

$$(S, H) \overset{Move'(n)}{\longrightarrow} (S'_n, H'_n) \overset{Move'(n-1)}{\longrightarrow} \cdots \overset{Move'(1)}{\longrightarrow} (S'_1, H'_1) \longrightarrow L$$

In the pair (S'_1, H'_1) S'_1 is the labelling L and H'_1 is the hook function with $h(v_j) = 1$ for all $1 \leq j \leq n$.
Move'(j): Consider v_j with its hook number $h(v_j)$. Set $k = j - (h(v_j) - 1)$. ($k < j$, so v_k is a successor of v_j.) Interchange the labels of the nodes of the unique walk from v_k to v_j step by step and set $h(v_j) = 1$. Our claim proofs the hook length formula for plane trees, Theorem 35.

Claim 36. *The map I and map II are inverse to each other.*

Proof. From the definition of Move(j) follows that S_j is a standard labelling of the subtree on $\{v_1, \ldots, v_j\}$. It is obvious that Move(j) and Move'(j) are inverse to each other. □

4. A second bijection

There are other possibilities to define a bijection between the set of pairs (S, H) and the set of labellings L. In the first bijection we moved the labellings of the nodes. In this bijection in some sense we fix the labellings and move the nodes.

We consider a labelling of the tree as a linear arrangement of the nodes and a label simple as the position of the node in this arrangement. A standard labelling is a special labelling which keeps the structure of the tree, the partial order of the nodes so it holds: if $v_i \leq v_j$ in the total order for some i and j then v_i stands before v_j in the linear arrangement.

We fix again a total order of the nodes (different from the one defined in the previous section). The *distance of two nodes v_i and v_j* is the number of the nodes of the unique path in the tree from v_i to v_j (involving v_i and v_j). We say that a node v_i is on the level k when the distance of the root and v_i is k. We fix the total order of the nodes according the following simple rule: we denote the node on the first level, the root by v_1. Let the right most node on the level $(k-1)$ be the node $v_{(k-1)^*}$. We denote the nodes on the level k from left to the right by $v_{(k-1)^*+1}, v_{(k-1)^*+2}, \ldots, v_{k^*}$. $k^* - (k-1)^*$ is the number of the nodes on the level k. (See Figure 8.)

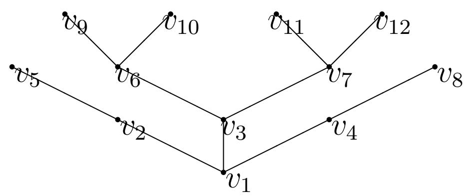
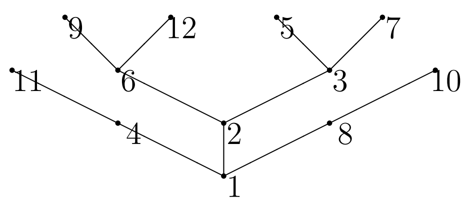


FIGURE 8. The total order

We represent a pair (S, H) as a sequence of $(v_j, h(v_j))$, where the order of $(v_j, h(v_j))$ is given by the standard labelling S.

The standard labelling S



The hook function H

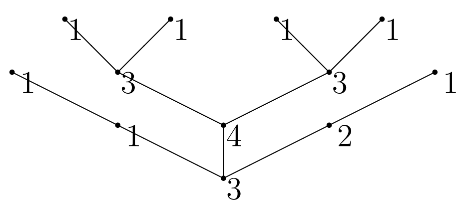


FIGURE 9. The standard labelling and hook function (S,H)

For instance the associated sequence to the standard labelling and hook function showed in Figure 9. is:

$$(S, H) = (v_1, 3)(v_3, 4)(v_7, 3)(v_2, 1)(v_{11}, 1)(v_6, 3)$$
$$(v_{12}, 1)(v_4, 2)(v_9, 1)(v_8, 1)(v_5, 1)(v_{10}, 1)$$

We define a map φ that transforms a pair (S, H) to a pair (S_{n+1}, H_{n+1}), a pair of a labelling and the hook function with $h(v_j) = 1$ for all $1 \leq j \leq n$ step by step. We associate to S_{n+1} the labelling L.

$$(S, H) = (S_1, H_1) \xrightarrow{\text{Step 1}} (S_2, H_2) \xrightarrow{\text{Step 2}} \cdots \xrightarrow{\text{Step } n} (S_{n+1}, H_{n+1}) \to L$$

We describe Step j: Consider the node v_j and the sequence of its successors $A(v_j)$. Move the node v_j to the position signed by its hook number $h(v_j)$ among the members of $A(v_j)$ and set $h(v_j) = 1$. We arrange the other members of $A(v_j)$ in the remaining positions keeping their relative relations. The nodes outside of $A(v_j)$ keep their previous positions.

We give next an example. We consider the tree which is shown in Figure 8. with the standard labelling and hook function given in Figure 9. We apply the map φ:

Step 1: Consider v_1. All the nodes are successors of v_1 and $h(v_1) = 3$. So v_1 moves to the third position of the whole sequence. We set $h(v_1) = 1$. The result of the first transformation is:

$$\begin{aligned}(S_2, H_2) \;&=\; (v_3, 4)(v_7, 3)(v_1, 1)(v_2, 1)(v_{11}, 1)(v_6, 3)\\ &\quad (v_{12}, 1)(v_4, 2)(v_9, 1)(v_8, 1)(v_5, 1)(v_{10}, 1).\end{aligned}$$

Step 2: Consider v_2. The sequence of the successors is $A(v_2) = v_2, v_5$ and $h(v_2) = 1$. So v_2 keeps its position and $h(v_2) = 1$. The result of this step:

$$\begin{aligned}(S_3, H_3) \;&=\; (S_2, H_2) = (v_3, 4)(v_7, 3)(v_1, 1)(v_2, 1)(v_{11}, 1)\\ &\quad (v_6, 3)(v_{12}, 1)(v_4, 2)(v_9, 1)(v_8, 1)(v_5, 1)(v_{10}, 1).\end{aligned}$$

Step 3: Consider v_3. The sequence of the successors is $A(v_3) = v_3, v_7, v_{11}, v_6, v_{12}, v_9, v_{10}$ and $h(v_3) = 4$. v_3 moves to the fourth position of the seven positions of the nodes from $A(v_3)$ and we set $h(v_3) = 1$. The result of this step is:

$$\begin{aligned}(S_4, H_4) \;&=\; (v_7, 3)(v_{11}, 1)(v_1, 1)(v_2, 1)(v_6, 1)(v_3, 1)\\ &\quad (v_{12}, 1)(v_4, 2)(v_9, 1)(v_8, 1)(v_5, 1)(v_{10}, 1).\end{aligned}$$

Step 4: Consider v_4. $A(v_4) = v_4, v_8$ and $h(v_4) = 2$. So

$$\begin{aligned}(S_5, H_5) \;&=\; (v_7, 3)(v_{11}, 1)(v_1, 1)(v_2, 1)(v_6, 1)(v_3, 1)\\ &\quad (v_{12}, 1)(v_8, 1)(v_9, 1)(v_4, 1)(v_5, 1)(v_{10}, 1).\end{aligned}$$

Step 5: $(S_6, H_6) = (S_5, H_5)$ since $h(v_5) = 1$.

Step 6: Consider the node v_6. $A(v_6) = v_6, v_9, v_{10}$ and $h(v_6) = 3$. So

$$\begin{aligned}(S_7, H_7) \;&=\; (v_7, 3)(v_{11}, 1)(v_1, 1)(v_2, 1)(v_9, 1)(v_3, 1)\\ &\quad (v_{12}, 1)(v_8, 1)(v_{10}, 1)(v_4, 1)(v_5, 1)(v_6, 1).\end{aligned}$$

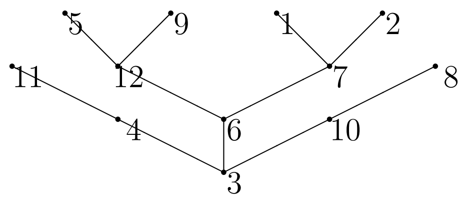
Step 7: Consider the node v_7. $A(v_7) = v_7, v_{11}, v_{12}$ and $h(v_7) = 3$. So

$$\begin{aligned}(S_8, H_8) &= (v_{11}, 1)(v_{12}, 1)(v_1, 1)(v_2, 1)(v_9, 1)(v_3, 1)\\ &\quad (v_7, 1)(v_8, 1)(v_{10}, 1)(v_4, 1)(v_5, 1)(v_6, 1).\end{aligned}$$

Step 8–12: The other hook numbers $h(v_8), h(v_9), \ldots, h(v_{12})$ are all 1, so we have

$$(S_8, H_8) = (S_9, H_9) = \cdots = (S_{13}, H_{13}) \to L.$$

FIGURE 10. The labelling L



Now we describe the map ψ that transforms a labelling L of the nodes into a pair of (S, H), a pair of a standard labelling and a hook function. The n steps of the map are:

$$L \to (S^*_{n+1}, H^*_{n+1}) \overset{\text{Step } n^*}{\longrightarrow} (S^*_n, H^*_n) \overset{\text{Step } (n-1)^*}{\longrightarrow} \cdots \overset{\text{Step } 1^*}{\longrightarrow} (S^*_1, H^*_1) = (S, H)$$

We consider the nodes in the reverse order. During the generic step Step j^* we replace the node v_j and change $h(v_j)$ when its necessary. The successors of v_j were already investigated. We denote this sequence by $A^*(v_j)$. We set $h(v_j)$ according the relative relation of v_j among the members of $A^*(v_j)$ and move v_j to the first position among $A^*(v_j)$. We arrange the other members of $A^*(v_j)$ in the remaining positions occupying by $A^*(v_j)$ keeping their relative relations. The nodes outside of $A^*(v_j)$ keep their previous positions.

We apply now map ψ to

$$\begin{aligned}L = (S^*_{13}, H^*_{13}) &= (v_{11}, 1)(v_{12}, 1)(v_1, 1)(v_2, 1)(v_9, 1)(v_3, 1)\\ &\quad (v_7, 1)(v_8, 1)(v_{10}, 1)(v_4, 1)(v_5, 1)(v_6, 1).\end{aligned}$$

Step 12^*: Consider the last node $v_n = v_{12}$. The subsequence of its successors is $A^*(v_{12}) = v_{12}$. v_{12} is the first node in $A^*(v_{12})$. So $(S^*_{13}, H^*_{13}) = (S^*_{12}, H^*_{12})$.

Step $11^* - 8^*$: $(S^*_{12}, H^*_{12}) = (S^*_{11}, H^*_{11}) = \cdots = (S^*_8, H^*_8)$

Step 7^*: Consider v_7. $A^*(v_7) = v_{11}, v_{12}, v_7$. v_7 is in the third position. We put it to the first position in $A^*(v_7)$, set $h(v_7) = 3$ and shift the other nodes in $A^*(v_7)$:

$$\begin{aligned}(S^*_7, H^*_7) &= (v_7, 3)(v_{11}, 1)(v_1, 1)(v_2, 1)(v_9, 1)(v_3, 1)\\ &\quad (v_{12}, 1)(v_8, 1)(v_{10}, 1)(v_4, 1)(v_5, 1)(v_6, 1).\end{aligned}$$

Step 6*: Consider v_6. $A^*(v_6) = v_9, v_{10}, v_6$. v_6 is in the third position. We put it to the first position in $A^*(v_6)$, set $h(v_6) = 3$ and shift the other nodes in $A^*(v_6)$:

$$\begin{aligned}(S_6^*, H_6^*) &= (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_6,1)(v_3,1)\\ &\quad (v_{12},1)(v_8,1)(v_9,1)(v_4,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 5*: $(S_5^*, H_5^*) = (S_6^*, H_6^*)$

Step 4*: Consider v_4. $A^*(v_4) = v_8, v_4$. We put v_4 to the first position in $A^*(v_4)$ and set $h(v_4) = 2$.

$$\begin{aligned}(S_4^*, H_4^*) &= (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_6,1)(v_3,1)\\ &\quad (v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 3*: Consider v_3. $A^*(v_3) = v_7, v_{11}, v_6, v_3, v_{12}, v_9, v_{10}$. v_3 is in the third position. We put it to the first position and set $h(v_3) = 4$.

$$\begin{aligned}(S_3^*, H_3^*) &= (v_3,4)(v_7,3)(v_1,1)(v_2,1)(v_{11},1)(v_6,1)\\ &\quad (v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 2*: $(S_2^*, H_2^*) = (S_3^*, H_3^*)$.

Step 1*: We consider v_1. $A^*(v_1)$ is the whole sequence. v_1 is in the third position. We put it to the first position and set $h(v_1) = 3$.

$$\begin{aligned}(S,H) = (S_1^*, H^*) &= (v_1,3)(v_3,4)(v_7,3)(v_2,1)(v_{11},1)(v_6,3)\\ &\quad (v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).\end{aligned}$$

We complete our second proof of the hook length formula for plane tree by showing that φ and ψ are inverse maps.

Theorem 37. *The map φ and the map ψ are inverse to each other.*

Proof. First we give several important properties of the maps:

(1) φ: The *Step* $j(S_j, H_j)$ change the position of v_j according to the hook number $h(v_j)$ in the subsequence $A(v_j)$.
ψ: The *Step* $j^*(S_{j+1}, H_{j+1})$ change the hook number $h(v_j)$ according to the position of v_j in $A^*(v_j)$ and moves v_j to the first position in $A^*(v_j)$.

(2) φ: After *Step* j of the map φ (in $\{S_i\}_{i>j}$) the node v_j keeps its position.
ψ: The node v_j keeps its position until *Step* j^* (in $\{S_i^*\}_{i>j}$).

It is obvious that given a set (S_j, H_j):

$$Step\ j^*(Step\ j(S_j, H_j)) = (S_j, H_j)$$

and given a set (S_{j^*}, H_{j^*}) :

$$Step\ j(Step\ j^*(S^*_{(j+1)}, H^*_{(j+1)})) = (S^*_{(j+1)}, H^*_{(j+1)}).$$

This means that *Step j* and *Step j^** are inverse to each other and the theorem follows. □

5. Related topics

In this section we would like to show some interesting further results in the area in order to emphasize the wealth of the topic of ,,hook formula".

The hook length formula for trees has received more attention since Postnikov presented his remarkable hook formula for binary trees in Stanley's 60–th Birthday Conference in 2004,

Theorem 38 ([**73**], Corollary 17.3.)**.**

$$(n+1)^{n-1} = \sum_{T\in\mathcal{B}(n)} \prod_{v\in T} \frac{n!}{2^n}\left(1+\frac{1}{h_v}\right),$$

where the sum ranges over all plane binary trees $\mathcal{B}(n)$ with n vertices and h_v is denotes as before the descendants of v including itself. (A binary tree is a rooted unlabelled tree in which each vertex has at most two children.)

Various proofs and generalizations inspired by this formula appeared. For instance Han [**40**] proved two further identities which have the interesting property that hook lengths appear as exponents.

Theorem 39. [**40**] *For each $n \geq 1$ we have*

$$\sum_{T\in\mathcal{B}(n)} \prod_{v\in T} \frac{1}{h_v 2^{h_v-1}} = \frac{1}{n!} \tag{9}$$

and

$$\sum_{T\in\mathcal{B}(n)} \prod_{v\in T} \frac{1}{2h_v+1} 2^{2h_v-1} = \frac{1}{(2n+1)!} \tag{10}$$

$B(n)$ denoted the set of all binary trees on n nodes.

Han proved this formulas by recurrence relations. Yang [**93**] generalized the second formula for k–ary trees using generating functions. (A k–ary tree is a rooted unlabelled tree where each vertex has exactly k subtrees in linear order where we allow a subtree to be empty.)

Theorem 40. [**93**]

$$\sum_{T\in\mathcal{K}(n)} \prod_{v\in T} \frac{1}{h_v k^{h_v-1}} = \frac{1}{n!}$$

Sagan proved these formulas using probabilistic approach and Chen, Gao and Guo [**19**] presented combinatorial proofs.

Han [**41**] developed an expansion technique for deriving hook formulas for binary trees. Chen, Gao and Guo [**20**] extended Han's technique and obtained expansion formulas for other classes of trees. Most of the identities that the authors derived are fairly complicated, but some are simpler, such as

$$\sum_{F\in\mathcal{F}([n])}\prod_{v\in F}\frac{1}{h_v^2}=\frac{(n+1)!}{2^n},$$

where the sum runs over all labelled forests with labels in $[n]$, and

$$n!\sum_{T\in\mathcal{T}(n)}\prod_{v\in T}\left(1-\frac{1}{h}\right)^{h-1}=(n-1)^{n-1}.$$

where the sum runs over all rooted trees with n vertices. In a note Eriksen [**25**] has given combinatorial proofs for these formulas, but we hope that our bijections can be used to find different bijective proofs.

CHAPTER 4

312–avoiding permutations

1. Introduction

Pattern avoidance is a central problem in recent research in enumerative combinatorics. The first surprising result ([**11**], [**87**]), that the number of permutations of $[n]$ that avoid a pattern of length 3 is equal to the n–th Catalan number C_n in all the 6 cases.

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

It is important to note that the number of all permutations of $[n]$ is $n!$ and c_n is a much smaller number, that can be easily bounded by an exponential function of n.

Stanley and Wilf formulated in the late 1980's independently the famous Stanley–Wilf conjecture, that states for all permutations pattern τ there exists a constant $c = c_\tau$ such that

$$|S_n(\tau)| \leq c_\tau^n,$$

where $S_n(\tau)$ denotes the set of all permutations of $[n]$ that avoid the pattern τ. The stronger form of the conjecture is that the following limit exists and

$$\lim_{n\to\infty} \sqrt[n]{S_n(\tau)} = c_\tau < \infty.$$

The two forms are by the result of Arratia [**3**] equivalent. Marcus and Tardos [**65**] has given an explicit bound exponential in the length of τ:

$$c_\tau \leq 15^{2k^4\binom{k^2}{k}},$$

where $\tau \in S_k$. The topic is still active. A recent result of Fox [**29**] showed that the c_τ limit is typically exponential.

The story of the first proof of the Stanley–Wilf conjecture is interesting. The Füredi–Hajnal conjecture [**32**] (1992) states that for all $n \times n$ permutation matrices the number of 1 entries is $O(n)$. Klazar [**52**] (2000) exhibited that the Füredi–Hajnal conjecture implies the Stanley–Wilf conjecture. Finally Marcus and Tardos [**65**] (2004) proved the Füredi–Hajnal conjecture. This story shows that research fields meet sometimes on crucial points an unexpected way and enhances the importance of considering problems from different point of views.

Extensions of the notion of pattern avoidance in permutations led to the study of pattern avoiding matchings, set partitions, ordered graphs, matrices etc.

Very often the basic ideas can be used to tackle the general problem. For this reason we studied the 312–avoiding permutations.

Our main result in this chapter is a simple bijection between 312–avoiding permutations and the other well known Catalan family, the triangulations of a polygon.

There are many bijections between Catalan families. (See for instance [**22**], [**54**], [**88**], [**86**]). However we can not find any direct description of a bijection between triangulations and 312–avoiding permutations in the literature. Though our bijection can be constructed as a composition of known ones, we think that it is worthwhile to formulate this direct bijection. On the one hand it has a nice, simple description: we label the vertices of the underlying $(n+2)$–gon and project this labelling to the triangles. We define a special code word (permutation) for the labelled triangles. The way triangles are attached to each other (without any intersection in their interior) corresponds to the fact that the code permutation can not contain the pattern 312. On the other hand our bijection emphasizes the role of the inversion table of a 312–avoiding permutation, which we think is a new observation in this area. See Observations 42. and 46.

Another feature of our coding is that it can be applied to k–triangulations ([**45**], [**70**], [**91**]). In [**70**] a geometric view of k–triangulations is presented in which $(2k+1)$–stars are substituted for the triangles. The projection of the vertex labelling of the underlying polygon to a star labelling is natural. Also there are several natural methods of coding the labelled stars into words. To exploit this observation remains an open problem.

The literature about Catalan families is very rich. We don't even attempt to give a complete picture of the previous work on the topic, but in this section we collect some related results.

Possibly the best known bijection involving triangulations is given in [[**88**], Prop. 6.2.1 and Cor. 6.2.3.]. This translation of a triangulation to a binary tree is very natural and elegant.

In [**54**] one finds nice correspondences between several Catalan families, such as binary trees, nested parentheses, Dyck–paths and certain integer sequences. Also included in Knuth's correspondences is a certain class of permutations which is characterized by conditions on its inversion table. This turns out to be the class of 312–avoiding permutations, though Knuth makes no note of that.

The bijection we present in our work is a composition of one given by Knuth and one given by Stanley. When we modify (by reflection) the binary tree that represents the colex forest in Knuth's work we obtain a binary tree that can be translated to a triangulation by the standard bijection given by Stanley. The bijection of our Theorem 44. is precisely the composition of these two bijections.

Inversions are central in the theory of permutations. We introduce the inversion diagram of a 312–avoiding permutation. The special form of the diagram visualizes the conditions of the inversions of a permutation in that the pattern 312 does not appear. We define the two natural orderings on the set of 312–avoiding permutations using the inversion tables (we call them s–resp. c–vectors) of the permutation. These are the Tamari resp. the Dyck lattices, two well known Catalan lattices. Thanks to our approach the known relationship between these two lattices are better understandable and the link between Tamari lattice (resp. Dyck lattice) and the weak (resp. strong) Bruhat order on the set of the symmetric group is quite clear. The importance of these lattices is shown by the fact that there are many combinatorial objects that are enumerated by the intervals of the Tamari or the Dyck lattice. Our unified interpretation allows to define bijections between these objects and appropriate pairs of 312–avoiding permutations. Since permutations are good candidates to code combinatorial objects these bijections could have a simple description. We present a bijection in this spirit in order to prove the result of Jelinek [**44**].

The outline of this chapter is as follows. We start with a brief introduction about the ubiquitous sequence of enumerative combinatorics, about the Catalan numbers. In Section 3. we describe the bijection and in Section 4. we give the proof of it. Section 5. is devoted to the description of the inversion tables of 312–avoiding permutations, that leads to definitions of Catalan lattices, which are the theme of Section 6. Finally in Section 7. we present a new bijection between *abccab*–avoiding matchings and certain pairs of 312–avoiding permutations.

2. Catalan numbers

The early history of Catalan numbers includes contradictory accounts. Catalan numbers are named after Eugene Charles Catalan (1814–1894), based on his work concerning the number of possible ways bracketing a product of terms. Leonhard Euler (1704–0783) proposed the problem of dissection of a polygon by means of noncrossing diagonals in a letter to Christian Goldbach (1690–1764). He computed the first few Catalan numbers, suggested an explicit product formula and a generating function. The hungarian mathematician János Segner (1704–1777) played also an important role in the story. He found and proved a recurrence relation between Catalan numbers.

Problems related to this sequence appear very often at various part of mathematics. Richard P. Stanley has given first a list of combinatorial objects that are enumerated by Catalan numbers in his book Enumerative Combinatorics 2 [**88**]. This list is continually supplemented on Stanley's homepage [**86**]. The number of known combinatorial interpretations of C_n is over 200.

In Sloane's Online Encyclopedy of Integers Sequences [**85**] A000108. The first values are:

$$1, \quad 1, \quad 2, \quad 5, \quad 14, \quad 42, \quad 132, \quad 429, \quad 1430, \quad 4862, \quad 16796, \quad 58786, \dots$$

The following short list includes some basic expressions, that can be viewed as definitions of the Catalan sequence.

- Explicit formulas:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \prod_{i=0}^{n} \frac{n+i}{i}$$

- Recursions:

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \tag{11}$$

  or

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n. \tag{12}$$

- The generating function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

We recall some well known enumeration problems counted by the Catalan numbers that will play a role in our latter work.

- Eugene Catalan raised the following question: given $a_0, a_1, \dots, a_n$ $n \geq 0$. How many different ways can the product completely parenthesized? (this is the same as the number of associating n applications of a binary operator.)

$$(((a_0)(a_1)a_2)((a_3)a_4)a_5)$$

  It is easy to establish a recursion: for $n = 0$ it is clearly one way to parenthesize one factor (a_0): $C_0 = 1$.

  Let $n \geq 1$. Let a_k be the first factor such that every opener parenthesis before a_k is closed right after a_k. Then there are C_k possibilities to parenthesize the first k factors and C_{n-k} ways to parenthesize the last $(n-k)$ factors. Hence the recursion (11) holds.

- The first enumeration problem for that the Catalan numbers as answer appeared is the *triangulation* of a polygon. Precisely: C_n is the number of different ways a convex polygon with $(n+2)$ sides can be cut into triangles by nonintersecting diagonals.

  Given a triangulated polygon P with $n+2$ mark one of the side as the *base*. Further choose one of the $2n+1$ diagonals and give an orientation of this diagonal. There are $(4n+2)C_n$ possibilities to do this. Given a triangulated polygon Q with $n+3$ sides, choose one side as a base and mark a further side. There are $(n+2)C_{n+1}$ ways to do this. There is a simple bijection between these two kinds of decorated triangulations: Collapse the triangle in Q, which has the chosen side common with the polygon in order to receive the triangulated polygon P with the same base and an oriented marked diagonal. In reverse expand the oriented diagonal in P to a triangle and mark the new side. Hence the recursion (12) holds.

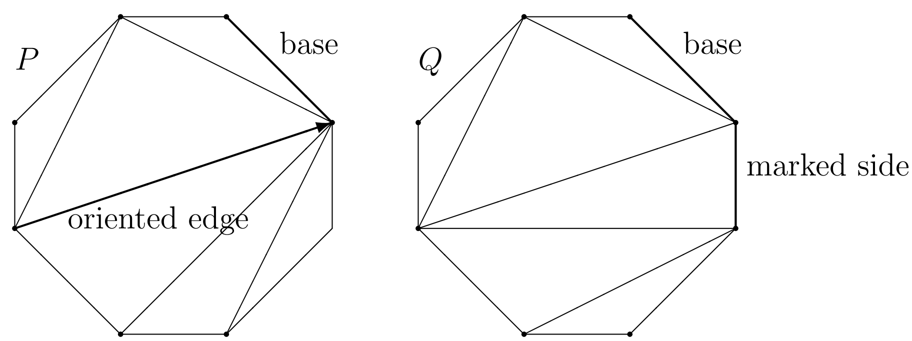


FIGURE 1. Illustration of the recursion for triangulations

- One of the most popular combinatorial interpretation of the Catalan sequence are Dyck–paths. A *Dyck path* of size n is a lattice path in the plane integer lattice $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ consisting of n up steps of the form $(1,1)$ and n down steps of the form $(1,-1)$ which never goes below the x–axis.

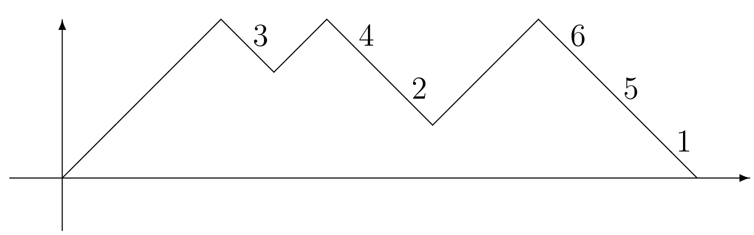


FIGURE 2. The Dyck path corresponding to $((()())(()))$

The close relation to parenthesis is obvious: an up step corresponds to an opener parenthesis and a down step to a closer parenthesis.

- Probably one of the funniest description of a combinatorial correspondence is the one that relates Dyck paths to plane trees. If we imagine

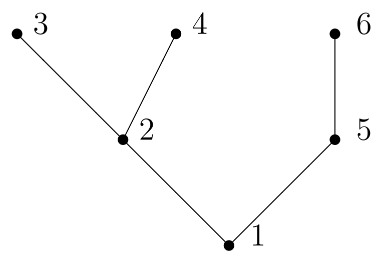


FIGURE 3. The plane tree corresponding to the Dyck path in Figure 2.

a worm that crawls around the periphery of the plane tree starting at the root drawing an up step whenever it passes the left side of an edge and drawing a down step whenever it passes the right side of an edge that worm will have reconstructed a corresponding Dyck–path.

- The last classical Catalan family we mention is the set of 312–avoiding permutations. The permutation π is called *312-avoiding* ($\pi \in S_n(312)$) if there are no indices $i < j < k$ with $\pi_j < \pi_k < \pi_i$. For instance $342651 \in S_6(312)$. The correspondence to trees are described in [**54**] without mentioning that these permutations are exactly the 312–avoiding permutations. See [**54**] for further details.

3. 312-avoiding permutations, inversion tables and triangulations

Let us give a quick review of the terminology.

We denote by S_n the symmetric group of all permutations of the set $[n]$. We write a permutation $\pi \in S_n$ in one-line notation as a word $\pi = \pi_1\pi_2\cdots\pi_n$ of length n where $\pi_i = \pi(i)$. A *subword* of π is a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ of π with $i_1 < i_2 < \cdots < i_k$. Let $\pi \in S_n$ and $\tau \in S_k$; then π is called *τ-avoiding* if π does not contain a subword of length k having the same relative order as τ. We denote the set of τ-avoiding permutations by $S_n(\tau)$.

We denote by $\mathcal{T}_n$ the set of triangulations of a convex $(n+2)$-gon into n triangles by diagonals that do not intersect in their interior. The number of such triangulations is equal to the n-th Catalan number.

A pair (π_i, π_j) is called an *inversion* of the permutation $\pi = \pi_1\pi_2\cdots\pi_n$ if $i < j$ and $\pi_i > \pi_j$. The *inversion table* of the permutation π is an n-tuple of integers $\underline{s} = (s_1, s_2, \ldots, s_n)$ where s_k is the number of elements that are greater than k and are to the left of it.

$$s_k = |\{\pi_i | \pi_i > k = \pi_j \text{ and } i < j\}|.$$

Clearly it is true that $0 \leq s_k \leq n - k$ for $1 \leq k \leq n$.

Observation 41. *The inversion table of a 312-avoiding permutation* $\pi = \pi_1\pi_2\cdots\pi_n$ *satisfies the following condition:*

$$s_{k+i} \leq s_k - i \quad \textit{for} \quad 1 \leq k \leq n-2 \quad \textit{and} \quad 1 \leq i \leq s_k.$$

Furthermore every ineteger sequence $(s_1, \ldots, s_n)$ *with the conditions*

(s.1) $0 \leq s_k \leq n - k$ *for* $1 \leq k \leq n$
(s.2) $s_{k+i} \leq s_k - i$ *for* $1 \leq k \leq n-2$ *and* $1 \leq i \leq s_k$.

defines uniquely a 312*–avoiding permutation of* $[n]$.

Namely, when $1 \leq i \leq s_k$, then $k+i$ is to the left of k, but also the elements $k+1, \ldots, k+i, \ldots, (k+i) + s_{k+i}$ are to the left of k.

We note that this observation is very crucial for us. It is implicit in several of our references but we couldn't find it stated explicitly.

In this section we give a simple bijection between the sets $\mathcal{T}_n$ and $S_n(312)$. We label the vertices of the $(n+2)$-gon with the numbers $\{0, 1, \ldots, n, n+1\}$ in clockwise order. We mark the vertices of the triangle Q by A_Q, B_Q, C_Q so that $l(A_Q) < l(B_Q) < l(C_Q)$ where $l(P)$ denotes the label of the vertex P in the $(n+2)$–gon. We refer to these as the *first* (A_Q), *middle* (B_Q) and *last* (C_Q) vertices of the triangle.

Lemma 42. *In each triangulation, for every* $i \in \{1, 2, \ldots, n\}$ *there is exactly one triangle* Q *where the middle vertex is* i *(*$l(B_Q) = i$*).*

Proof. Assume that there exist two triangles P and Q with $l(B_P) = l(B_Q)$. Then without loss of generality

(1) $l(A_P) = l(A_Q) < l(B_P) = l(B_Q) < l(C_P) < l(C_Q)$ or
(2) $l(A_P) < l(A_Q) < l(B_P) = l(B_Q) < l(C_P), l(C_Q)$.

In the first case the sides $[A_P, C_P]$ and $[B_Q, C_Q]$ cross each other. In the second case the sides $[A_P, B_P]$ and $[A_Q, C_Q]$ cross each other. This is a contradiction to the fact that P and Q are triangles in a triangulation. □

With the help of this observation we can define a map w from the set of triangulations to the set of 312–avoiding permutations.

Coding algorithm

Input: a triangulation T

Output: a permutation of $\{1, 2, \ldots, n\}$

1. Label the triangles according to their middle vertex
2. For $i = 2, \ldots, n+1$ do the following:

   Consider the labels of the triangles such that the last vertex has label i. List these labels in decreasing order (this is the same as the counter clockwise order of the triangles meeting at vertex i).

The length of the listing increases as we process the vertices. The output ($w(T)$) is the list after examining the last vertex ($n+1$). Then $w(T)$ contains the labels of all the triangles in some order. An example is given in Figure 4.

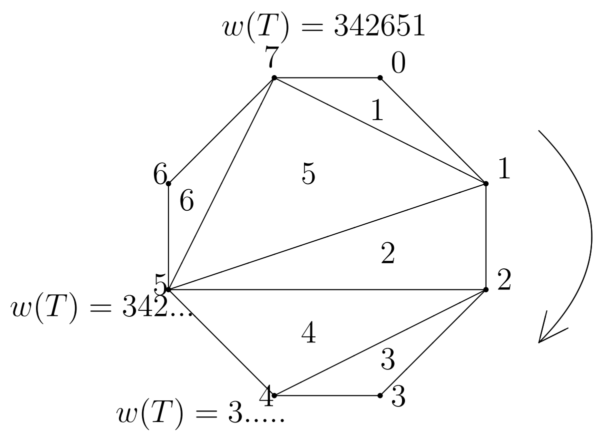


Figure 4. The bijection

Theorem 43 (Main theorem)**.** *The map w is a bijection between the set of triangulations of a convex $(n+2)$-gon and the set of 312-avoiding permutations of $[n]$.*

4. Proofs and remarks

Lemma 44. *The word $w(T)$ is a* 312*-avoiding permutation of $[n]$ (we consider permutations as words).*

Proof. Note that for two triangles P and Q with $l(B_P) < l(B_Q)$ and $l(C_P) = l(C_Q)$ the algorithm records the label of Q before the label of P. Assume that $w(T)$ is not 312-avoiding. Then there are three triangles P, Q, R with

$$l(B_P) < l(B_Q) < l(B_R) \text{ and } l(C_R) \leq l(C_P) < l(C_Q).$$

But then the sides $[B_P, C_P]$ and $[B_Q, C_Q]$ cross each other, a contradiction. $\square$

Observation 45. *Let T be a triangulation. Take the triangle labelled by i. Its $[B_i, C_i]$ side determines the i–th condition of the inversion table of the permutation $w(T)$:*

$$s_i = l(C_i) - l(B_i) - 1.$$

Proof of the main theorem. We prove the theorem by defining the inverse map of w. We use Observation 46. in constructing the decoding algorithm.

Decoding algorithm:

Input: a 312-avoiding permutation π
Output: a triangulation T
0. Compute the inversion table $(s_1, s_2, \ldots, s_n)$ of π.
1. Let the triangle with label 1 be: $l(A_1) = 0$, $l(B_1) = 1$, $l(C_1) = s_1 + 2$
i. $(i = 2, \ldots, n)$ Let the triangle with label i be $l(B_i) = i$, $l(C_i) = s_i + (i+1)$, and $l(A_i)$ minimal so that the $A_iB_iC_i$ can be a triangle of a triangulation ($[A_i, B_i]$ and $[A_i, C_i]$ do not create any new crossings).

From the properties of the inversion table of a 312–avoiding permutation it follows that the sides of the triangles do not cross each other and so the algorithm determines a unique triangulation.

Remark 1. Using the complement and reverse operations we can modify the bijection w. Recall that for a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ the *reverse* of π is defined as $\pi^r = \pi_n\pi_{n-1}\ldots\pi_1$ and the *complement* of π denoted by π^c as the permutation whose i-th entry is $\pi_i^c = n + 1 - \pi_i$.

We generate the reverse permutation when we read off the word moving around the polygon counter-clockwise, and we generate the complement permutation when we modify the labelling of the triangles so that the triangle Q gets the label $n + 1 - k$ when its middle vertex is k ($l(B_Q) = k$).

With these modifications we generate bijections between the set of triangulations and $S_n(213)$, $S_n(132)$, $S_n(231)$, since if a permutation avoids the pattern 312, then its reverse avoids the pattern 213, its complement avoids the pattern 132 and the reverse of its complement avoids the pattern 231 (see [**11**]).

The permutations in S_3 form actually two basic sets:

$$\{123, \quad 321\} \qquad \text{and} \qquad \{132, \quad 213, \quad 231, \quad 312\}.$$

It is obvious that if a permutation avoids 123, then its reverse avoids 321 thus

$$|S_n(123)| = |S_n(321)|.$$

Similarly if a permutation avoids 132, then its reverse avoids 231, its complement avoids 312 and the reverse of its complement avoids 213. Therefore we have

$$|S_n(312)| = |S_n(231)| = |S_n(213)| = |S_n(132)|.$$

So if we pick one candidate from each of the two basic classes and define a bijection between them, it is proven that all permutation patterns of length 3 are wilf–equivalent.

In the survey of Claesson and Kitaev [**51**] the authors described the known bijections, that have been given to prove that the permutations that avoid one of the two basic classes are equinumerous, with statistical characterizations and relations to each other.

Supplementing our bijection with a similar direct simple bijection between triangulations and 123–avoiding permutations would enrich the garden of the proofs of the wilf–equivalence of $\tau \in S_3$.

With a further modification of the algorithm we can generate the inversion of a permutation. To this end we have to label the triangles according their last vertex and define the code word according the last vertices moving around the polygon clockwise.

Remark 2. As we already noted our coding algorithm can be used in several different ways to map k–triangulations to permutations. The most natural way is to label $(2k+1)$–stars by their middle vertex. The code words can be based on the $(k+2)$nd, $(k+3)$rd, ..., last vertices. This suggests two open problems. Is it true that the above defined map of k–triangulations to k–tuples of permutations is an injection? Can one characterize the image of this map?

5. The inversion tables of 312–avoiding permutations

We have seen that the inversion table $(s_1, s_2, \ldots, s_n)$ played a crucial role in the bijection we presented in the previous sections.

Actually there are two different ways to record the inversions in a permutation and hence two different kind of integer sequences arise: we can take in account the number of greater elements to the left of the actual element, or the number of less elements to the right of it. In both cases the permutation is uniquely defined by the recorded sequence. In the case of 312–avoiding permutations both sequences are determined by two conditions. In the previous section we defined the inversion table $(s_1, \ldots, s_n)$. The similar way we define the other type of inversion table $(c_1, \ldots, c_n)$ (in [**87**] this associated sequence is called *the code* of a permutation) where c_k the number of elements in π that are to the right of k and are less than k. Formally

$$c_k = |\{\pi_i : \pi_i < k = \pi_j \quad \text{and} \quad i > j\}|.$$

We refer to this kind of inversion table as the *c–vector* of the 312– avoiding permutation. It fulfils the following two conditions:

(c.1) $c_1 = 0$
(c.2) $0 \le c_{k+1} \le c_k + 1$ for $1 \le k < n$.

The condition (c. 1) is trivial, the second is easy to see. In a 312–avoiding permutation it is forbidden that whenever the element $(k+1)$ is to the left of k then an element less than k appears between k and $(k+1)$. This condition is (c. 2).

The *inversion diagram* is a triangular shape with 1 cell in the top–most row, 2 cells in the row below, etc. and $n-1$ cells in the $(n-1)$th row. We mark every inversion (i, j) of the permutation by an X entry in the cell (i, j). Hence s_k is simply the number of the X entries in the k'th column and c_k is the number of the X entries in the k'th row. The diagram we obtained has a

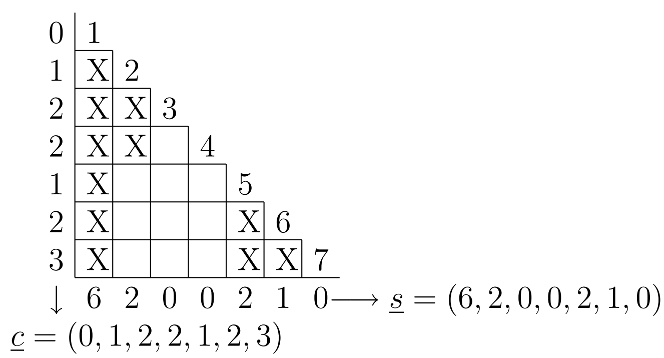


FIGURE 5. The sequences for $\pi = 3427651$

special form that rooted on the fact that in the permutation the pattern 312 can not appear. The X entries are arranged in each column in one block of the form $B_{i,l} = \{(i, j) : 1 \le i \le l\}$ and each such block has a ,,shadow" in that the occurence of X any entry is forbidden. The block of length l in the i'th column (denoted by $B_{i,l}$) has the shadow:

$$\{i + l + 1, i + l + 2, \ldots, n\} \times \{i + 1, i + 2, \ldots, i + l\}.$$

We describe a bijection between 312–avoiding permutations and Dyck paths using the following trick on the diagram: slide all X entries to the right of the diagram. The boarder of the X entries determines the corresponding Dyck path. (See Figure 6.)

Our definition reveals the fact that the area of a Dyck path (the number of full squares ,,below" the path) coincides with the number of inversions in the corresponding 312–avoiding permutations.

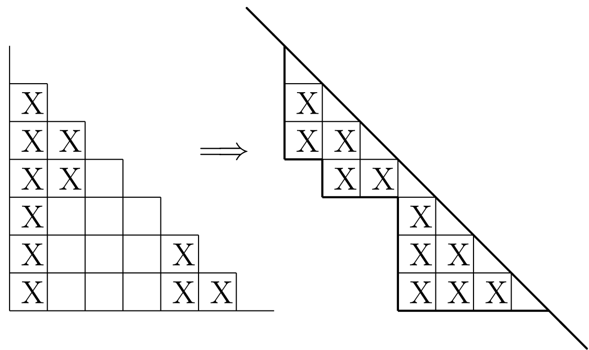


FIGURE 6. Inversion diagram and Dyck–path

6. Intervals in Catalan lattices

The study of Catalan sets led to definitions of different orderings based on natural parameters in the case of a special Catalan family as for instance:

- Tamari lattice – defined on triangulations
- Dyck–lattice – defined on Dyck–paths
- Kreweras lattice – defined on noncrossing partitions
- phagocyte lattice – defined on Dyck words
- pruning–gafting lattice – defined on binary trees.

It is well known that the three classical lattices can be defined on the set of plane trees (see [**54**]) or on the set of Dyck–paths (see [**8**]) in such a way that the Dyck-lattice is an extension of the Tamari lattice which in turn is an extension of the Kreweras lattice.

A rather complicated description can be found in [**24**] recovering the well known fact that the weak and the strong Bruhat order on 312–avoiding permutations are isomorphic to the Tamari resp. Dyck lattice.

We supplement the study of relations between these lattices with the observation on the inversion tables of a 312–avoding permutations. The natural ordering on the set of sequences $(s_1, \ldots, s_n)$ resp. $(c_1, \ldots, c_n)$ defines the Tamari lattice resp. the Dyck lattice and hence this unified approach contributes to the understanding of the connections between them.

6.1. Tamari lattice. We consider first the Tamari lattice. Tamari lattice was introduced by D. Tamari (1962) as a partial order on the set of different ways of setting parenthesis. (One grouping is ordered before another if the second grouping may be obtained by only rightward application of the associative law $(xy)z = x(yz)$.)

Here we give the definition of the Tamari lattice on the set of 312–avoiding permutations using the inversion tables $(s_1, \ldots, s_n)$ (We denote by $s(\pi)$ the s–vector of the permutation π).

Definition 7. Consider the set of 312–avoiding permutations with the order relation

$$\pi \leq_s \sigma \quad \text{iff} \quad s(\pi) \leq s(\sigma),$$

where $s \leq s'$ is the usual relation on vectors: $(s_1, \ldots, s_n) \leq (s'_1, \ldots, s'_n)$ iff $s_j \leq s'_j$ for all $1 \leq j \leq n$.

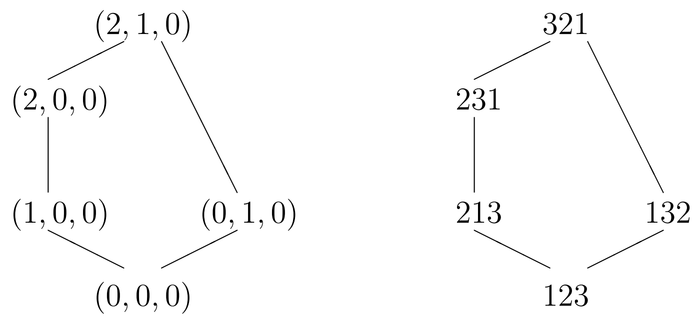


FIGURE 7. The Tamari lattice with five elements

We find different definitions of the Tamari lattice in the literature. Probably the best known is the poset of triangulations $\{\mathcal{T}_l\}$ ordered by diagonal flip operation that substitute one diagonal of the polygon for another. Other examples are:

- the poset of rooted trees $\{\mathcal{F}_l\}$, in which $F_1 \leq F_2$ if for every k the k–th node in a preorder traversal of F_1 has at least as many descendants as the k–th node in a preorder traversal of F_2. [**54**]
- the poset of Dyck paths $\{\mathcal{D}_l\}$ in which $D_1 \leq D_2$ for every k, the length of the k–th tunnel in D_1 is less than in D_2. [**8**]

By bijections it is clear that the s–vector plays the crucial role in each case. This is obvious for rooted trees: s_k is the number of descendants of the k–th vertex (in preorder). According the correspondence between Dyck–paths and rooted trees the interpretation of the s–vector considering Dyck–paths is also clear: s_k is the length of the k–th tunnel (the tunnel with the k–th up step).

Remark 1. In [[**10**], Section 9] Björner and Wachs studied the Tamari lattice. They discuss – as they remark – a surprisingly close connection that exists between Tamari lattices and weak order on the symmetric group. By defining a map from permutations to binary trees they showed that the sublattice of the weak order consisting of 312–avoiding permutations (and thus the Tamari lattice) is a quotient of the weak order in the order theoretic sense.

We remark here that in view of our bijection this fact is a natural observation. It is interesting that though in [**10**] the Tamari lattice is actually defined on the set of inversion tables of 312–avoiding permutations, this simple fact is not mentioned there.

Remark 2. Reading [**76**] continued to study the Tamari lattice. He defined a direct map from permutations to triangulations that is identical to the one given in [**10**] up to the standard bijection from triangulations to binary trees. A generalization of this concept led to the introduction of Cambrian lattices.

From the combinatorial point of view the enumeration of intervals (pairs of comparable elements) of a poset is often interesting because they may correspond to combinatorial objects.

Chapoton [**18**] determined the number of intervals of the Tamari lattice using generating function approach. Let denote $I_n^{\mathcal{T}}$ the set of intervals in the Tamari lattice .

$$|I_n^{\mathcal{T}}| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

with the first values ([**85**] A000260):

$$1, \quad 3, \quad 13, \quad 68, \quad 399, \quad 2530, \quad 16965, \quad 118668, \ldots$$

Chapoton noticed that this is the number of maximal planar maps and asked for an explanation. Since the faces of a maximal planar map including the outer one are bounded by three edges maximal planar maps are called alternatively ,,triangulation".

Bernardi and Bonichon [**8**] investigated realizers of maximal planar maps and established a bijection between special pairs of Dyck–paths and minimal realizers of size n.

Open Problem 4. *Is it possible to give a simple bijective proof using pairs of* 312*–avoiding permutations (or equivalently inversion tables), or eventually pairs of triangulations of polygons to give a combinatorial explanation to answer Chapoton's question?*

6.2. Dyck lattice. The Dyck lattice or Stanley lattice appears naturally on the set of Dyck–paths considering the ordering relation $D_1 \le D_2$ if D_1 stays below the path D_2. We give here the definition of Dyck lattice on the set of 312–avoiding permutations using the c–vectors.

Definition 8. Consider the set of 312–avoiding permutations with the order relation

$$\pi \le_c \sigma \quad \text{iff} \quad c(\pi) \le c(\sigma),$$

where $c \le c'$ is defined as usual $(c_1, \ldots, c_n) \le (c'_1, \ldots, c'_n)$ iff $c_j \le c'_j$ for all $1 \le j \le n$.

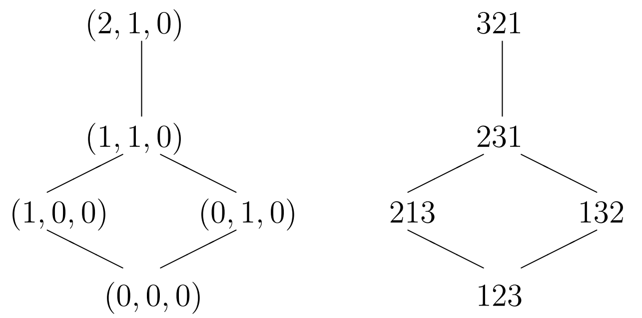


FIGURE 8. The Dyck lattice with five elements

Our description of the well known bijection between Dyck–paths and 312–avoiding permutations make it obvious that this definition is equivalent the one above.

Noncrossing Dyck paths are well studied combinatorial objects. The number of k noncrossing Dyck paths of size n is given by the determinant formula of Lindström–Gessel–Viennot [**34**]:

$$\det\begin{pmatrix} C_{n-2} & \dots & C_{n-k-1} \\ \vdots & \ddots & \vdots \\ C_{n-k-1} & \dots & C_{n-2k} \end{pmatrix} = \prod_{1\le i\le j<n-2k} \frac{i+j+2k}{i+j}.$$

Hence $|I_n^{\mathcal{D}}|$, the number of intervals of the Dyck lattice, which are pairs of noncrossing Dyck–paths, is:

$$|I_n^{\mathcal{D}}| = \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix} = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}, \tag{13}$$

The first values are ([**85**] A005700):

$$1, \quad 3, \quad 14, \quad 84, \quad 594, \quad 4719, \quad 40898, \quad 379236, \dots$$

The list of combinatorial objects that are enumerated by this sequence is long and seems to grow. We give here a short (not complete) list with references: 2-triangulations [**45**][**70**], reduced pipe dreams of $\pi_{2,n}$ [**91**], 3-noncrossing and 3-nonnesting matchings, oscillating tableaux [**21**], realizers of triangulations [**8**]. (See references for details.)

Open Problem 5. *Is it possible to give a simple bijective proof using pairs of* 312*–avoiding permutations (or equivalently inversion tables) to any of the results above?*

In the next section we present one example of such a bijection.

7. Bijection between pattern avoiding matchings and pairs of permutations

We define a simple bijection between matchings of $[2n]$ that avoid the matching $\{\{1,5\},\{2,6\},\{3,4\}\}$ which we denote by $M_n(abccab)$ and pairs of 312–avoiding permutations, which represents an interval in the Dyck–lattice. (This bijection coincides with the one described in Jelinek's [**44**] work when we turn the pair of 312-avoiding permutations into pairs of noncrossing Dyck–paths.)

Now we recall some basic definitions.

A *matching* M of size n is a graph on the vertex set $[2n]$ whose every vertex has degree one. For an arbitrary edge $e = \{i,j\}$ of M $i < j$ we say that i is an *l–vertex* and j is an *r–vertex* of M.

The linear order of the left and right vertices of the matching M defines a word $w \in \{0,1\}^{2n}$ such that $w_i = 0$ if i is an l–vertex of M and $w_i = 1$ if i is an r–vertex of M. We say that $w = b(M)$ is the *base* of the matching M. The base of a matching of $[2n]$ is a Dyck–word, since it fulfils the conditions:

- The word w contains exactly n of both bits.
- In every prefix w' of w is true: $|w'|_0 \geq |w'|_1$, where $|w|_i$ denotes the number of occurences of the bit i.

A matching $M = (V,E)$ *contains* a matching $M' = (V',E')$ if there is a monotone edge-preserving injection from V' to V; in other words, M contains M' if there is a function $f : V' \to V$ such that $u < v$ implies $f(u) \to f(v)$ and $\{u,v\} \in E'$ implies $\{f(u), f(v)\} \in E$.

Let $M_n(abccab)$ denote the matchings that avoid the matching pattern $\{\{1,5\},\{2,6\},\{3,4\}\}$.

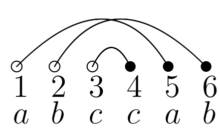


FIGURE 9. The matching pattern $\{\{1,5\},\{2,6\},\{3,4\}\}$

We present a new proof of the following theorem.

Theorem 46. [44]

$$|\mathcal{M}_n(abccab)| = |I_n^{\mathcal{D}}| = \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix}$$

Every matching $M \in \mathcal{M}_n(abccab)$ defines two permutations. As a preparation we label the l–vertices from left to right in their forecoming order.

First we ignore the edges and consider only the base as an arrangement of l– and r–vertices or Dyck–word. It is well known that a Dyck–word defines a unique noncrossing perfect matching on the $2n$ points [**88**] and a noncrossing matching corresponds to a 312–avoiding permutation.

We describe this correspondence by the way how we can recognize the permutation σ directly. We extract the inversion table $c(\sigma)$ from the base. We associate to each l–vertex a number the following way: c_k is $k-1$ decreased by the number of r–vertices to the left of the kth l–vertex. Let σ be the permutation with the so defined c–vector. Since this c–vector fulfills the two conditions $(c.1)$ and $(c.2)$ the permutation σ avoids the pattern 312. Alternatively c_k counts the arcs above the arc incident to the kth l–vertex in the corresponding noncrossing matching.

We obtain the permutation π if we consider M as a perfect matching of the bipartite graph. We label the edges according to their l–vertices, then we move from left to right and at each r–vertex we record the label of the incident edge. It is clear that π is a 312–avoiding permutation.

Lemma 47. *For the permutations π and σ that are ordered to $M \in \mathcal{M}_n(abccab)$ by the bijection described above is true that*

$$c(\pi) \leq c(\sigma).$$

Proof. Hence in a correspoding matching c_k is the number of arcs that are above the kth arc. Since the noncrossing matching is maximal in this sense the statement holds. □

The Figure 10. shows the bijection with

$$M = \{\{1,9\},\{2,3\},\{4,7\},\{5,6\},\{8,14\},\{10,12\},\{11,13\},\{15,16\}\}$$
$$(\pi,\sigma) = (24316758, 24357618)$$
$$c(\pi), c(\sigma) = (0,1,1,2,0,1,1),(0,1,1,2,1,1,2)$$

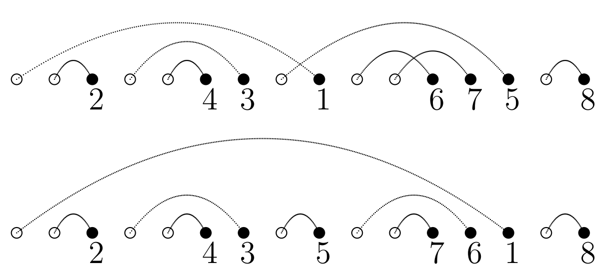


Figure 10. The bijection

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Summary

In this thesis we obtained combinatorial proofs in three different areas. Since the three topics are separated problems in enumerative combinatorics, we found it necessary to make the reader acquainted with the issues of the given problems. Hence each chapter started with important informations about the considered question, beside the definitions and notations significant results of the actual research. Our own results followed these introductions. At the end of each chapter we presented related, open problems that require further investigations. Considering these questions in our opinion is a natural extension of our work. It is possible that the methods we used will help in attacking these problems.

After a first introductory chapter we investigated combinatorial questions that arise in connection to poly–Bernoulli numbers. The poly–Bernoulli number $B_n^{(k)}$ is a natural generalization of the classical Bernoulli number and was introduced in 1997 by Kaneko [**47**].

It turned out that for negative parameter $B_n^{(k)}$ are positive integers. First we presented a summary of known combinatorial descriptions that can be found sporadic in the literature. Some of the interpretations were born before poly–Bernoulli numbers were introduced, some other are remarks without proof on the internet. Our survey is the first self–contained, complete list of these problems, where all the bijections are described or sketched.

Our starting point was an obvious/trivial interpretation as an ordered pair of partitions of two sets. The first known non–trivial interpretation of $B_n^{(k)}$ was given by Brewbaker [**13**] as the number of $n \times k$ lonesum matrices. There exist three interpretations in terms of permutations. We illustrated the relation between lonesum matrices and Callan permutations with a bijection. The ascending–to–max permutations, that play a crucial role in the characterization of suffix arrays, can be viewed as the value–position dual of Callan permutations. A substantially different class of permutations is the Vesztergombi permutation, that are enumerated by the poly–Bernoulli numbers as well. The connection between these permutation classes is not obvious. Graph theory plays a role in this connection. Finally we give a very recent interpretation in terms of graphs. The poly–Bernoulli numbers enumerate the acyclic

orientations of the complete bipartite graphs. This graph theoretical interpretation has close links to lonesum matrices. This connection is not surprising: graphs and 01 matrices has a long history of close connections. Using this interpretation we presented a new proof a formula of poly–Bernoulli numbers.

The numerous drastically different interpretations suggest that poly–Bernoulli numbers may play/occupy of central role.

Followed by this survey we presented a new interpretation: the Γ–free matrices. We established a bijection between Γ–free matrices and our obvious interpretation of poly–Bernoulli numbers. The importance of this new poly–Bernoulli family is shown in the fact that this is the only interpretation that gives a transparent combinatorial explanation for the recursive formula of poly–Bernoulli numbers. This recursion was only proved before by algebraic methods.

Our further result is a combinatorial proof of a summation formula for poly–Bernoulli numbers that was derived only algebraically. Our proof is based on an appropriate interpretation, on Callan permutations.

We closed this chapter with open problems from the intensive research on this area. These studies are mostly number theoretic or analytic. It includes generalizations and analogously defined number sequences. It remains an open question whether an analytical generalization as for instance special multi poly–Bernoulli numbers can be associated to a natural generalization of any combinatorial interpretation? Another interesting open problem is to find a combinatorial interpretation of the poly–Cauchy numbers.

In the third chapter we considered hook formula for plane trees. The hook formula is a compact formula that enumerates Standard Young Tableaux. Shifted Standard Young Tableaux and plane trees can be enumerated by a similar formula. In this thesis we considered plane trees.

In this case the hook formula counts the number of monotone labellings of the nodes of the tree or equivalently the number of linear extensions of the partial order that is naturally defined by a plane tree.

The hook formula is a nice combinatorial formula but it took time to find a satisfying bijection. Novelli, Stoyanovskii and Pak [**68**] presented the celebrated bijection for the case of Standard Young Tableaux, in which they used the *jeu de taquin* principle. We supplemented this line of research with the bijection for plane trees in this spirit. We ordered to a labeling without any restriction a pair of a monotone labelling and a hook function. As a first step we defined a total order on the nodes of the tree. Our algorithm visited the nodes in this linear order and changed the actual label step by step in order to obtain a monotone labelling. The necessary moves are coded in a hook function.

We presented another bijection in which the monotone labelling is regarded as a restricted permutation on the set of the nodes. Our bijection emphasized and used the fact that this restriction is hidden in the hook lengths of the nodes.

At the end of this chapter we collected some identities for different classes of trees that involve the hook lengths. The relation of our bijection and these formulas can be the subject of further studies.

The fourth chapter of this thesis contributed new results to the wealth of results on the Catalan sequence. Over 200 objects are known to be enumerated by the Catalan numbers. Numerous bijections can be defined between these sets. In our work we constructed a bijection between 312–avoiding permutations and triangulations. It was based on the inversion table of 312–avoiding permutation. This approach is important, because it shed a light of the structure of Tamari lattice.

Our main observation was that in a triangulation the triangles can be labelled according to the incidence of their middle vertex. Hence if we visit the vertices of the polygon in clockwise order and record at each vertex the label of the triangles which third vertex is incident with the actual vertex of the polygon, we obtain a permutation. We proved that this permutation avoids the pattern 312.

The $2k+1$–stars that build up a k–triangulation can be labelled the same way according to their middle vertex. In this case we obtain after recording the labels of the $k+1$th, $k+2$th, ... a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$. This thesis doesn't contain a description of this map.

We characterized 312–avoiding permutations by their inversion tables. The relation of the two tables are illustrated by the inversion diagram that we introduced.

In several enumeration problems arise the number of the intervals of the Tamari or Dyck lattice. We recalled some of these known results and defined the Tamari and Dyck lattice using the inversion tables of 312–avoiding permutations. It is a natural question whether there are bijections between these sets and appropriate pairs of 312–avoiding permutations. We presented one instance in order to show the usefulness of our idea. We defined a bijection between the set of pairs of 312–avoiding permutations and *abccab* avoiding matchings of ordered graphs.

Összefoglalás

A tézisben három különböző területen adtunk kombinatorikai bizonyításokat. Mivel ezek különálló területek, fontosnak tartottuk, hogy az olvasót minden esetben bevezessük az adott kérdéskörbe. Minden fejezetet a témára jellemző fontosabb információkkal kezdtünk. Így a definíciók és a jelölésrendszer ismertetése mellett a kutatás aktuális eredményeit is megemlítettük. Ezután olvashatóak a saját eredmények. Minden témát olyan kérdésekkel, nyitott problémákkal zártunk, melyek megvá- laszolása ugyan még további vizsgálatokat igényel, de eredményeink hozzájárul- hatnak a megoldásukhoz.

a bevezető fejezet után a második fejezetben a poly–Bernoulli számok kapcsán felmerülő kombinatorikai kérdéseket vizsgáltunk. A poly–Bernoulli számok $B_n^{(k)}$ a klasszikus Bernoulli számok egy természetes általánosítása, melyet 1997-ben vezetett be Kaneko [**47**]. Ha a k paraméter negatív szám, akkor a $B_n^{(k)}$ pozitív egész szám.

Először egy összefoglalást adtunk azokról a többé kevésbé ismert kombinatorikai interpretációkról, melyek az irodalomban igen szórványosan találhatóak csak meg. Némely kombinatorikai eredmény még a poly–Bernoulli számok bevezetése előtt született, némely eredmény csupán egy megjegyzés az interneten. Munkánk az első teljes összefoglalása a témához kapcsolódó problémáknak, melyekben az interpretációkat összekötő bijekciók is megtalálhatók részletesen vagy vázolva.

Kiindulópontunk a poly–Bernoulli számok triviális értelmezése volt. Az első ismert nem triviális interpretáció Brewbaker [**13**] munkájában olvasható, melyben megmutatta, hogy $B_n^{(k)}$ a $n \times k$ lonesum mátrixok számával egyezik meg.

Három lényegesen különböző permutációosztályt ismerünk, melyek számosságát a poly–Bernoulli számok adják. A Callan permutációk duálisaként fogható-ak fel a maximumhoz tartó permutációk, melyek a suffix array adatstruktúra jellemzésekor játszanak kulcsszerepet. Végül bár a Vesztergombi permutációk összeszámlálásakor is a poly–Bernoulli számokat kapjuk, már nem nyílvánvaló, hogy ez a permutációosztály milyen módon kapcsolható az előbb említettekhez. Bizonyításunkban a gráfelmélet néhány elemét használtuk.

A gráfelmélet egy másik kérdésköréhez is kapcsolatot találtunk. A poly–Bernoulli számok adják ugyanis a teljes páros gráf aciklikus orientációinak számát. Ez könnyen látható a lonesum mátrixokkal való szoros kapcsolatból. Felhasználva ezt az interpretációt kombinatorikailag bebizonyítottuk a poly–Bernoulli számok egy másik ismert formuláját.

A sokféle és lényegében eltérő megjelenési forma jelzi, hogy ez a számsor központi jelentőségű lehet.

Az ismert interpretációk után megmutattuk, hogy az ún. Γ–mentes bináris mátrixok összeszámlálásakor szintén a poly–Benoulli számok lépnek fel. Bijektív módon bizonyítottuk állításunkat erre vonatkozólag. Eredményünk fontossága abban is megmutatkozik, hogy ez az egyetlen olyan interpretáció, mely a poly–Bernoulli számok rekurzív definíciójára ad világos kombinatorikai magyarázatot.

Egy további eredményünk egy kombinatorikai bizonyítása egy olyan összefüggésnek, mely a poly–Bernoulli számok vizsgálatakor szembetűnik.

A fejezet lezárásaként a jelenleg igen intenzív kutatásokból szemezgettünk. Ezek a kutatások főként algebrai illetve számelméleti oldalról vizsgálják ezeket a poly–Bernoulli, ehhez kapcsolódó további általánosításokat, számsorozatokat.

Érdekes kérdés marad például, hogy az algebrai úton történő általánosítások és valamelyik kombinatorikai interpretáció természetes általánosítása összekapcsolható–e. Egy másik nyitott kérdés, hogy azok a számsorok, melyek valamilyen módon a poly–Bernoulli számokhoz kapcsolhatóak, mint például az analógiával definiált poly–Cauchy számok, szintén értelmezhetőek–e kombinatorikailag.

A második témakör középpontjában a hook formula állt. A hook formula egy meglepően kompakt képlet, mely a standard Young tablókat számolja össze. A ferde Young tablók valamint gyökereztetett fák esetére is felírhatóak hasonló képletek. A dolgozatban a gyökereztetett fák esetében adtunk a hook formulára bijektív bizonyításokat.

A fák esetében a hook formula a csúcsok leszármazottjainak számával határozza meg, hogy egy adott gyökereztetett fának hányféle monoton címkézése létezik.

A hook formula kombinatorikai értelmezése nyílvánvaló, sokáig nem volt ismert egy olyan bijektív bizonyítás, mely ezt a kombinatorikai jelleget kielégítő módon magyarázta volna meg. Az első ilyen bijekciót Novelli, Stoyanovskii és Pak [**68**] fogalmazta meg a standard Young tablók esetére.

Első bijekciónk megfogalmazásakor Novelli, Stoyanovskii és Pak módszerét alkalmaztuk a rendezett fák esetére. A fa egy tetszőleges címkézéséhez rendeltünk egy párt, amely a fa egy monoton címkézéséből és egy hook függvényből áll. Előkészítésként definiáltunk a fa csúcsain egy céljainknak megfelelő teljes rendezést. Algoritmusunk az általunk definiált sorrendben végigjárva a csúcsokat a tetszőleges címkézésből kiindulva lépésről lépésre ,,javítja" a rendezést monotonná, s az ehhez szükséges mozgatásokat a hook függvényben kódolja.

Egy további bijektív bizonyítást is megadtunk, melyben erősebben megmutatkozik az a szemlélet, mely a monoton címkézést egy olyan permutációnak tekinti, amely bizonyos feltételeknek eleget tesz. Ezek a feltételek, mely a fa struktúrából származnak, a csúcsok hook számában vannak kódolva. Ezt hangsúlyozza második bijekciónk.

A fejezetet szintén azzal zártuk, hogy rámutattunk a jelenleg aktívan folyó kutatásokra. Érdekes hook formulák, azaz olyan képletek, melyekben a leszármazottak száma szerepel, adódnak a fák különböző speciális osztályait vizsgálva. Bijekcióink specializálását és ezeknek a formuláknak a kapcsolatát még későbbi kutatásaink témájaként tűztük ki.

A negyedik fejezet a Catalan számok gazdag területéhez járult hozzá új bijektív bizonyításokkal. A Catalan számoknak több mint 200 interpretációja ismert, s ezek között az interpretációk között számos bijekció definiálható. Tézisünkben mi a 312–elkerülő permutációk és a triangulációk között konstruáltunk egy megfeleltetést. Ez a megfeleltetés egy egyszerű algoritmus, melyben a permutációk inverziótáblája játsza a kulcsszerepet.

Bijekciónk alapjául az a megfigyelés szolgált, hogy egy triangulációban a háromszögek címkézhetőek az alapján, hogy a középső csúcsuk a sokszög hányadik csúcsára illeszkedik. Így ha a sokszöget körbejárjuk, s minden háromszög címkéjét a harmadik csúcsának illeszkedése alapján jegyezzük fel, - külön szabályozva azt az esetet, amikor több háromszög harmadik csúcsa ugyanarra a sokszögcsúcsra illeszkedik - egy permutációt kapunk. Ez a permutáció, éppen azzal jellemezhető, hogy nem fordul elő benne a 312 részpermutáció.

Hasonlóan címkézhetőek a közepső csúcsuk alapján a k–triangulációkat felépítő $2k + 1$–csillagok is. Az analógiát követve a különböző csúcsok leolvasásával egy k–triangulációkhoz rendelhető a $\{1^k, 2^k, \ldots, n^k\}$ multihalmaz egy permutácója. Ennek a leképezésnek a pontos jellemzését nem tartalmazta tézisünk.

Az általtánosítások szempontjából fontosnak tartottuk, hogy a 312–elkerülő permutációkat mindkét inverzótáblájukkal jellemezzük. Ezek kapcsolatát jól szemlélteti az általunk bevezetett inverziódiagram, mely egyben a Tamari és Dyck hálók kapcsolatára is egyszerű magyarázatot ad.

Azért is tartottuk fontosnak kiemelni az inverziótáblák és a Dyck illetve Tamari hálók kapcsolatát, mert ezen hálók intervallumainak száma többféle összeszámláláskor is megjelenik. Természetesen merül fel a kérdés, hogy adható–e bijektív magyarázat ezekben az esetekben.

Egy példát mutattunk arra, hogyan használható fel szemléletmódunk a *abccab* mintát elkerülő teljes párosítások számának meghatározására. Egy egyszerű bijekciót definiálunk a szóban forgó párosítások és azon 312–elkerülő permutációpárok között, melyek egy intervallumot határoznak meg a Dyck hálóban.