On Teaching Mathematical Problem-Solving and Problem Posing

PhD thesis

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2012
Szeged

I. Relevance of research topic and goals

1. Problem solving and problem posing for elementary education majors

According to the National Core Curriculum (NAT), one of the central goals of Hungarian mathematics education is the development of the problem solving skills of the students. Competence based teaching, and the solution of practical problems call for teaching professionals who themselves are capable and knowledgeable in solving problems, and are able to rephrase real life situations in the language of mathematics. In order to train professionals who can adapt to the contemporary challenges, higher education needs to concentrate on the development of problem solving and problem posing skills. Future teachers need to attain problem solving skills and experiences to train the students in problem solving taking in to account varying student skill sets and preparedness. Teachers need to provide multiple representations of problems including graphical approaches, as well as activities that fit the students' developmental stage and conceptual understanding. These different representations promote/foster understanding and discovery of the underlying connections necessary for successful problem solving. It is especially important for future teachers to learn and practice the methods of problem posing, which are even more significant in the ever changing circumstances.
The Mathematical Problem Solving course is an elective course in the elementary education major curriculum. Its syllabus was developed based on the future teachers' mathematical skills and the needs of their pupils with the goal of developing their skills in the areas of problem solving and problem posing. The focused set of course materials combines relevant concepts in a pedagogically rich context. (This is the first time such material is put together focus that combines relevant concepts in a pedagogically rich context.) The development of problem solving competencies is a long and complex process, and problem-based learning should be emphasized throughout the higher education curriculum, e.g., in methods classes, Elementary Mathematics courses, and in Probability and Combinatorial Games courses, which provide an opportunity to practice problem solving and problem posing.

2. Objective of the research

Our research focuses on the investigation of the development of problem solving skills of elementary education majors.
Research objectives:

- Identify the theoretical background and basis for problem solving and problem posing;
- Assess students' problem solving and problem posing skills; determine goals and areas for improvement, and compare problem solving skills of different groups;
- Develop, implement and assess a course that focuses on developing problem solving and problem posing skills.
- Test specific hypotheses concerning the improvement of problem solving skills:

- The problem solving skills of elementary education majors can be improved in a course specially designed for and focused on this purpose.
- Problem solving strategies can be successfully taught to elementary education majors.
- Reasoning skills of students can be improved;
- Problem posing skills of students can be successfully developed;

II. Theoretical aspects of the research

During this research we surveyed the relevant literature, and determined the framework of the research. In our work, a ‘problem’ is defined to be a situation when the path to a certain goal is hidden [36]. Levels of the difficulty of a problem start at the application of a recently learned method. At the next stage, students need to choose between known methods, or sometimes need to combine several different methods, while at the highest level they discover new solution methods. We strived to select problems of increasing difficulty that require mathematical and other kinds of thinking for their solution. Thus we covered all levels of difficulty. To use previously taught techniques, students needed to rephrase or reformulate the problem, and with proper guidance they could discover new solution methods during experimentation. The basis of our research is the following variant of the model of Polya and Schoenfeld [70] [78] that contains cognitive as well as metacognitive elements of mathematical problem solving.

Step 1. Understand the problem, determine the objective.

- Read the problem or task, and restate it in your own words.
- Interpret, visualize or simulate the situation.
- Find relevant assumptions, data, and introduce notations.
- Draw a figure, or a diagram to organize the given data.
- Specify what you need to find.
- Determine whether enough data is provided, or there is need for more. Are there any redundant information?
- If possible/needed reformulate the problem to clarify it.

Step 2. Devise a plan and strategy for solution.

- Detangle the problem, find its crucial elements, and focus on how to get at them.
- Simplify the problem (choose smaller numbers, change assumptions, consider special cases).
- Identify a pattern by judicious guess and check.
- Decompose the problem, and attempt to identify a step-by-step strategy.
- Find an analogous problem, and attempt to use a similar solution strategy.
- Determine a specific approach, and try to take it as far as you can.
- Identify where we are, and what the goal is, and try to push them closer by reformulating/rephrasing either the current situation or the objective.

Step 3. Carry out the plan, check and modify, if necessary.

- Record and explain the steps to the solution.
- Determine the tools (methods and techniques) needed for the solution.
- Check our work step-by step.
- If the plan does not lead to a successful solution, evaluate it and find another, more suitable plan.

Step 4. Review and extend the problem.

- Evaluate and critique the result, check whether it makes sense. (put in context)
- Find a way to check the solution with an independent way.
- Check the validity of our conclusions.
- Write down the solution in a clear and concise way, evaluate the method.
- Find an alternative way of solving the problem.
- Find generalizations and extensions.
- Pose new questions, create a new problem by changing the data or the assumptions in the problem.

Besides the emphasis on the problem solving steps (internal dialogue) and heuristic strategies we strongly focus on multiple and varied representations, and the affective and metacognitive elements of problem solving.

During our research we utilize multiple ways of problem posing:

- We create problems from games and activities.
- We formulate specific steps and new representation during the process of problem solving, which form subproblems along the way of solving the original task.
- As a continuation of the original problem we ask the question 'What if …' and create a new set of problems.
- We create a problem from a given or imagined situation.
- We create a problem for a given solution method or solution.

Problem posing is always accompanied by problem solving, and thus problem posing is not solely the means of generating many more problems, but it fits organically into the web of complex activities that surrounds problem solving.

In the process of developing problem solving skills we need to take into account several key aspects of this development:

- Cognitive domain
  - Creation of multiple representations of the problem, selection of the best ones fitting the current situation;
  - Teach and recognize different types of problems
  - Teach problem solving strategies
- Metacognitive domain
  - Develop consciousness of the solution steps
  - Development of self-check and control during the solution process
- Affective domain
  - Foster creative and problem solving attitudes and activities
  - Foster beliefs in successful problem solving, and positive attitudes
  - Provide pedagogical strategies and positive examples for students that foster successful problem solving

Development of problem solving skills is especially/more effective in a group-based cooperative learning setting, so we frequently utilize this method.

III. Methods of the research

We created and administered a survey instrument/pre-test at the beginning of this research to provide a base-line for the investigation, and to determine the major areas for improvement/intervention in students' problem solving skills. Based on the results of this test we identified ten focus areas that are described in section xx below. A survey at the end of the research was administered to assess its results. This research was conducted in real-life situation, in small classes. Students' problem solving behavior is analyzed based on different criteria through case studies. Samples are comprised of self-selected students who enrolled in the problem solving course. Since the samples are small, conclusions cannot be interpreted as diagnostic metric.

1. The pre-test

1.1. Objectives of the pre-test

- Assess students' problem solving skills, determine major areas that need improvement;
- Compare problem solving attitudes and behavior of different groups of students (future high school teachers, college students, high school students in special/gifted and talented mathematics classes, elementary education majors participating in the research and in a control group).

1.2. Survey method

Students are given a list of seven problems to solve that focus on different aspects of problem solving. The most important among them are:

- Recognition of the model of the problem in contexts of varying difficulty; the effects of different representations;
- Students' problem solving strategies; Do they have a need/motivation to find and execute/apply different solution methods;
- Execute and reason out the steps in word problems;
- Application of checking strategies;
- Motivation/need for proving statements;
- Creative steps, constructions;
- Creation of new problems;

1.3. The pre-test and its outcome

Problems

Instructions: An important aspect of problem solving is recording the steps towards devising a solution strategy. Please record your steps and ideas even if they do not lead you to a successful solution, since this might serve as a venue to trace, modify, and possibly reformulate them. Attempt to find multiple solution strategies for a given problem. You can solve the problems in any order you please.

1. Two circles are given with radius 3 and 4 units, respectively. The two circles intersect in such a way that at both intersection points the tangent to circle A is perpendicular to the tangent to circle B. Compute how much larger is the area of circle A not covered by circle B than the area of circle B not covered by circle A.
2. Books are arranged in three bookcases. There are twice as many books in the second one than the first, and three times as many books in the third than in the first one. If we put 460 books from the third bookcase to the first one, then there will be 310 more books there than in the second one. How many books are there in the first bookcase now?

3. In a class 18 students study English, and 15 students study French. How many more student study English only than French only, if some students study both languages, and nobody studies any other language than English or French?
4. In basketball competition between high schools one team has already won 10 games and lost 5. They get into the finals if they win at least 4/5 (80%) of all their games. What is the least number of games they still need to play to get into the finals?
5. The difference of two numbers is 548. What will be their difference if we subtract 496 from both of them?
6. Laci said: “I thought of a number. I multiplied it by itself, and added the original number to this product. The sum ends in the digit 5. What was my number?” Eve replied: “No, this is impossible.” Who is right/correct and why?
7. Can a square be dissected into six, not necessarily congruent, squares? Change the ‘givens’ in the problem, and formulate new problems.

The model that underlies problems 1, 3, and 5 is identical: the difference of two numbers does not change if both are decreased by the same amount. In general, students recognized this model in the explicit arithmetic problem (5.), and a majority of them solved this problem correctly. In problem 3, which involved sets, there were much fewer correct solutions, and even some of those who solved the problem correctly did not realize the common model. Very few correct solutions were given for problem 1, which concerns the intersecting circles. This shows that the redundant information given in the problem hid the underlying model.
Relatively few students solved the second problem drawing a figure (line segments), and many students did not check their result, so they did not realized that no solution exists for this problem.
Elementary education majors solved the fourth problem predominantly using guess and check, some of them using the complementary situation successfully. Future high school teachers were more inclined to construct an equation/ inequality(?) based on the problem.
In problem 6 elementary education majors predominantly checked the possible digits, while a larger portion of future high school teachers proved their statements using parity.
After finding the solution by construction for problem 7 some of the students did try to posea new question. However, the formulation of their questions was generally imprecise/ ambiguous, and their question generally did not constitute a new problem [59].

1.4. Results of the pre-test

Results of the pre-test determined the main areas for development. Based on these results the main areas of problem solving skills that need improvement are the following

- Consideration of the steps of problem solving.
- Conscious approach to problem solving.
- Learning of problem solving strategies.
- Finding different approaches, strategies for solving a given problem.
- Recognition of a common underlying model in problems, formulation of new representations for a given model.
- Narrative description of the solution.
- Development of the need and motivation to verify statements and steps in the practice of problem solving.
- Posing new problems.

Comparison of the different groups participating/taking in the survey/pre-test we can say that in general

- Elementary education majors have a lower level of abstraction than future high school teachers, and use symbolic methods less often, resorting to guess and check as their method of problem solving.
- Elementary education majors have an active attitude towards problem solving, they make attempts even if they do not see whether these attempts will lead to a final solution.
- Members of the control group were more successful in solving the problems, they were more creative in new constructions.

2. Methods of development

10 different topics were considered in the problem solving course. In case of topics aimed at the development of problem solving skills the students were give a list of problems. Structure of the list of problems:

- Example problems: typically the first few problems are easier, but getting more and more difficult in the interest of differentiation and development. Example problems were always solved as a class starting in actual experimentation, simulation or activity. Activities were done in pairs or groups of three or four students. Based on their experimentation students formulated hypotheses, and tried to verify them. Complete solutions were devised based on these suggestions under the instructor's direction. During the solution process we made students conceptualize the steps toward solving the problem, and demonstrated how to pose questions to help develop their own critical thinking skills and the need and motivation to check their steps. Solution of a problem was followed by the review of the solution, which not only meant an evaluation of the method used, but the extension of the problem, and the posing of new problems as well.
- Problems for homework: these are practice problems that are similar to the example problems, and were assigned as homework to be handed in. The objective is to use the methods learned, the development of self-guided activity, the practice of writing down detailed reasoning in a narrative. Solutions were evaluated for each topic, and conclusions were drawn.
- Puzzle: development of creative, out-of-the-box thinking skills.

In case of topics concerning the development of problem posing skills we determined the initial question, and demonstrated the methods of posing new problems. We solved these new problems together as a whole class activity. These topics were also followed by homework connected to classroom examples, and the homework was evaluated as before.

IV. The process of development and assessment of the work

1. The steps of problem solving

- Description of the topic: We presented the steps of problem solving in different representations from concrete activities and figures to symbolic manipulations through the detailed study of a few problems. We practiced the decomposition of problems to steps, and the creation of new problems during the process of problem solving. We demonstrated the kind of questions that arise during this process and that can help it to get along if students get stuck. Students practiced understanding new problems, recording relevant data and connections in different formats, and checking and evaluation strategies.
- Assessment of this topic:
  - Concrete activities facilitated better understanding and the solution of the problems. Students found new ideas, discovered more representations by experimentation in this extended problem domain.
  - Students were helped by formulating the steps as questions in the problem solving process. Questions like “Where are we? / What do we know now?” and “What is the goal?” facilitated the correct evaluation of the situation, and students decided to continue or abandon their current strategy faster.
  - Students were motivated to create new questions. Precise formulation of these questions was initially difficult, but improved on practice. Students were interested in finding answers to their own questions.
  - Students found useful the review and evaluation of their problem solving experiences at the end of each problem.

2. Experimentation, examples and counterexamples

- Description of the topic: We emphasized the importance of experimentation in the process of problem solving. Students instinctively do the guess and check method, and we developed its judicious use through the creation of examples and counterexamples. Students had the opportunity to recognize the need for proving/verifying their hypotheses, and practiced reasoning out their steps. Students experienced cases when a counterexample suffices to show the validity of a statement, and when a series of conclusions are needed.
- Assessment of the topic:
  Finding examples and counterexamples as a whole class activity was successful, but students had difficulties doing it on their own. Students were able to formulate hypotheses based on examples. Some students were able to prove simple statements, but very few were able to use algebraic reasoning.

3. Let’s draw

- Description of the topic/area: It is very useful to create graphical representations (diagrams, figures) before introducing symbolic representations. Creation of graphical representations is aided by problems that use the strategy of drawing diagrams and figures. We emphasized graphs, Venn-diagrams and tables, since these occur already in grades 1-4 [61].
- Assessment of the topic:
  - Students were open to new kinds of representations, and were motivated to use this methods on different kinds of problems.
  - It was not easy for students to transform a problem to a new representation, and then translate the solution back to the original situation.
  - Students experienced how these representations made the solution richer/more valuable by revealing the underlying structure of the problem. Thus they could discover when a problem had multiple solutions or none at all.
  - Based on homework problems we observed that students used Venn diagrams, tables, and graphs correctly on enumeration problems, but they are not used to describing their solutions in a narrative.

4. Solving word problems by using line segments

- Description of the topic/area: The most important strategy for solving word problems in grades 1-4 involves the use of line segments. However, students are not well-versed in this technique, and have difficulties applying it. We practiced this strategy on more difficult problems as well [66]. We created text for given model involving line segments.
- Assessment of the topic:
  - Students often have difficulties using line segments in their solutions, and think that it would be easier for them to use equations. However, for more difficult problems, they had a hard time making up their equations as well. If the strategy using line segments is not practiced on easier problems, then they are not able to it on more difficult problems either. Thus we emphasized using figures or line segments and conclusions based on them, even if they already solved the problem using an equation. Unfortunately, only a few students even tried this in the homework, and their efforts were unsuccessful.
  - We observed that as students run into problems in creating an equation or a graphical representation they resort to the method of guess and check. It is very important for students to understand that while guess and check is a valuable method for better understanding the problem and for formulating hypotheses, they should not stop at finding just one suitable solution. Instead, they should strive to produce a complete, reasoned-out solution, since this serves the development of their pupils reasoning skills the best.
  - It greatly helped to acquire this technique that we showed multiple solution methods for the problems, and completed the students’ solution ideas even if they could not arrive at a correct solution themselves. Thus students realize that there is not just a single solution method that needs to found, and they can start in multiple ways to produce a solution.
  - The biggest difficulty proved to be the representation of relationships among ratios.

5. Thinking backwards

- Description of the topic: the solution strategy of 'thinking backward' appears in the lower grades, hence it is important that elementary education majors become familiar with it, and could apply it in the context of more complex problems. We presented multiple graphical representations for this strategy, e.g., tables, line segments, and bubbles [63].
- Assessment of topic:
  - Students applied the thinking backwards strategy in connection with arithmetic operations more successfully than in the context of general situations(?).
  - Conscious decomposition of problems into steps was developed/ by using dialogues since this approach is particularly important for future educators.
  - Many students still applied the strategy of making equations instead of visual representations, even though in these type of problems equations tend to become quite complicated and long.

6. Create problems based on Harry Potter

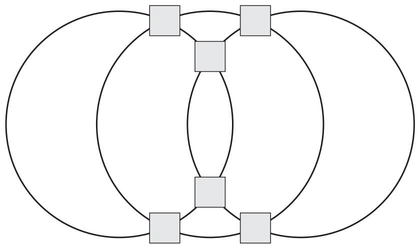
- Description of the topic: We showed an example that popular children's literature can serve as a springboard to pose new problems. We created problems concerning a puzzle in the first Harry Potter book (), and also drew the accompanying figure, missing from the book, that would help the reader to solve the riddle herself. Some students were able to pose creative problems for similar situations, but this kind of an activity was a novelty for them [62].
- Assessment of the topic:
  - Students experienced the importance of precise formulations of problems.
  - Some students were motivated and able to pose new, creative problems based on a given pattern.
  - There were a few students who were not inclined to create their own problems, but enjoyed solving the ones created by others.
  - While the problem posing activity was a novelty for the students, it is a very useful skill for future educators especially in strengthening connections between different academic subjects.

7. Problem sets by asking 'What if?'

- Description of the topic: We presented the strategy of posing problems by asking the question 'What if…?' By changing the given data or the assumptions in the problem we created new problems, and initiated a 'mini-research' project in the given area.
- We show the example problems of this topic shortly for an example of the lessons.

Magic circles

Put the numbers 1; 2; 3; 4; 5; 6 into the squares so that the numbers on each circle add up to the same amount. When this happens, the circles are said to be magic.



We found some solutions. We realized that the numbers have to be in three pairs, and the sum of the numbers in a pair is the same for the pairs.

Further problems

1. **How many arrangements** are there which produce magic circles?
2. **Change the numbers**: Can you find six other numbers which could be put into original squares to make the circles magic?
   We found a necessary and sufficient condition for the sum of the numbers which could be put into the squares to make the circles magic.
3. **Change the method of filling the squares**:
   a. The numbers 1; 2; 3; 4; 5; 6 are written on six slips (on each slip there is one number) and these slips are put into a hat. We draw a slip from the hat and write the number in the first square. We draw the slips without replacement, and fill the squares, in this order, with the numbers on the slips. What is the probability that we will get magic circles?
   b. The numbers in the squares are thrown by a regular die in the order of the squares. What is the probability that we will get magic circles?
   c. There are six numbers thrown by six regular die. What is the probability that we can put them into the squares to make the circles magic?
   In which case is the probability the greatest?
   d. Find six different Fibonacci numbers which can be put into the squares to get magic circles.
   We proved that its impossible to find such numbers.
4. **Change the number of circles to four**
   Put one of the numbers 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12 at each intersection of four circles so that the circles are magic.
5. **Find another twelve numbers** which can be put at the intersections of four circles so that the circles are magic.
6. **Change the circle to straight lines**
   Put one of the numbers 1; 2; 3; 4; 5; 6 at each intersection of four straight lines so that the sums of the numbers on the straights are the same.
   We got that it’s not possible to put six different numbers on the intersections so that the sums on the lines are equal.
7. **Change the operation**
   Put the numbers 1; 2; 3; 4; 5; 6 into the original squares so that the products of the numbers on each circle are the same.

This is impossible because the products of the numbers of pairs must be equal but the prime number 5 occurs only once. It’s possible to find six different number for ‘product magic’ circles, for example:1; 2; 3; 4; 6; 12.

- Assessment of the topic:
  Students worked well in groups, but did not feel capable to perform similar investigations by themselves. Very few students formulated hypotheses, and even fewer solved any new problems.

8. One model – several representations

- Description of the topic:
  We presented multiple representations of problems that share a common underlying model. This practice helps students to recognize similarities in new problems, and to apply an already know solution method for them. We practiced describing word problems in arithmetic operations. Based on the problems presented students needed to collect problems for a given model, which is an important skill for future teachers.
- Assessment of the topic:
  - Word problems leading to a sequence of arithmetic operations frequently proved to be complicated for the students.
  - There is a considerable difference in the recognition of addition and multiplication as an underlying model. Addition and subtraction did not prove to be difficult, but many students stumbled on multiplication, division, and fractions.
  - Few students were able to find good problems for a given model.

9. Create a games from a mathematical problems

- Description of the topic: We presented methods that can be used to create games from given problems involving enumeration, the pigeon-hole principle or coloring [67].
- Assessment of the topic:

Games helped to achieve a deeper understanding of the problems, and to find ways of solving them. Students were not yet able to create games by themselves.

10. Role-playing to imitate the steps of problem posing and problem solving

- Description of the topic: Role-playing games were performed in groups of six to seven students who had roles corresponding to the steps of problem solving. Thus they experience the difficulties of problem solving in a different context, and become more conscious of the steps involved. We had the following roles:
  Captain: determines the rules, and makes everyone obey them.
  Snoop: has the task of questioning
  – asks the question in the problem,

– asks about the data

– helps the work of the Wise Ones (What is suspicious? What is missing? Are we heading in the right direction? Why does this work?)

Computer: provides the data asked for if they cannot be found by otherwise; executes the calculations asked for by the Wise Ones.

Wise Ones: solve the problem.

Scribe: records the activity of the group, the rules made by the Captain, the answers and the solutions.

- Assessment of the topic:
  - Personalization of the steps of problem solving has a positive effect on understanding problem solving and problem posing on a deeper level by dividing the steps involved according to different roles. Students paid more attention to certain elements of these processes that they typically neglected before, e.g., the detailed narrative description of the solution, or the formulation of new questions.
  - More attention need to be paid to questions concerning metacognitive direction, and the recording of their reasoning.
  - The cooperative activity motivated the students, they enjoyed working together, they argued, exchanged opinions, listened to each other, divided up the tasks, and built on each others work.
  - This activity provided a strong motivating factor, students were more inclined to solve mathematical problems in a situation created by themselves.
  - This role playing game for problem posing and problem solving could be used in other courses as well.

V. Post-test

1. Objective of the post-test

At the end of the course students solved a list of problems to assess their problem solving skills, and to compare those participating in the course with a control group.

2. Methods of the post-test

Problems on the post-test were not simple routine problems. Some of them were of moderate difficulty that required the selection of a suitable strategy and its use in a new situation. The other problems were more difficult, their solution called for several steps and the use of different representations. Our goal was to assess the level students attained at the end of the course. We did not intend to show that students learned to solve problems that were very similar to the ones in the pre-test, rather we tested their changing perspective and understanding.

The post-test assessed the following skills:

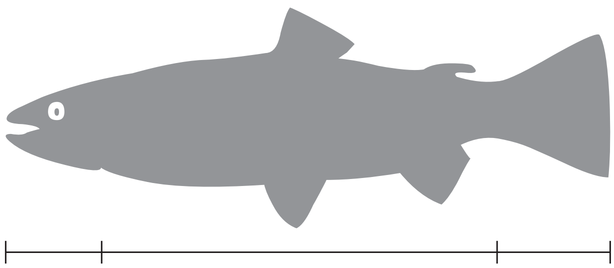
- formulation and interpretation of different representations based on a given problem;

– decomposition of problems into steps, formulation of an internal dialogue, and its written record;
– narrative record of reasoning;
– strategy of using line segments;
– strategy of thinking backwards;
– formulation of new constructions;
– posing new problems.

3. The problems and their assessment

Problems

1. A square shaped paper ABCD is white on one side, black on the other. The area of the square is 3 dm^2. We fold vertex A to a point A’ on the diagonal AC in such a way that half of the visible part of the paper is white and half of it is black. What is the distance of A’ from the line of folding?
2. The sum of two three-digit numbers is 1000. Is it possible to tell what the sum of the sums of the digits of the two numbers is, if none of the digits is 0?
3. Janka had a bag of marbles. She gave halfof them and two more to Bori. Then she gave half of the remainder and two more marbles to Blanka. Next, Teri got half of the remainder and two more, and Janka had only one marble left. How many marbles did she have at the beginning?
4. Robi, Csaba and Denes are playing a game of cards. They agree that at the end of each round the loser divides up his money between the two other players. In three rounds each of the players lost one game. Robi ended up with 400ft, Csaba with 1000ft, while Denes did not have any money left. Who lost the first round?
5. There are apples and pears in a box. The same number of apples and pears are spoilt. Two-thirds of apples, and three-fourth of the pears are spoilt. What fraction of the fruit in the box is spoilt?
6. The tail of a fish is as long as its head and one quarter of its body. Its body is three-quarters of the total length of the fish. If the fish’s head is 4cm long, what is the total length of the fish?



7. Determine how we can measure 6 centimeters if we have a rectangular piece of paper with dimensions 17cm and 22cm.
8. Choose a 3x 3 block of dates containing 9 numbers on a page of a calendar. Multiply together the numbers diagonally opposite to each other. What is the absolute value of the difference of these products?

November						
Sunday	**Monday**	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Saturday**
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

In problem 1 recognition of fractional parts and geometric correspondences proved to be very difficult. The second problem was solved formally by most of the control group, while many of the students participating in the course produced a detailed reasoning. Students used the ‘thinking backwards’ strategy successfully in the third problem with bubbles (control group) and with bubbles and line segments (students in the course). The fourth problem was solved mostly by a table and the thinking backwards strategy, while relatively fewer students were able to find a solution in the control group. The result was similar in the fifth and sixth problems, many students in the course developed their skill of drawing line segments, and representing ratios. The seventh problem called for a creative construction, and students in the course had a better result than in the pre-test. Few students were able to follow the transition between different representations. In the eighth problem the conjecture/hypothesis based on calculations needed to be proved by a symbolic method. Students in the course realized this, and did not accept the conjecture without a proof. The given page of the calendar provided a good opportunity to pose new problems, and several good ones were created. These problems were also solved by the students.

4. Outcomes of the post-test

The post test served as a comparison between students completing the problem solving course and a control group, and showed the effects of the targeted development.

- Students in the course
  - were able to solve problems of medium difficulty,
  - represented the given word problems by line segments,
  - successfully used the strategy of ‘thinking backwards’,
  - showed improvement in using a narrative reasoning when solving word problems,
  - had better success in solving problems involving ratios,
  - improved in creating new constructions,
  - showed considerable improvement in posing new problems both qualitatively and quantitatively.
- Outcome of the course:
  Compared with the control group, a larger portion of students in the course
  - solved the problem s successfully
  - wrote down their reasoning in a narrative
  - drew line segments for solving word problems
  - represented the thinking backwards strategy in a table
  - checked their solution,
  - posed new problem, and solved them.

VI. Conclusions

1. Validity of hypotheses

The research supported the hypotheses:

- Students who completed a course specifically designed for developing their problem solving skills had improved problem solving skills, and were able to solve more difficult problems. Students had a positive attitude toward problem solving, developed a need for demonstrating problems with a concrete activity or experimentation, and were able to create such activities. They listened to each other, and worked to complete each others' ideas. They were able to work together after some individual thinking.
- Students successfully used the strategy of thinking backwards, and showed marked improvement in the strategy of using line segments.
- Improvement can be observed in the recording of their solutions, students wrote down their reasoning in more detail.
- Students showed a considerable improvement in posing problems, they were able to create problems for a given situation, or as a continuation of a given problem. The formulation of their problems became more precise and concise as well.

2. Future work

The development of the students' problem solving skills cannot/should not stop at the end of the course. On one hand, several more strategies can be taught, for example, coloring, parity, or invariance methods [60]. Results confirm that students need more practice in using proofs and reasoning with the strategy of drawing line segments. Students in primary grades can solve word problems using this method, which can provide a basis for their calculations, so their future teachers need to be proficient in using it as well. Another important point of the curriculum in primary grades is the development of the need for providing reasoning for statements. This reasoning serves as proof, so conclusions are not drawn based on one or two examples. Thus teachers need to be able to provide clear explanations and proofs for their statements.

These needs encompass more than what a single course could provide, and the development of problem solving skills should be included in all courses in an integrated fashion, even if not much time can be explicitly devoted to it. Posing problems can be successfully included in combinatorial problems, when different representations of a given model play an important role in the solution [65], or in word problems when we create a situation for given line segments.

Another direction of the development of problem solving skills concerns the students in elementary grades. An important aspect of this work is the fostering of creative thinking and discovery learning, in which teachers have a central role. A special curriculum can also support the development of problem solving skills. For example, problem solving steps for word problems can be explicitly taught [64]. Development of problem solving skills can occur in each unit with a variety of representations and questions. Students' attitudes can be positively affected by active questioning and problem posing.

The author's publications according to the thesis

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