

Uniform convergence of single and double sine series and integrals

Summary of Ph.D. thesis

PÉTER KÓRUS

Supervisor:
FERENC MÓRICZ D.SC.
professor emeritus

Doctoral School in Mathematics and Computer Science
University of Szeged, Bolyai Institute

Szeged
2012

In the dissertation we study the uniform convergence of single and double sine series together with single and double sine integrals. We use the well-known notion of convergence for single series and integrals, while for double series and integrals we are interested in the uniformity of regular convergence. For sine series, we generalize results from [13] of S. P. Zhou, P. Zhou and D. S. Yu, for double sine series, ones from [12] of Žak and Šneider, for sine integrals, ones from [10] of Móricz. Finally we prove similar results for double sine integrals. The monotone nonincreasing and the so-called general monotone sequences and functions play an important role in our discussion.

The dissertation is based on papers [4], [5], [6], [7], [8] of the author.

1 Sine series

Known results

The fundamental theorem in the theory of the uniform convergence of sine series is due to Chaundy and Jolliffe from 1916.

Theorem 1.1. [1] *If the sequence $\{c_k\} \subset \overline{\mathbb{R}}_+ := [0, \infty)$ is monotonically nonincreasing and tends to 0, then the sine series of the form*

$$(1) \quad \sum_{k=1}^{\infty} c_k \sin kx$$

converges uniformly in x if and only if

$$(2) \quad kc_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This theorem has been generalized by a number of authors. S. P. Zhou, P. Zhou and D. S. Yu proved in 2006 the most general extension of theorem. They introduced the general monotone class of sequences called MVBVS: $\{c_k\} \in \text{MVBVS}$ (*Mean Value Bounded Variation Sequences*), if there exist constants C and $\lambda \geq 2$ such that

$$\sum_{k=n}^{2n-1} |\Delta c_k| := \sum_{k=n}^{2n-1} |c_k - c_{k+1}| \leq \frac{C}{n} \sum_{k=\lfloor n/\lambda \rfloor}^{\lfloor \lambda n \rfloor} |c_k|$$

is satisfied for all $n \geq 1$.

Theorem 1.2. [13] *Let $\{c_k\} \subset \mathbb{C}$ belong to class MVBVS.*

(i) *If (2) holds, then (1) converges uniformly in x .*

(ii) *Conversely, if $\{c_k\} \subset \overline{\mathbb{R}}_+$ and (1) is uniformly convergent in x , then (2) holds.*

New results

Our goal is to extend Theorem 1.2 by defining larger class(es) than MVBVS.

Definition. A sequence $\{c_k\} \subset \mathbb{C}$ is said to be a *Supremum Bounded Variation Sequence*, shortly $\{c_k\} \in \text{SBVS}$, if there exist constants C and $\lambda \geq 1$ depending only on $\{c_k\}$ such that

$$\sum_{k=n}^{2n-1} |\Delta c_k| \leq \frac{C}{n} \sup_{m \geq [n/\lambda]} \sum_{k=m}^{2m} |c_k|.$$

Remark. A similar class to SBVS was introduced in [3] using max instead of sup in the definition.

Definition. We say that $\{c_k\} \subset \mathbb{C}$ sequence is a *Supremum Bounded Variation Sequence of 2nd type*, shortly $\{c_k\} \in \text{SBVS}_2$, if there exist constant C and $\{b(k)\}_{k=1}^{\infty} \subset \overline{\mathbb{R}}_+ = [0, \infty)$ converging to infinity depending only on $\{c_k\}$ such that

$$\sum_{k=n}^{2n-1} |\Delta c_k| \leq \frac{C}{n} \sup_{m \geq b(n)} \sum_{k=m}^{2m} |c_k|.$$

Theorem 1.3. [4] $\text{SBVS}_2 \supsetneq \text{SBVS} \supsetneq \text{MVBVS}$.

Theorem 1.4. [4] *Let $\{c_k\} \subset \mathbb{C}$ belong to class SBVS_2 .*

(i) *If (2) holds, then (1) converges uniformly in x .*

(ii) *Conversely, if $\{c_k\} \subset \overline{\mathbb{R}}_+$ and (1) is uniformly convergent in x , then (2) holds.*

Corollary 1.5. [4] *If $\{c_k\} \subset \overline{\mathbb{R}}_+$ belongs to SBVS_2 , then (2) is a necessary and sufficient condition for the uniform convergence of (1) in x .*

Theorem 1.6. [4] *For any sequence from SBVS_2 which is not in SBVS , (2) is satisfied. Hence sine series with coefficients from $\text{SBVS}_2 \setminus \text{SBVS}$ are uniformly convergent.*

Auxiliary results

For the proof of the first (sufficiency) part of our main result, Theorem 1.4, it is worth to mention a lemma, which can be proved by using dyadic blocks.

Lemma 1.7. *If $\{c_k\} \in \text{SBVS}_2$ and (2) holds, then*

$$n \sum_{k=n}^{\infty} |\Delta c_k| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The second (necessity) part of Theorem 1.4 is based on the following lemma, whose proof was given with the use of summation by parts.

Lemma 1.8. *Let $\{c_k\} \subset \overline{\mathbb{R}}_+$ from SBVS_2 . Then*

$$nc_n \leq C \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k + \sum_{k=n}^{2n} c_k$$

where C and $b(n)$ are from the definition SBVS_2 belonging to $\{c_k\}$.

Formal derivative and integral

We study the uniform convergence of the series of the form

$$(3) \quad \sum_{k=1}^{\infty} k^r c_k \sin kx,$$

which is the r times formally differentiated series of (1) in case r is a positive even integer and is the r times formally integrated series of (1) in case r is a negative even integer.

Theorem 1.9. [5] *If $\{c_k\}$ belongs to class MVBVS , then the sequence $\{d_k = k^r c_k\}$ also belongs to MVBVS for any fixed integer r .*

Corollary 1.10. [5] *Suppose that $\{c_k\} \in \text{MVBVS}$ and r is a positive even integer.*

(i) *If $k^{r+1} c_k \rightarrow 0$, then (3) is uniformly convergent in x .*

(ii) *Conversely, if $\{c_k\} \subset \overline{\mathbb{R}}_+$ and (3) is uniformly convergent in x , then $k^{r+1} c_k \rightarrow 0$.*

Theorem 1.11. [5] *If $\{c_k\}$ belongs to SBVS , then $\{d_k = k^r c_k\}$ also belongs to SBVS for any fixed integer r .*

Corollary 1.12. [5] *For $\{c_k\} \in \text{SBVS}$ and any positive even integer r , statements (i) and (ii) of Corollary 1.10 hold.*

2 Double sine series

Regular convergence

Let $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ and consider the double series of the form

$$(4) \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk} \sin jx \sin ky.$$

This series is regularly convergent in case of a fixed (x, y) if the rectangular sums

$$\sum_{j=1}^m \sum_{k=1}^n c_{jk} \sin jx \sin ky$$

converge to a finite number as m and n independently tend to infinity, moreover the row and column series

$$\sum_{j=1}^{\infty} c_{jn} \sin jx \sin ny, \quad n = 1, 2, \dots \quad \text{and} \quad \sum_{k=1}^{\infty} c_{mk} \sin mx \sin ky, \quad m = 1, 2, \dots$$

are convergent. The last definition is equivalent to the following: for any $\varepsilon > 0$ there exists a positive number $m_0 = m_0(\varepsilon)$ such that

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| < \varepsilon$$

is satisfied if $m + n > m_0$, $1 \leq m \leq M$ and $1 \leq n \leq N$, see in [9] by F. Móricz.

Žak and Šneider proved an analogous theorem to Theorem 1.1 for double sine series with coefficients of nonnegative, nonincreasing double sequences. A double sequence $\{c_{jk}\}$ of nonnegative numbers is called nonincreasing if

$$\Delta_{10}c_{jk} \geq 0, \Delta_{01}c_{jk} \geq 0, \Delta_{11}c_{jk} \geq 0, \quad j, k = 1, 2, \dots$$

where

$$\begin{aligned} \Delta_{10}c_{jk} &:= c_{jk} - c_{j+1,k}, & \Delta_{01}c_{jk} &:= c_{jk} - c_{j,k+1}, \\ \Delta_{11}c_{jk} &:= \Delta_{10}(\Delta_{01}c_{jk}) = \Delta_{01}(\Delta_{10}c_{jk}) = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}. \end{aligned}$$

Theorem 2.1. [12] *If the double sequence $\{c_{jk}\} \subset \mathbb{R}_+$ is nonincreasing, then the regular convergence of (4) is uniform in (x, y) if and only if*

$$(5) \quad jkc_{jk} \rightarrow 0, \quad \text{as } j+k \rightarrow \infty.$$

New results

We extend Theorem 2.1 by defining and using classes of general monotone double sequences.

Definition. A double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to class MVBVDS (*Mean Value Bounded Variation Double Sequences*), if there exist C and $\lambda \geq 2$ constants, depending only on $\{c_{jk}\}$, for which

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| &\leq \frac{C}{m} \sum_{j=[m/\lambda]}^{[\lambda m]} |c_{jn}|, \quad m \geq \lambda, n \geq 1, \\ \sum_{k=n}^{2n-1} |\Delta_{01} c_{mk}| &\leq \frac{C}{n} \sum_{k=[n/\lambda]}^{[\lambda n]} |c_{mk}|, \quad m \geq 1, n \geq \lambda, \\ \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| &\leq \frac{C}{mn} \sum_{j=[m/\lambda]}^{[\lambda m]} \sum_{k=[n/\lambda]}^{[\lambda n]} |c_{jk}|, \quad m, n \geq \lambda. \end{aligned}$$

Theorem 2.2. [7] *Let $\{c_{jk}\} \subset \mathbb{C}$ belong to MVBVDS.*

- (i) *If (5) is satisfied, then the regular convergence of (4) is uniform in (x, y) .*
- (ii) *Conversely, if $\{c_{jk}\} \subset \overline{\mathbb{R}}_+$ and the regular convergence of (4) is uniform in (x, y) , then (5) holds.*

Moreover, in [7], we also proved that class MVBVDS contains class NBVDS defined there, which class contains the nonnegative, nonincreasing double sequences. Later, in [6], we generalized MVBVDS to extend the results of Theorem 2.2.

Definition. $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ is said to be a *Supremum Bounded Variation Double Sequence of 1st type*, in symbols: $\{c_{jk}\} \in \text{SBVDS}_1$, if there exist constants C and integer $\lambda \geq 2$ and sequences $\{b_1(l)\}_{l=1}^{\infty}$, $\{b_2(l)\}_{l=1}^{\infty}$, $\{b_3(l)\}_{l=1}^{\infty}$, each one converges to infinity, all of them depend only on $\{c_{jk}\}$, such that

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| &\leq \frac{C}{m} \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} |c_{jn}|, \quad m \geq \lambda, n \geq 1, \\ \sum_{k=n}^{2n-1} |\Delta_{01} c_{mk}| &\leq \frac{C}{n} \max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} |c_{mk}|, \quad m \geq 1, n \geq \lambda, \\ \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| &\leq \frac{C}{mn} \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}|, \quad m, n \geq \lambda. \end{aligned}$$

Definition. $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ is said to be a *Supremum Bounded Variation Double Sequence of 2nd type*, shortly $\{c_{jk}\} \in \text{SBVDS}_2$, if there exist constants C and integer $\lambda \geq 1$ and $\{b(l)\}_{l=1}^{\infty}$ converging to infinity, depending only on $\{c_{jk}\}$, such that

$$(6) \quad \begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| &\leq \frac{C}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}|, \quad m \geq \lambda, n \geq 1, \\ \sum_{k=n}^{2n-1} |\Delta_{01} c_{mk}| &\leq \frac{C}{n} \sup_{N \geq b(n)} \sum_{k=N}^{2N} |c_{mk}|, \quad m \geq 1, n \geq \lambda, \\ \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| &\leq \frac{C}{mn} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}|, \quad m, n \geq \lambda. \end{aligned}$$

Theorem 2.3. [6] $\text{SBVDS}_2 \supsetneq \text{SBVDS}_1 \supsetneq \text{MVBVDS}$.

Theorem 2.4. [6] (i) If $\{c_{jk}\} \subset \mathbb{C}$ belongs to SBVDS_2 and (5) is satisfied, then the regular convergence of (4) is uniform in (x, y) .

(ii) Conversely, if $\{c_{jk}\} \subset \overline{\mathbb{R}}_+$ belongs to SVBVD_1 and the regular convergence of (4) is uniform in (x, y) , then (5) holds.

Auxiliary results

We proved the sufficiency part of Theorem 2.4 with the following lemma, which can be proved by using double dyadic blocks.

Lemma 2.5. Suppose that $\{c_{jk}\} \subset \mathbb{C}$ satisfies conditions (5) and (6). Then

$$mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} c_{jk}| \rightarrow 0, \quad mn \sum_{j=m}^{\infty} \sup_{k \geq n} |\Delta_{10} c_{jk}| \rightarrow 0, \quad mn \sum_{k=n}^{\infty} \sup_{j \geq m} |\Delta_{01} c_{jk}| \rightarrow 0$$

as $m+n \rightarrow \infty$ and $m, n \geq \lambda$.

The proof of the necessity part of Theorem 2.4 is based on the next lemma, which was proved by double summation by parts.

Lemma 2.6. Let $\{c_{jk}\} \subset \overline{\mathbb{R}}_+$ be from class SBVDS_1 with constants $C, \lambda \geq 2$ and sequences $\{b_1(l)\}, \{b_2(l)\}, \{b_3(l)\}$. Then

$$mnc_{mn} \leq C \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} + C \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n} c_{jk}$$

$$+ C \sum_{j=m}^{2m} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 2 \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{jk} \quad \text{if } m, n \geq \lambda.$$

Similar statements were proved for class MVBVDS in [7].

3 Sine integrals

Known results

Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be a Lebesgue measurable function, where $\mathbb{R}_+ := (0, \infty)$. Consider the sine integrals of the form

$$(7) \quad \int_0^{\infty} f(x) \sin tx \, dx.$$

For the uniform convergence of (7) in t ($t \in \mathbb{R}$), F. Móricz proved the following.

Theorem 3.1. [10] *Suppose that $f(x) : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ is nonincreasing for which*

$$(8) \quad xf(x) \in L^1_{\text{loc}}(\overline{\mathbb{R}}_+).$$

Then the sine integral (7) is uniformly convergent in t if and only if

$$(9) \quad xf(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Furthermore he introduced class $\text{MVBVF}(\mathbb{R}_+)$, which class contains the nonnegative, nonincreasing, locally absolutely continuous functions defined on \mathbb{R}_+ . He partly generalized the previous theorem.

Definition. A function $f(x) \in \text{AC}^1_{\text{loc}}(\mathbb{R}_+)$ is said to belong to class $\text{MVBVF}(\mathbb{R}_+)$ (*Mean Value Bounded Variation Functions*) if there exist constants $C, A > 0$ and $\lambda \geq 2$ which depend only on f and satisfy condition

$$\int_a^{2a} |f'(x)| \, dx \leq \frac{C}{a} \int_{a/\lambda}^{\lambda a} |f(x)| \, dx, \quad a > A.$$

Theorem 3.2. [10] *Assume that $f(x) \in \text{MVBVF}(\mathbb{R}_+)$ and (8) holds.*

- (i) *If $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ and (9) is satisfied, then integral (7) is uniformly convergent in t .*
- (ii) *Conversely, if $f : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ and (7) is uniformly convergent in t , then (9) holds.*

Extension of the theorem of F. Móricz

The generalization is made by defining general monotone function classes similar to the classes SBVS and SBVS₂ which were introduced in the first section.

Definition. A function $f(x) \in AC_{\text{loc}}^1(\mathbb{R}_+)$ is said to belong to class SBVF(\mathbb{R}_+) (*Supremum Bounded Variation Functions*) if there exist constants C , $A > 0$ and $\lambda \geq 2$ which depend only on f and satisfy condition

$$\int_a^{2a} |f'(x)| dx \leq \frac{C}{a} \sup_{b \geq a/\lambda} \int_b^{2b} |f(x)| dx, \quad a > A.$$

Remark. A similar class to SBVF(\mathbb{R}_+) was defined in [2].

Definition. A function $f(x) \in AC_{\text{loc}}(\mathbb{R}_+)$ is said to be in class SBVF₂(\mathbb{R}_+) (*Supremum Bounded Variation Functions of 2nd type*), if there exist constants C , $A > 0$ and function $B(x) \subset \overline{\mathbb{R}}_+$ tending to infinity, depending only on f , such that

$$\int_a^{2a} |f'(x)| dx \leq \frac{C}{a} \sup_{b \geq B(a)} \int_b^{2b} |f(x)| dx, \quad a > A.$$

Theorem 3.3. [5] SBVF₂(\mathbb{R}_+) \supsetneq SBVF(\mathbb{R}_+) \supsetneq MVBVF(\mathbb{R}_+). Moreover, if a function $f(x)$ belongs to class SBVF₂(\mathbb{R}_+), but is not in SBVF(\mathbb{R}_+), then (9) holds.

Theorem 3.4. [5] Suppose that $f(x) \in \text{SBVF}_2(\mathbb{R}_+)$ and (8) is satisfied.

- (i) If $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ and (9) holds, then integral (7) converges uniformly in t .
- (ii) Conversely, if $f: \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ and (7) converges uniformly in t , then (9) holds.

Remark. The sufficiency part of Theorem 3.4 hinges on the following. If $f(x) \in \text{SBVF}_2(\mathbb{R}_+)$ and (9) holds, then

$$a \int_a^\infty |f'(x)| dx \rightarrow 0, \quad \text{if } a \rightarrow \infty.$$

The necessity part is based on the fact that for any $f(x) \in \text{SBVF}_2(\mathbb{R}_+)$

$$a |f(a)| \leq \int_a^{2a} |f(x)| dx + C \sup_{b \geq B(a)} \int_b^{2b} |f(x)| dx.$$

Formal derivative and integral

We describe the uniform convergence of the sine integrals

$$(10) \quad \int_0^{\infty} x^r f(x) \sin tx \, dx,$$

which are the r times formally differentiated integrals of (7) with respect to t for positive even integer r and are the r times formally integrated series of (7) with respect to t in case r is a negative even integer.

Theorem 3.5. [5] *Suppose that $f(x) \in \text{MVBVF}(\mathbb{R}_+)$. Then $g_r(x) = x^r f(x)$ also belongs to $\text{MVBVF}(\mathbb{R}_+)$ for any fixed integer r .*

Corollary 3.6. [5] *Assume that $f(x) \in \text{MVBVF}(\mathbb{R}_+)$ with property (8) and r is a positive even integer.*

(i) *If $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ and $x^{r+1} f(x) \rightarrow 0$ as $x \rightarrow \infty$, then the sine integral (10) converges uniformly in t .*

(ii) *Conversely, if $f : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ and (10) converges uniformly in t , then $x^{r+1} f(x) \rightarrow 0$ as $x \rightarrow \infty$.*

For any negative even integer r the above statements remain true if $x^{r+1} f(x) \in L^1_{\text{loc}}(\overline{\mathbb{R}}_+)$.

Theorem 3.7. [5] *Suppose that $f(x) \in \text{SBVF}(\mathbb{R}_+)$. Then $g_r(x) = x^r f(x) \in \text{SBVF}(\mathbb{R}_+)$ for any fixed positive integer r .*

Corollary 3.8. [5] *If we assume $f(x) \in \text{SBVF}(\mathbb{R}_+)$ with property (8) and r is an even positive integer, then statements (i) and (ii) of Corollary 3.6 hold.*

4 Double sine integrals

Main results

Consider the double sine integrals of the form

$$(11) \quad \int_0^{\infty} \int_0^{\infty} f(x, y) \sin ux \sin vy \, dx \, dy, \quad (u, v) \in \mathbb{R}^2$$

where $f(x, y) : \mathbb{R}_+^2 \rightarrow \mathbb{C}$ is a Lebesgue measurable function and

$$(12) \quad xyf(x, y) \in L_{\text{loc}}^1(\overline{\mathbb{R}_+^2}).$$

We say that the double integral (11) is regularly convergent at a fixed (u, v) if

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) \sin ux \sin vy \, dx \, dy \rightarrow 0 \quad \text{as } a_1 + a_2 \rightarrow \infty, \, b_1 > a_1 \geq 0, \, b_2 > a_2 \geq 0.$$

Definition. A function $f(x, y) \in AC_{\text{loc}}(\mathbb{R}_+^2)$ is said to be in class $MVBVF(\mathbb{R}_+^2)$, if there exist constants C and $\lambda \geq 2$ depending only on f , such that

$$\begin{aligned} \int_{a_1}^{2a_1} |f_x(x, y)| \, dx &\leq \frac{C}{a_1} \int_{a_1/\lambda}^{\lambda a_1} |f(x, y)| \, dx, \quad a_1, y > 0, \\ \int_{a_2}^{2a_2} |f_y(x, y)| \, dy &\leq \frac{C}{a_2} \int_{a_2/\lambda}^{\lambda a_2} |f(x, y)| \, dy, \quad x, a_2 > 0, \\ \int_{a_1}^{2a_1} \int_{a_2}^{2a_2} |f_{xy}(x, y)| \, dx \, dy &\leq \frac{C}{a_1 a_2} \int_{a_1/\lambda}^{\lambda a_1} \int_{a_2/\lambda}^{\lambda a_2} |f(x, y)| \, dx \, dy, \quad a_1, a_2 > 0. \end{aligned}$$

Theorem 4.1. Suppose that $f \in MVBVF(\mathbb{R}_+^2)$ with property (12).

(i) If $f : \mathbb{R}_+^2 \rightarrow \mathbb{C}$ and

$$(13) \quad xyf(x, y) \rightarrow 0 \quad \text{as } x + y \rightarrow \infty,$$

then the regular convergence of (11) is uniform in $(u, v) \in \mathbb{R}_+^2$.

(ii) Conversely, if $f : \mathbb{R}_+^2 \rightarrow \overline{\mathbb{R}_+}$ and the regular convergence of (11) is uniform in (u, v) , then (13) holds.

Furthermore, we defined class $NBVF(\mathbb{R}_+^2)$ and proved that $NBVF(\mathbb{R}_+^2)$ is a subclass of $MVBVF(\mathbb{R}_+^2)$, however $NBVF(\mathbb{R}_+^2)$ contains the nonnegative, nonincreasing, locally absolutely continuous functions defined on \mathbb{R}_+^2 .

Corollary 4.2. If $f : \mathbb{R}_+^2 \rightarrow \overline{\mathbb{R}_+}$ is monotonically nonincreasing and $f \in AC_{\text{loc}}(\mathbb{R}_+^2)$, then (13) is necessary and sufficient for the regular convergence of (11) to be uniform in (u, v) .

Remark. Part (i) of Theorem 4.1 and Corollary 4.2 can be found in [8] with a slight difference, since the theorems in [8] contain a plus condition, which is surplus as we proved in our dissertation.

Remark. At the same time as article [8], F. Móricz got a similar result to Corollary 4.2 in [11] for nonnegative, nonincreasing (not necessarily locally absolutely continuous) functions defined on \mathbb{R}_+^2 .

Auxiliary results

The necessity and sufficiency parts of Theorem 4.1 are based on the following lemmas.

Lemma 4.3. *Assume that $f : \mathbb{R}_+^2 \rightarrow \mathbb{C}$, $f \in \text{MVBVF}(\mathbb{R}_+^2)$ satisfies (12). If (13) holds, then*

$$\begin{aligned} a_1 y \int_{a_1}^{\infty} |f_x(x, y)| dx &\rightarrow 0, \quad \text{ha} \quad a_1 + y \rightarrow \infty, \quad a_1, y > 0, \\ x a_2 \int_{a_2}^{\infty} |f_y(x, y)| dy &\rightarrow 0, \quad \text{ha} \quad x + a_2 \rightarrow \infty, \quad x, a_2 > 0, \\ a_1 a_2 \int_{a_1}^{\infty} \int_{a_2}^{\infty} |f_{xy}(x, y)| dx dy &\rightarrow 0, \quad \text{ha} \quad a_1 + a_2 \rightarrow \infty, \quad a_1, a_2 > 0. \end{aligned}$$

Lemma 4.4. *Suppose that $f : \mathbb{R}_+^2 \rightarrow \overline{\mathbb{R}}_+$, $f \in \text{MVBVF}(\mathbb{R}_+^2)$ with constants C and λ . Then for any $a_1, a_2 > 0$ we have*

$$a_1 a_2 f(a_1, a_2) \leq (3C + 1) \int_{a_1/\lambda}^{\lambda a_1} \int_{a_2/\lambda}^{\lambda a_2} f(x, y) dx dy.$$

References

- [1] T. W. CHAUNDY and A. E. JOLLIFFE, The uniform convergence of a certain class of trigonometric series, *Proc. London Math. Soc.*, **15** (1916), 214-216.
- [2] M. DYACHENKO, E. LIFLYAND and S. TIKHONOV, Uniform convergence and integrability of Fourier integrals, *J. Math. Anal. Appl.*, **372** (2010), 328-338.

- [3] M. DYACHENKO and S. TIKHONOV, Integrability and continuity of functions represented by trigonometric series: coefficients criteria, *Studia Math.*, **193** (2009), 285-306.
- [4] P. KÓRUS, Remarks on the uniform and L^1 -convergence of trigonometric series, *Acta Math. Hungar.*, **128** (4) (2010), 369-380.
- [5] P. KÓRUS, On the uniform convergence of special sine integrals, *Acta Math. Hungar.*, **133** (1) (2011), 82-91.
- [6] P. KÓRUS, On the uniform convergence of double sine series with generalized monotone coefficients, *Periodica Math. Hungar.*, **63** (2) (2011), 205-214.
- [7] P. KÓRUS and F. MÓRICZ, On the uniform convergence of double sine series, *Studia Math.*, **193** (2009), 79-97.
- [8] P. KÓRUS and F. MÓRICZ, Generalizations to monotonicity for uniform convergence of double sine integrals over $\overline{\mathbb{R}}_+^2$, *Studia Math.*, **201** (2010), 287-304.
- [9] F. MÓRICZ, Some remarks on the notion of regular convergence of multiple series, *Acta Math. Acad. Sci. Hungar.*, **41** (1983), 161-168.
- [10] F. MÓRICZ, On the uniform convergence of sine integrals, *J. Math. Anal. Appl.*, **354** (2009), 213-219.
- [11] F. MÓRICZ, On the uniform convergence of double sine integrals over $\overline{\mathbb{R}}_+^2$, *Analysis (München)*, **31** (2011), 191-204.
- [12] I. E. ŽAK and A. A. ŠNEIDER, Conditions for uniform convergence of double sine series (russian), *Izv. Vysš. Učebn. Zaved. Mat.*, **4** (1966), 44-52.
- [13] S. P. ZHOU, P. ZHOU and D. S. YU, Ultimate generalization to monotonicity for uniform convergence of trigonometric series,
online: <http://arxiv.org/abs/math/0611805v1>.