Periodic solutions for delay differential equations with monotone feedback

Outline of Ph.D. Thesis

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1 Introduction

In the dissertation we study delay differential equations of the form

$$\dot{x}(t) = -\mu x(t) + f(x(t-\tau)), \quad t > 0, \tag{1.1}$$

where $\mu \geq 0$, $\tau > 0$ and the feedback function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. If $xf(x) \geq 0$ for all $x \in \mathbb{R}$, then the feedback is said to be positive, and if $xf(x) \leq 0$ for all $x \in \mathbb{R}$, then the feedback is said to be negative.

The phase space for delay differential equations is usually $C = C([-\tau, 0], \mathbb{R})$, which is the Banach space of continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}$ with the supremum norm $\|\varphi\| = \sup_{s\in[-\tau,0]} |\varphi(s)|$. If for some $t \in \mathbb{R}$, the interval $[t-\tau, t]$ is in the domain of a continuous function x, then the segment $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for $-\tau \leq s \leq 0$.

All $\varphi \in C$ define a unique continuous function $x^\varphi : [-\tau, \infty) \to \mathbb{R}$, that is differentiable on $(0, \infty)$ and satisfies (1.1) for all $t > 0$ and $x_0^\varphi = \varphi$. That x^φ is the solution of (1.1) correspond to initial function φ. A differentiable function $x\colon \mathbb{R} \to \mathbb{R}$ is a solution if it satisfies (1.1) for all $t \in \mathbb{R}$.

In the doctoral dissertation, we study problems related to periodic solutions in case of positive and negative feedback. The dissertation is based on the following two publications:

- Sz. Beretka, G. Vas, Saddle-node bifurcation of periodic orbits for a delay differential equation, *J. Differential Equations* 269 (2020), no. 5, 4215-4252.
- Sz. Beretka, G. Vas, Stable periodic solutions for Nazarenko's equation, *Communications on Pure & Applied Analysis* 19 (2020), no. 6, 3257-3281.

2 Bifurcation of periodic orbits in case of positive feedback

Numerous scientific works have studied the Hopf-bifurcation of periodic orbits for delay differential equations. In the Chapter 2 of the dissertation we study an uncommon phenomenon, the saddle-node bifurcation of periodic orbits.

Consider the delay differential equation

$$\dot{x}(t) = -x(t) + f_K(x(t-1)), \quad t > 0, \tag{2.2}$$

where the feedback function f_K is a nondecreasing continuous function depending on parameter K.

If χ is a fixed point of f_K (i.e., $f_K(\chi) = \chi$), then

$$\widehat{\chi} : [-1, 0] \ni t \to \chi \in \mathbb{R}$$

is an equilibrium. This equilibrium is asymptotically stable if $f'_K(\chi) < 1$, and unstable if $f'_K(\chi) > 1$.

If f_K has more fixed points at which its derivative is greater than 1 (and hence the dynamical system has more unstable equilibria), then we say that a periodic solution has large amplitude if it oscillates about at least two such fixed points. Krisztin and Vas introduced the definition of large-amplitude periodic solution and showed the existence of a pair of large-amplitude periodic orbits for special f_K in [2]. The paper [3] of Krisztin and Vas described the complicated geometric structure of the unstable set of a large-amplitude periodic orbit in detail. In [11] Vas proved that all configurations of large-amplitude periodic orbits indeed exist that Mallet-Paret and Sell allowed in [5].

In Chapter 2. we prove that the large-amplitude periodic orbits of (2.2) arise via a saddle-node bifurcation if

$$f_K(x) = \begin{cases} K, & x \geq 1 + \varepsilon, \\ \frac{K}{\varepsilon}(x-1), & 1 \leq x < 1 + \varepsilon, \\ 0, & -1 \leq x < 1, \\ \frac{K}{\varepsilon}(x+1), & -1-\varepsilon \leq x < -1, \\ -K, & x < -1 - \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a small fixed number, and $K \in (6, 7)$ is the bifurcation parameter.

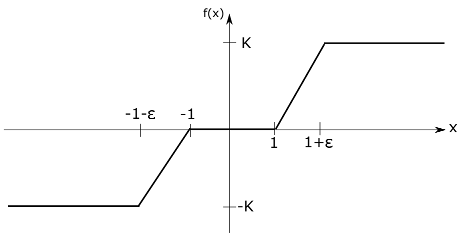

Figure 2.1: The plot of f_K

The following theorem has already appeared in [2] as a conjecture.

Theorem 2.3. *(Saddle-node bifurcation of periodic orbits) For all sufficiently small positive* ε*, one can give a threshold parameter* $K^* = K^*(\varepsilon) \in (6.5, 7)$*, a large-amplitude periodic solution* $p = p(\varepsilon) : \mathbb{R} \to \mathbb{R}$ *of (2.2)* *for the parameter $K = K^*$, an open neighborhood $B = B(\varepsilon)$ of its initial segment p_0 in C and a constant $\delta = \delta(\varepsilon) > 0$ such that*

(i) if $K \in (K^* - \delta, K^*)$, then no periodic orbit for (2.2) has segments in B;

(ii) if $K = K^*$, then $\mathcal{O} = \{p_t : t \in \mathbb{R}\}$ is the only periodic orbit with segments in B;

(iii) if $K \in (K^*, K^* + \delta)$, then there are exactly two periodic orbits with segments in B, and both of them are of large-amplitude.

To our knowledge, only López Nieto has a similar result for delay differential equations: he proved saddle-node bifurcation of periodic orbits for another class of delay equations. His result is awaiting publication [4].

The proof is organized as follows. We introduce a one-dimensional map F which depends also on parameters K and ε. We show that there is a bijection locally between the fixed points of $F(\cdot, K, \varepsilon)$ and the large amplitude periodic solutions of (2.2). Then we show that F undergoes a saddle-node bifurcation as K varies if ε is a fixed and sufficiently small positive number.

In the saddle-node bifurcation of F, a neutral fixed point splits into two fixed points, one attracting and one repelling. This does not imply that we have one stable and one unstable periodic orbit for $K > K^*$. We know that if f_K is a C^1-function with nonnegative derivative, then all periodic orbits for equation (2.2) are unstable, see e.g., Proposition 7.1 in [11]. Hence we presume that the periodic orbits given by the above theorem are also unstable.

The steps for proof are described in detail below.

The map F

In Section 2.3 of the dissertation we consider a special periodic function p as the concatenation of certain auxiliary functions $y_1, y_2, ..., y_{10}$ in such way, that if p is a solution of the delay equation (2.2), then $y_1, y_2, ..., y_{10}$ satisfy a system of ordinary differential equations with boundary conditions. Then we reduce this ODE system to a single fixed point equation of the form $F(L_2, K, \varepsilon) = L_2$, where L_2 is a parameter corresponding to p. The details of the construction are as follows.

Let $\varepsilon \in (0, 1)$ and $K \in (6.5, 7)$. Assume that

(H1) $L_i > 0$ for $i \in \{1, 2, ..., 5\}$,

(H2) $2L_1 + 5L_2 + 5L_3 + 3L_4 + 3L_5 = 1$,

(H3) $\theta_i > 1 + \varepsilon$ for $i \in \{1, 2, 3, 4\}$, and $\theta_i \in (1, 1 + \varepsilon)$ for $i \in \{5, 6\}$.

Consider the subsequent continuous functions (their horizontal translations are shown in Fig. 2.2):

(H4) $y_1 \in C([0, L_1], \mathbb{R})$ with $y_1(0) = 1 + \varepsilon$ and $y_1(L_1) = \theta_1$,
$y_2 \in C([0, L_2], \mathbb{R})$ with $y_2(0) = \theta_1$ and $y_2(L_2) = \theta_2$,
$y_3 \in C([0, L_3], \mathbb{R})$ with $y_3(0) = \theta_2$ and $y_3(L_3) = \theta_3$,
$y_4 \in C([0, L_4], \mathbb{R})$ with $y_4(0) = \theta_3$ and $y_4(L_4) = \theta_4$,
$y_5 \in C([0, L_5], \mathbb{R})$ with $y_5(0) = \theta_4$ and $y_5(L_5) = 1 + \varepsilon$,
$y_6 \in C([0, L_2], \mathbb{R})$ with $y_6(0) = 1 + \varepsilon$ and $y_6(L_2) = \theta_5$,
$y_7 \in C([0, L_3], \mathbb{R})$ with $y_7(0) = \theta_5$ and $y_7(L_3) = \theta_6$,
$y_8 \in C([0, L_4], \mathbb{R})$ with $y_8(0) = \theta_6$ and $y_8(L_4) = 1$,
$y_9 \in C([0, L_2 + L_5], \mathbb{R})$ with $y_9(0) = 1$ and $y_9(L_2 + L_5) = -1$,
$y_{10} \in C([0, L_3], \mathbb{R})$ with $y_{10}(0) = -1$ and $y_{10}(L_3) = -1 - \varepsilon$,

(H5) if $i \in \{1, 2, ..., 5\}$, then $y_i(s) > 1 + \varepsilon$ for all s in the interior of the domain of y_i,
if $i \in \{6, 7, 8\}$, then $y_i(s) \in (1, 1 + \varepsilon)$ for all s in the interior of the domain of y_i,
$y_9(s) \in (-1, 1)$ for all $s \in (0, L_2 + L_5)$,
$y_{10}(s) \in (-1 - \varepsilon, -1)$ for all $s \in (0, L_3)$.

Set $0 < \tau_1 < \tau_2 < \tau_3 < \omega < 1$ as

$$\begin{aligned} \tau_1 &= \sum_{i=1}^{5} L_i, \\ \tau_2 &= \tau_1 + L_2 + L_3 + L_4, \\ \tau_3 &= \tau_2 + L_2 + L_5, \\ \omega &= \tau_3 + L_3. \end{aligned}$$

Introduce a 2ω-periodic function $p : \mathbb{R} \to \mathbb{R}$ as follows. Set p on $[-1, -1+\omega]$ such that

$$\begin{aligned} p(t-1) &= y_1(t) \quad \text{for} \quad t \in [0, L_1], \\ p(t-1+L_1) &= y_2(t) \quad \text{for} \quad t \in [0, L_2], \\ p(t-1+L_1+L_2) &= y_3(t) \quad \text{for} \quad t \in [0, L_3], \\ p(t-1+L_1+L_2+L_3) &= y_4(t) \quad \text{for} \quad t \in [0, L_4], \\ p(t-1+L_1+L_2+L_3+L_4) &= y_5(t) \quad \text{for} \quad t \in [0, L_5], \\ p(t-1+\tau_1) &= y_6(t) \quad \text{for} \quad t \in [0, L_2], \\ p(t-1+\tau_1+L_2) &= y_7(t) \quad \text{for} \quad t \in [0, L_3], \\ p(t-1+\tau_1+L_2+L_3) &= y_8(t) \quad \text{for} \quad t \in [0, L_4], \\ p(t-1+\tau_2) &= y_9(t) \quad \text{for} \quad t \in [0, L_2+L_5], \\ p(t-1+\tau_3) &= y_{10}(t) \quad \text{for} \quad t \in [0, L_3], \end{aligned} \tag{P.1}$$

see Fig. 2.2.

Let

$$p(t) = -p(t-\omega) \quad \text{for all } t \in [-1+\omega, -1+2\omega]. \tag{P.2}$$

Then extend p to the real line 2ω-periodically.

In Section 2.3 we investigate what is the relationship between parameters $L_1, ..., L_5$, $\theta_1, ...,$ θ_6 and functions $y_1, ..., y_{10}$, if p satisfies the equation (2.2) for all $t \in \mathbb{R}$. First we apply hypothesis (H2), (H5) and the fact that p is a solution of (2.2). Then we get a system of ten ordinary differential equations:

$$\begin{aligned}
\dot{y}_1(t) =& - y_1(t) + K \quad \text{for } t \in [0, L_1], \\
\dot{y}_2(t) =& - y_2(t) + \frac{K}{\varepsilon}(y_6(t) - 1) \quad \text{for } t \in [0, L_2], \\
\dot{y}_3(t) =& - y_3(t) + \frac{K}{\varepsilon}(y_7(t) - 1) \quad \text{for } t \in [0, L_3], \\
\dot{y}_4(t) =& - y_4(t) + \frac{K}{\varepsilon}(y_8(t) - 1) \quad \text{for } t \in [0, L_4]. \\
\dot{y}_5(t) =& - y_5(t) \quad \text{for } t \in [0, L_5], \\
\dot{y}_6(t) =& - y_6(t) \quad \text{for } t \in [0, L_2], \\
\dot{y}_8(t) =& - y_8(t) - K \quad \text{for } t \in [0, L_4], \\
\dot{y}_9(t) =& - y_9(t) - K \quad \text{for } t \in [0, L_2 + L_5], \\
\dot{y}_{10}(t) =& - y_{10}(t) - K \quad \text{for } t \in [0, L_3].
\end{aligned} \tag{S.1}$$

If we use the boundary conditions from (H4) for the solutions of the system, then we get a system of 10 algebraic equations for eleven unknowns $L_1, ..., L_5$, $\theta_1, ...,$ θ_6. The eleventh equation comes from (H2). We reduce this system to an equation of the form $F(L_2, K, \varepsilon) = L_2$, where

$$\begin{aligned}
F : U \ni (L_2, K, \varepsilon) \mapsto & \frac{K}{\varepsilon}(K+1)\left(1 - (1 - L_4)e^{L_4}\right) \\
& + \theta_3 - (1+\varepsilon)\frac{K + \theta_6}{K - 1}e^{-L_2} + L_2 \in \mathbb{R}.
\end{aligned}$$

The domain of F is

$$U = \left\{(L_2, K, \varepsilon) \in \mathbb{R}^3 \colon\ \varepsilon \in (0,1),\ K \in (6.5, 7),\ L_2 \in (-\varepsilon, \varepsilon)\right\}.$$

It is easy to verify that F is well-defined and continuous on U. The above reasoning gives the following proposition.

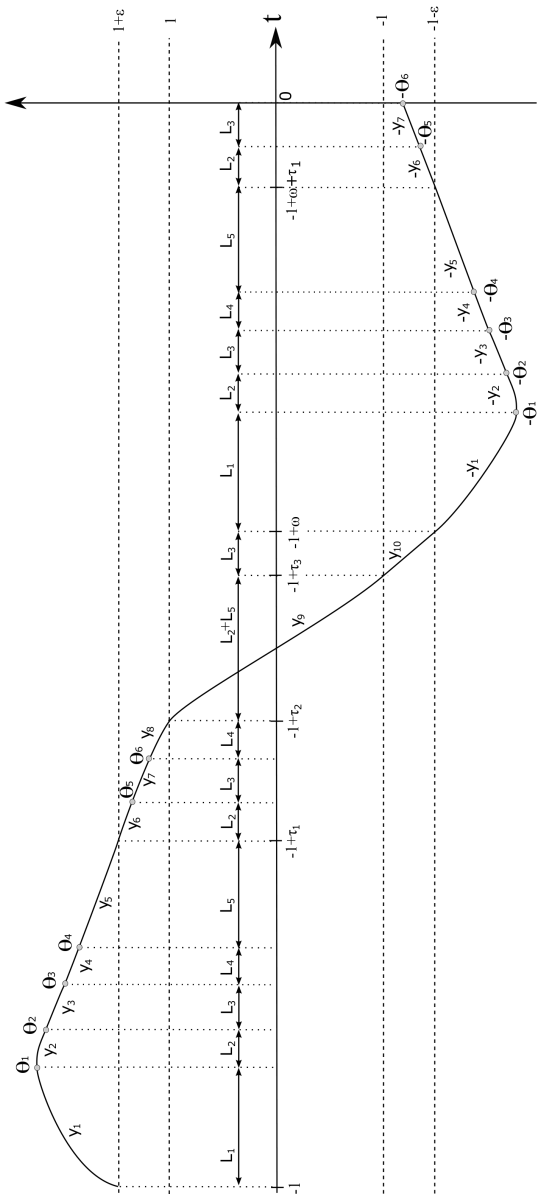

Figure 2.2: The plot of p on [-1,0]

Proposition 2.4. *Let $\varepsilon \in (0,1)$ and $K \in (6.5, 7)$. Suppose that a 2ω-periodic function $p\colon \mathbb{R} \to \mathbb{R}$ is a solution of* (2.2)*, p is the concatenation of functions $y_1, y_2, ..., y_{10}$ as in* (P.1)-(P.2)*, and the functions $y_1, y_2, \ldots, y_{10}$ satisfy (H1)-(H5) for some parameters $L_i > 0$, $i \in \{1, 2, ..., 5\}$, and θ_i, $i \in \{1, \ldots, 6\}$. Then $L_2 \in (0, \varepsilon)$ and $F(L_2, K, \varepsilon) = L_2$.*

Based on the above reasoning, we express parameters L_1, L_3, L_4, L_5 and $\theta_1, \theta_2, ..., \theta_6$ as functions of L_2, K and ε. We use this in the following section.

The fixed points of F yield periodic solutions

By Section 2.3, if (H1)-(H5) hold, and $p\colon \mathbb{R} \to \mathbb{R}$ is a 2ω-periodic solution of (2.2) given by (P.1)-(P.2), then $L_2 \mapsto F(L_2, K, \varepsilon)$ has a fixed point. We devote Section 2.4 to verify the converse statement: if $\varepsilon > 0$ is small enough and $K \in (6.5, 7)$, then all sufficiently small positive fixed points of $L_2 \mapsto F(L_2, K, \varepsilon)$ yield periodic solutions of (2.2).

We need to consider L_1, L_3, L_4, L_5 and θ_i, $1 \le i \le 6$, as functions of L_2, K and ε (and not as parameters given by hypotheses (H1)-(H5)). So assume that

(H6) L_i, $i \in \{1, 3, 4, 5\}$, and θ_i, $1 \le i \le 6$, are functions of L_2, K, ε on U as given in Section 2.3 (see (C.1)-(C.10) in the dissertation).

One can easily check that L_i, $i \in \{1, 3, 4, 5\}$, and θ_i, $1 \le i \le 6$, are continuous functions of (L_2, K, ε) on U.

In this section we also need the assumption that

(H7) the functions $y_1, ..., y_{10}$ are that solutions of the system of (S.1), which are given by (Y.1)-(Y.10) in the dissertation.

Let $\widehat{L}_2$ be that value of L_2 for which $L_4 = 0$, i.e., for which $\theta_6 = 1$. Consider the following subset of U:

$$V = \Big\{(L_2, K, \varepsilon)\colon\ \varepsilon \in (0,1),\ K \in (6.5, 7) \text{ and } L_2 \in \Big(0, \widehat{L}_2(K, \varepsilon)\Big)\Big\} \subset U.$$

The most important result of Section 2.4:

Corollary 2.11. *Assume that (H6) and (H7) hold, $(L_2, K, \varepsilon) \in V$, $F(L_2, K, \varepsilon) = L_2$ and $\varepsilon > 0$ is small enough. Then the 2ω-periodic function p given by (P.1)-(P.2) satisfies the delay differential equation (2.2) for all $t \in \mathbb{R}$.*

The saddle-node bifurcation of F

In Section 2.5 we show that F undergoes a saddle-node bifurcation. For $\varepsilon \in (0,1)$, let

$$U_\varepsilon = (-\varepsilon, \varepsilon) \times (6.5, 7)$$

and define

$$F_\varepsilon : U_\varepsilon \ni (L_2, K) \mapsto F(L_2, K, \varepsilon) \in \mathbb{R}.$$

Proposition 2.14. *For all sufficiently small positive ε, one can give $K^* = K^*(\varepsilon) \in (6.5, 7)$ and $L_2^* = L_2^*(\varepsilon) \in (0, \widehat{L}_2(K, \varepsilon))$ such that F_ε undergoes a saddle-node bifurcation at (L_2^*, K^*): there exist a neighborhood $\mathcal{U}$ of L_2^* in $(0, \widehat{L}_2(K^*, \varepsilon))$ and a constant $\delta_1 > 0$ such that*

- *the map $F_\varepsilon(\cdot, K)$ has no fixed point in $\mathcal{U}$ for $K \in (K^* - \delta_1, K^*)$,*
- *L_2^* is the unique fixed point of $F_\varepsilon(\cdot, K^*)$ in $\mathcal{U}$,*
- *$F_\varepsilon(\cdot, K)$ has exactly two fixed points in $\mathcal{U}$ for $K \in (K^*, K^* + \delta_1)$.*

The delay equation has no other types of periodic solutions locally

To prove the main theorem, we still need to show that all periodic solutions of the delay differential equation (2.2) come from fixed points F - at least locally, in an neighborhood of p_0, where p_0 now denotes the initial segment of the periodic solution p constructed for parameter K^*. This step is detailed in Section 2.6.

The main result of Chapter 2, Theorem 2.3 easily follows from these partial results, see Section 2.7.

3 Periodic orbits for an equation with negative feedback

In Chapter 3 of the dissertation we study the delay differential equation

$$\dot{y}(t) + py(t) - \frac{qy(t)}{r + y^n(t-\tau)} = 0, \qquad t > 0, \tag{3.6}$$

under the assumption that

$$p, q, r, \tau \in (0, \infty), \quad n \in \mathbb{N} = \{1, 2, \ldots\} \quad \text{and} \quad \frac{q}{p} > r. \tag{3.7}$$

This equation was proposed by Nazarenko in 1976 to study the control of a single population of cells [6]. The quantity $y(t)$ is the size of the population at time t. The rate of change $y'(t)$ can be given as the difference of the production rate $qy(t)/(r+y^n(t-\tau))$ and the destruction rate $py(t)$. We see that the destruction rate at time t depends only on the present state $y(t)$ of the system, while the production rate also depends on the past of y. This is a typical concept in population dynamics; delay appears due to the fact that organisms need time to mature before reproduction. In the most widely studied Mackey-Glass equation the production rate is similar to the production rate in Nazarenko's equation:

$$\dot{y}(t) = -py(t) + \frac{qy(t-1)}{r+y^n(t-\tau)}, \qquad t > 0.$$

In this model the production rate is very similar to the one considered by Nazarenko.

In accordance with the previous chapters, the phase space is the Banach space $C = C([-\tau, 0], \mathbb{R})$ with the supremum norm. The solutions of the equation (3.6) and the segments of the solutions are defined as in the Introduction. Under condition (3.7), the functions $\mathbb{R} \ni t \mapsto 0 \in \mathbb{R}$ and $\mathbb{R} \ni t \mapsto K = (q/p - r)^{1/n} \in \mathbb{R}$ are the only constant solutions, i.e., there exists a unique positive equilibrium besides the trivial one.

In this chapter we focus on those positive periodic solutions of (3.6) that oscillate slowly about K. A solution y is called slowly oscillatory about K if all zeros of $y-K$ are spaced at distances greater than the delay τ.

If we restrict our examinations only to positive solutions, then we can apply the transformation $x = \log y - \log K$. Thereby we obtain the equation

$$x'(t) = -f(x(t-\tau)), \tag{3.1}$$

where the feedback function $f \in C^1(\mathbb{R}, \mathbb{R})$ is defined as

$$f(x) = p - \frac{q}{r + \left(\frac{q}{p} - r\right) e^{nx}} \text{ for all } x \in \mathbb{R}, \tag{3.8}$$

see Fig. 3.1. Then we focus on those periodic solutions which oscillate slowly about 0 (SOP solutions), i.e. on those periodic solutions which has zeros spaced at distances greater than τ.

Nussbaum verified the global existence of SOP solutions for equations of the form (3.1) and for a wide class of feedback functions containing (3.8), see [7] and also [8]. By [7, 8], equation (3.1) has at least one SOP solution for

$$\tau > \tau_0 = \frac{\pi}{2f'(0)} = \frac{q\pi}{2np(q-pr)}.$$

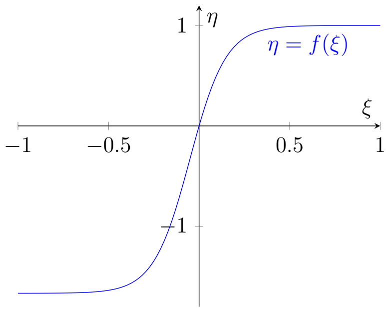

Figure 3.1: The plot of f if $p = 1$, $q = 4$, $r = 1,5$ and $n = 10$

Song, Wei and Han studied the equation in the form (3.6) in [10]. They showed that a series of Hopf bifurcations takes place at the positive equilibrium as τ passes through the critical values

$$\tau_k = \frac{1}{f'(0)}\left(\frac{\pi}{2} + 2k\pi\right) = \frac{q}{np(q-pr)}\left(\frac{\pi}{2} + 2k\pi\right), \qquad k \geq 0.$$

Song and his coauthors could not determine the stability of the periodic orbits for τ far away from the local Hopf bifurcation values. Uniqueness of the slowly oscillatory periodic solution has not been studied either. In the dissertation we focus on these questions and verify the following two theorems.

Theorem 3.8. *Set p, q, r and n as in (3.7).*
(i) If $\tau > 0$ is large enough, then equation (3.6) has a unique positive periodic solution $\bar{y} : \mathbb{R} \to \mathbb{R}$ oscillating slowly about K. The corresponding periodic orbit is asymptotically stable, and it attracts the set

$$\left\{\phi : y^{\phi}(t) > 0 \text{ for } t \geq -\tau, y_t^{\phi} - K \text{ has at most one sign change for large } t\right\}.$$

(ii) If $\bar{\omega}$ denotes the minimal period of $\bar{y}$, and

$$\omega = \left(2 + \frac{q-pr}{pr} + \frac{pr}{q-pr}\right)\tau, \tag{3.9}$$

then $\lim_{\tau\to\infty} \bar{\omega}/\omega = 1$.

Uniqueness of the periodic solution is always meant up to time translation.

If we fix p, q, r and τ, then we can determine the asymptotic shape of the periodic solution as $n \to \infty$.

Theorem 3.9. *Set p, q, r and τ such that (3.7) and $\tau \min\{p, q/r - p\} > 8$ hold.*
(i) Theorem 3.8.(i) is true for all sufficiently large n.
(ii) Define $v : \mathbb{R} \to \mathbb{R}$ as the ω-periodic extension of the piecewise linear function

$$[0,\omega] \ni t \mapsto \begin{cases} -pt, & 0 \le t < \tau, \\ \left(\frac{q}{r} - p\right) t - \frac{q}{r}\tau, & \tau \le t < \left(2 + \frac{pr}{q-pr}\right)\tau, \\ -pt + \left(\frac{q}{r} + p + \frac{p^2 r}{q-pr}\right)\tau, & \left(2 + \frac{pr}{q-pr}\right)\tau \le t < \omega \end{cases} \in \mathbb{R},$$

where ω is given by (3.9)*. Let $\eta_1 > 0$ and $\eta_2 > 0$ be arbitrary. If n is large enough, then there exists $T \in \mathbb{R}$ for the $\bar{\omega}$-periodic solution $\bar{y}$, such that $|\bar{\omega} - \omega| < \eta_1$, and*

$$\left| \log \frac{\bar{y}(t+T)}{K} - v(t) \right| < \eta_2 \qquad \textit{for all } t \in [0, \bar{\omega}].$$

The proofs of these theorems are similar, and they are organized as follows. We examine equation (3.6) in the form of (3.1) with feedback function (3.8). First we calculate an SOP solution v for the "limit equation"

$$v'(t) = -g(v(t-\tau)),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a piecewise constant function chosen so that (3.8) is close to g outside a neighborhood of 0. Then we consider (3.8) as a perturbation of g and follow the technique used by Walther in [12] (for a slightly different class of equations) to obtain information about those solutions of equation (3.1) which has initial segments in

$$\mathcal{A}(\beta) = \{\phi \in C : \ \phi(t) \ge \beta \text{ for all } -\tau \le t \le 0, \ \phi(0) = \beta\} \subseteq C.$$

We show that these solutions return to $\mathcal{A}(\beta)$ (for appropriately chosen β). Thereby a Poincaré-map $P \colon \mathcal{A}(\beta) \to \mathcal{A}(\beta)$ can be introduced. Next we explicitly evaluate a Lipschitz constant $L(P)$ for P. If τ or n is large enough, then $L(P) < 1$, i.e., P is a contraction. The unique fixed point of P is the initial segment of an SOP solution. Besides this, we need the results of paper [9] of Nussbaum to show that all SOP solutions have segments in $\mathcal{A}(\beta)$, and hence the SOP solution is unique up to time translation. Stability comes from work [1] of Kaplan and Yorke. The rest of the theorems will follow easily.

The details of the proof are described below.

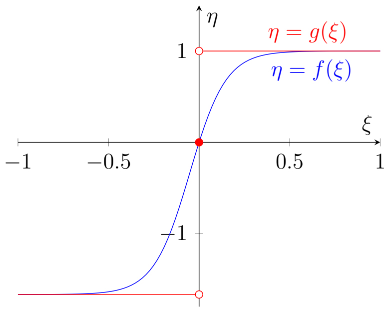

Figure 3.2: The plot of $g^{A,B}$

The limit equation

Consider equation (3.1) with feedback function (3.8). Let $A = q/r - p > 0$ and $B = p > 0$.

Note that if p, q, r are fixed according to (3.7), then

$$f(x) \to p - \frac{q}{r} = -A \quad \text{if } nx \to -\infty,$$

and

$$f(x) \to p = B \quad \text{if } nx \to \infty.$$

Therefore in Section 3.3 we examine the "limit equation"

$$v'(t) = -g^{A,B}(v(t-\tau)), \tag{3.11}$$

where $g^{A,B} : \mathbb{R} \to \mathbb{R}$ is defined as

$$g^{A,B}(v) = \begin{cases} -A, & v < 0, \\ 0, & v = 0, \\ B, & v > 0. \end{cases}$$

Proposition 3.10. *Equation* (3.11) *admits a periodic solution* $v : \mathbb{R} \to \mathbb{R}$ *defined as follows:*

$$v(t) = \begin{cases} -Bt, & t \in [0, \tau], \\ At - (A+B)\tau, & t \in [\tau, \sigma + \tau], \\ -Bt + \left(A + 2B + \frac{B^2}{A}\right)\tau, & t \in [\sigma + \tau, \omega], \end{cases}$$

where $\sigma = (1 + B/A)\tau$ *is the first positive zero, and* $\omega = (2 + A/B + B/A)\tau$ *is the second positive zero and the minimal period of* v.

Preliminary estimates

For $A > 0$, $B > 0$, $\beta > 0$, $0 < \varepsilon < \min\{A, B\}/2$, let $\mathcal{N}(A, B, \beta, \varepsilon)$ denote the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with

$$-A \leq f(x) \leq -A + \varepsilon \quad \text{for } x < -\beta,$$

$$-A \leq f(x) \leq B \quad \text{for } -\beta \leq x \leq \beta,$$

and

$$B - \varepsilon \leq f(x) \leq B \quad \text{for } x > \beta.$$

Fig. 3.3 shows an element of $\mathcal{N}(A, B, \beta, \varepsilon)$. Function (3.8) is an element of $\mathcal{N}(A, B, \beta, \varepsilon)$ if $A = q/r - p$, $B = p$, $0 < \varepsilon < \min\{A, B\}/2$ and

$$\beta \geq \max\left\{f^{-1}(B - \varepsilon), -f^{-1}(-A + \varepsilon)\right\}.$$

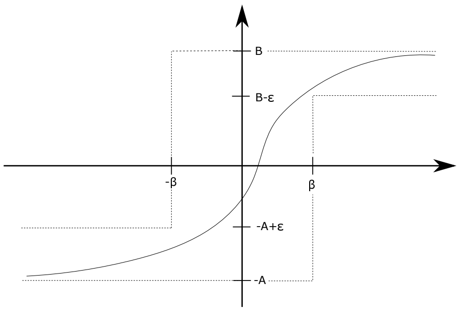

Figure 3.3: An element of $\mathcal{N}(A, B, \beta, \varepsilon)$

In Section 3.4 of the dissertation we examine the solutions $x = x^{\phi}$ of the equation (3.1) if $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ and

$$\phi \in \mathcal{A}(\beta) = \{\phi \in C : \ \phi(t) \geq \beta \text{ for all } -\tau \leq t \leq 0, \, \phi(0) = \beta\} \subseteq C.$$

We prove that if β and ε are chosen correctly, then there are $q = q(\phi) > 0$ and $\tilde{q} = \tilde{q}(\phi)$ such that

$$x_q \in -\mathcal{A}(\beta) = \{\phi \in C : \phi(t) \leq -\beta \text{ if } -\tau \leq t \leq 0, \ \phi(0) = -\beta\},$$

and $x_{q+\tilde{q}} \in \mathcal{A}(\beta)$.

We can prove that $x_q \in -\mathcal{A}(\beta)$ and $x_{q+\tilde{q}} \in \mathcal{A}(\beta)$ for some $q, \tilde{q} > 0$ by giving estimates for the $\left|x^{\phi} - v\right|$, where v is special periodic function constructed in the previous section. This technique is based on Walther's paper [12].

Lipschitz continuous return maps

Based on the results of the previous section, we can introduce the Poincaré-map $P : \mathcal{A}(\beta) \to \mathcal{A}(\beta)$. The next step is to determine a Lipschitz constant for P, see Section 3.5 of the dissertation.

Suppose in addition that $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ is Lipschitz-continuous, and $L(f)$ is a Lipschitz constant for f. Let $L_\beta = L_\beta(f)$ and $L_{-\beta} = L_{-\beta}(f)$ be the Lipschitz constants for the restrictions $f|_{[\beta,\infty)}$ and $f|_{(-\infty,-\beta]}$, respectively.

In this section Φ denotes the semiflow corresponding to (3.1):

$$\Phi\colon [0,\infty) \times C \ni (t,\phi) \mapsto x_t^\phi \in C.$$

Consider the map

$$R : \mathcal{A}(\beta) \ni \phi \mapsto \Phi(q(\phi), \phi) = x_{q(\phi)}^\phi \in -\mathcal{A}(\beta).$$

Chose ε and β as in the previous section. Then the following is true.

Corollary 3.20. *The constant*

$$L(R) = 3\tau L_\beta \left(1 + \delta L(f)\right) \left(1 + (N-1)\tau L_{-\beta}(1 + \tau L_{-\beta})^{N-2}\right)$$

is a Lipschitz constant for R, where $N = \lceil 1 + B/A \rceil$ and $\delta = 2\beta/(B - \varepsilon)$.

Now consider the map

$$Q : -\mathcal{A}(\beta) \ni \phi \mapsto \Phi(\tilde{q}(\phi), \phi) \in \mathcal{A}(\beta).$$

It is clear that $P = Q \circ R$.

Proposition 3.21. *The constant*

$$L(Q) = 3\tau L_{-\beta} \left(1 + \tilde{\delta} L(f)\right) \left(1 + (\tilde{N}-1)\tau L_\beta(1 + \tau L_\beta)^{\tilde{N}-2}\right)$$

is a Lipschitz constant for Q, where $\tilde{N} = \lceil 1 + A/B \rceil$ and $\tilde{\delta} = 2\beta/(A - \varepsilon)$.

As a consequence, the following can be stated.

Proposition 3.22. *The Poincaré map $P : \mathcal{A}(\beta) \ni \phi \mapsto Q\left(R(\phi)\right) \in \mathcal{A}(\beta)$ is Lipschitz continuous, and*

$$\begin{aligned} L(P) =& L(R)L(Q) \\ =& 3\tau L_\beta(1 + \delta L(f)) \left(1 + (N-1)\tau L_{-\beta}(1 + \tau L_{-\beta})^{N-2}\right) \\ & \times 3\tau L_{-\beta} \left(1 + \tilde{\delta} L(f)\right) \left(1 + (\tilde{N}-1)\tau L_\beta(1 + \tau L_\beta)^{\tilde{N}-2}\right). \end{aligned}$$

is a Lipschitz constant for P.

If $L(P) < 1$, then P is a contraction and P has only one fixed point, which is the initial segment of a slowly oscillatory periodic solution.

On the ranges of the SOP solutions

In Section 3.6 we show that if τ is large enough and β is small enough, then any SOP solution $x : \mathbb{R} \to \mathbb{R}$ of (3.1) has segments in $\mathcal{A}(\beta)$.

As we are going to apply paper [9] of Nussbaum, we consider equation (3.1) in form

$$\tilde{x}'(t) = -\tau f(\tilde{x}(t-1)), \tag{3.33}$$

where $\tilde{x}(t) = x(\tau t)$ and f is given in (3.8). Set

$$d = \frac{1}{2} \min\{-f(-1), f(1), f'(0)\}.$$

In [9] Nussbaum gives specific estimates for the ranges of the slowly oscillatory periodic solutions on different subintervals of a real line. The immediate consequence of these estimates is the following proposition.

Proposition 3.26. *If $\tau d > 4$ and B is an upper bound for f, then for each SOP solution $\tilde{x} : \mathbb{R} \to \mathbb{R}$ of (3.33), one can give an interval I of length* 1 *such that*

$$\tilde{x}(t) \geq \frac{\tau(\sqrt{B^2 + d^2} - B)}{2} \quad \textit{for} \quad t \in I.$$

Corollary 3.27. *If $\tau d > 4$ and $\beta \leq \tau(\sqrt{B^2 + d^2} - B)/2$, where B is an upper bound for f, then any SOP solution of (2.2) has a segment in $\mathcal{A}(\beta)$.*

Proofs of the main theorems in this chapter

The proof of Theorem 3.8 is the following. We show that ε and β can be chosen such that the propositions described in the previous sections of the dissertation are satisfied: let ε be a fixed, small positive number, and let $\beta = \alpha\tau$, where $\alpha > 0$ is also a fixed small number. Then for all sufficiently large τ, $L(P) < 1$, i.e., P is a contraction, and the assumptions of Consequence 3.27 are also satisfied. From this we obtain the existence and uniqueness of the slowly oscillatory periodic solution. Stability follows from Theorem 2.1 and Remark 2.5 of Kaplan and Yorke's paper [1]. The statement for the minimal period of a slowly oscillatory periodic solution is obtained from Theorem 1 of [9].

Theorem 3.9 can be proved in a similar way. The statement describing the asymptotic form of the periodic solution comes immediately from the estimates given for $\left|x^{\phi} - v\right|$ in Section 3.4.

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