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Outline of Ph.D. thesis

Constructions, classifications and embeddings of abstract unitals

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1 Introduction

The classification of combinatorial structures has been a significant research topic for a long time and our main concern is designs with parameters $t = 2$ and $\lambda = 1$. Important classes of such 2-designs are affine and projective planes, Steiner triple systems and abstract unitals. The thesis concentrates on abstract unitals, especially on the embeddability of unitals into the classical projective plane taking advantage of the full points of unitals and on creating new unitals via paramodification. Chapter 1 of the thesis serves as an introduction: it outlines the structure of the thesis.

In this section we recall some essential concepts and results regarding unitals, the reader may find more details in Chapter 2 of the thesis and in the monographs [1, 2, 3].

Definition 1.1. Let t and λ be positive integers and $\mathbf{D} = (\mathcal{P}, \mathcal{B}, I)$ a finite incidence structure. Then $\mathbf{D}$ is called a *t-design* with parameters k and λ if and only if

(i) any t-subset $\mathcal{Q}$ of the point set $\mathcal{P}$ is incident with exactly λ blocks, and

(ii) any block is incident with exactly k points.

A t-design on v points is called an $\mathrm{S}_\lambda(t, k, v)$ or a t-(v, k, λ) design. In case $\lambda = 1$, it is called a *Steiner system* $\mathrm{S}(t, k, v)$. A 2-design on v points is called a *block design*.

Theorem 1.2. *Let* $\mathbf{D}$ *be an* 2-(v, k, λ) *design. Then we have:*

(i) every point P is incident with r blocks and

$$r = \frac{\lambda\,(v-1)}{k-1};$$

(ii) for the number of blocks $|\mathcal{B}|$

$$|\mathcal{B}| = \lambda \frac{v\,(v-1)}{k\,(k-1)}.$$

The projective plane $\mathrm{PG}(2, \mathbb{F})$, constructed from a three-dimensional vector space over over some field (or skew field) $\mathbb{F}$ is called the *classical projective plane*. In the finite setting $\mathbb{F} = \mathrm{GF}(q)$ for some prime power q. In this case we denote $\mathrm{PG}(2, \mathbb{F})$ by $\mathrm{PG}(2, q)$.

Definition 1.3. Let Π be a projective plane and let Π^* denote its dual plane. Then an $\alpha\colon \Pi \to \Pi^*$ bijection which preserves containment, is called a *correlation*. Any α correlation is also a $\Pi^* \to \Pi$ correlation as well. If the correlation α has order two, namely $\alpha \circ \alpha$ is the identity collineation of Π, then α is called a *polarity*.

Definition 1.4. Let ρ be a unitary polarity of $\mathrm{PG}\left(2, q^2\right)$. The set of absolute points of ρ is called a *nondegenerate Hermitian curve* denoted by $\mathcal{H}(q)$.

Using the combinatorial properties of nondegenerate Hermitian curves we define abstract unitals of order n.

Definition 1.5. Let n be an integer, $n \geq 3$. A *unital of order n* is any 2-$\left(n^3 + 1, n + 1, 1\right)$ design.

2 Paramodifications of unitals

Chapter 3 of the thesis is based on the paper *New Steiner 2-designs from old ones by paramodifications* by Mezőfi and Nagy [6].

Let $\mathbf{D} = (\mathcal{P}, \mathcal{B}, I)$ be a t-(v, k, λ) design. The integer

$$r = \frac{|\mathcal{B}|\, k}{n}$$

is the number of blocks through a given point. The map $\chi\colon \mathcal{B} \to X$ is called a *proper block coloring* of $\mathbf{D}$, if for different blocks b, b', $b \cap b' \neq \emptyset$ implies $\chi(b) \neq \chi(b')$. If $|X| = m$ and $\mathbf{D}$ has a proper block coloring $\chi\colon \mathcal{B} \to X$ then we say that $\mathbf{D}$ is *block m-colorable*.

Lemma 2.1 (Lemma 3.1.1 (iv))**.** *Let $\mathbf{D} = (\mathcal{P}, \mathcal{B}, I)$ be a t-(v, k, λ) design. Then $\mathbf{D}$ is block r-colorable if and only if it is resolvable.*

From now on, $\mathbf{D} = (\mathcal{P}, \mathcal{B}, I)$ denotes a 2-$(v, k, 1)$ design. The incidence relation $I = \in$, that is, the blocks of $\mathbf{D}$ are subsets of size k of $\mathcal{P}$. Fix a block $b \in \mathcal{B}$ and consider the subset

$$C(b) = \left\{ b' \in \mathcal{B} \colon |b' \cap b| = 1 \right\}$$

of blocks. We write $\mathbf{D}_b$ for the subsystem $(\mathcal{P} \setminus b, C(b), I)$. We define the map $\chi_b \colon C(b) \to b$ by

$$\chi_b \colon b' \mapsto b' \cap b;$$

this is clearly a block coloring of $\mathbf{D}_b$.

Lemma 2.2 (Lemma 3.1.2)**.** $\mathbf{D}_b$ *is a resolvable* 1-$(v-k, k-1, k)$ *design.*

Section 3.1 of the thesis aims to show that any parallelism of $\mathbf{D}_b$ leads to a block design $\mathbf{D}'$ such that $\mathbf{D}$ and $\mathbf{D}'$ have the same parameters.

Definition 2.3. Let $\mathbf{D} = (\mathcal{P}, \mathcal{B}, I)$ be a Steiner 2-$(v, k, 1)$ design. Let $b \in \mathcal{B}$ be a block and $\chi \colon C(b) \to b$ a block coloring of the subsystem $\mathbf{D}_b$ with k colors. Define the incidence relation $I^* \subseteq \mathcal{P} \times \mathcal{B}$ by

$$P \; I^* \; b' \Leftrightarrow \begin{cases} P \; I \; b', & \text{if } b' \notin C(b) \text{ or } P \not I b \\ P = \chi(b'), & \text{if } P \; I \; b \text{ and } b' \in C(b). \end{cases}$$

We call the incidence structure

$$\mathbf{D}^* = \mathbf{D}^*_{\chi, b} = (\mathcal{P}, \mathcal{B}, I^*)$$

the (χ, b)-*paramodification* of $\mathbf{D}$.

Theorem 2.4 (Theorem 3.1.4)**.** *Let* $\mathbf{D} = (\mathcal{P}, \mathcal{B}, I)$ *be a* 2-$(v, k, 1)$ *design. Let* $b \in \mathcal{B}$ *be a block and* $\chi \colon C(b) \to b$ *a block coloring of the subsystem* $\mathbf{D}_b$ *with* k *colors. Then,* $\mathbf{D}^*_{\chi, b}$ *is a Steiner* 2*-design with the same parameters.*

Section 3.2 describes the effect of a paramodification on the incidence matrix of the 2-$(v, k, 1)$ design $\mathbf{D}$.

Proposition 2.5 (Proposition 3.2.1)**.** *Let* $\mathbf{D}$ *be a Steiner* 2-$(v, k, 1)$ *design and* $\mathbf{D}^* = \mathbf{D}^*_{\chi,b}$ *be a* (χ, b)*-paramodification of* $\mathbf{D}$*. Let* $r = (v-1) / (k-1)$*. Then, the respective incidence matrices* $\mathbf{M}$ *and* $\mathbf{M}^*$ *differ at most in a* $k \times k(r-1)$ *submatrix.*

It is shown in Proposition 3.2.2, that switchings are special cases of paramodifications. In a 2-$(v, k, 1)$ design, a Pasch configuration consists of six points $P_1, \ldots, P_6$ such that the triples $\{P_1, P_3, P_4\}$, $\{P_1, P_5, P_6\}$, $\{P_2, P_3, P_5\}$, $\{P_2, P_4, P_6\}$ are collinear. The design is anti-Pasch if it does not contain any Pasch configuration.

Proposition 2.6 (Proposition 3.2.3)**.** *Let* $\mathbf{D}$ *be an anti-Pasch* 2-$(v, k, 1)$ *design. If*

$$v < 2k^3 - 8k^2 + 13k - 6,$$

then no switching can be carried out for $\mathbf{D}$*.*

Section 3.3 of the thesis discusses the paramodification of certain well-known classes of Steiner 2-designs.

Proposition 2.7 (Proposition 3.3.1 (i))**.** *Paramodifications of a finite projective plane are isomorphic. In other words, finite projective planes are para-rigid.*

A Steiner triple system STS(v) is a 2-$(v, 3, 1)$ design; an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$. Steiner triples systems, cubic graphs (regular graphs of degree 3), and edge colorings are much connected from different points of view. Let $\mathbf{T} = (\mathcal{P}, \mathcal{B}, I)$ be an STS(v) and fix a triple $b = \{x, y, z\} \in \mathcal{B}$. Then $\mathbf{T}_b$ is a simple cubic graph whose edges can be colored by three colors. The Steiner triple system $\mathbf{T}$ is para-rigid, if the cubic graph $\mathbf{T}_b$ has a unique edge 3-coloring for each block b. We close the paramodifications of Steiner triple systems by formulating the open problem of the existence of para-rigid Steiner triple systems.

Unitals with many translation centers constructed by Grundhöfer, Stroppel and Van Maldeghem [4] are discussed (only in the finite case), as the idea of the paramodification of Steiner 2-designs has been motivated by their construction.

We close Section 3.3 of the thesis with an observation on finite Hermitian unitals.

Proposition 2.8 (Proposition 3.3.4)**.** *Finite Hermitian unitals have no switchings, but they do have non-trivial paramodifications.*

In Section 3.4 some methods to compute block colorings of a subsystem $\mathbf{D}_b$ are presented. We are interested in the computation of all block colorings of in order to construct new Steiner 2-designs by paramodification. We formulate the problem in the language of vertex colorings of simple graphs, which is known to be NP-complete in general. The *line graph* $\Gamma = (V, E)$ of $\mathbf{D}_b$ is defined by $V = C(b)$, and $(b_1, b_2) \in E$ if and only if b_1 and b_2 have a unique point $P \notin b$ in common. A straightforward consequence of Lemma 2.2 is that Γ is a $(k-1)^2$-regular simple graph.

Definition 2.9. Let $G = (V, E)$ be a simple graph and $\chi\colon V \to C$ a proper vertex coloring. The vertex $v \in V$ is called *dominant,* if for any color $c' \in C \setminus \{\chi(v)\}$ there is a neighbor v' of v such that $\chi(v') = c'$. The coloring χ is said to be a *b-coloring* if there is at least one dominant vertex in each color class.

Lemma 2.10 (Lemma 3.4.2)**.** *The map $\chi\colon C(b) \to b$ is a proper block coloring of $\mathbf{D}_b$ if and only if it is a b-coloring of the line graph Γ of $\mathbf{D}_b$.*

One way to compute all b-colorings of the graph Γ is to find all solutions of a set cover problem of independent sets. In fact, a color class is an independent set of size $K = (v-k) \,/\, (k-1)$ and the k color classes of a coloring χ are pairwise disjoint. Using the GRAPE package of GAP this approach is easy to implement.

The b-coloring problem can be formulated as an integer linear programming (ILP) problem. Most of the ILP solvers are optimized to find one solution to each problem. However, for our block coloring problem, we are interested in finding all solutions. Up to our knowledge, this is only possible with the MILP solver SCIP. There are many ways to give the ILP formulation of a graph coloring problem. The assignment-based model is the standard formulation of the vertex coloring problem. This formulation uses only binary variables, one for each color and one for each vertex-color pair, and the objective is to minimize the number of used colors. There are other approaches as well, based on partial ordering, like POP and POP2. The idea is to introduce a partial ordering on the union of the vertices and the color set.

A drawback of the ILP formulations is that, in contrast to the set cover method, it is hard to make use of the symmetry of the underlying graph. We conclude that since GRAPE is very efficient in coping with symmetries of a line graph, it is better suited to compute all paramodifications of a given Steiner 2-design.

Section 3.5 of the thesis presents computational results on paramodifications of known small unitals (of order up to 6). This way we construct 173 new unitals of order 3, and 25 712 new unitals of order 4.

The *paramodification graph* Ψ_n for a given order n consists of one vertex for each equivalence class of unitals of order n and with edges between two vertices whenever one can get from one equivalence class to the other via a paramodification. The connected components of the paramodification graph are called *paramodification classes*.

We carried out computations to determine the paramodification classes of Ψ_3 and Ψ_4, containing at least one unital from the classes BBT, KRC or KNP. For the case of order 3, we found all such classes, resulting 173 new unitals of order 3. This subgraph of Ψ_3 is complete in the sense that all paramodifications of all vertices are known, see Table 2.1 (Table 3.1 in the thesis). As switches are special cases of paramodifications, the switching graph is a subgraph of the graph Ψ_3 mentioned above. By restricting the type of transformations to switches, we lose 623 edges between the unitals in contrast to paramodifications, and only 131 of the new 173 unitals are reachable via switching.

In the case of order 4, out of the 1777 unitals of KNP, 1458 turn out to be isolated vertices of Ψ_4. By repeating the paramodification step, we produced 36 878 new unitals of order 4. However, the

Table 2.1: Distribution of the sizes of the paramodification classes

Class size	Ψ_3	Ψ_4
1	3182	1458
2–5	466	99
6–10	35	13
11–100	13	16
101–1000		14
1001–2000		2
2001–7595		2
7596		1^*
12 887		1^*

graph is incomplete as it has unfinished vertices; these are unitals whose paramodifications have not been computed yet. Not counting the isolated vertices, the number of complete paramodification classes is 146. The remaining 2 classes are all incomplete (see the starred entries in Table 2.1), with 16 518 unfinished vertices in total. The largest component with 12 887 known vertices has 6 vertices of KNP type, and its growth computed until the fifth layer of the breadth-first tree is

$$6, \quad 28, \quad 445, \quad 3008, \quad 9400,$$

but the search stopped there, and probably there are more unitals in further layers.

Appendix A contains the source code of the implementation of paramodifications using the package UnitalSZ.

3 Full points of abstract unitals

In Chapter 4 of the thesis the results of the paper *On the geometry of full points of abstract unitals* by Mezőfi and Nagy [7] are presented.

Definition 3.1. Let $U = (\mathcal{P}, \mathcal{B})$ be an abstract unital of order n and fix two blocks b_1, b_2. We say that $P \in \mathcal{P}$ is a *full point with respect to* (b_1, b_2) if $P \notin b_1 \cup b_2$ and for each $Q \in b_1$, the block connecting P and Q intersects b_2.

In other words, there is a well defined projection π_{b_1,P,b_2} from b_1 to b_2 with center P. We denote by $F_U(b_1, b_2)$ the set of full points of U with respect to the blocks b_1, b_2. By definition, any full point P of the blocks b_1, b_2 defines a bijective map $\pi_{b_1,P,b_2} \colon b_1 \to b_2$; we call it the *perspectivity with center P*.

Definition 3.2. Let b_1, b_2 be blocks of the abstract unital U. Define the *group of perspectivities of* b_1 as

$$\mathrm{Persp}_{b_2}(b_1) = \left\langle \pi_{b_1,P,b_2} \pi_{b_2,Q,b_1} \colon P, Q \in F_U(b_1, b_2) \right\rangle .$$

Section 4.1 covers some combinatorial properties of the sets of full points, and it introduces the concept of embedded dual k-nets.

Lemma 3.3 (Lemma 4.1.2)**.** *Let $U = (\mathcal{P}, \mathcal{B})$ be an abstract unital of order $n \geq 2$. Then*

$$|F_U(b_1, b_2)| \leq \begin{cases} n^2 - n & \text{if } b_1, b_2 \text{ have a point in common,} \\ n^2 - 1 & \text{if } b_1, b_2 \text{ are disjoint.} \end{cases}$$

Definition 3.4. Let $U = (\mathcal{P}, \mathcal{B})$ be an abstract unital of order n and $k \geq 3$ an integer. We say that the blocks $b_1, \ldots, b_k$ form an *embedded dual k-net* in U, if the following hold for all $1 \leq i < j \leq k$:

(i) $b_i \cap b_j = \emptyset$.

(ii) For all $P \in b_i$, $Q \in b_j$, the block containing P, Q intersects all $b_1, \ldots, b_k$ in a point.

For embedded dual k-nets, the trivial bound is $k \leq n + 1$. We can improve this to $k \leq n - 1$, implying that an abstract unital of order 3 has no embedded dual 3-nets.

Proposition 3.5 (Proposition 4.1.7)**.** *Let U be an abstract unital of order $n \geq 3$.*

(i) If U has an embedded dual k-net $\{b_1, \ldots, b_k\}$, then $k \leq n-1$.
(ii) For two blocks b_1, b_2, $F_U(b_1, b_2)$ cannot contain more than $n-3$ blocks.

The questions on the embeddings of abstract unitals in projective planes are long studied problems, with special focus on the embeddings of abstract unitals of order q in the desarguesian plane $\mathrm{PG}(2, q^2)$. Korchmáros, Siciliano and Szőnyi [5] introduced the concept of *full point* to study the embedding problem. Their approach was to look at the group of perspectivities with respect to blocks.

Definition 3.6. Let $U = (\mathcal{P}, \mathcal{B})$ be an abstract unital and $b_1, b_2 \in \mathcal{B}$ disjoint blocks.

(i) The triple (U, b_1, b_2) is *full point regular* if the set of full points $F_U(b_1, b_2) \subseteq c$ for some block $c \in \mathcal{B}$ such that $b_1 \cap c = b_2 \cap c = \emptyset$.
(ii) If (U, b_1, b_2) is a full point regular triple and $\mathrm{Persp}_{b_2}(b_1)$ is a cyclic semi-regular permutation group of b_1, then (U, b_1, b_2) is said to be *strongly full point regular.*
(iii) The abstract unital U is *strongly full point regular* if for any two disjoint blocks b_1, b_2 the triple (U, b_1, b_2) is strongly full point regular.

The main theorem of Section 4.2 in the thesis is about the embeddability of abstract unitals into $\mathrm{PG}(2, q^2)$.

Theorem 3.7 (Theorem 4.2.6)**.** *If the unital U of order q is embedded in $\mathrm{PG}(2, q^2)$ then it is strongly full point regular.*

In Section 4.3 of the thesis we show that for an even prime power q, the blocks of the Hermitian unital $\mathcal{H}(q)$ contained in a polar triangle form an embedded dual 3-net (cf. Proposition 4.3.1 in the thesis), and that embedded dual 3-nets and Baer sublines are closely related.

Proposition 3.8 (Proposition 4.3.3)**.** *Let $U = (\mathcal{P}, \mathcal{B})$ be an abstract unital of order q, embedded in $\mathrm{PG}(2, q^2)$. Let b_1, b_2, b_3 form an embedded dual 3-net. Then b_1, b_2, b_3 are Baer sublines.*

Section 4.4 of the thesis presents computational results on the structure of full points of known small unitals. The number of unitals of order 3 and 4 according to the he number of full points and to the structure of the group of perspectivities are shown in the Tables 4.1 and 4.2. The number of (strongly) full point regular unitals in the examined libraries are shown in Table 3.1 (Table 4.4 in the thesis). Note that unitals which are not strongly full point regular cannot be embedded into $\mathrm{PG}(2, q^2)$.

Table 3.1: Full point regularity

Library	Unitals	FPR	SFPR
BBT	909	815	815
KRC	4466	4081	4081
P3M	173	166	166
KNP	1777	1586	1582
P4M	25 641	9196	8980

4 The GAP package UnitalSZ

In Chapter 5 of the thesis the features of the GAP package UnitalSZ [8] developed by the author of the thesis and his supervisor dr. Gábor Péter Nagy are presented, along with some implemented algorithms. The current version of the package is version 0.6, available in a tarball on the website `https://nagygp.github.io/UnitalSZ`, and the source code can be found on GitHub under the GNU General Public License v3.0. The package requires a GAP version 4.8 or higher, and the GAP packages GAPDoc, Digraphs and IO to be installed. Throughout the chapter there are example GAP code snippets to illustrate the described functions.

Section 5.1 presents how one can create a unital object using the package via boolean and incidence matrices and via the list of blocks. Algorithm 5.1 shows, how the check of the conditions is implemented in the package. Methods computing some basic properties of a unital are also demonstrated, e.g. the points, the list of blocks, the automorphism group of the unital, and one may check whether to unitals are isomorporphic or not. In Section 5.2 the commands regarding the available classes and libraries of unitals in the package are shown. Algorithms 5.2 and 5.3 illustrate how the construction of Hermitian unitals and Buekenhout–Metz unitals are implemented in the package.

In Section 5.3 we describe the commands regarding full points of unitals: one can compute the full points of a unital U with respect to the distinct blocks b_1 and b_2, and not just the full points, but the group of perspectivities as well, see Algorithms 5.4 and 5.6. Algorithm 5.5 shows how the computation of all full points of a unital U up to the automorphism group of U are implemented in the package. The functions for determining the embedded dual 3-nets, and the (strong) full point regularity of a unital are also presented.

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