Local weak limits of random graphs and parameter continuity

Ph.D. Thesis

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Introduction

The theory of graph limits is motivated by the aim of understanding the behavior of large graphs. Such graphs can arise from natural models of random graphs, and also in many applications, such as computer science, statistical physics, biological studies, social networks.

One goal of graph limit theory is to define proper notions of distance that reflects the local similarities observed between the members of various families of large (random) graphs and to find a proper limit object for graph sequences that are convergent in this metric. This aim is also motivated by the earlier observations that large graphs chosen from the same family behave very similar to each other, even when one considers random graphs [41, 22]. Local weak convergence provides a tool to understand this phenomenon and study properties of large graphs through the properties of the limit object.

It turned out, that in many cases, it is easier to work with the limit graph as with the large graphs converging to it [5]. One can hope that the examined parameter of the graphs converges and a proper probabilistic parameter of the limit object can be defined that the parameters of the sequence converge to. One example for such a parameter is the matching ratio. This phenomenon illustrate why graph limit theory can provide a useful approach to various questions about parameters of large graphs. Our work on the matching ratio presented in Chapter 3 fits into the series of studies in this direction.

Local weak convergence of random graphs extends naturally to possibly infinite rooted random graphs. The same questions arise for this wider class: in what sense are infinite random graphs determined by their local structures? Do certain parameters of the graphs converge along a local weak convergent sequence? We will examine this type of questions for parameters related to Bernoulli percolation in Chapter 4.

Historical background

The main subject of the thesis, *local weak convergence*, also called *Benjamini-Schramm convergence*, was introduced by Benjamini and Schramm in [17], originally for sequences of finite graphs. Local weak convergence applies for *sparse* graphs, where the average degree of the sequence is bounded uniformly, hence the graphs have $O(|V(G)|)$ edges. The theory behaves especially nice when we consider graphs with a uniform bound on the vertices. There is another approach to local weak convergence of bounded degree graphs which joins combinatorics and analysis. In this concept, the limit object is not a rooted graph but rather an analytic object, a *graphing*, see [4, Example 9.9], [36]. An approach to local weak convergence via homomorphism densities is found in [29].

The above limit concepts do not work for *dense* graphs, i.e., graphs where the number of edges is proportional to the number of all possible edges, that is, graphs with $O(|V(G)|^2)$ edges: if $\lim |E(G_n)|/|V(G_n)|^2 > 0$, then there is no local weak limit of the graph sequence. However, there is a proper notion for convergence in the class of dense graphs. In this case, the limit object is a symmetric measurable function $W : [0,1] \times [0,1] \to \mathbb{R}$ called *graphon*; see [63], [27], [28]. This notion of convergence is trivial for sparse graphs: a sequence with $o(|V(G_n)|^2)$ edges converges to the $W \equiv 0$ graphon.

In the thesis, we work with the local weak convergence defined by Benjamini and Schramm. This concept applies well for the graph models that are most widely used in the applications and have unbounded degrees with bounded expected value.

The convergence of a graph parameter along a local weak convergent sequence means that the parameter is essentially determined by the local structure of the graph. If this is the case, one may hope to find the limit value of the parameter via a properly defined parameter on the limit graph. There are several parameters of graphs that turned out to converge along a local weak convergent sequence of finite graphs: the normalized logarithm of the number of spanning trees [65], the normalized number of self-avoiding walks [49], the relative size of a maximal matching [40, 26], chromatic measures [1], various graph polynomials [31]. We refer to [5] for a survey on the application of graph convergence for parameters of finite graphs. We examine questions about the convergence of the relative size of the maximal matching in Chapter 3.

Similar questions concerning parameters of infinite graphs arise naturally: do certain parameters of infinite graphs converge whenever the sequence of graphs converges in the local weak sense? A central open problem in this field is the conjecture of Schramm [16] about the locality of the percolation critical probability of transitive graphs. We investigate this problem in Chapter 4 in the class of unimodular random graphs.

We note that by the nature of local weak convergence, it cannot be used to gain information about some global behavior, such as connectedness, bipartiteness, etc. To see an example, consider the following sequence: let F_n be random d-regular graphs on n vertices. Let G_n be the following sequence: for n odd, let $G_n = F_n$, for n even, let G_n be the union of two copies of $F_{n/2}$. This sequence converges to the infinite d-regular tree in the local weak sense, but the even and odd members of the sequence differ fundamentally because of their connectedness. A possible approach to refine the notion of local weak convergence to be sensitive also for global properties of the graphs is *local-global convergence*, introduced in [24] and further studied in [52]. A sequence of bounded degree graphs G_n converges in the local-global sense, if for any k and ε there is an l such that for every $n, m > l$ and any fixed coloring of G_n with k colors, there is a coloring of G_m that is at distance at most ε to the fixed coloring of G_n. This notion is sensitive for both local and global properties of the graph sequence.

Parameter and property testing, local algorithms

Given a large graph, besides the value of certain parameters, one may be interested whether the graph satisfies certain property. If a small error in the accuracy of the estimation of the parameter or in the probability of the correct guess is allowed, then in some cases a *constant time randomized algorithm* suffices. For both questions, the goal is to give an answer by looking at a sufficiently large neighborhood around a bounded number of random vertices of the graph, where the radius and the number of the examined neighborhoods is independent of the size of the graph. The existence of such an algorithm is closely related to the continuity of the parameter along local weak convergent graphs sequences, as explained below.

Call the neighborhoods of radius k of k independently uniformly chosen vertices of a finite graph G a *k-sample.* A parameter of finite graphs is called *testable (or estimable)* if it can be estimated with arbitrary small error with arbitrary large probability via a k-sample of a graph for a sufficiently large k depending only on the errors. Elek [38] showed that a bounded parameter of finite graphs is estimable if and only if its value is convergent along local weak convergent sequences of graphs. This theorem shows that the convergent parameters listed above are estimable. An example of a parameter that is not estimable is the independence ratio, see [62, Example 22.5].

We say that a property $\mathcal{P}$ of bounded degree graphs is *testable* if the following holds. Whenever a finite graph G has the property $\mathcal{P}$ or $\mathcal{P}$ does not hold even for any graph G' obtained from G by modifying (adding or removing) $\varepsilon|V(G)|$ edges, then we can decide with high probability between these two cases just by looking at a k-sample of the finite graph G, where k depends only on the errors. Benjamini, Schramm and Shapira [18] showed that every minor-closed property (e.g., planarity) is testable. Elek [39] and Newman and Sohler [72] proved that every property and a large class of parameters are testable for every class of hyperfinite graphs.

Local algorithms can be used to compute not just the relative size but also the structure of e.g., an almost maximal matching or flow of a graph [73, 40, 32]. These algorithms can be deterministic [32] or they use random labels on the vertices (e.g., [40] or our Lemma 3.3.5) to break the symmetries of the graphs. For an overview of estimable parameters, testable properties and randomized local algorithms on bounded degree graphs we refer to [62, Chapter 22].

Aims of the thesis

The thesis is intended to present our results on parameter continuity along local weak convergent sequences of random graphs. We recall the basic notions related to random graphs in Chapter 1 and illustrate them by examples which will be also used later in the thesis.

Chapter 2 is devoted to the definition of local weak convergence of random graphs and the illustration of it by several examples. In Section 2.2.1 we present a partially unpublished result about unimodular Galton–Watson trees that provides an example of the local weak convergence of infinite graphs with unbounded degrees. We define almost sure local weak convergence, a stronger version of local weak convergence for sequences of finite random graphs in Section 2.3. We show that preferential attachment graphs converge in this stronger sense, which will be important for the almost sure convergence of the matching ratio in Chapter 3.

In Chapter 3 we examine the concentration and limiting properties of the (directed) matching ratio, which is closely related to an important parameter in control theory, as shown by Liu, Slotine and Barabási [61]. Our main motivation was that the observations of [61] suggested that the matching ratio of directed graphs is in some cases essentially determined by the degrees of the graphs and converges along local weak convergent sequences. We give the precise forms and proofs of two main observations of [61], which were based on numerical results and heuristics from statistical physics. First, we show in Section 3.2 that the directed matching ratio of directed random networks given by a fix sequence of degrees or by the preferential attachment rule is concentrated around its mean. Second, we study the convergence of the (directed) matching ratio of a random (directed) graph sequence that converges in the local weak sense, and generalize the result of Elek and Lippner [40]. We prove in Theorem 3.3.3 that the mean of the directed matching ratio converges to the properly defined matching ratio parameter of the limiting graph. We further show in Theorem 3.3.10 the almost sure convergence of the matching ratios for the most widely used families of scale-free networks, which was the main motivation of Liu, Slotine and Barabási. The results of this chapter have been published in Beringer and Timár [21].

In Chapter 4 we investigate parameters of infinite graphs related to Bernoulli percolation. We study generalizations of the classical percolation critical probabilities p_c, p_T and the critical probability $\tilde{p}_c$ defined by Duminil-Copin and Tassion (2016), introduced originally for transitive graphs, to bounded degree infinite unimodular random graphs. We show in Theorem 4.2.1 that the equation $p_c = \tilde{p}_c$ holds also in this class of graphs. However, there are unimodular graphs with sub-exponential volume growth and $p_T < p_c$ (Example 4.2.10); i.e., the classical sharpness of phase transition does not hold. We further examine Schramm's conjecture in the case of unimodular random graphs: does $p_c(G_n)$ converge to $p_c(G)$ if G_n converge to G in the local weak sense? In Proposition 4.3.4 we give conditions which imply $\lim p_c(G_n) = p_c(\lim G_n)$. We provide examples of sequences of unimodular graphs such that G_n converges to G but $p_c(G) > \lim p_c(G_n)$ or $p_c(G) < \lim p_c(G_n) < 1$ (Examples 4.3.7, 4.3.8 and 4.3.9). This shows that the locality conjecture does not hold in the generality of bounded degree unimodular random graphs. As a corollary to our positive results, we show in Proposition 4.4.1 that for any transitive graph with sub-exponential volume growth there is a sequence $\mathcal{T}_n$ of large girth bi-Lipschitz invariant subgraphs such that $p_c(\mathcal{T}_n) \to 1$. It remains open whether this holds whenever the transitive graph has cost 1. The results of Chapter 4 have been published in Beringer, Pete and Timár [20].

Chapter 1

Random graphs

1.1 Notations and basic definitions

In this section, we give our notations and the basic definitions used throughout the thesis.

Given a (multi)set F (of edges or vertices) we denote by $|F|$ the number of elements of the set (counted with multiplicity). When a graph G is given, $|G|$ denotes the number of vertices of the graph. Let $[n]$ be the set $\{1, \dots, n\}$.

1.1.1 Graphs

We always consider locally finite graphs, i.e. graphs with finite degrees. Multiple edges and loops are allowed. The graphs can be undirected or directed. We denote the vertex set of the graph G by $V(G)$ and the edge set by $E(G)$. We write $x \sim y$ if x and y are adjacent vertices in G. We denote by $\{x, y\}$ the undirected edge connecting the vertices x and y, and by (x, y) the directed edge pointing from x to y. We denote by e^- and e^+ the endpoints of the (directed) edge e. Given a directed edge $e = (e^-, e^+)$, we call e^- the **tail** and e^+ the **head** of the edge. When a subgraph S is given (maybe implicitly) and it contains exactly one endpoint of the undirected edge e, then we denote that endpoint by e^-. Given a set F of edges, let $V(F)$ be the set of vertices that are incident to an edge in F. In the directed case, let $V^-(F)$ and $V^+(F)$ be the set of the tails and the heads of the edges in F, respectively. We write $\deg_G x$ for the degree of a vertex x in a graph G. If the graph G is directed, denote by $\deg_G^{in} x$ and $\deg_G^{out} x$ the in- and out-degree of the vertex x, respectively.

For any subset S of the vertices, let $\partial_E S := \{e \in E(G) : e^- \in S, e^+ \notin S\}$ be the **edge boundary** of S, let $\partial_V^{\text{in}} S := \{x \in S : \exists y \sim x, y \notin S\}$ be the **internal vertex boundary** of S, and let $\partial_V^{\text{out}} S := \{x \notin S : \exists y \sim x, y \in S\}$ be the **outer vertex boundary** of S.

We use $\text{dist}_G(x, y)$ for the graph distance between the vertices x and y in the graph G. Let $B_G(x, r) := \{y \in V(G) : \text{dist}_G(x, y) \le r\}$ be the ball of radius r around a vertex x in the graph G induced by the graph metric. With a slight abuse of notation, we often regard $B_G(x, r)$ as the *subgraph* of G induced by the vertices at distance at most r from x.

We often use rooted graphs (G, o), where G is a graph and o is a distinguished vertex of G, the **root** of the graph.

Given two graphs G_1 and G_2 we denote by $G_1 \simeq G_2$ that **the graphs are isomorphic**, i.e. there is a graph isomorphism $\phi : V(G_1) \to V(G_2)$ such that ϕ is a bijection and $\{x, y\} \in E(G_1)$ if and only if $\{\phi(x), \phi(y)\} \in E(G_2)$. If (G_1, o_1) and (G_2, o_2) are rooted graphs, we say that they are **rooted isomorphic**, denoted by $(G_1, o_1) \simeq (G_2, o_2)$ if there is a graph isomorphism $\phi : G_1 \to G_2$ such that $\phi(o_1) = o_2$.

We often consider **labeled graphs**: in this case, we assume (directly or implicitly) that there are some labels $c : V(G) \cup E(G) \to \Theta$, where Θ is usually $[0, 1]$ or a countable set. Whenever we consider labeled graphs, we require that the graph isomorphism ϕ between G_1 and G_2 also preserves the labels: if G_1 has labels c_1 and G_2 has labels c_2 then $c_1(v) = c_2(\phi(v))$ for every $v \in V(G_1)$ and $c_1(\{x, y\}) = c_2(\{\phi(x), \phi(y)\})$ for every edge $\{x, y\}$.

We say that a graph is **(vertex-)transitive** if for every pair of vertices x and y there is an automorphism ϕ of the graph such that $\phi(x) = y$. The graph is quasi-transitive if there are finitely many orbits of the vertices under the action of the automorphism group of the graph.

Directed graphs

In Chapter 3 we examine directed graphs as well. There is a useful tool for this purpose: the bipartite representation of a directed graph. This allows us to easily generalize many theorems from undirected graphs to directed graphs in a standard way. Any directed graph can be represented as a bipartite graph.

Definition 1.1.1 (Bipartite representation of a directed graph)**.** *The* ***bipartite representation*** *of a directed graph $G = (V, E)$ is the bipartite graph $\bar{\bar{G}} = (V^-, V^+, \bar{\bar{E}})$ with $V^- = \{v^- : v \in V\}$, $V^+ = \{v^+ : v \in V\}$ and $\bar{\bar{E}} := \{\{v^-, w^+\} : (v, w) \in E\}$.*

We lose information when we convert a directed graph into its bipartite representation, namely the connection between the vertices v^- and v^+. This has consequences when we want to translate a statement back from the bipartite representation to the directed graphs, as we will see in Proposition 2.1.4.

1.1.2 Finite random graphs

We consider two types of random graphs in the thesis. First, the easier concept of random graphs is finite random graphs. We use this family in Chapter 3. If we say that G is a finite random graph, we mean that there is a probability distribution on the set of isomorphism classes of (possibly disconnected) finite graphs. Note that this is a countable set, and we consider the discrete topology on it. For the sake of simplicity, we usually speak about graphs instead of isomorphism classes. By the **realization of the random graph** G we mean an arbitrary graph from the isomorphism class chosen randomly according to the distribution of G.

Let $\mathcal{G}_f$ be the **set of isomorphism classes of finite (possibly disconnected) labeled graphs**. We define **random finite labeled graphs** in two steps. First we consider a random finite graph G, and then we define the joint distribution of the labels of the vertices and edges given the realization of G.

Remark 1.1.2 (Labels)**.** *We usually chose the labels to be elements of* $[0, 1]$*, but one can define them more generally, as elements of a complete separable metric space. When we consider labeled graphs, we always specify the set of labels. If this set is countable, then we mean without further mentioning that the distance of two distinct elements are one.*

Given a random (labeled) graph G, we denote by $\mathbb{P}_G$ the **probability with respect to the distribution of** G and by $\mathbb{E}_G$ the expectation taken with respect to $\mathbb{P}_G$. We omit the index G from this notation if it is clear what the measure is.

Examples of finite random graphs

In this section, we define the most widely used random graph models. These graphs will be examined in Chapter 3. We work there with both undirected and directed graphs.

Definition 1.1.3 (Random d-regular graphs)**.** *The* ***random*** d***-regular graph*** *is a random graph chosen uniformly at random from the set of graphs on the vertex set* $[n]$ *with all degrees equal* d*.*

There are two natural ways to define random directed regular graphs*. The first one is if* G_n *is a uniformly chosen directed graph on* $[n]$ *such that each vertex has in- and out-degrees* d*. The second way to define directed graphs* G_n *is if we choose a uniform random non-directed* d*-regular graph on* $[n]$ *and orient each edge uniformly at random independently from each other. This model is a special case of the random configuration model defined in the sequel.*

Definition 1.1.4 (Erdős–Rényi random graphs)**.** *The* ***Erdős–Rényi random graphs*** $\mathcal{G}_{n,p}$ *are defined in the following way: consider the complete graph on* n *vertices and keep each edge with probability* p*, and delete each edge with probability* $1 - p$ *independently from each other. The resulting random graph is* $\mathcal{G}_{n,p}$*.*

We define the directed *Erdős–Rényi random graphs* $\vec{\mathcal{G}}_{n,p}$ *by orienting each edge of* $\mathcal{G}_{n,p}$ *uniformly at random independently for the edges.*

The next two graphs have become increasingly important in applications, because they can grab important characteristics of real-world networks, such as scale-free degree distribution. This is the reason why in [61], which was motivated by applications of controllability, these graphs were studied.

Definition 1.1.5 (Random configuration model)**.** *We fix a non-negative integer valued probability distribution* ξ*. We define the graph* G_n *in the following way: let* $\xi_1, \ldots, \xi_n$ *be i.i.d. variables with distribution* ξ*. Given* $\xi_1, \ldots, \xi_n$ *let* $\mathcal{E} := \{(k, j) :$

$k \in [n], j \in [\xi_k]\}$ be the ***set of the half-edges****. Let H be a uniform random perfect matching of the set $\mathcal{E}$ (if $|\mathcal{E}|$ is odd, then put off one half-edge uniformly at random before choosing a perfect matching). The* ***random configuration model*** *is the graph $G_n = G_n(H)$ on $[n]$ given naturally by the random perfect matching H.*

If we want to define a directed *graph, then we orient each edge uniformly at random independently from the other edges. We get the same distribution if after fixing the degree sequence $\xi_1, \dots, \xi_n$ we select a subset $\mathcal{E}_T \subseteq \mathcal{E}$ of size $\lfloor |\mathcal{E}|/2 \rfloor$ uniformly at random. Then we set $\xi_k^- := |\{j \in [\xi_k] : (k,j) \in \mathcal{E}_T\}|$, $\xi_k^+ := \xi_k - \xi_k^-$ and we denote by $\mathcal{T} := \{(k,j,-) : k \in [n], j \in [\xi_k^-]\}$ the set of the tail-type half-edges and by $\mathcal{H} := \{(k,j,+) : k \in [n], j \in [\xi_k^+]\}$ the set of the head-type half-edges. Let $\mathcal{N}$ be the set of the perfect matchings of $\mathcal{T}$ to $\mathcal{H}$ and denote by N a uniform random element of $\mathcal{N}$. Then N defines the random directed graph $G_n = G_n(N)$ on the vertex set $[n]$.*

Our second graph model, the preferential attachment graph was introduced by Barabási and Albert in [13] and the precise construction was given by Bollobás and Riordan in [23]. There are several versions of the definition of this family of random graphs which have turned out to be asymptotically the same: they all converge to the same infinite limit graph; see [19]. Although in the original definitions the preferential attachment graphs are not directed, there is a natural way to give each edge an orientation and these orientations extend to the limit graph as well. We will examine the maximal size of the directed matchings of preferential attachment graphs in Chapter 3. We use the following definition from [19] completed with the natural orientation of the edges.

Definition 1.1.6 (Preferential attachment graphs)**.** *Fix a positive integer r and $\alpha \in [0,1)$. For each n the* ***preferential attachment graph*** *on n vertices is the random graph $G_n = G^{PA}_{r,\alpha,n}$ on the vertex set $[n]$ defined by the following recursion: let G_0 be the graph with one vertex and no edges. Given G_{n-1} we construct G_n by adding the new vertex n and r new edges with tails n. We choose the heads $w_1, \dots w_r$ of the new edges independently from each other in the following way: with probability α we choose w_j uniformly at random among $[n-1]$, and with probability $1-\alpha$ we choose w_j proportional to $\deg_{G_{n-1}}$. Note that each vertex except the starting vertex has out-degree r and each vertex has a random in-degree such that the mean of the average in-degree converges to r as $n \to \infty$.*

1.1.3 Random rooted graphs

The case of possibly infinite random graphs is more difficult. The set of locally finite graphs is uncountable and it is not a priori clear, what is the natural topology on it.

We need the concept of rooted graphs to define an appropriate topology on the set of isomorphism classes of possibly infinite graphs. There is a natural way to define the distance of rooted graphs: we say that two graphs are close, if they are isomorphic in a large neighborhood of the root. More precisely, let the **distance**

of two rooted graphs (G_1, o_1) and (G_2, o_2) be 2^{-k} if k is the largest integer such that $B_{G_1}(o_1, k) \simeq B_{G_2}(o_2, k)$. If the balls of radius k are rooted isomorphic for all k, then let the distance be 0. It is easy to see that this defines a metric on *the space of isomorphism classes of locally finite, connected rooted graphs* and this space is complete and separable with this metric, see [17] or [25, Lemma 3.4] for a proof.

In the case of *labeled* graph, we have to refine the definition of the distance of two graphs. We do not specify the labels yet, but we assume that they are chosen from a complete separable metric space. See Remark 1.1.2 for more on the labels. Let $\mathcal{G}_\star$ be the **space of isomorphism classes of locally finite, connected labeled rooted graphs**. We equip the space $\mathcal{G}_\star$ with the metric defined by the following distance: we say that two graphs are close, if they are rooted isomorphic in large balls around the roots and the corresponding labels are close to each other. More precisely, let (G_1, o_1) and (G_2, o_2) be two labeled rooted graphs with labels c_1 and c_2, respectively. Let their **distance** be $\mathrm{dist}_\star\big((G_1, o_1), (G_2, o_2)\big) := \inf\{2^{-k}\}$ where the infimum is taken over all k which satisfy the following: there is a rooted isomorphism $\phi_k : B_{G_1}(o_1, k) \to B_{G_2}(o_2, k)$ such that $\min\{1; |c_1(a) - c_2(\phi_k(a))|\} \leq 2^{-k}$ for every $a \in V(B_{G_1}(o_1, k)) \cup E(B_{G_1}(o_1, k))$.

The space of isomorphism classes of locally finite connected graphs without labels can be viewed as the subspace of $\mathcal{G}_\star$ with labels $c \equiv 1$, so we need to give the further definitions only in the more general settings of labeled rooted graphs.

Remark 1.1.7 (Compactness)**.** *If the labels are chosen from a compact separable metric space and we fix a positive integer D, then the subspace $\mathbb{G}_\star^D$ of $\mathbb{G}_\star$ consisting of the graphs with degrees bounded above by D form a compact separable metric space. There are several advantages of working with this space, and many results on convergent sequences in $\mathbb{G}_\star^D$ make use of the compactness of the space, see e.g. [17], [40].*

However, there are several widely used random graph models that do not have a uniform bound on the degrees, hence we cannot use compactness when studying them.

Definition 1.1.8 (Random rooted graphs)**.** *By a* ***random rooted graph*** *we mean a probability distribution on the space $(\mathcal{G}_\star, \mathcal{B}(\mathcal{G}_\star))$, where $\mathcal{B}(\mathcal{G}_\star)$ is the Borel σ-algebra given by the metric defined above. For the sake of simplicity, instead of working with random isomorphism classes, we always work with a representative graph from the isomorphism class, which we call the* ***realization*** *of the random rooted graph. We often consider a realization that naturally arise from the definition. One can also chose the canonical representative of the isomorphism class, as defined in [4, Section 2].*

Given a random (labeled, directed) rooted graph G, we denote the ***probability*** *and* ***expected value*** *with respect to its distribution by $\mathbb{P}_G$ and $\mathbb{E}_G$, respectively. We sometimes omit the subscript from the notation when we examine only a single random graph. When we need to use integrals with respect to the distribution of a random graph, then we introduce a notion for the distribution of it.*

Remark 1.1.9. *We will speak about random rooted (labeled) graphs without mentioning the root in two cases.*

First, if the random graph is almost surely finite*, then we turn it into a random rooted graph by choosing the root uniformly at random among the vertices of the realization of the graph.*

Second, if the random graph is a transitive graph *and the distribution of the labels is invariant under the automorphisms of the graph. In this case, it does not matter which vertex we choose to be the root, the distribution of the isomorphism class of the random rooted graph will be the same.*

1.2 Unimodular random graphs

The notion of unimodularity comes from group theory. It was showed by Schlichting [76] and Trofimov [82] that the unimodularity of a closed group of automorphism of a fixed graph G can be described by an easy formula using the stabilizers of the vertices. Namely, a subgroup Γ of the automorphism group of the graph is unimodular iff for each x and y on the same orbit

$$|Stab_{\Gamma}(x)y| = |Stab_{\Gamma}(y)x|.$$

See [14] for more on fixed graphs and unimodularity. It turned out that Cayley graphs of finitely generated groups are unimodular (see Definition 1.2.5 and Proposition 1.2.6), and the definition can be generalized to quasi-transitive graphs and invariant random subgraphs of them [4]. We give two equivalent definitions below that work in the generality of random rooted graphs.

Many theorems can be extended from Cayley graphs of finitely generated groups to invariant random subgraphs of them; see e.g. [14], [15]. This suggests using techniques developed for invariant subgraphs of unimodular transitive graphs for studying even more general unimodular random graphs. Another motivation for using these techniques is that they apply for local weak limits of finite graphs, because these graphs are also unimodular (Proposition 1.2.4).

One may think that unimodular random graphs are generalizations of transitive graphs, however, not every transitive graph is unimodular, as we will see in Example 1.2.9. Unimodular graphs should satisfy a property that is stronger than stationarity, see Definition 1.2.1.

There are several equivalent definitions of unimodularity. One can approach unimodularity through random walks, see [4, Section 4], [15, Section 2.2], [75, Definition 14.1]. When a random rooted graph has finite expected degree (which will be the case throughout the thesis), one can define unimodularity as follows. Note that the notion of *reversibility* of Definition 1.2.1 used throughout this section differs from the usual definition in the theory of Markov chains.

Definition 1.2.1 (Unimodularity and random walks)**.** *Let (G, o) be a random rooted graph and let $(X_0, X_1, \dots)$ be a simple random walk on G starting from o. We say* *that (G,o) is **reversible**, if (G,X_0,X_1) and (G,X_1,X_0) have the same distribution. The random rooted graph (G,o) with distribution μ and $\mathbb{E}(\deg o) < \infty$ is called **unimodular**, if the random rooted graph obtained from (G,o) by biasing by the degree of o is reversible; i.e., the measure $\tilde{\mu}$ on $\mathcal{G}_\star$ is reversible, where $\tilde{\mu}$ is absolutely continuous with respect to μ and $\frac{d\tilde{\mu}}{d\mu} = \deg_\mu o$.*

Note that this definition says that a transitive graph (or any regular graph) is unimodular iff it is reversible. This property holds for Cayley graphs, but does not hold for every transitive graph, see Example 1.2.9.

Unimodularity of random rooted graphs is closely related to measured equivalence relations and this concept could be also used to give an equivalent definition. We do not use this approach in the thesis. For more on the connection of unimodular random graphs and measured equivalence relations, see [15, Section 2.3], [4, Example 9.9]. For more equivalent definitions of unimodularity see [75, Definition 14.1].

Now, we define unimodularity using the so called *Mass Transport Principle*. The Mass Transport Principle was introduced by Häggström [50] to study percolation on regular trees. This definition is useful in the proofs, even beyond percolation theory: finding a proper mass transport, one can prove (in)equalities concerning unimodular graphs.

To define unimodular random graphs, we need to define **the space of isomorphism classes of locally finite labeled graphs with an ordered pair of distinguished vertices**, denoted by $\mathcal{G}_{\star\star}$. (We have already used this space implicitly in Definition 1.2.1.) We equip this space with the natural topology, which is similar to the topology of the space $\mathcal{G}_\star$: two doubly rooted graphs are close if they are isomorphic in large neighborhoods of the roots and the corresponding labels are close to each other.

Definition 1.2.2 (Unimodular random graphs)**.** *[4, Definition 2.1] We say that a random rooted (labeled) graph (G,o) with distribution μ is **unimodular** if it obeys the **Mass Transport Principle**:*

$$\int \sum_{x\in V(\omega)} f(\omega,o,x)d\mu(\omega,o) = \int \sum_{x\in V(\omega)} f(\omega,x,o)d\mu(\omega,o)$$

for each measurable function $f : \mathcal{G}_{\star\star} \to [0,\infty]$.

The Mass Transport Principle can be interpreted as follows. One can think of $f(\omega,o,x)$ to be *the mass sent from o to x given the labeled graph ω*, and unimodularity is equivalent with the property that the expected value of the total mass sent out by the root equals the expected value of the total mass received by the root.

Aldous and Steele [5] considered a similar definition of unimodularity but they required the principle to hold for functions $f(\omega,x,y)$ that take positive values only when $x \sim y$. That definition is equivalent with the above general form of the Mass Transport Principle, see [4, Proposition 2.2].

It follows easily from the Mass Transport Principle that the class of unimodular probability measures is convex. A unimodular probability measure is called **extremal** if it cannot be written as a convex combination of other unimodular probability measures. Extremal unimodular random graphs will be important in Chapter 4.

1.2.1 Examples of unimodular random graphs

We present a few basic examples to illustrate the concept of unimodular graphs. As mentioned earlier, this class is convex, hence every convex combination of the following examples is also unimodular.

Example 1.2.3 (Finite random graphs)**.** *As we mentioned in Remark 1.1.9, if we do not define the root of a finite graph, we mean that the root is chosen uniformly at random among the vertices of the graph. It is easy to check that a finite graph with a uniform random root is unimodular. It follows by the convexity of the class of unimodular graphs that every finite random graph with a root chosen uniformly at random is unimodular.*

An important property of unimodular random graphs is that this class is *closed under taking local weak limits* of random graphs. The concept of local weak convergence will be defined in Chapter 2. Also, it is an open question whether the class of unimodular random graphs is strictly larger than the class of sofic measures; i.e., the closure of the set of finite random graphs with a uniformly chosen root under local weak convergence. For further discussion on this question, see [4, Section 10] or [75, Question 14.2].

Proposition 1.2.4 (Local weak limits of unimodular graphs [17])**.** *The space of unimodular random rooted graphs is closed under taking local weak limits (see Chapter 2 for the definition). It follows that random rooted graphs that arise as limits of finite graphs are unimodular.*

An important class of unimodular graphs consists of Cayley graphs of finitely generated groups and of invariant random subgraphs of a Cayley graph. We also use this family of random graphs in several examples in Chapter 4.

Definition 1.2.5 (Cayley graph of a finitely generated group)**.** *Given a finitely generated group $\mathbb{G}$ with generator set $\mathcal{S}$, its (left) Cayley graph Γ is the directed graph with vertex set $V(\Gamma) = \mathcal{G}$ and edge set $E(\Gamma) = \{(g, sg) : g \in \mathbb{G}, s \in \mathcal{S}\}$. Usually, it is assumed that $\mathcal{S}$ is symmetric, i.e., $s \in \mathcal{S} \Leftrightarrow s^{-1} \in \mathcal{S}$. We often forget about the orientations of the edges and view the Cayley graph as an undirected graph.*

When we talk about invariant random subgraphs of a Cayley graph Γ of a group $\mathbb{G}$, we always mean that the measure on subgraphs is invariant under the natural action of $\mathbb{G}$: the action of the elements of the group by multiplication from the right.

Proposition 1.2.6 (Unimodularity of invariant subgraphs of Cayley graphs)**.** *[4, Remark 3.3] Let Γ be a Cayley graph of a finitely generated group and let o be a vertex of Γ. Then Γ is a unimodular graph. Furthermore, if G is a random subgraph of Γ that is invariant under the action of the group, then (G, o) is unimodular.*

An important special case of Proposition 1.2.6 is the cluster of the root of a Cayley graph obtained by the *Bernoulli percolation* on the graph. Here we give just the basic concept needed for the example. We define percolation in more detail in Chapter 4, which is devoted to our results on percolation critical probabilities.

Example 1.2.7 (Bernoulli percolation on a unimodular transitive graph)**.** *Let G be a unimodular graph, fix a vertex o and let $p \in [0, 1]$. Let ω be the random subgraph of G obtained by keeping each edge with probability p and removing it with probability $1 - p$, independently for the edges. Denote by $\mathcal{C}_o$ the connected component of ω containing o. Then $(\mathcal{C}_o, o)$ is a unimodular random graph [4].*

As we mentioned earlier, not every transitive graph is unimodular. For the most basic example of a non-unmodular graph, we need the notion of ends of a tree. There is a more general notion for ends of general graphs [68, Section 7.3], but we do not need this notion in our thesis, hence we omit the general definition.

Definition 1.2.8 (Ends of a tree)**.** *[68, Example 7.1] Let T be an infinite tree. An* ***end*** *of T is an equivalence class of infinite non-backtracking paths, where two paths are equivalent if the symmetric difference of their vertex sets is finite.*

Example 1.2.9 (A non-unimodular transitive graph: the grandmother graph)**.** *[68, Example 7.1] Let $\mathbb{T}_3$ be the infinite 3-regular tree and let ξ be a distinguished end of $\mathbb{T}_3$. Given an end of the tree T, for every vertex x there is a unique path in the equivalence class ξ starting at x. We denote this path by $P_x = (x, x_1, x_2, \dots)$. We obtain the grandmother graph G by adding to $\mathbb{T}_3$ extra edges of the form $\{x, x_2\}$ for all vertices $x \in V(\mathbb{T}_3)$.*

The graph G is clearly transitive but it is not unimodular.

Proof. We show the non-unimodularity of G by giving a measurable function $f : \mathcal{G}_{\star\star} \to [0, \infty)$ that does not satisfy the Mass Transport Principle.

First, we note that for every vertex x one can determine the vertex x_1 (the second vertex of the path in ξ starting from x) just by looking at $B_G(x, 1)$, as follows. There are three vertices in $B_G(x, 1)$ with $\deg_{B_G(x,1)} y = 4$, and it is easy to see that x_1 is the unique vertex among them that is connected to both of the others by an edge. This shows that the distinguished end ξ is encoded in the structure of G. Let f be the following mass transport:

$$f(\omega, x, y) = \begin{cases} 1 & \text{if } \omega = G, y = x_1 \\ 0 & \text{otherwise} \end{cases}$$

Then every vertex sends out mass 1 and receives mass 2 almost surely, hence the probability measure on $\mathcal{G}_\star$ concentrated on (G, o) (where o is any vertex of G) does not obey the Mass Transport Principle:

$$\int \sum_{x \in V(\omega)} f(\omega, o, x) d\mu(\omega, o) = 1 \neq 2 = \int \sum_{x \in V(\omega)} f(\omega, x, o) d\mu(\omega, o).$$

□

For further examples of transitive, non-unimodular graph see [14, Example 3.2] and [80].

There is an important class of unimodular random graphs, the so called *unimodular Galton–Watson trees* which provide examples of unimodular random graphs with unbounded degrees. These graphs arise naturally as local weak limits of finite random graphs, see Section 2.2. For the unimodularity of the Galton–Watson trees see [4, Example 1.1] or [25, Lemma 3.10].

Definition 1.2.10 (Unimodular Galton–Watson tree)**.** *Let ξ be a non-negative integer valued random variable with $\mathbb{E}\xi < \infty$. The **unimodular Galton–Watson tree** with degree distribution ξ (denoted by $UGW(\xi)$) is a random rooted tree with root o. We say that a vertex y is the* child *of the vertex x, if they are adjacent and $dist(y, o) = dist(x, o) + 1$. In this case, we say that x is the* parent *of y. The distribution of the graph $UGW(\xi)$ is given by the following recursive definition:*

- *The probability that o has $k \geq 0$ children is $\mathbb{P}(\xi = k)$.*
- *For each vertex x, the probability that x has $k \geq 0$ children is $\frac{(k+1)\mathbb{P}(\xi=k+1)}{\mathbb{E}\xi}$, independently for each vertex.*

Let the directed *unimodular Galton–Watson tree $\overrightarrow{UGW}(\xi)$ be the random rooted directed graph obtained from $UGW(\xi)$ by orienting each edge independently.*

1.2.2 Factor of IID labels

Factor of IID processes provide a useful tool to give bounds on combinatorial quantities of large graphs or show the convergence of certain parameters. We will use this concept in Chapter 3 to prove the convergence of the matching ratio of random graphs.

A fundamental related question in this theory is whether an invariant random subgraph of a Cayley graph can be obtained as a factor of IID, as defined in the next paragraph. Positive answers for this questions were given for (perfect) matchings on certain Cayley graphs by Lyons and Nazarov [67], Timár [78], Csóka and Lippner [34], Lyons [66], for labeling with uniform marginals and infinite equivalence classes on $\mathbb{T}_3$ by Mester [71], for large independent sets on $\mathbb{T}_d$ by Gamarnik and Sudan [45], Csóka, Gerencsér, Harangi and Virág [33], Harangi and Virág [51]. Nevertheless, there are still many open questions in this field; see e.g., Lyons [66]. Results about properties of factor of IID processes that can also be used to prove a negative answer for questions of the above type can be found in Backhausz, Szegedy and Virág [12] and Lyons [66]. Factors of Poisson point processes have been studied by Holroyd and Peres [54], Timár [79, 81], Holroyd, Pemantle, Peres and Schramm [53].

A sequence (or set) of *independent, identically distributed random variables* is abbreviated as **IID**. Informally, a *factor of IID* is a random labeling of a graph obtained by the following process. We assign to each vertex or edge of the graph IID labels, and then we obtain new labels for every vertex v or edge e by applying a deterministic rule, the *factor* to the graph and the IID labels around v or e.

Definition 1.2.11 (Factor of IID)**.** *A measurable function $f : \mathcal{G}_\star \to \mathbb{R}$ is called a* ***factor (map)****. In some cases we need to define factor labels also on the edges, hence we extend the notion of factor to measurable functions $f : \mathcal{G}_\star \cup \mathcal{G}_{\star\star} \to \mathbb{R}$.*

Let G be a (random directed) graph, let $c : V(G) \cup E(G) \to [0,1]$ be IID uniform random $[0,1]$-labels on the vertices and edges, and let $G(c)$ be the random labeled graph given by the labels c. The collection of random variables $\{X_a = f((G(c), a)) : a \in V(G) \cup E(G)\}$ is called a ***factor of IID process****, if f is a factor.*

The easiest examples of factor of IID use a factor map that takes into account only the IID label of the edge or vertex, and ignores the other labels. One can obtain Bernoulli percolation on a graph in such a way, even for more parameters simultaneously, coupled independently or by the standard coupling (see [68, Section 5.2]). We will define random matchings as factor of IID in Section 3.3.

Remark 1.2.12 (Unimodularity of factor of IID)**.** *When we equip a unimodular random graph with the IID labels, then the resulting labeled graph and also the factor of IID process is unimodular. This allows us to use the Mass Transport Principle for factor of IID processes which will be the case in Section 3.3.1.*

1.2.3 Operations preserving unimodularity

Some of our examples in Chapter 4 arise from Cayley graphs using operations from $\mathcal{G}_\star$ to $\mathcal{G}_\star$. One of this operations is the *edge replacement* defined in [4, Example 9.8]. In the paper [20], we defined further operations, called *vertex replacement* and *contraction*. In the first two operations, we replace each edge or vertex of a unimodular graph by a finite random graph with an appropriate number of distinguished vertices corresponding to the endpoints of the edge or to the neighbors of the vertex. For the third operation, we consider a unimodular random subgraph of a unimodular random graph and identify the vertices that are in the same connected component of the subgraph.

Before presenting the details of the general concepts, we start with easier examples. Given IID labels to a unimodular random graph turns it into a unimodular labeled graph. Applying a factor map (Definition 1.2.11) to a (labeled) unimodular graph also obtains a unimodular labeled graph. Factor of IID labelings and in particular, invariant random labelings of a unimodular random graph preserves unimodularity, see Remark 1.2.12 and [4, Theorem 3.2].

We can obtain a new graph by adding extra edges using a deterministic rule, i.e., adding an edge between pairs of vertices with a fixed distance that belong to the same vertex class of a (quasi-)transitive graph. However, as we have seen in Example 1.2.9, when the rule of adding the new vertices breaks the symmetries of the original unimodular graph, the resulting graph is not necessarily unimodular.

Another way to obtain a new graph is to removing edges using a deterministic rule, i.e., removing edges that belong to a particular edge-class of a quasi-transitive graph. Removing edges independently at random with the same probability, i.e. Bernoulli percolation on a unimodular graph results in a subgraph that is a unimodular random graph, see [4, Example 9.4]. More generally, if the removal of the edges depends on a factor of IID process or on any unimodular labeling, then the resulting graph is also unimodular, see Remark 1.2.12.

The easiest example of the *edge replacement* operation is when we substitute each edge by a path of a fixed length; or more generally, by a path having a random length according to some fixed distribution. Similarly, we get an example of *vertex replacement* when we substitute each vertex x by a complete graph with $\deg x$ vertices; or more generally, with a complete graph with a random number of vertices. It is not a priori clear, how to choose the root of the resulting graph, even when the original graph is a transitive unimodular graph. We describe the proper way of chosing the random root of the resulting graph in the detailed discussion of the general forms of edge and vertex replacement below.

Reversing the above examples gives an example of *contraction*: we can substitute each path of a fixed length, or more generally, each path of length at least two by a single edge. Another natural application of contraction is when each connected component of Bernoulli bond percolation (Definition 4.1.3) is replaced by a single vertex. For further applications of the operations of this section see Examples 4.2.10, 4.2.11, 4.3.9. Further operations on unimodular graphs can be found in [4, Section 9].

We note that in the edge and vertex replacement, the space of labels has a special form, but the topology on them can be defined similar to the topology of $\mathcal{G}_{\star\star}$. The proper measurability conditions of the labels are ensured by Definition 1.1.8 of the random labeled rooted graph.

Edge replacement [4, Example 9.8]. Let (Γ, o) be a unimodular random labeled graph with distribution μ, where the label $(G_{(x,y)}, o_x^{(x,y)}, o_y^{(x,y)})$ of each edge (x, y) is a finite graph with two distinguished vertex. We assume that the labels corresponding to the reverse orientation of the same edge satisfy $G_{(x,y)} = G_{(y,x)}$, $o_y^{(x,y)} = o_y^{(y,x)}$ and $o_x^{(x,y)} = o_x^{(y,x)}$, so the orientation of a particular edge does not matter. If the labeling satisfies $\mathbb{E}_\mu \left(\sum_{x\sim o} \left(|V(G_{(o,x)})| - 2\right)\right) < \infty$, then we can define the rooted random graph $H(\Gamma, o)$ as follows. Let $A(\Gamma, o) := 2 + \sum_{x\sim o} \left(|G_{(o,x)}| - 2\right)$. Choose (Γ, o) with probability distribution μ biased by $A(\Gamma, o)$, and obtain $H(\Gamma, o)$ by replacing each edge (x, y) by the graph $G_{(x,y)}$, and for each fixed vertex x of Γ, identifying x with the vertices $o_x^{(x,y)}$ for each $y \sim x$. Let the root o' of $H(\Gamma, o)$ be o with probability $\frac{2}{A(\Gamma,o)}$ and any vertex from $\cup_{x\sim o}\left(E(G_{(o,x)}) \setminus \{o_o^{(o,x)}, o_x^{(o,x)}\}\right)$ with probability $\frac{1}{A(\Gamma,o)}$. For the proof that the resulting graph is unimodular, see [4, Example 9.8].

Vertex replacement. Let (Γ, o) be a unimodular random labeled graph with distribution μ, where the labels are in the form (G_x, φ_x), where G_x is a finite graph and φ_x is a map from $\{(x, y) \in E(\Gamma) : y \sim x\}$ to $V(G_x)$. If the labeling satisfies $\mathbb{E}_\mu|V(G_o)| < \infty$, then we can define the following rooted random graph $H(\Gamma)$: we choose $(\Gamma, o, \{(G_x, \varphi_x) : x \in V(\Gamma)\})$ with respect to the probability measure μ biased by $|V(G_o)|$, and replace each vertex x of Γ by the graph G_x and each edge e of Γ by the edge $\{\varphi_{e^-}(e), \varphi_{e^+}(e)\}$. Let the root o' of $H(\Gamma)$ be a uniform random vertex of $V(G_o)$. Denote the law of $(H(\Gamma), o')$ by μ'.

We claim that if μ is unimodular with $\mathbb{E}_\mu|V(G_o)| < \infty$, then μ' is also unimodular. Let $f(\omega, u, v)$ be a Borel function from $\mathcal{G}_{\star\star}$ to $[0, \infty]$ and let

$$\bar{f}(\bar{\omega}, x, y) := \frac{1}{\mathbb{E}_\mu|V(G_o)|} \sum_{u\in V(G_x), v\in V(G_y)} f(H(\bar{\omega}), u, v)$$

which is a Borel function on the subspace of $\mathcal{G}_{\star\star}$ that consists of graphs with labels of the above form. We show that μ' obeys the Mass Transport Principle:

$$\begin{aligned}
\int \sum_{v\in V(\omega)} f(\omega, o', v) d\mu'(\omega, o') &= \int \sum_{o'\in V(G_o), v\in V(H(\bar{\omega}))} \frac{1}{|V(G_o)|} f(H(\bar{\omega}), o', v) \frac{|V(G_o)|}{\mathbb{E}_\mu|V(G_o)|} d\mu(\bar{\omega}, o) \\
&= \int \sum_{x\in V(\bar{\omega})} \sum_{o'\in V(G_o), v\in V(G_x)} \frac{1}{\mathbb{E}_\mu|V(G_o)|} f(H(\bar{\omega}), o', v) d\mu(\bar{\omega}, o) \\
&= \int \sum_{x\in V(\bar{\omega})} \bar{f}(\bar{\omega}, o, x) d\mu(\bar{\omega}, o) \\
&= \int \sum_{x\in V(\bar{\omega})} \bar{f}(\bar{\omega}, x, o) d\mu(\bar{\omega}, o) \\
&= \int \sum_{v\in V(\omega)} f(\omega, v, o') d\mu'(\omega, o').
\end{aligned}$$

Contraction. Let (Γ, o) be a unimodular random edge-labeled graph with distribution μ, where the labels of the edges are 0 or 1. We denote by G the random subgraph of Γ spanned by all the vertices and the edges with label 1. For a vertex x of Γ let $\mathcal{C}_x$ be the connected component of x in G. We define the contracted graph $H(\Gamma)$: in practice, this is what we get by identifying every vertex in the same component of G. More formally, first we choose (Γ, o, G) with respect to the distribution μ biased by $\frac{1}{|\mathcal{C}_o|}$. The vertices of $H(\Gamma)$ are the connected components of G and we join two vertices by an edge iff there is an edge in Γ which connects the two components. Let the root o' of $H(\Gamma)$ be the connected component $\mathcal{C}_o$. Denote the law of $(H(\Gamma), o')$ by μ'.

We claim that if μ is unimodular then μ' is also unimodular. Let $f(\omega, u, v)$ be a Borel function from $\mathcal{G}_{\star\star}$ to $[0, \infty]$ and let

$$\bar{f}(\bar{\omega}, x, y) := \frac{1}{|\mathcal{C}_x||\mathcal{C}_y|} f(H(\bar{\omega}), \mathcal{C}_x, \mathcal{C}_y)$$

which is a Borel function on the subspace of $\mathcal{G}_{\star\star}$ that consists of graphs with edges labeled by 0 or 1, such that the subgraph defined by the edges with label 1 consists of finite components. We show that μ' obeys the Mass Transport Principle:

$$\begin{aligned}
\int \sum_{v \in V(\omega)} f(\omega, o', v) d\mu'(\omega, o') &= \int \sum_{x \in V(\bar{\omega})} \frac{1}{|\mathcal{C}_x|} f(H(\bar{\omega}), \mathcal{C}_o, \mathcal{C}_x) \frac{1}{|\mathcal{C}_o| \mathbb{E}_\mu \left(\frac{1}{|\mathcal{C}_o|} \right)} d\mu(\bar{\omega}, o) \\
&= \frac{1}{\mathbb{E}_\mu \left(\frac{1}{|\mathcal{C}_o|} \right)} \int \sum_{x \in V(\bar{\omega})} \bar{f}(\bar{\omega}, o, x) d\mu(\bar{\omega}, o) \\
&= \frac{1}{\mathbb{E}_\mu \left(\frac{1}{|\mathcal{C}_o|} \right)} \int \sum_{x \in V(\bar{\omega})} \bar{f}(\bar{\omega}, x, o) d\mu(\bar{\omega}, o) \\
&= \int \sum_{v \in V(\omega)} f(\omega, v, o') d\mu'(\omega, o').
\end{aligned}$$

Chapter 2

Local weak convergence of graph sequences

This chapter is devoted to the presentation of the concept of local weak convergence introduced by Benjamini and Schramm [17].

In Section 2.1 we give the definitions and basic remarks on local weak convergence. In Section 2.2 we illustrate the definitions with examples of convergent graph sequences. We present a partially unpublished result in Section 2.2.1. In Section 2.3 we examine a stronger notion, the *almost sure local weak convergence* of graph sequences and give examples.

2.1 Definitions

The local weak convergence of random rooted graphs is basically the weak convergence of their distributions in the space $\mathcal{G}_\star$ of isomorphism classes of connected, locally finite rooted graphs. The following definition is a more convenient description of local weak convergence, and captures the property that the local statistics of the graphs converge to that of the limit graph.

Definition 2.1.1 (Local weak convergence of graphs)**.** *We say that the sequence (G_n, o) of locally finite random rooted graphs* ***converge in the local weak sense*** *to the locally finite connected random rooted graph (G, o) if for any positive integer r and any finite rooted graph (H, o) we have $\mathbb{P}\big(B_{G_n}(o,r) \simeq (H,o)\big) \to \mathbb{P}\big(B_G(o,r) \simeq (H,o)\big)$.*

Let $\mathcal{G}_\star^D$ be the subspace of $\mathcal{G}_\star$ consisting of **the isomorphism classes of rooted graphs with degrees bounded by** D. It is not hard to show that this space with the topology given by the distance defined in Section 1.1.3 is a compact space. It follows, that every sequence in $\mathcal{G}_\star^D$ has a convergent subsequence with a limit in $\mathcal{G}_\star^D$. Many known results about convergent sequences have been proven assuming the stronger property that the sequence is in $\mathcal{G}_\star^D$ for some D. For some questions, as in our Chapter 4, basic examples show that they make sense only in this class. However, many natural graph models do not satisfy the uniformly bounded degree property. Nevertheless, all graph sequences we examine have bounded expected average degrees. Such sequences also have a nice behavior from the point of view of local weak convergence, hence the bounded degree assumption can be removed from certain results about convergent sequences, as we will see in Chapter 3.

Remark 2.1.2 (Disconnected graphs)**.** *For the sake of simplicity, we often work with possibly disconnected rooted graphs with countably many connected components (or the ismorphism classes of them). From the point of view of local weak convergence, only the component of the root matters in this case and we can regard our graph as a convex combination of random rooted graphs.*

Remark 2.1.3 (Convergence of finite graphs)**.** *We often examine sequences of non-rooted random finite graphs. The natural way of turn them into rooted graphs is to choose a random uniform vertex to be the root. When we consider a local weak convergent sequence G_n of random finite graphs, we usually do not mention the root. If this is the case, we always mean that the root o_n of G_n is chosen uniformly at random among the vertices of the random graph G_n (after choosing the graph itself with respect to its distribution).*

In Chapter 3, we will examine the matching ratio of convergent sequences of both undirected and directed graphs. In the next proposition, we analyze the relationship between a convergent graph sequence and its bipartite representation (see Definition 1.1.1). As we mentioned after Definition 1.1.1, we lose information when we examine the bipartite representation $\bar{\bar{G}}$ of a directed graph instead of G. This explains the phenomenon described in the second statement of the next proposition.

Proposition 2.1.4. *1) If a sequence G_n of random directed graphs converges to the random rooted directed graph (G, o), then the bipartite representations $\bar{\bar{G}}_n$ converge to $(\bar{\bar{G}}, \bar{\bar{o}})$, where $\bar{\bar{G}}$ is the bipartite representation of G with root $\bar{\bar{o}}$ being o^- or o^+ with probability 1/2-1/2.*

2) The converse does not hold: the convergence of the sequence of bipartite representations $\bar{\bar{G}}_n$ does not imply the convergence of G_n. In fact, there are different random directed rooted graphs (G_1, o_1) and (G_2, o_2) that are limits of sequences of finite random rooted graphs such that $(\bar{\bar{G}}_1, \bar{\bar{o}}_1)$ is isomorphic to $(\bar{\bar{G}}_2, \bar{\bar{o}}_2)$.

Proof. Denote the distribution of $B_{G_n}(o, r)$ and $B_G(o, r)$ in the space of locally finite rooted directed graphs by $\mu_{n,r}$ and μ_r, respectively. Similarly, denote the distribution of $B_{\bar{\bar{G}}_n}(\bar{\bar{o}}, r)$ and $B_{\bar{\bar{G}}}(\bar{\bar{o}}, r)$ in the space of locally finite rooted graphs by $\bar{\bar{\mu}}_{n,r}$ and $\bar{\bar{\mu}}_r$, respectively. The random uniform root $\bar{\bar{o}}$ of a bipartite representation $\bar{\bar{G}}_n$ of a finite directed graph G_n is o^- or o^+ with probability 1/2-1/2, where o is a uniform random root of G. It follows that $\bar{\bar{\mu}}_{n,r} = 1/2\bar{\bar{\mu}}_{n,r,o^-} + 1/2\bar{\bar{\mu}}_{n,r,o^+}$, where $\bar{\bar{\mu}}_{n,r,o^-}$ and $\bar{\bar{\mu}}_{n,r,o^+}$ are the distributions of $B_{\bar{\bar{G}}_n}(o^-, r)$ and $B_{\bar{\bar{G}}_n}(o^+, r)$, respectively. The first statement of the remark follows.

An example to the second statement is the following. Let G_1 be the graph with vertex set $V(G_1) = \mathbb{Z}$ and edge set $E(G_1) = \{(2k, 2k-1), (2k, 2k+1) : k \in \mathbb{Z}\}$, i.e.

the usual graph of $\mathbb{Z}$ with an alternating orientation of the edges. Let the random root o_1 be $2k$ or $2l-1$ for some $k, l \in \mathbb{Z}$ with probability 1/2-1/2 (the isomorphism class of (G_1, o) does not depend on the actual choice of the integers k and l). This graph is the limit of the cycles C_{2n} with $2n$ vertices and edges with alternating orientations. Let G_2 be the one-point graph without edges with probability 1/2 and with probability 1/2 let G_2 be the infinite regular tree with in- and out-degrees 2. This graph is the limit of the sequence of random graphs on n vertices where with probability 1/2 there are no edges and with probability 1/2 the graph is uniformly randomly chosen from the set of graphs on n vertices with all in- and out-degrees 2. Then $(\bar{\bar{G}}_1, \bar{\bar{o}}_1)$ and $(\bar{\bar{G}}_2, \bar{\bar{o}}_2)$ are both isomorphic to the random graph that is the one-point graph without edges or $\mathbb{Z}$ with probability 1/2-1/2. □

2.2 Examples of convergent sequences

In this section, we list a few basic examples that illustrate the definition of local weak convergence. In Sections 2.2.1 and 2.3.2 we present our results on convergent sequences of graphs. Most of the examples and families of random graphs defined in this section will be used later in the thesis.

As we mentioned in Remark 1.1.9, finite (random) graphs are considered with a random root chosen uniformly among the vertices. In the case of transitive graphs we can chose any vertex to be the root. The notations $\mathbb{Z}^d$ and $\mathbb{Z} = \mathbb{Z}^1$ stand for the graph with vertex set $\mathbb{Z}^d$ and edges between the pairs of vertices $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ with $\sum_{j=1}^d |x_j - y_j| = 1$. Denote by Q_n **the subgraph of** $\mathbb{Z}^d$ **spanned by the box** $[-n, n]^d$. We will use these notations throughout the thesis.

In the first example, the members of the converging graph sequences are deterministic transitive graphs, hence the r-ball around every vertex looks the same, thus the ball around the random root is deterministic. In this case, the r-ball around the root of G_n looks *exactly the same* as the r-ball in the limit graph for n large enough.

Example 2.2.1 (Transitive finite graphs converging to $\mathbb{Z}^d$)**.** *1) Let C_n be the cycle of length n. Then $C_n \to \mathbb{Z}$ in the local weak sense.*

2) More generally, fix a dimension d and sequences $(a_{n,k}), k \in [d]$ satisfying $\lim_{n\to\infty} a_{n,k} = \infty$ for every $k \in [d]$ and let T_n be the torus of size $a_{n,1} \times \cdots \times a_{n,d}$. Then $T_n \to \mathbb{Z}^d$ in the local weak sense.

Our next example describes a general phenomenon: transitive amenable graphs can be approximated by properly chosen subgraphs. The notion of amenability comes from group theory, but we will need only the definition of amenable transitive graphs, which does not involve groups. For the following definition of amenability we need only the subgraphs of a graph.

Definition 2.2.2 (Amenable graphs)**.** *We say that an infinite graph G is* ***(edge) amenable*** *if*

$$\inf\left\{\frac{|\partial_E F|}{|F|} : F \text{ is a finite connected subset of } V(G)\right\} = 0 \qquad (2.2.1)$$

A sequence of finite connected subsets satisfying $|\partial_E F_n|/|F_n| \to 0$ is called a ***Følner sequence****. We call the infimum on the left hand side of* (2.2.1) *the* ***Cheeger constant*** *of the graph G. If the Cheeger constant is positive, then we say that the graph is* ***nonamenable****. In the case of bounded degree graphs the ∂_E in the above definition can be substituted by ∂_V^{in} or ∂_V^{out} without changing the notion of amenability.*

Amenability can be defined in more generality, see [14] and [4, Section 8] for more on the amenability of the automorphism group of a graph. In the thesis, we examine only *transitive* amenable graphs. If a transitive graph is amenable, then one can choose a *Følner exhaustion* for a graph, defined in the next lemma.

Lemma 2.2.3 (Følner exhaustion in transitive graphs)**.** *[75, Lemma 5.3] If G is a transitive amenable graph, then there is a* ***Følner exhaustion****: there exists connected subsets F_n of the vertices such that they form a Følner sequence, $F_n \subset F_{n+1}$ and $\cup_n F_n = V(G)$.*

Example 2.2.4. *1) For any fixed dimension d, the sequence Q_n of boxes converges to $\mathbb{Z}^d$ in the local weak sense.*

2) Let G be a transitive amenable graph and let F_n be a Følner exhaustion of G. Then F_n converges to G in the local weak sense.

3)[75, Exercise 14.1] A transitive graph G has a sequence of subgraphs converging to it in the local weak sense iff G is amenable.

Our next example illustrates that the above phenomenon does not hold for nonamenable graphs: the balls $B_G(v, r)$ around a vertex in a transitive non-amenable graph G do not converge to the graph G. It is easy to see that the infinite d-regular tree is nonamenable for $d \geq 3$. Although the balls do not converge to the d-regular tree, we will see in Example 2.2.7, that the d-regular infinite tree also arises as the local weak limit of finite graphs.

We note that even for amenable graphs, $B_G(o, r)$ is not necessary a proper choice for a sequence in Part 3) of Example 2.2.4. An example of an amenable transitive graph where the balls do not form a Følner exhaustion is defined in Example 4.3.7: the Cayley graph of the *lamplighter group* is amenable, but the size of the balls grows exponentially. This shows that amenability is not equivalent with sub-exponential volume growth.

Example 2.2.5 (Limit of the balls in the 3-regular tree)**.** *Let $\mathbb{T}_3$ denote the 3-regular infinite tree and let $G_n = B_{\mathbb{T}_3}(v, n)$ be the ball of radius n around a vertex of $\mathbb{T}_3$. Then the graphs G_n converge to an infinite graph Λ, referred to it as the* ***canopy tree*** *and defined rigorously after the example.*

Proof. The graph G_n is not transitive: it consists of vertices of degree 3 and of degree 1. Note, that almost half of the vertices have degree 1, and in fact the probability that the uniform random root has degree 1 tends to $1/2$. Furthermore, it is easy to compute that the probability that a uniform random root has distance k from the vertices with degrees 1 tends to 2^{-k-1}. Thus, the limit of the sequence G_n is an infinite tree Λ with a random root. The vertex set of the connected graph Λ can be partitioned into countably many sets: let $L(0)$ be the set of vertices with degree 1, and for each $k > 0$, let $L(k)$ be the set of vertices that are at distance k from $L(0)$. Each set $L(j)$ in the partition has infinitely many vertices and each vertex in $L(j)$ is connected to one vertex in $L(j+1)$ and to two vertices in $L(j-1)$. The root of Λ is a vertex in $L(j)$ with probability 2^{-j-1}. □

Now we give the precise definition of the limit in 2.2.5. We will use this graph in several examples in Chapter 4.

Definition 2.2.6 (Canopy tree). *Let $\mathbb{T}$ be the 3-regular infinite rooted tree with a distinguished end ξ (see Definition 1.2.8). Let $\mathfrak{h} : \mathbb{T} \to \mathbb{Z}$ be a* Busemann function *(see [83]) that gives the levels w.r.t. to ξ. More precisely, to define $\mathfrak{h}$, fix a root $o \in \mathbb{T}$. For any vertex x, let (ξ, x) be the unique infinite simple path from x which is in the equivalence class ξ. Denote by $o \wedge x$ the unique vertex in $\mathbb{T}$ such that $(\xi, x \wedge o) = (\xi, x) \cap (\xi, o)$. Finally, let $\mathfrak{h}(x) := dist(o, x \wedge o) - dist(x, x \wedge o)$.*

*Let $\Lambda \subset \mathbb{T}$ be the subgraph spanned by the vertices x with $\mathfrak{h}(x) \geq 0$. This tree Λ is called the **canopy tree**. Denote by $L(n) := \{x \in V(\mathbb{T}) : \mathfrak{h}(x) = n\}$ the n^{th} vertex level and by $L_E(n) := \{e \in E(\mathbb{T}) : e^- \in L(n), e^+ \in L(n+1)\}$ the n^{th} edge level of $\mathbb{T}$, or, for $n \geq 0$, of Λ. If we choose the root o of Λ such that $\mathbb{P}(o \in L(n)) = 2^{-n-1}$, we get a unimodular random graph.*

The sequences of finite random graphs defined in Section 1.1.2 are examined in Chapter 3. We work there with both undirected and directed graphs. We present here the results about the local weak convergence of both the undirected and directed versions of these sequences. The case of directed graphs follows trivially from the undirected case by the definition of local weak convergence. We will see in Section 2.3 that these sequences converge in an even stronger sense: they *converge almost surely in the local weak sense.* We postpone the definition of this notion and the corresponding statements to Section 2.3.

Example 2.2.7 (Random d-regular graphs). *Let G_n be the random graph chosen uniformly at random from the set of graphs on the vertex set $[n]$ with all degrees equal d. It is standard (see e.g., [22, Corollary 2.19]), that the local weak limit of G_n as $n \to \infty$ is the infinite d-regular tree $\mathbb{T}_d$.*

Recall from Definition 1.1.3 that there are two natural ways to define random directed regular graphs. When the vertices are oriented in a way that each vertex has in- and out-degrees d, then the local weak limit is a regular tree with in- and out-degrees d. When the edges of the d-regular graphs are oriented independently, then the model is a special case of the directed random configuration model. The limit of that graph sequence is the d-regular tree with independently oriented edges.

When we examine the local weak convergence of Erdős–Rényi random graphs (see Definition 1.1.4), we have to chose the parameters properly. If we fix p and let n tend to infinity, then the expected number of edges is proportional to n^2, hence the local weak limit of the graph does not exists. To have a nice limiting behavior, we need a uniformly bounded expected average degree which can be achieved by choosing $p = c/n$ with some fixed positive constant c.

Example 2.2.8 (Erdős–Rényi random graphs)**.** *The local weak limit of* $\mathcal{G}_{n,c/n}$ *is* $UGW(Poi(c))$*, that is the unimodular Galton–Watson tree (see Definition 1.2.10) with Poisson(c) degree distribution.*

The local weak limit of the sequence of oriented Erdős–Rényi random graphs $\vec{\mathcal{G}}_{n,c/n}$ *is* $\overrightarrow{UGW}(Poi(c))$*, the same tree with edges oriented independently.*

The random configuration model and the preferential attachment graph converges in the local weak sense when the parameters are fixed and only the size tends to infinity.

Example 2.2.9 (Random configuration model)**.** *Let* G_n *is a sequence of graphs given by the random configuration model (Definition 1.1.5) with degree distribution* ξ *with* $\mathbb{E}(\xi^2) < \infty$*. Then* G_n *converge to* $UGW(\xi)$ *in the local weak sense; see [4, Example 10.2], [25, Theorem 3.15].*

The local weak limit of the directed graphs $\overrightarrow{G_n}$ *is* $\overrightarrow{UGW}(\xi)$*, the directed unimodular Galton–Watson tree with the same degree distribution.*

Berger, Borgs, Chayes and Saberi proved in [19] that the local weak limit of $G^{PA}_{r,\alpha,n}$ as $n \to \infty$ is the Pólya-point graph with parameters r and α. This graph is a unimodular random infinite tree with directed edges; see [19, Section 2.3] for the definition.

Theorem 2.2.10 (Convergence of the preferential attachment graphs [19])**.** *The sequence of preferential attachment graphs with fixed parameters* r *and* α *converges in the local weak sense to the Pólya-point graph with the same parameters.*

The convergence also holds for the directed versions of the members of the sequence and the limit graph with the natural orientations of the edges.

We finish this section with a few interesting results and an open question on converging graph sequences. We note that the following examples have no uniform bound on the degrees.

The sequence of planar maps with n triangle or quadrangle faces converge in the local weak sense as showed by Angel [7] and Benjamini and Curien [15], respectively.

The sequence of uniform trees on the vertex set $[n]$ converges to the unimodular Galton–Watson tree $UGW(Poisson(1))$ [56].

It can be shown easily, that the *largest component* of the Erdős–Rényi random graph $\mathbb{G}(n, c/n)$ converge to the *infinite* unimodular Galton–Watson tree $UGW_\infty(Poisson(c))$ whenever $c > 1$. It is expected that the same holds for $c = 1$, but there is no known proof on this statement yet.

2.2.1 Local weak convergence of unimodular Galton–Watson trees

In this section, we present a result on sequences of *infinite* random rooted graphs that converge in the local weak sense. We examine unimodular Galton–Watson trees conditioned to be infinite. First we note that if we regard the unimodular Galton–Watson tree as the limit of finite graphs, it is natural to use the parameterization given by the distribution of the degree of the root, see Definition 1.2.10 and Example 2.2.9. Contrary to this, when we investigate percolation on the graph (which will be the case in Example 4.3.2), it is more natural to work with the distribution of the number of children of the other vertices. Since we investigate Galton–Watson trees conditioned to be infinite, we only need to consider the degree distribution of the root conditioned to be positive. It follows from Definition 1.2.10, that there is a one to one correspondence between the distribution of the degree of the root conditioned to be positive and the distribution of the number of children of the other vertices. In this section, it is more convenient to work with the latter, which we denote by X and call the **offspring distribution** of the Galton–Watson tree. We also restrict our attention to degree distributions of the root that are almost surely positive. Given the offspring distribution X, i.e., the distribution of the number of children of any vertex except the root, the degree distribution of the tree, i.e., the distribution of the root is denoted by $\hat{X}$, where

$$\mathbb{P}(\hat{X} = k) = \frac{\mathbb{P}(X = k - 1)}{k\,\mathbb{E}(\frac{1}{X+1})} \tag{2.2.2}$$

for $k \geq 1$. If $\mathbb{E}X > 1$, then $\mathbb{P}(|UGW(\hat{X})| = \infty) > 0$ [68, Proposition 5.4], thus we can consider the measure $UGW_\infty(\hat{X})$ which is $UGW(\hat{X})$ conditioned on the event $\{|UGW(\hat{X})| = \infty\}$. The random graph UGW_∞ is unimodular, being an ergodic component of a unimodular measure. Moreover, it is an extremal unimodular graph [4, Section 4], wich will be important in Chapter 4.

If $\mathbb{E}X = 1$ and $\mathbb{P}(X = 1) < 1$, then $UGW(\hat{X})$ is almost surely finite, but there is still a natural definition of the infinite tree $UGW_\infty(\hat{X})$ as follows. Let $UGW_n(\hat{X})$ be the tree $UGW(\hat{X})$ conditioned to have n vertices and choose the root of $UGW_n(\hat{X})$ uniformly at random. We denote by $UGW_\infty(\hat{X})$ the local weak limit of the trees $UGW_n(\hat{X})$, which is a unimodular infinite tree with one end; see [3, Example 3.4 and Proposition 11].

The following proposition describes the convergence of unimodular Galton–Watson trees. The first part is trivial, it follows from the definition of local weak convergence, while the proof of the second part needs more effort and is based on the decomposition of the infinite Galton–Watson tree given in [68, Section 5.5]. This is an extended version of the proof published in Beringer, Pete and Timár [20] by the case $\mathbb{E}X_n = 1$.

Proposition 2.2.11. *Let X_n and X are non-negative integer valued random variables and define $\hat{X}_n$ and $\hat{X}$ as in* (2.2.2)*.*

1)The sequence of unimodular Galton–Watson trees $UGW(\hat{X}_n)$ converge in the local weak sense to $UGW(\hat{X})$ iff $X_n \to X$ in distribution.

2) Let $UGW_\infty(\hat{X})$ be the unimodular Galton–Watson tree with degree distribution $\hat{X}$, conditioned to be infinite. If the offspring distributions satisfy $\mathbb{E}X_n \geq 1$ and $\mathbb{E}X \geq 1$, then $UGW_\infty(\hat{X}_n) \to UGW_\infty(\hat{X})$ in the local weak sense iff $X_n \to X$ in distribution.

Proof. Part 1) is trivial, it follows from the definitions of unimodular Galton–Watson tree and local weak convergence.

For the proof of Part 2), let $f_X(t) := \sum_{k=0}^{\infty} \mathbb{P}(X = k)t^k$ be the probability generating function of a non-negative integer valued random variable X. Denote by $GW(X)$ the Galton–Watson tree with offspring distribution X, and let $q = q(X) := \mathbb{P}(|GW(X)| < \infty)$, which is the smallest non-negative number that satisfies $f_X(q) = q$; see [68, Section 5.1].

According to [68, Section 5.5], the distribution of $UGW_\infty(\hat{X})$ can be described as follows. Each vertex except the root has a type: the vertices that have an infinite line of descendants (i.e. there is an infinite path from this vertex that does not contain the parent of this vertex) are called *special*, the other vertices are called *normal*. Each special vertex has at least one special child and each normal vertex has only normal children. Denote the number of special and normal children of a vertex v by $\deg_s v$ and $\deg_n v$, respectively. We claim that the distribution of $UGW_\infty(\hat{X})$ equals the distribution of the family tree of a branching process, where the number of children of the vertices are independent, they depend only on the type of the vertex, and their distribution is described by the following equations for $k \geq 0$ and $l \geq 1$:

$$\mathbb{P}(\deg_n o = k, \deg_s o = l) = \frac{\binom{k+l}{k}\mathbb{P}(X = k+l-1)q^k(1-q)^{l-1}}{(k+l)\sum_{j=1}^{\infty}\mathbb{P}(X = j-1)(1+\cdots+q^{j-1})/j}, \tag{2.2.3}$$

$$\mathbb{P}(\deg_n v = k, \deg_s v = l \,|v \text{ is special}) = \binom{k+l}{k}\mathbb{P}(X = k+l)q^k(1-q)^{l-1},$$

$$\mathbb{P}(\deg_n v = k \,|v \text{ is normal}) = \mathbb{P}(X = k)q^{k-1}.$$

Note, that the offspring distribution of the normal vertices has mean at most 1, thus the subtree of the descendants of a normal vertex is almost surely finite. If $\mathbb{E}X = 1$ and $\mathbb{P}(X = 1) < 1$, then $q = 1$ thus the root and all special vertices have only one special child, hence the special vertices form a unique infinite path in $UGW_\infty(\hat{X})$. If $\mathbb{E}X > 1$, then the distribution of the subtrees containing the special vertices equals the distribution of the union of a random number of independent Galton–Watson trees with an appropriate offspring distribution with mean larger then one. (Recall that we did not assign any type to the root.) It follows, that in this case $UGW_\infty(\hat{X})$ has infinitely many ends.

If $\mathbb{E}X = 1$ and $\mathbb{P}(X = 1) < 1$, then $q = 1$ and [56, Remark 7.13] shows that $UGW_\infty(\hat{X})$ has the claimed distribution. If $\mathbb{P}(X = 1) = 1$, then the root has 2 special children and each special vertex has one child almost surely, which gives that the family tree is $\mathbb{Z}$, as claimed. If $\mathbb{E}X > 1$, then the number of children is independent for each vertex; see [68, Proposition 5.23]. To prove the above formulas for the offspring distributions of the types let $l \geq 1$. Using Bayes' Rule we have

$$\mathbb{P}_{UGW_\infty(\hat{X})}(\deg_n o = k, \deg_s o = l) = \frac{\mathbb{P}_{UGW(X)}(\deg_n o = k, \deg_s o = l, |UGW(X)| = \infty)}{\mathbb{P}(|UGW(X)| = \infty)}$$
$$= \frac{\frac{\mathbb{P}(X=k+l-1)\binom{k+l}{k}q^k(1-q)^l}{(k+l)\mathbb{E}((X+1)^{-1})}}{\sum_{j=1}^\infty \frac{\mathbb{P}(X=j-1)(1-q^j)}{j\mathbb{E}((X+1)^{-1})}}$$
$$= \frac{\binom{k+l}{k}\mathbb{P}(X = k+l-1)q^k(1-q)^{l-1}}{(k+l)\sum_{j=1}^\infty \mathbb{P}(X = j-1)(1+\cdots+q^{j-1})/j}.$$

For the special vertices we have

$$\mathbb{P}_{UGW_\infty(\hat{X})}(\deg_n v = k, \deg_s v = l \,|v \text{ is special})$$
$$= \mathbb{P}_{GW(X)}(\deg_n o = k, \deg_s o = l \,||GW(X)| = \infty)$$
$$= \frac{\mathbb{P}_{GW(X)}(\deg_n o = k, \deg_s o = l, |GW(X)| = \infty)}{\mathbb{P}(|GW(X)| = \infty)}$$
$$= \frac{\mathbb{P}(X = k+l)\binom{k+l}{k}q^k(1-q)^l}{1-q}$$

and a similar argument shows that the offspring distribution of the normal vertices also equals the claimed distribution.

Assume that $X_n \to X$ in distribution, first with $\mathbb{P}(X = 1) < 1$. Then the uniform convergence of the convex functions f_{X_n} to the strictly convex function f_X on $[0, 1]$ implies that $q_n = q(X_n) \to q(X)$. It follows that the offspring distribution of each type associated to X_n converges in distribution to the offspring distribution of the same type associated to X. Thus $UGW_\infty(\hat{X}_n) \to UGW_\infty(\hat{X})$.

Now assume that $X_n \to X$, $\mathbb{P}(X = 1) = 1$ and $\mathbb{P}(X_n = 1) \to 1$. Using (2.2.3),

$$\mathbb{P}_{UGW_\infty(\hat{X}_n)}(\deg o = 2) = \frac{(1-q_n^2)\mathbb{P}(X_n = 1)}{2\sum_{j=1}^\infty \mathbb{P}(X_n = j-1)(1-q_n^j)/j}. \tag{2.2.4}$$

We claim that $\mathbb{P}_{UGW_\infty(\hat{X}_n)}(\deg o = 2) \to 1$. If q_n converges to some $q_\infty < 1$, then plugging $\mathbb{P}(X_n = 1) \to 1$ into (2.2.4) yields the claim immediately. If $q_n \to 1$, then a trivial bound on the denominator gives

$$\frac{(1+q_n)\,\mathbb{P}(X_n = 1)}{2\sum_{j=1}^\infty \mathbb{P}(X_n = j-1)(1+q_n+\cdots+q_n^{j-1})/j} \geq \frac{(1+q_n)\mathbb{P}(X_n = 1)}{2} \to 1\,. \tag{2.2.5}$$

Finally, if q_n does not converge, we can still apply one of these two arguments to any convergent subsequence, and obtain the claim. Therefore, in the local weak limit, the root has degree 2 almost surely. By unimodularity, this limit must be $\mathbb{Z}$. This is also $UGW_\infty(\hat{X})$, thus we have $UGW_\infty(\hat{X}_n) \to UGW_\infty(\hat{X})$.

For the other direction of Part 2), suppose that there are X_n and X such that $UGW_\infty(\hat{X}_n) \to UGW_\infty(\hat{X})$, but $X_n \nrightarrow X$. The set $\{X_n\}$ of probability distributions must be tight: otherwise, a uniform random neighbor of o in $UGW_\infty(\hat{X}_n)$, whose offspring distribution stochastically dominates X_n because of the conditioning on $\big\{|UGW(X_n)| = \infty\big\}$, would have arbitrarily large degrees with a uniform positive probability, and thus $UGW_\infty(\hat{X}_n)$ could not converge to the locally finite graph $UGW_\infty(\hat{X})$. It follows from this tightness that there is a subsequence $\{X_{k(n)}\}$ that converges in distribution to a random variable $Y \neq X$. We show that $\mathbb{E}Y \geq 1$. Suppose $\mathbb{E}Y < 1$, then $\lim q_n = q(Y) = 1$, hence

$$\mathbb{P}_{UGW_\infty(\hat{X}_n)}(\deg o = k) = \frac{\mathbb{P}(X_n = k-1)(1 + \cdots + q_n^{k-1})}{k \sum_{j=1}^{\infty} \mathbb{P}(X_n = j-1)(1 + \cdots + q_n^{j-1})/j} \to \mathbb{P}(Y = k-1).$$

It follows that the expected degree of the root in the limit graph is $\mathbb{E}Y + 1 < 2$. The local weak limit of the graphs $UGW_\infty(\hat{X}_n)$ is almost surely infinite, hence the expected degree of the root is at least 2 (see [4, Theorem 6.1]), a contradiction.

Since $\mathbb{E}Y \geq 1$, the first direction of Part 2) implies that $UGW_\infty(\hat{X}_{k(n)}) \to UGW_\infty(\hat{Y})$. If we prove that the distribution of $UGW_\infty(\hat{X})$ determines X, then we must have $X = Y$, a contradiction.

First, if $UGW_\infty(\hat{X}) = \mathbb{Z}$, then $\mathbb{P}(X = 1) = 1$. If $UGW_\infty(\hat{X})$ has one end (i.e. there is a unique non-backtracking infinite path from the root), then it must be the case that $\mathbb{E}X = 1$ and $\mathbb{P}(X = 1) < 1$ by the notes after (2.2.3). In this case $\mathbb{P}(\deg o = k + 1) = \mathbb{P}(X = k)$, hence the degree distribution of the root determines the distribution of X. The last case is $\mathbb{E}X > 1$. If we consider the components of the subgraphs formed by the special and normal vertices, respectively; and we choose for each such tree the vertex closest to o to be the root of that component, then the distributions of the components are equal to Galton–Watson tree measures with offspring distributions which have probability generating functions $f^*(t) := \frac{f_X(q+(1-q)t)}{1-q}$ and $\bar{f}(t) = \frac{f(qt)}{q}$, respectively; see [68, Theorem 5.28]. It follows that the distribution of $UGW_\infty(\hat{X})$ determines $(f^*, \bar{f})$. We get the function f from $(f^*, \bar{f})$ by the transform $f(s) = q\bar{f}\left(\frac{s}{q}\right)$, if $0 \leq s \leq q$ and $f(s) = (1-q)f^*\left(\frac{s-q}{1-q}\right)$, if $q \leq s \leq 1$. There is a unique q for which the resulting $f(s)$ has the same second derivative from the left and from the right at $s = q$. Since $f(s)$ has to be analytic, we see that $(f^*, \bar{f})$ uniquely determines f and hence X. $\square$

2.3 Almost sure local weak convergence

Previous results about the continuity of graph parameters state that a certain parameter of finite *deterministic* graphs converges when the sequence of graphs converges in the local weak sense [65, 49, 40, 26, 1, 31]. When we examine parameters of convergent sequences of finite *random* graphs, the parameters are also random variables. It is a natural question in this setting, in what sense does the parameter converge if it does converge. It turned out, that the matching ratio, which we introduce in Chapter 3, converges almost surely for the most widely used random graph models. This result relies on the fact that the examined directed graph models converge in a stronger sense: the sequence of random graphs given by the random configuration model or the preferential attachment rule converges almost surely in the local weak sense.

First we give the precise definition of the almost sure local weak convergence and illustrate it with a few examples. Then in Section 2.3.1 we generalize the result of Elek [38] to the case of directed random graphs with no uniform bound on the degrees. In Section 2.3.2 we present an unpublished result on the almost sure local weak convergence of the preferential attachment graphs.

Definition 2.3.1 (Almost sure local weak convergence)**.** *Let G_n be a sequence of finite (labeled) random graphs defined on a common probability space with joint distribution μ. We say that G_n* ***converges almost surely in the local weak sense*** *if μ-almost every sequence $(G_1, G_2, \dots)$ converges in the local weak sense, in other words, almost every realizations of the sequence (G_n) satisfy that the sequence of the deterministic graphs converges in the local weak sense.*

First we note that the local weak convergence of a sequence G_n of random graphs defined on a common probability space does not imply automatically that the sequence converges almost surely in the local weak sense, as shown by the next example.

Example 2.3.2. *Let G_n be the path of length n^2 or the $n \times n$ square grid, with probability 1/2-1/2. Let the joint distribution of the sequence G_n given by the product measure. Then G_n converges in the local weak sense to the infinite rooted graph G which is $\mathbb{Z}$ or $\mathbb{Z}^2$ with probability 1/2-1/2, but there is almost surely no local weak limit of the deterministic graph sequence given by the product measure. This follows from the Strong Law of Large Numbers which implies that almost surely both $\mathbb{Z}$ and $\mathbb{Z}^2$ are accumulation points of the sequence.*

This example also shows that if we consider a continuous graph parameter that has different limiting values along the sequences of finite paths and and finite square grids, then this parameter does not converge almost surely for general sequences of random finite graphs.

Our aim is to use the results of this section in Chapter 3: if a sequence G_n of finite random graphs converges almost surely in the local weak sense, then Theorem 3.3.3 implies the almost sure convergence of the matching ratio, which will be the case for the examined random graph models.

Remark 2.3.3. *Skorohod's Representation Theorem states that for a weakly convergent sequence $\mu_n \to \mu$ of probability measures on a complete separable metric space S there is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and S-valued random variables X_n and X with distributions μ_n and μ respectively, such that $X_n \to X$ almost surely.*

One could think that Skorohod's Theorem could be applied for the graph sequences that we consider, and get the convergence of the matching ratio for almost every sequence, using Theorem 3.3.3. This argument does not work for our purpose, because in Skorohod's Theorem, the coupling between the finite graphs is coming from the theorem, while in the case of the preferential attachment graphs there is given a joint *probability space by construction, that contains them all. In fact, we prove an even stronger statement for preferential attachment graphs: the almost sure convergence for any joint distribution of the sequence.*

Let us present a few examples of sequences of finite random rooted graphs known to converge almost surely in the local weak sense. The convergence in the next two examples relies on a certain concentration phenomenon: given a functional on finite graphs that does not change much when changing just one edge of the graphs (e.g., a density of vertices with a given neighbourhood), the random value of the functional is strongly concentrated around its mean when considering random graphs from one of the families in the following two examples; see [25, Section 3.6] for more detail. We will use a similar method to that in Section 2.3.2 to show the almost sure convergence of the preferential attachment graphs.

Example 2.3.4 (Erdős–Rényi random graphs)**.** *[25, Theorem 3.23] The sequence $\mathcal{G}_{n,c/n}$ of Erdős–Rényi random graphs converges to $UGW(Poi(c))$ almost surely in the local weak sense.*

The next example also covers the case of random d-regular graphs as a special case.

Example 2.3.5 (Random configuration model)**.** *[25, Theorem 3.28] Let ξ be a non-negative integer valued random variable with $\mathbb{E}(\xi^p) < \infty$ with some $p > 2$, and let G_n be a random graph given by the random configuration model on n vertices with degree distribution ξ. Then for any joint distribution of the sequence G_n, the sequence converges to $UGW(\xi)$ almost surely in the local weak sense.*

2.3.1 Almost surely convergent graph sequences with independently oriented or labeled edges

Our next lemma will be used in Chapter 3 to prove the almost sure convergence of the directed matching ratio of certain families of graphs. Lemma 2.3.6 states that when the edges of a local weak convergent deterministic graph sequence are oriented independently at random, then the resulting sequence of random directed graphs converges almost surely in the local weak sense. The proof of Lemma 2.3.6 essentially follows the proof of Proposition 2.2 in [38]. The main difference is that we had to generalize the proof for graph sequences without a uniform bound on the degrees. Lemma 2.3.7 is more general than Lemma 2.3.6: it states that giving IID random labels to a convergent sequence of deterministic graphs turns the sequence into an almost surely local weak convergent sequence of random labeled graph. This lemma can be proven similarly, hence we omit its proof here.

An important consequence of Lemmas 2.3.6 and 2.3.7 is that when IID random orientations or labels are added to an almost surely local weak convergent sequence of random graphs, then the resulting sequence also converges almost surely in the local weak sense.

Lemma 2.3.6. *Let G_n be a sequence of deterministic undirected graphs on n vertices that converges to the random rooted graph (G, o) in the local weak sense. Let $\overrightarrow{G_n}$ be the sequence of random directed graphs obtained from G_n by giving a random uniform orientation to each edge uniformly independently. Then the sequence $\overrightarrow{G_n}$ converges almost surely in the local weak sense to $(\overrightarrow{G}, o)$, which is the random rooted graph obtained from (G, o) by orienting each edge independently.*

Lemma 2.3.7. *Let G_n be a sequence of deterministic graphs on n vertices that converges to the random rooted graph (G, o) in the local weak sense. Let G_n^{iid} be the sequence of random directed graphs obtained from G_n by giving a random uniform $[0, 1]$ label to each edge and vertex independently. Then the sequence G_n^{iid} converges almost surely in the local weak sense to (G^{iid}, o), which is the random rooted graph obtained from (G, o) by giving each edge and vertex a random uniform $[0, 1]$ label independently.*

Now we prove Lemma 2.3.6. The proof of Lemma 2.3.7 is similar.

Proof. To handle the case of unbounded degrees, we consider the following neighborhoods of the vertices: for any graph G and $v \in V(G)$ denote by $B_G^-(v, r)$ the subgraph of G obtained from $B_G(v, r)$ by removing all edges with both endpoints being at distance r from v. Then the local weak convergence of the sequence of the finite (directed) random graphs G_n to the rooted random (directed) graph (G, o) is equivalent with the following: for any r and any finite (directed) rooted graph H we have $\lim_{n\to\infty} \mathbb{P}(B_{G_n}^-(o_n, r) \simeq H) = \mathbb{P}(B_G^-(o, r) \simeq H)$, where o_n is a uniform random vertex of G_n.

Fix any positive integer r and any finite directed rooted graph $\overrightarrow{H}$. Let H be the rooted non-directed graph obtained from $\overrightarrow{H}$ by forgetting the orientations of the edges, and suppose that $\mathbb{P}(B_G^-(o, r) \simeq H) > 0$. Denote by $b(G_n)$ and $b(\overrightarrow{G_n})$ the number of vertices v of G_n and $\overrightarrow{G_n}$ such that $B_{G_n}^-(v, r) \simeq H$ and $B_{\overrightarrow{G_n}}^-(v, r) \simeq \overrightarrow{H}$, respectively. We show that $\mathbb{P}\left(B_{\overrightarrow{G_n}}^-(o, r) \simeq \overrightarrow{H}\right) = \frac{b(\overrightarrow{G_n})}{n}$ almost surely converges to $\mathbb{P}\left(B_{\overrightarrow{G}}^-(o, r) \simeq \overrightarrow{H}\right)$. Since this holds for any $\overrightarrow{H}$, the lemma follows.

Let h be the probability that the graph obtained from H by giving each edge a random orientation independently is isomorphic to $\overrightarrow{H}$. Then $\mathbb{E}(b(\overrightarrow{G_n})) = hb(G_n)$. We will show that

$$\frac{b(\overrightarrow{G_n})}{b(G_n)} \to h \text{ almost surely.} \tag{2.3.1}$$

The statement of the lemma follows from this, because the assumption on the convergence of G_n implies that $\frac{hb(G_n)}{n}$ converges to $h\mathbb{P}(B_G^-(o, r) \simeq H) = \mathbb{P}(B_{\overrightarrow{G}}^-(o, r) \simeq \overrightarrow{H})$.

To show (2.3.1), we note that if two vertices x, y in G_n satisfy $B_{G_n}^-(x, r) \simeq B_{G_n}^-(y, r) \simeq H$ and $\text{dist}_{G_n}(x, y) \geq 2r$, then the orientations of the edges in $B_{\overrightarrow{G_n}}^-(x, r) \cup B_{\overrightarrow{G_n}}^-(y, r)$ are independent. Let D be the maximum degree of the graph H. We claim that we can define a partition $(R_j^n)_{j=1}^{D^{2r}+1}$ of the set $\{x \in V(G_n) : B_{G_n}^-(x, r) \simeq H\}$ such that the distance between any two points of R_j^n is at least $2r$ for every j and

n. Indeed, if $\text{dist}_{G_n}(x,y)$ is less than $2r$ and $B^-_{G_n}(x,r) \simeq B^-_{G_n}(y,r) \simeq H$, then there is a path of length at most $2r-1$ such that every vertex of that path has distance at most $r-1$ from the set $\{x,y\}$, and hence every vertex in the path has degree at most D. It follows, that for any fixed x, the number of such paths and hence the number of vertices y with $\text{dist}_{G_n}(x,y) < 2r$ is at most D^{2r}.

We conclude as in the proof of Proposition 2.2 in [38]. The further part of the proof is essentially the same as the proof of that proposition, but for the sake of completeness we present it here. The graph with vertex set $\{x \in V(G_n) : B_{G_n}(x,r) \simeq H\}$ and edge set $\{\{x,y\} : \text{dist}_{G_n}(x,y) < 2r\}$ has maximal degree at most D^{2r}, thus there is a coloring of its vertices with $D^{2r}+1$ colors, that gives the partition (R^n_j).

Let $0 < \varepsilon < \frac{1}{2}\mathbb{P}(B^-_G(o,r) \simeq H)$ and $0 < \delta$ be arbitrary, and let $R^n_1, \dots, R^n_{k(n)}$ be the list of the sets R^n_j that satisfy $|R^n_j| \geq \varepsilon^2|V(G_n)|/(D^{2r}+1)$. Denote by $\overrightarrow{b}(R^n_j)$ the number of vertices v in R^n_j such that $B_{\overrightarrow{G_n}}(v,r) \simeq \overrightarrow{H}$. By the Strong Law of Large Numbers

$$\left| \frac{\overrightarrow{b}(R^n_j)}{|R^n_j|} - h \right| < \varepsilon \tag{2.3.2}$$

holds for all n large enough and $j \leq k(n)$ with probability at least $1-\delta$. Using (2.3.2), the assumptions on ε and the size of the sets R^n_j we have

$$\left| \frac{b(\overrightarrow{G_n})}{b(G_n)} - h \right| < 2\varepsilon$$

for all large enough n with probability at least $1-\delta$. Since ε and δ was arbitrary, this implies (2.3.1). □

2.3.2 Preferential attachment graphs

Berger, Borgs, Chayes and Saberi [19] showed that the preferantial attachment graphs (Theorem 2.2.10) converge in the local weak sense. They also showed, that the distribution of the r-ball around the random root converges in probability. The orientations of the edges of this class of graphs are given naturally by the recursive definition, and the proof in [19] shows implicitly the convergence also for the sequence of the *directed* graphs. Using this result, we prove a strong concentration phenomenon, which implies that the convergence holds almost surely.

There is a natural joint distribution for a sequence of preferential attachment graphs, given by the definition (see Definition 1.1.6). However, it follows from the concentration shown in the proof of Theorem 2.3.8, that the almost sure local weak convergence holds not just for this joint distribution, but for any, provided the marginals have the required distributions.

Theorem 2.3.8 (Almost sure local weak convergence of the preferential attachment graphs)**.** *Let G_n be the sequence of random directed graphs given by the preferential attachment rule with parameter m, and consider any joint distribution of the graphs G_n. Then the sequence G_n converges almost surely in the local weak sense.*

Before proving the theorem, we state a version of the Azuma–Hoeffding inequality, that will be used also in Chapter 3. This theorem provides a powerful tool for proving the concentration of a random variable around its expected value.

Theorem 2.3.9 (Azuma–Hoeffding inequality)**.** *[68, Theorem 13.2] Let $X_1, \dots, X_n$ be a series of martingale differences. Then*

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k > \varepsilon\right) \le \frac{\varepsilon^2}{2\sum_{k=1}^{n} \|X_k\|_\infty^2}.$$

Proof of Theorem 2.3.8. We denote by $\mathbb{P}_{G_n}$ the probability correspondig to the fixed graph G_n but considering a random uniform root o_n. The notations $\mathbb{P}$ and $\mathbb{E}$ stand for the joint probability of the sequence of random graphs G_n with the independently chosen uniform random roots o_n. The statement of the theorem holds for any joint distribution of the sequence (G_n), provided that the graph G_n has the distribution of the preferential attachment graph on n vertices.

To handle the case of unbounded degrees, we consider the following neighborhoods of the vertices: for any graph G and $v \in V(G)$ denote by $B_G^+(v, r)$ the subgraph $B_G(v, r)$ together with the set of labels $L_r(G_n, v) := \{(w, \deg_{G_n} w) : \operatorname{dist}_{G_n}(v, w) = r\}$. Then the local weak convergence of the sequence of the finite (directed) random graphs G_n to the rooted random (directed) graph (G, o) is equivalent with the following: for any positive integer r and any finite (directed) rooted graph $H = (H, o)$ and label set $L_r(H) := \{(w, d_w) : \operatorname{dist}_H(o, w) = r\}$ with positive integer valued labels d_w, we have

$$\lim_{n\to\infty} \mathbb{P}(B_{G_n}^+(o_n, r) \simeq^+ H) = \mathbb{P}(B_G^+(o, r) \simeq^+ H), \tag{2.3.3}$$

where $\simeq^+$ means that there is a rooted graph isomorphism ϕ between the directed graphs such that the labels of w and $\phi(w)$ are equal. Since the sequence of preferential attachment graphs converges in the local weak sense (see [19]), the limit in (2.3.3) exists.

Given that the sequence of random graphs G_n converges to (G, o) in the local weak sense, the almost sure local weak convergence of G_n is equivalent with the following: for every r and rooted directed labeled graph $H = (H, o) \cup L_r(H)$ we have

$$\mathbb{P}\left(\lim_{n\to\infty} \mathbb{P}_{G_n}\left(B_{G_n}^+(o_n, r) \simeq^+ H\right) = \mathbb{P}_G\left(B_G^+(o, r) \simeq^+ H\right)\right) = 1. \tag{2.3.4}$$

We will show that for any fixed r and $H = (H, o) \cup L_r(H)$ there is a positive constant $c = c(\varepsilon, H)$ such that

$$\mathbb{P}\left(\left|\mathbb{P}_{G_n}\left(B_{G_n}^+(o_n, r) \simeq^+ H\right) - \mathbb{P}\left(B_{G_n}^+(o, r) \simeq^+ H\right)\right| > \varepsilon\right) \le 2e^{-cn}. \tag{2.3.5}$$

This and the existence of the limit in (2.3.3) imply (2.3.4) by the Borel–Cantelli lemma, hence the statement of the theorem follows.

It remains to prove (2.3.5). Fix any positive integer r and any finite directed rooted labeled graph $H = (H, o) \cup L_R(H)$. Denote $A(G_n) = \{v \in [n] : B_{G_n}^+(v, r) \simeq^+$

$H\}$. We fix n and denote by $G_n[k]$ the subgraph of G_n spanned by the vertices $\{1,\dots,k\}$. We define

$$X_k := \mathbb{E}\left(|A(G_n)| \,\Big|\, G_n[k]\right) - \mathbb{E}\left(|A(G_n)| \,\Big|\, G_n[k-1]\right).$$

The random variables X_k, $k \in [n]$ form a series of martingale differences and satisfy

$$\begin{aligned}\sum_{k=1}^{n} X_k =& |A(G_n)| - \mathbb{E}|A(G_n)| \\ =& n\Big(\mathbb{P}_{G_n}\left(B^+_{G_n}(o_n,r) \simeq^+ H\right) - \mathbb{P}\left(B^+_{G_n}(o_n,r) \simeq^+ H\right)\Big).\end{aligned} \tag{2.3.6}$$

Note that in (2.3.6), the quantities $\mathbb{P}_{G_n}\left(B^+_{G_n}(o_n,r) \simeq^+ H\right)$ are random variables which depend on the value of the random G_n. We will show that there is an almost sure bound $|X_k| \le 2ma$ for every k, where m is the parameter of the graph, i.e., the out-degree of every vertex except the first, and $a = \sum_{j=0}^{r} d^j$, where d is the maximum of the degrees and labels of H. We express the left hand side of (2.3.5) using (2.3.6), and bound it by applying the Azuma–Hoeffding inequality to the variables X_k:

$$\begin{aligned}\mathbb{P}\left(|A(G_n) - \mathbb{E}(A(G_n))| > \varepsilon n\right) &= \mathbb{P}\left(|X_1 + \dots + X_n| > \varepsilon n\right) \\ &\le 2\exp\left\{-\frac{(\varepsilon n)^2}{2\sum_{k=1}^{n}\|X_k\|_\infty^2}\right\} \\ &\le 2\exp\left\{-\frac{\varepsilon^2 n^2}{8nm^2a^2}\right\}.\end{aligned}$$

This implies (2.3.5) and hence the statement of the theorem.

It remains to show the bound $|X_k| \le 2ma$. We show that for any fixed pair of directed graphs F and F' on the vertex set $[k]$ with $F[k-1] = F'[k-1]$, the inequality

$$\left|\mathbb{E}\Big(|A(G_n)| \,\Big|\, G_n[k] = F\Big) - \mathbb{E}\Big(|A(G_n)| \,\Big|\, G_n[k] = F'\Big)\right| \le 2ma \tag{2.3.7}$$

holds. This implies the bound $|X_k| \le 2ma$.

Fix F and F' as above and let

$$C := \{j : (k,j) \in E(F) \cup E(F')\}.$$

For any possible configuration of G_n, denote by

$$h(G_n) := \{(\ell,j) \in E(G_n) : \ell > k, j \in [n] \setminus C\}$$

the subset of the edges of G_n with tails in $\{k+1,\dots,n\}$ that do not have a common head with the edges in the graphs F or F' with tail k. The proof of inequality (2.3.7) is based on two observations:

1. The distribution of $h(G_n)$ conditioned on $\{G_n[k] = F\}$ is the same as conditioned on $\{G_n[k] = F'\}$ by the definition of the preferential attachment graph.

2. For any configuration of G_n with $G_n[k] = F$, the size of $A(G_n)$ changes by at most $2ma$ if we fix $h(G_n)$, set $G_n[k] := F'$ and vary arbitrary the heads of the edges with tails in $\{k+1, \ldots, n\}$ that are not in $h(G_n)$.

In order to prove the second observation, let $G[S]$ be the graph on the vertex set $[n]$ with edge set $E(G[S]) := E(F[k-1]) \cup S$. We obtain any possible configuration of the graph G_n with $G_n[k] = F$, $h(G_n) = S$ by adding to $G[S]$ new edges with heads in C in such a way that after adding the new edges, the out-degree of every vertex (except for the first) equals m. We denote the set of such configurations by

$$\mathcal{N}_{F,S} := \big\{G : G[k] = F, h(G) = S, \deg_G^{out} j = m \quad \forall j \in [n] \setminus \{1\}\big\}.$$

Decompose $A(G_n) = A_1(G_n) \cup A_2(G_n)$, where

$$A_1(G_n) = \{v \in A(G_n) : \mathrm{dist}_{G_n}(v, C) > r\}, \quad A_2(G_n) = \{v \in A(G_n) : \mathrm{dist}_{G_n}(v, C) \leq r\}.$$

If $G_n \in \mathcal{N}_{F,S} \cup \mathcal{N}_{F',S}$, then the size of the set $A_1(G_n)$ depends only on $G[S]$ and it does not depend on the choice of F or F'. This is because if $v \in A_1(G_n)$, then $B_{G_n}(v, r)$ lies entirely in $G_n[S]$ and the degrees of the boundary vertices are also determined by $G_n[S]$. If $G_n \in \mathcal{N}_{F,S}$ then the size of $A_2(G_n)$ is bounded: if $v \in A_2(G_n)$, then there is a path of length at most r from v to a vertex in C, such that every vertex in the path has degree at most d by the definition of $B^+_{G_n}(v, r)$. Since $|C| \leq 2m$ by the definition of the set C, we have that $|A_2(G_n)| \leq |C|a \leq 2ma$ for every $G_n \in \mathcal{N}_{F,S}$. By symmetry, the same holds for any $G_n \in \mathcal{N}_{F',S}$. Note, that $|A_2(G_n)|$ can be arbitrary large if we consider the usual neighborhood $B_{G_n}(v, r)$ instead of $B^+_{G_n}(v, r)$. The bounds on the size of the sets A_1 and A_2 imply the second observation. It follows that

$$\left|\mathbb{E}\left(|A(G_n)| \,\middle|\, G_n \in \mathcal{N}_{F,S}\right) - \mathbb{E}\left(|A(G_n)| \,\middle|\, G_n \in \mathcal{N}_{F',S}\right)\right| \leq 2ma. \tag{2.3.8}$$

We give a trivial upper bound on the left hand side of (2.3.7), and then use the first observation and (2.3.8) to prove the desired bound. This finishes the proof of the theorem.

$$\begin{aligned}
&\sum_S \Big|\mathbb{E}\left(|A(G_n)| \middle| G_n[k] = F, h(G_n) = S\right) \mathbb{P}\left(h(G_n) = S \middle| G_n[k] = F\right) \\
&\qquad - \mathbb{E}\left(|A(G_n)| \middle| G_n[k] = F', h(G_n) = S\right) \mathbb{P}\left(h(G_n) = S \middle| G_n[k] = F'\right)\Big| \\
&\leq \sum_S \left|\mathbb{E}\left(|A(G_n)| \middle| G_n \in \mathcal{N}_{F,S}\right) - \mathbb{E}\left(|A(G_n)| \middle| G_n \in \mathcal{N}_{F',S}\right)\right| \mathbb{P}\left(h(G_n) = S \middle| G_n[k-1] = F[k-1]\right) \\
&\leq \sum_S 2ma \mathbb{P}\left(h(G_n) = S \,\middle|\, G_n[k-1] = F[k-1]\right) \\
&= 2ma.
\end{aligned}$$

□

Chapter 3

The matching ratio of large graphs

3.1 Introduction

3.1.1 Motivation

There is an important parameter in control theory which is closely related to the directed matching ratio of the network, as shown in the paper of Liu, Slotine and Barabási [61]. Informally, the controllability parameter of a network is defined as the minimum number N_D of nodes needed to control a network, e.g., the number of nodes that can shift molecular networks of the cell from a malignant state to a healthy state. In [61], it was showed that the proportion $n_D = N_D/|V(G)|$ of nodes needed to control a finite network G equals one minus *the relative size of the maximal directed matching* which we call the **directed matching ratio**. This allows one to prove results on n_D by proving the corresponding statement for the directed matching ratio.

Our main motivation was the two further observations in [61], which are the followings. First, simulations run on both real networks and network models suggested that the matching ratio is mainly determined by the degree sequence of the graph; more precisely, if the edges are randomized in a way that does not change the degrees, then the matching ratio does not change significantly. Second, arguments based on methods from statistical physics and numerical results suggested that for the most widely used families of scale-free networks, the directed matching ratio converges to a constant. The models that were most relevant to them are the so-called scale-free networks, which are known to exhibit several characteristics, such as a power-law degree decay, of the networks observed in real-world applications. Our aim was to give rigorous mathematical proofs of these observations of [61], by extending the result of Elek and Lippner [40] on the convergence of the matching ratio.

In this chapter, we formalize the above statements and give the proofs of them. In Section 3.2, we show that the directed matching ratio of directed random networks given by a fix sequence of degrees is concentrated around its mean. In Section 3.3 we examine the convergence of the (directed) matching ratio of a random (directed) graph sequence that converges in the local weak sense, and generalize the result of Elek and Lippner [40]. We prove that the mean of the directed matching ratio converges to the properly defined matching ratio parameter of the limiting graph. We further show the almost sure convergence of the matching ratios for the most widely used families of scale-free networks, which was the main motivation of [61]. The results of this chapter have been published in Beringer and Timár [21].

3.1.2 Definitions

First we define directed matchings and the matching ratio of directed graphs.

Definition 3.1.1 (Directed matching and directed matching ratio)**.** *A **directed matching** M of a directed graph G is a subset of the edges such that the in- and out-degrees in the subgraph induced by M are at most one. The **directed matching ratio** of the finite directed graph G is $m(G) := \frac{|V^-(M_{max}(G))|}{|V(G)|} = \frac{|M_{max}(G)|}{|V(G)|}$, where M_{max} is a maximal size directed matching of G. For undirected finite graphs G we define the **matching ratio** as $m(G) := \frac{|V(M_{max}(G))|}{|V(G)|} = \frac{2|M_{max}(G)|}{|V(G)|}$, where M_{max} is a maximal size matching of G.*

For possibly disconnected graphs (for instance Erdős–Rényi graphs (Definition 1.1.4) or graphs defined by the random configuration (Definition 1.1.5)), there is another natural way to define the directed matching ratio. Viewing them as a unimodular random graph, one takes a uniformly chosen random root, and only keeps the *connected component* of this root. Then one could define the matching ratio as the size of the maximal matching of this component divided by the size of the component. Contrary to connected graphs, this later definition can give a random variable even if we consider deterministic but disconnected graphs. The reason of using Definition 3.1.1 in this paper is coming from our motivating applications in controllability. In a finite directed graph the minimum number of nodes needed to control the network equals the number of vertices that have in-degree 0 in a maximal directed matching M_{max} (which equals $|V(G)| - |M_{max}(G)|$); see [61]. We are thus interested in the directed matching ratio $m(G)$ of a finite directed graph G provided by Definition 3.1.1, which takes the proportion of vertices of the (possibly disconnected) network that are not needed to control the dynamics of the system.

Our statements about the *directed* matching ratio follow from the corresponding statements about the matching ratio of the *undirected* graphs. The next remark describes the relationship between the matching ratio of directed and undirected graphs.

Remark 3.1.2. *There is a natural bijection between the directed matchings of G and the matchings of the bipartite representation $\bar{\bar{G}}$ of it (see Definition 1.1.1) which preserves the size of the matching, namely if M is a directed matching of G then $M \mapsto \bar{\bar{M}} = \{\{v^-, w^+\} : (v, w) \in M\}$. Furthermore, M is a directed matching of maximal size if and only if $\bar{\bar{M}}$ is a maximal size matching of $\bar{\bar{G}}$. It follows that $m(G) = m(\bar{\bar{G}})$.*

Recall, that a matching M of G has maximal size if and only if there is no augmenting path in G for M. By an **augmenting path** of length k we mean a sequence of disjoint vertices $(v_0, \dots, v_{2k+1})$ such that $\{v_{2j-1}, v_{2j}\} \in M$ for $j \in [k]$, $\{v_{2j}, v_{2j+1}\} \notin M$ for $j \in \{0, \dots, k\}$ and $\deg_M v_0 = \deg_M v_{2k+1} = 0$.

3.1.3 Our contribution to Controllability of complex networks

We devote this subsection to the presentation of our results in a network theoretical language. We also highlight the relevance of the present chapter for the network theoretical research community; in particular, as a follow-up of the work [61] of Liu, Slotine and Barabási.

Denote by n_D the proportion of the minimum number of driver nodes needed to control the network G to the number of nodes, as defined in [61]. Our Theorems 3.2.1, 3.3.3 and 3.3.10 translate to Theorems 3.1.3 and 3.1.5 by $n_D(G) = 1 - m(G)$.

Previous results in [40] and [26] on the matching ratio and our Theorem 3.1.5 imply that the parameter n_D is a quantity determined by the local structure of the graph. It is a natural question whether even the most local structure, namely the degrees of the vertices, determines this parameter. Liu, Slotine and Barabási approached this question through simulation: in their paper [61] some particular networks are taken from the real world, and then the edges are randomly reshuffled while keeping all in- and out-degrees unchanged. More precisely, the randomization used in [61] generates a random graph $\bar{G}$ with the same distribution as given by the random configuration model with the fixed sequence of in- and out-degrees, conditioned to be a simple directed graph (see Definition 1.1.5 for the definition of the model). The parameter n_D of several real-world networks was compared with the average of this parameter over the randomized copies. (Generating a large number of random copies provides an average n_D that is close to the expectation of the random n_D by the Law of Large Numbers.) Their results suggested that for random graphs given by the above model, the degrees essentially determine n_D.

The first part of our Theorem 3.1.3 shows that for a random directed graph G given by the random configuration model with fixed in- and out-degrees, the difference between the parameter n_D of the random G and the expectation of n_D is small with large probability. This result confirms the phenomenon observed in [61]: with large probability, the parameter n_D of a randomized graph is close to the average, even with the randomization used in [61], see Corollary 3.1.4. Our theorem compared with the results in [61] also shows that the examined real-world networks could have been produced by a process that results in a random graph with distribution given by the random configuration model with fixed in- and out-degrees. Our theorem can also be used to substitute generating random networks in order to show concentration.

The second part of Theorem 3.1.3 shows that even the total degrees of the network encode the controllability parameter n_D if we assume that the orientation of the edges are independent. In this model, we only fix the total degrees of the vertices, generate the random undirected graph, and orient each edge independently. We proved that for this model the parameter n_D of the random graph G is close to the expectation of n_D with large probability. We can use this theorem only for graphs generated by the above algorithm. Such graphs have the property that the in- and out-degrees have the same distribution.

Theorem 3.1.3 (Concentration of the controllability parameter)**.** *Consider a sequence of in- and out-degrees $d_1^+, \dots, d_n^+$ and $d_1^-, \dots, d_n^-$ with $\sum_{j=1}^n d_j^- = \sum_{j=1}^n d_j^+$ and let $d_j = d_j^+ + d_j^-$. Denote by $e(G) := \sum_{j=1}^n d_j^-$ the number of the edges.*

1) Let G be a random directed network on n vertices given by the random configuration model conditioned on the event that the in- and out-degrees are $d_1^+, \dots, d_n^+$ and $d_1^-, \dots, d_n^-$, respectively. Then the controllability parameter $n_D(G)$ of G satisfies

$$\mathbb{P}\left(|n_D(G) - \mathbb{E}(n_D(G))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.$$

2) Let G be a random directed network on n vertices given by the random configuration model conditioned on the event that the total degrees of the vertices are $d_1, \dots, d_n$. Then the controllability parameter $n_D(G)$ of G satisfies

$$\mathbb{P}\left(|n_D(G) - \mathbb{E}(n_D(G))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.$$

Corollary 3.1.4 (Concentration of the controllability parameter in conditioned graphs)**.** *Let the random directed graph G defined as in Part 1) of Theorem 3.1.3 and let $\bar{G}$ have the same distribution, but conditioned to be a simple graph. Then the controllability parameter of $\bar{G}$ is concentrated around $\mathbb{E}(n_D(G))$:*

$$\begin{aligned}\mathbb{P}\Big(\Big|n_D(\bar{G}) - \mathbb{E}(n_D(G))\Big| > \varepsilon\Big) \\ &= \frac{\mathbb{P}\left(|n_D(G) - \mathbb{E}(n_D(G))| > \varepsilon, G \textit{ is simple}\right)}{\mathbb{P}(G \textit{ is simple})} \\ &\leq \frac{2}{\mathbb{P}(G \textit{ is simple})}\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.\end{aligned}$$

Our Theorem 3.1.5 confirms the observations of [61] that the parameter n_D converges for some particular network models when the size of the networks tends to infinity. In fact, Part 1) shows that the expectation of the parameter n_D converges for any sequence of directed graphs that converges in the local weak sense. There are several network models that are known to perform this type of convergence, e.g., Erdős–Rényi random graphs, random d regular graphs, networks given by the random configuration model, preferential attachment graphs; see Examples 2.2.8, 2.2.7, 2.2.9 and Theorem 2.2.10. Parts 2) and 3) of Theorem 3.1.5 imply that the graph sequences listed above have an even stronger property: their controllability parameter n_D converges almost surely to a constant. For the first three networks one can derive the limit of $n_D(G_n) = 1 - m(G_n)$ using the formula of Theorem 2 in [26], see Corollaries 3.3.13, 3.3.12, 3.3.11 in Subsection 3.3.2.

It was observed in [61] that the nodes with low degree are more likely to be driver nodes, i.e., nodes with in-degree zero in the maximal size directed matching. The method of the proof of the first part of our Theorem 3.1.5 shows that this feature of the driver nodes follows naturally from the construction of a maximal size matching: the nodes with higher degrees are more likely to be matched.

Theorem 3.1.5 (Almost sure convergence of the controllability parameter for scale–free graphs). *1) Let G_n be a sequence of random directed finite graphs that converges to a random rooted graph (G, o) in the local weak sense. Then $\mathbb{E}(n_D(G_n))$ converges, namely*

$$\lim_{n\to\infty} \mathbb{E}(n_D(G_n)) = \inf_M \mathbb{P}_G\big(o \notin V^+(M)\big),$$

where the infimum is taken over all directed matchings M of G such that the distribution of (G, M, o) is unimodular, and $V^+(M)$ denotes the set of the heads of the edges in M.

2) Let G_n be a sequence of undirected finite graphs defined on a common probability space that converges almost surely in the local weak sense and let $\overrightarrow{G_n}$ be a sequence of random directed graphs obtained from G_n by giving each edge a random orientation independently. Then $n_D(\overrightarrow{G_n})$ converges almost surely to the constant $\lim_{n\to\infty} \mathbb{E}(n_D(\overrightarrow{G_n}))$.

3) Let G_n be the sequence of random directed graphs given by the preferential attachment rule. Then $n_D(G_n)$ converges almost surely to the constant $\lim_{n\to\infty} \mathbb{E}(n_D(G_n))$.

3.2 Concentration of the matching ratio in randomized networks

In this section, we prove Theorem 3.2.1, which gives a quantitative version of the following observation: if we consider a large directed graph, and randomize the edges in such a way that does not change the in- and out-degrees of the graph, then the matching ratio does not alter significantly. Our main motivation was to give rigorous proof on the experimental results of Liu, Slotine and Barabási in [61]. We further show in Section 3.2.1 that the matching ratio of preferential attachment graphs also concentrates strongly around its expected value.

Our theorems extends the series of results on the concentration of certain parameters of random graphs. If there is a functional on graphs that does not change much if we add or remove an edge, then for Erdős–Rényi random graphs, the value of the functional is in some sense concentrated around its mean, see [6, Section 7], [25, Remark 3.25]. The almost sure local weak convergence of Erdős–Rényi random graphs can be shown by an argument using this type of concentration [25, Theorem 3.23]. These results made use of the independence in the definition of the Erdős–Rényi random graph. There is a similar but stronger concentration inequality for the undirected random configuration model with given degree sequence [25, Remark

3.31], which implies the almost sure local weak convergence of the random configuration model with a degree distribution that has finite p^{th} moment for some $p > 2$ [25, Theorem 3.28]. Our Theorems 3.2.1 and 3.2.3 extend these type of results for *directed matchings of directed graphs* given by the random configuration model and preferential attachment rule, respectively. In this models, one has to deal with a stronger dependence than in the models referred above.

Part 1) of Theorem 3.2.1 shows the concentration for randomized graphs with the in- and out-degrees left unchanged. This is the result that was observed through simulations in [61]. Part 2) of the theorem shows that a very similar concentration phenomenon holds even after a randomizing that does not require the in- and out-degrees to be unchanged but only the total degree to remain the same for every vertex. In particular, Theorem 3.2.1 shows that if a graph sequence satisfies that the empirical second moment of the degree sequence is $o(n)$ with probability tending to 1 (as $n \to \infty$), then the directed matching ratios of the graphs with randomized edges are strongly concentrated around their mean with high probability. Erdős–Rényi graphs with parameters $(n, c/n)$ or graphs given by the random configuration model with degree distribution ξ with $\mathbb{E}\xi < \infty$ have this property.

Theorem 3.2.1 (Concentration of the matching ratio)**.** *Consider sequences of in- and out-degrees $d_1^+, \ldots, d_n^+$ and $d_1^-, \ldots, d_n^-$ with $\sum_{j=1}^n d_j^- = \sum_{j=1}^n d_j^+$ and let $d_j = d_j^+ + d_j^-$. Denote by $e(G) := \sum_{j=1}^n d_j^-$ the number of the edges.*

1) Let G be a random directed graph on n vertices given by the random configuration model conditioned on the event that the in- and out-degrees are $d_1^+, \ldots, d_n^+$ and $d_1^-, \ldots, d_n^-$, respectively. Then the directed matching ratio $m(G)$ of G satisfies

$$\mathbb{P}\left(|m(G) - \mathbb{E}(m(G))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.$$

2) Let G be a random directed graph on n vertices given by the random configuration model conditioned on the event that the total degrees of the vertices are $d_1, \ldots, d_n$. Then the directed matching ratio $m(G)$ of G satisfies

$$\mathbb{P}\left(|m(G) - \mathbb{E}(m(G))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.$$

First we need a lemma that shows that modifying a (directed) graph just around a few vertices cannot alter the size of the maximal matching too much.

Lemma 3.2.2. *Adding some new edges with a common endpoint to an undirected finite graph or adding edges with a common head (respectively tail) to a directed finite graph can increase the size of the maximal matching by at most one.*

Proof. For directed graphs the statement follows from the undirected case, using the bipartite representation (see Definition 1.1.1). For undirected graphs let F be the set of new edges with common endpoint x and let G_2 be the graph with vertex set $V(G)$ and edge set $E(G_2) = E(G) \cup F$. If M_2 is a maximal size directed matching of G_2, then there is at most one edge in $M_2 \cap F$ by the definition of the matching. Then $M_2 \setminus F$ is a matching of G, hence $|M_{max}(G)| \geq |M_2| - 1$. □

As mentioned at the beginning of this section, the proof of Theorem 3.2.1 uses similar methods to that of Corollary 3.27 in [25], which implies the concentration of matching ratio for undirected graphs.

Proof of Theorem 3.2.1. We prove both parts of the theorem in the following way: we define random variables X_k, $k \in [e(G)]$ which form a series of martingale differences and satisfy $\sum_{k=1}^{e(G)} X_k = n\big(m(G) - \mathbb{E}(m(G))\big)$. We will show that there is an almost sure bound $|X_k| \leq 2$, hence we have by the Azuma–Hoeffding inequality (Theorem 2.3.9)

$$\begin{aligned}\mathbb{P}\left(|m(G(N)) - \mathbb{E}(m(G(N)))| > \varepsilon\right) &= \mathbb{P}\left(|X_1 + \cdots + X_{e(G)}| > \varepsilon n\right) \\ &\leq 2\exp\left\{-\frac{(\varepsilon n)^2}{2\sum_{k=1}^{e(G)} \|X_k\|_\infty^2}\right\} \\ &\leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.\end{aligned}$$

Part 1). Recall the second definition of the directed random configuration model from Definition 1.1.5, conditioned on the fixed sequences of in- and out-degrees. Denote by N a uniform random element of the set $\mathcal{N}$ of perfect matchings of $\mathcal{T}$ to $\mathcal{H}$. For a half-edge $h = (i, j, \pm) \in \mathcal{T} \cup \mathcal{H}$ let $N(h)$ be the pair of the half-edge h by the matching N. Let $\{h_i : i \in [e(G)]\}$ be an enumeration of $\mathcal{T}$ and denote by $N[k] := \{(h_i, N(h_i)) \in N : i \in [k]\}$ the partial matching that consists of the pairs of half-edges of N with the first k tails. Let $\mathcal{F}_0$ be the trivial σ-algebra, let $\mathcal{F}_k$ be the σ-algebra generated by $N[k]$ and define

$$X_k := \mathbb{E}\left(|M_{max}(G(N))| \,\Big|\, \mathcal{F}_k\right) - \mathbb{E}\left(|M_{max}(G(N))| \,\Big|\, \mathcal{F}_{k-1}\right). \tag{3.2.1}$$

The variables X_k clearly form a series of martingale differences, and we claim that $|X_k| \leq 2$ almost surely for all $k \in [e(G)]$.

Fix an arbitrary partial matching $F_0 = \{(h_i, F_0(h_i)) : i \in [k]\}$ and let $\mathcal{N}_k := \{S \in \mathcal{N} : S[k] = F_0\}$ and $\mathcal{N}_{k-1} := \{S \in \mathcal{N} : S[k-1] = F_0[k-1]\}$ be the set of perfect matchings of $\mathcal{H}$ to $\mathcal{T}$ with $S[k] = F_0$ and $S[k-1] = F_0[k-1]$, respectively. Denote $h' := F_0(h_k)$ and for a configuration $S \in \mathcal{N}[k-1]$ let

$$f(S) := \Big(S \setminus \Big\{\big(h_k, S(h_k)\big), \big(S(h'), h'\big)\Big\}\Big) \cup \Big\{(h_k, h'), \big(S(h'), S(h_k)\big)\Big\}. \tag{3.2.2}$$

For each $S \in \mathcal{N}_{k-1}$ there is a unique $f(S) \in \mathcal{N}_k$ and for each $S' \in \mathcal{N}_k$ the size of the set $\{S \in \mathcal{N}_{k-1} : f(S) = S'\}$ is equal, namely $e(G) - k = \frac{|\mathcal{N}_{k-1}|}{|\mathcal{N}_k|}$. We have

$$\begin{aligned}
&\left|\mathbb{E}\Big(|M_{max}(G(N))| \;\Big|\; N[k] = F_0\Big) - \mathbb{E}\Big(|M_{max}(G(N))| \;\Big|\; N[k-1] = F_0[k-1]\Big)\right| \\
&\qquad = \left| \sum_{S' \in \mathcal{N}_k} \frac{|M_{max}(G(S'))|}{|\mathcal{N}_k|} - \sum_{S \in \mathcal{N}_{k-1}} \frac{|M_{max}(G(S))|}{|\mathcal{N}_{k-1}|} \right| \\
&\qquad = \left| \sum_{S \in \mathcal{N}_{k-1}} \frac{|M_{max}(G(f(S)))|}{(e(G)-k)|\mathcal{N}_k|} - \frac{|M_{max}(G(S))|}{|\mathcal{N}_{k-1}|} \right| \\
&\qquad \leq \sum_{S \in \mathcal{N}_{k-1}} \Big| |M_{max}(G(S))| - |M_{max}(G(f(S)))| \Big| \frac{1}{|\mathcal{N}_{k-1}|}. \qquad (3.2.3)
\end{aligned}$$

For any $S \in \mathcal{N}_{k-1}$ the graphs $G(S)$ and $G(f(S))$ differ by at most four edges in such a way that the size of the set of the heads of these vertices is at most two. By Lemma 3.2.2 we have in this case $\Big| \, |M_{max}(G(S))| - |M_{max}(G(f(S)))| \, \Big| \leq 2$ which combined with (3.2.3) proves the bound $|X_k| \leq 2$.

Part 2). Recall the notations and the first definition of the directed random configuration model from Definition 1.1.5, conditioned on the fixed sequence of total degrees. Let $\mathcal{R}$ be the set of perfect matchings of the set $\mathcal{E}$ of half-edges together with an ordering on each matched pair (this ordering gives the orientation of the corresponding edge). Denote by R a uniform random element of $\mathcal{R}$ and let $\{h, R(h)\}_R = (h, R(h))$ or $(R(h), h)$, i.e., the unique ordered pair containing the half-edge h given by the matching R. Let $\mathcal{E} = \{h_i : i \in [2e(G)]\}$ be an enumeration of the set of half-edges. Let $R[k]$ be the partial matching that contains the first k edges with respect to the ordering given by the minimum of the indices of the two half-edges, i.e., let $R[0] := \emptyset$ and for $k \geq 1$ let

$$R[k] := R[k-1] \cup \Big\{ \{h_i, R(h_i)\}_R : i = \min\{j : h_j \notin \mathcal{E}(R[k-1])\} \Big\},$$

where $\mathcal{E}(R[k-1])$ is the set of half-edges that are paired by the partial matching $R[k-1]$.

Let $\mathcal{F}_0$ be the trivial σ-algebra, let $\mathcal{F}_k$ be the σ-algebra generated by $R[k]$ and define

$$X_k := \mathbb{E}\left(|M_{max}(G(R))| \;\Big|\; \mathcal{F}_k\right) - \mathbb{E}\left(|M_{max}(G(R))| \;\Big|\; \mathcal{F}_{k-1}\right).$$

We claim that $|X_k| \leq 2$ almost surely for all k, which implies the statement of the theorem.

Let F_0 be any fixed partial matching of $\mathcal{E}$ consisting of k ordered pairs. We will show that

$$\left| \mathbb{E}\left(|M_{max}(G(R))| \;\Big|\; R[k] = F_0\right) - \mathbb{E}\left(|M_{max}(G(R))| \;\Big|\; R[k-1] = F_0[k-1]\right) \right| \leq 2, \qquad (3.2.4)$$

which implies the bound $|X_k| \leq 2$.

Let $\mathcal{R}_{k-1} := \{S \in \mathcal{R} : S[k-1] = F_0[k-1]\}$ and $\mathcal{R}_k := \{S \in \mathcal{R} : S[k] = F_0\}$. Denote by (h, h') the unique ordered pair in $F_0 \setminus F_0[k-1]$ and define

$$\mathcal{S} := \Big\{(S, S') \in \mathcal{R}_{k-1} \times \mathcal{R}_k : \\ S(h) = h' \text{ and } S \setminus \big\{\{h, h'\}_S\big\} = S' \setminus \{(h, h')\} \\ \text{or } S(h) \neq h' \text{ and } |S \triangle S'| = 4\Big\}.$$

Note that the pairs $(S, S') \in \mathcal{S}$ that satisfy $S(h) \neq h'$,$|S \triangle S'| = 4$, have the properties $S'(S(h)) = S(h')$ and

$$S \setminus \Big\{\{h, S(h)\}_S, \{h', S(h')\}_S\Big\} = S' \setminus \Big\{(h, h'), \{S(h), S(h')\}_{S'}\Big\}.$$

For each $S \in \mathcal{R}_{k-1}$ with $S(h) = h'$ there is a unique $S' \in \mathcal{R}_k$ with $(S, S') \in \mathcal{S}$ and for each $S \in \mathcal{S}$ with $S(h) \neq h'$ the cardinality of the set $\{S' \in \mathcal{R}_k : (S, S') \in \mathcal{S}\}$ is 2. For each $S' \in \mathcal{R}_k$ the sets $\{S \in \mathcal{R}_{k-1} : S(h) = h', (S, S') \in \mathcal{S}\}$ and $\{S \in \mathcal{R}_{k-1} : S(h) \neq h', (S, S') \in \mathcal{S}\}$ have 2 and $8(e(G) - k)$ elements, respectively. To see that the last claim is true, note that given an $S' \in \mathcal{R}_k$ one can obtain an $S \in \mathcal{R}_{k-1}$ with $S(h) \neq h'$, $(S, S') \in \mathcal{S}$ by choosing a pair $(f, f') \in S' \setminus S'[k]$ and let $\{S(h), S(h')\} = \{f, f'\}$. The term 8 comes from the two possible choices of $S(h) = f$ or f' and the ordering of the pairs. Define a function c on $\mathcal{R}_{k-1}$ by $c(S) := 2$ if $S(h) = h'$ and $c(S) := 1$ otherwise. Denote $s := \sum_{(S,S') \in \mathcal{S}} c(S)$, which satisfies $s = 2|\mathcal{R}_{k-1}| = (4 + 8(e(G) - k))|\mathcal{R}_k|$ by the cardinalities of the sets mentioned above. Using these notations we obtain

$$\begin{aligned}
&\left|\mathbb{E}\Big(|M_{max}(G(R))| \;\Big|\; R[k] = F_0\Big) - \mathbb{E}\Big(|M_{max}(G(R))| \;\Big|\; R[k-1] = F_0[k-1]\Big)\right| \\
&= \left| \sum_{S' \in \mathcal{R}_k} \frac{|M_{max}(G(S'))|}{|\mathcal{R}_k|} - \sum_{S \in \mathcal{R}_{k-1}} \frac{|M_{max}(G(S))|}{|\mathcal{R}_{k-1}|} \right| \\
&= \left| \sum_{S' \in \mathcal{R}_k} \frac{\big(4 + 8(e(G) - k)\big)|M_{max}(G(S'))|}{s} - \sum_{S \in \mathcal{R}_{k-1}} \frac{2|M_{max}(G(S))|}{s} \right| \\
&\leq \sum_{(S,S') \in \mathcal{S}} \Big| |M_{max}(G(S))| - |M_{max}(G(S'))| \Big| \frac{c(S)}{s} \qquad (3.2.5)
\end{aligned}$$

For any pair $(S, S') \in \mathcal{S}$ the graphs $G(S)$ and $G(S')$ differ by at most four edges in such a way that both of the graphs can be obtained from the same graph by adding at most two edges to it. By Lemma 3.2.2 we have in this case $\Big| |M_{max}(G(S))| - |M_{max}(G(S'))| \Big| \leq 2$ which combined with (3.2.5) proves the bound $|X_k| \leq 2$. □

3.2.1 Concentration of the matching ratio in preferential attachment graphs

In the next theorem, we show a strong concentration of the matching ratio around its expected value in preferential attachment graphs. The orientations of the edges of this class of graphs are given naturally by the recursive definition, and differ significantly from the independent random orientation. Thus we cannot apply the results of Theorem 3.2.1.

Theorem 3.2.3 (Concentration of the matching ratio of preferential attachment graphs)**.** *Let G_n be a random graph sequence obtained by the preferential attachment rule with parameter r. Then $m(G_n)$ is concentrated around its expected value: for any $c > 0$ we have*

$$\mathbb{P}\left(|m(G_n) - \mathbb{E}(m(G_n))| > c\right) \leq 2\exp\left\{-\frac{c^2 n}{8r^2}\right\}.$$

In the proof, we use similar methods to that of Theorem 3.2.1, but we should take into account the different probabilities of the configurations given by the preferential attachment rule.

Proof. Fix n and denote by $G_n[k]$ the subgraph of G_n spanned by the vertices $\{1, \ldots, k\}$. Let

$$X_k := \mathbb{E}\left(|M_{max}(G_n)| \,\Big|\, G_n[k]\right) - \mathbb{E}\left(|M_{max}(G_n)| \,\Big|\, G_n[k-1]\right). \tag{3.2.6}$$

We will show that $|X_k| \leq 2r$ almost surely for all $k \in [n]$. Since $Y_k := \mathbb{E}\left(|M_{max}(G_n)|\Big|G_n[k]\right)$ is a martingale, we can apply the Azuma–Hoeffding inequality (Theorem 2.3.9) to the random variables X_k. It follows that for any $c > 0$ we have

$$\begin{aligned}\mathbb{P}\left(|m(G_n) - \mathbb{E}(m(G_n))| > c\right) &= \mathbb{P}\left(|X_1 + \cdots + X_n| > cn\right) \\ &\leq 2\exp\left\{-\frac{(cn)^2}{2\sum_{k=1}^n \|X_k\|_\infty^2}\right\} \\ &\leq 2\exp\left\{-\frac{c^2 n^2}{8nr^2}\right\}.\end{aligned}$$

What remains to show is that for any fixed pair of directed graphs F and F' on the vertex set $[k]$ with $F[k-1] = F'[k-1]$, the inequality

$$\left|\mathbb{E}\Big(|M_{max}(G_n)| \,\Big|\, G_n[k] = F\Big) - \mathbb{E}\Big(|M_{max}(G_n)| \,\Big|\, G_n[k] = F'\Big)\right| \leq 2r \tag{3.2.7}$$

holds. This implies $|X_k| \leq 2r$.

Fix F and F' as above. For any possible configuration of G_n, denote by

$$h(G_n) := \{(\ell, j) \in E(G_n) : \ell > k, (k, j) \notin E(F) \cup E(F')\}$$

the subset of the edges of G_n with tails in $\{k+1, \ldots, n\}$ that do not have a common head with the edges in the graphs F or F' with tail k. The proof of inequality (3.2.7) is based on two observations: first, by the definition of the preferential attachment graph, the distribution of $h(G_n)$ conditioned on $\{G_n[k] = F\}$ is the same as conditioned on $\{G_n[k] = F'\}$ (note the symmetry in F and F' in the definition of $h(G_n)$).

Second, for any configuration of G_n with $G_n[k] = F$, the size of the maximal matching changes by at most $2r$ if we fix $h(G_n)$, set $G_n[k] := F'$ and vary arbitrarily the heads of the edges with tails in $\{k+1, \ldots, n\}$ that are not in $h(G_n)$. This follows from Lemma 3.2.2 by the following argument. For any fixed H we obtain any graph in the set $\{G_n : G_n[k] = F, h(G_n) = H\}$ by adding new edges with heads in the set $\{j : (k,j) \in E(F) \cup E(F')\}$ of size at most $2r$ to the graph G_H with $V(G_H) := [n]$ and $E(G_H) := E(F[k-1]) \cup H$. It follows from Lemma 3.2.2 that

$$\begin{aligned} |M_{max}(G_H)| &\leq \mathbb{E}\Big(|M_{max}(G_n)| \;\Big|\; G_n[k] = F, h(G_n) = H\Big) \\ &\leq |M_{max}(G_H)| + 2r, \end{aligned} \tag{3.2.8}$$

and the same holds with F' in the place of F. This proves the second observation.

Using the first observation and (3.2.8) the left hand side of (3.2.7) can be estimated from above by

$$\begin{aligned} &\sum_H \Big| \mathbb{E}\left(|M_{max}(G_n)| \Big| G_n[k] = F, h(G_n) = H\right) \mathbb{P}\left(h(G_n) = H \Big| G_n[k] = F\right) \\ &\qquad - \mathbb{E}\left(|M_{max}(G_n)| \Big| G_n[k] = F', h(G_n) = H\right) \mathbb{P}\left(h(G_n) = H \Big| G_n[k] = F'\right) \Big| \\ &\leq \sum_H \mathbb{P}\left(h(G_n) = H \;\Big|\; G_n[k-1] = F[k-1]\right) \\ &\qquad\qquad \cdot \Big| \mathbb{E}\left(|M_{max}(G_n)| \;\Big|\; G_n[k] = F, h(G_n) = H\right) \\ &\qquad\qquad\quad - \mathbb{E}\left(|M_{max}(G_n)| \;\Big|\; G_n[k] = F', h(G_n) = H\right) \Big| \\ &\leq \sum_H \mathbb{P}\left(h(G_n) = H \;\Big|\; G_n[k-1] = F[k-1]\right) \cdot 2r \\ &= 2r \end{aligned}$$

□

3.3 Convergence of the matching ratio

The goal of this section is to prove the convergence of the directed matching ratio for convergent sequences of random directed graphs. This convergence is understood in the stronger sense of almost sure convergence for the most widely used graph models, as we will see. This fact follows from the almost sure convergence of the examined graph sequences, but the convergence in expectation holds also for more general models. For a fixed deterministic non-directed graph sequence that is locally convergent when a uniform root is taken, the convergence of the matching ratio is proven by Elek and Lippner in [40] in the uniformly bounded degree case and by Bordenave, Lelarge and Salez in [26] in the unbounded case. To prove the results of Liu, Slotine and Barabási in [61], we need to generalize these results for *directed random* graphs.

In Subsection 3.3.1 we use the method of Elek and Lippner to prove Theorem 3.3.3 on the convergence of the *expected value* of the directed matching ratio of sequences of random graphs. In Definition 3.3.1 we give an extension of the definition of the expected matching ratio to unimodular random rooted graphs. By [26][Theorem 1] and our Theorem 3.3.3, our definition of the expected matching ratio equals twice the parameter γ defined in [26].

In Subsection 3.3.2 we prove the almost sure convergence of the directed matching ratios for the network models defined in Subsection 2.2.

3.3.1 Convergence of the mean of the matching ratio

Elek and Lippner proved that the non-directed matching ratio converges if G_n is a convergent sequence of finite deterministic graphs with uniformly bounded degree; see [40, Theorem 1.1]. There are three properties of our examined models, that do not let us apply this theorem directly: our graphs do not have bounded degrees, and they are directed and random graphs. Although the degrees are not bounded in the examined models of convergent graph sequences, the expected value of the degree of the uniform random root of the random graphs has a uniform bound in each model. In Theorem 3.3.3 we prove the convergence of the mean of the matching ratio for convergent sequences of random directed graphs using the method of Elek and Lippner.

One can extend the (expected) matching ratio to the class of unimodular random (directed) graphs in a natural way. For finite random graphs, the following definition gives the expected value of the matching ratio.

Definition 3.3.1 (Matching ratio of an infinite graph and unimodular matchings)**.** *Let (G, o) be a unimodular random (directed) rooted graph. Then the **(expected) matching ratio** of (G, o) is*

$$m_E(G, o) = \sup_M \mathbb{P}_G(o \in V^{(-)}(M)),$$

*where the supremum is taken over all random (directed) matchings of G such that the law of the labeled graph of (G, M, o) with labels $c(e) = \chi_M(e)$ is unimodular and M is almost surely a (directed) matching of (G, o). Matchings with this property will be called **unimodular matchings**.*

The result of Timár [78] shows that $m_E(\mathbb{Z}^d) = 1$ and the same holds for every bipartite Cayley graph of a non-amenable group by [67]. Both results are obtained by factor of IID constructions; see Definition 1.2.11.

Remark 3.3.2. *Let (G, o) be a random directed rooted unimodular graph and let $(\bar{\bar{G}}, \bar{\bar{o}})$ be its bipartite representation (see Definition 1.1.1). Then Remark 3.1.2 implies that $m_E(G, o) = m_E(\bar{\bar{G}}, \bar{\bar{o}})$.*

Theorem 3.3.3. *Let G_n be a sequence of random finite (directed) graphs that converges to the random (directed) rooted graph (G, o) that has finite expected degree. Then*

$$\lim_{n\to\infty} \mathbb{E}(m(G_n)) = m_E(G, o).$$

The proof of Theorem 3.3.3 follows the method of [40]. The main differences to that proof come from the lack of uniform bound on the degrees. We will define the matchings $M(T)$ in Lemma 3.3.5 as *factor of IID* (Definition 1.2.11), which helps us handle the case of unbounded degrees. For graphs with unbounded degrees, Lemma 4.1 of [40] (Lemma 3.3.8) does not apply, hence we will have to proceed through Lemma 3.3.9.

Recall from Remark 1.2.12, that given a unimodular random rooted graph (G, o), any factor of IID process on G gives a unimodular labeled rooted graph. In particular, every factor of IID matching M of a unimodular graph satisfies that (G, M, o) is unimodular.

Definition 3.3.4 (Factor of IID matching)**.** *A random subset $M \subseteq E(G)$ is called a* ***factor of IID (directed) matching*** *if there is a factor of IID process (X_a) such that an edge e is in M if and only if $X_e = 1$ and M is a matching of G with probability 1 with respect to the distribution of G equipped with the IID labels.*

Lemma 3.3.5. *(1) For any locally finite graph G and any $T > 0$ there is a factor of IID matching $M(T)$ that has no augmenting paths of length at most T.*
(2) If (G, o) is a random unimodular rooted graph, then

$$\lim_{T\to\infty} \mathbb{P}_G\left(o \in V(M(T))\right) = m_E(G, o).$$

Remark 3.3.6. *The above lemma holds for directed graphs as well: the statements of the lemma remain true for the pre-images of the matchings $M(T)$ by the bijection defined in Remark 3.1.2.*

The proof of part 1) of Lemma 3.3.5 is similar to that of Lemma 2.2 of [40], but for the sake of completeness we present it here. The main difference is that for graphs with unbounded degrees we cannot define the matchings $M(T)$ using Borel colorings, which were used in [40]. To handle the case of unbounded degrees we define $M(T)$ as factor of IID matchings. Our language is also different, although all the claims stated for Borel matchings in [40] hold for factor of IID matchings as well.

We need the following lemma for the proof of part 2) of Lemma 3.3.5.

Lemma 3.3.7. *Let (G, o) be a unimodular random rooted graph. Then if a unimodular matching M of G satisfies that there are no augmenting paths of length at most k, then*

$$\mathbb{P}(o \in V(M)) \geq m_E(G, o) - 1/k.$$

Proof. We show that for every ε and k, any unimodular matching M that has no augmenting path of length at most k satisfies

$$\mathbb{P}(o \in V(M)) \geq m_E(G,o) - \varepsilon - 1/k. \tag{3.3.1}$$

This implies the statement of the lemma. Let M_ε be a fixed unimodular matching that satisfies $m_E(G,o) - \mathbb{P}(o \in V(M_\varepsilon)) \leq \varepsilon$. Consider the symmetric difference $M \triangle M_\varepsilon$, that is a disjoint union of paths and cycles, which alternately consists of edges of M and M_ε by the definition of matchings. We will bound $\mathbb{P}(o \in V(M_\varepsilon) \setminus V(M))$ from above by $1/k$, which implies (3.3.1) by

$$\mathbb{P}(o \in V(M)) \geq \mathbb{P}(o \in V(M_\varepsilon)) - \mathbb{P}(o \in V(M_\varepsilon) \setminus V(M)).$$

If a vertex x of G is in $V(M_\varepsilon) \setminus V(M)$, then there is an alternating path consisting of at least $2k+2$ edges in $M \triangle M_\varepsilon$ starting from x with an edge of M_ε by the assumption on M. Define the following mass transport: let $f(x, y, (G, M \triangle M_\varepsilon))$ be 1, if $x \in V(M_\varepsilon) \setminus V(M)$ and y is at distance at most $k-1$ from x in the graph metric induced by $M_\varepsilon \triangle M$ (there is exactly k such y, by our previous observation on the alternating path starting from x). Let $f(x, y, (G, M \triangle M_\varepsilon))$ be 0 otherwise. Note that each vertex receives mass at most 1. The labeled graph $(G, M \triangle M_\varepsilon, o)$ is unimodular, hence we have by the Mass Transport Principle that

$$\begin{aligned} k\mathbb{P}(o \in V(M_\varepsilon) \setminus V(M)) &= \mathbb{E}\left(\sum_{x \in V(G)} f(o, x, (G, M \triangle M_\varepsilon))\right) \\ &= \mathbb{E}\left(\sum_{x \in V(G)} f(x, o, (G, M \triangle M_\varepsilon))\right) \leq 1. \end{aligned}$$

This gives the desired bound on $\mathbb{P}(o \in V(M_\varepsilon) \setminus V(M))$. □

Proof of Lemma 3.3.5. We assign to each vertex x of G a uniform random $[0,1]$-label $c(x)$. First we note that with probability 1 all the labels are different, so we can assume this property. Furthermore, we can decompose each label $c(x)$ into countably many labels $(c_{i,j}(x))_{i,j=0}^{\infty}$ whose joint distribution is IID uniform on [0,1]. First we construct partitions $\mathcal{V}_T = \{V_{T,j} : j \geq 1\}, T \geq 1$ of V such that for each T and j $\inf\{\mathrm{dist}(x,y) : x, y \in V_{T,j}\} \geq 6T$ holds. Let

$$\begin{aligned} V_{T,1} &:= \{x \in V : c_{T,1}(x) < c_{T,1}(y) \text{ for every } y \in B_G(x, 6T)\}, \\ V_{T,j} &:= \left\{ x \in V \setminus \left(\bigcup_{l=1}^{j-1} V_{T,l} \right) : c_{T,j}(x) < c_{T,j}(y) \text{ for every } y \in B_G(x, 6T) \right\}, \end{aligned}$$

for $j \geq 2$. Since the labels are uniform in $[0,1]$, we get a partition with probability one.

We define the matchings $M_n(T)$ in the following way. Let $M_0(T) = M(T-1)$ (and the empty matching if $T = 1$) and let $k(n)$ be a fixed sequence that consists of positive integers and contains each of them infinitely many times. To define $M_n(T)$ we improve the matching $M_{n-1}(T)$ in all the balls $B(x, 3T)$ with $x \in V_{T,k(n)}$: we improve using the augmenting path of length at most T lying in $B(x, 3T)$ with the maximal sum of $c_{T,0}$-labels of the vertices and we repeat this as long as there are short augmenting paths. The number of vertices in $B(x, 3T)$ that are incident to edges of the matching increases in each step, hence we can make only a finite number of improvements in each ball. Since for all n the balls in $\{B(x, 3T) :\in V_{T,k(n)}\}$ are disjoint, $M_n(T)$ is a well defined matching for every n and T.

Let $M(T)$ be the edge-wise limit of $M_n(T)$ as $n \to \infty$. We claim that $M(T)$ is well defined and has no augmenting paths of length at most T. Indeed, an edge $e = \{x, y\}$ changes its status of being in the matching or not only if there is an improvement in $B(x, 3T)$. Such an improvement increase the number of vertices incident to edges of the matching in $B(x, 3T)$, which is bounded above by the number of vertices in the ball, thus the number of changes is bounded above as well. The lack of short augmenting paths follows trivially from the construction of $M(T)$.

We note that every factor of IID matching M of a unimodular random rooted graph (G, o) satisfies that (G, M, o) is unimodular, hence Lemma 3.3.7 implies the second statement of the theorem. □

Since we do not assume the existence of a uniform bound on the degrees, we need a lemma that plays the role of Lemma 4.1 of [40], which can be formalized in our language as follows.

Lemma 3.3.8. *[40, Lemma 4.1] Let (G, o) be a labeled unimodular random graph with law μ and degrees bounded above by d. Then for any n and any measurable event H, we have $\mu(H^n) < (d + 1)^n \mu(H)$, where $H^n := \{(\omega, x) : (\omega, o) \in H, dist_\omega(o, x) \leq n\}$.*

Our Lemma 3.3.9 extends Lemma 3.3.8 to unimodular graphs with a bound only on the expected degree, which is the case in the graph models examined in the thesis.

Lemma 3.3.9. *Let (G, o) be a labeled (directed) unimodular random graph with law μ and finite expected degree. Then for any $\varepsilon > 0$ and any n there is a δ such that if a measurable event H satisfies $\mu(H) < \delta$, then $\mu(H^n) < \varepsilon$, where $H^n := \{(\omega, x) : (\omega, o) \in H, dist_\omega(o, x) \leq n\}$.*

Proof. Fix ε and define $D = D(\varepsilon)$ to be the smallest positive integer that satisfies $\mathbb{E}\left(\mathbf{1}_{\{\deg o > D\}} \deg o\right) < \varepsilon/4$. We define the following mass transport: let $f(x, y, \omega) = 1$, if $(\omega, x) \in H, (\omega, y) \notin H, \{x, y\} \in E(\omega)$ (or in the directed case (x, y) or $(y, x) \in E(\omega)$), and let $f(x, y, \omega) = 0$ otherwise. Then by the Mass Transport Principle

$$\begin{aligned}
\mu(H^1 \setminus H) &\leq \int \sum_{x \in V(G)} f(x, o, \omega) d\mu(\omega, o) = \int \sum_{x \in V(G)} f(o, x, \omega) d\mu(\omega, o) \\
&\leq \mathbb{E}\left(\deg o \cdot \mathbf{1}_{\{o \in H\}}\right) \\
&\leq \mathbb{E}\left(D \cdot \mathbf{1}_{\{o \in H, \deg o \leq D\}}\right) + \mathbb{E}\left(\deg o \cdot \mathbf{1}_{\{o \in H, \deg o > D\}}\right) \\
&\leq D\mu(H) + \varepsilon/4,
\end{aligned}$$

which is less then $\varepsilon/2$ if $\mu(H) < \frac{\varepsilon}{4D(\varepsilon)} := \varepsilon_1$. It follows that $\mu(H^1) < \varepsilon$. We define recursively $\varepsilon_k := \frac{\varepsilon_{k-1}}{4D(\varepsilon_{k-1})}$ for $k \geq 2$. Then the same argument shows that if $\mu(H) < \varepsilon_n$, then $\mu(H^n) < \varepsilon$. □

Proof of Theorem 3.3.3. First we note that by Remark 3.1.2 and Proposition 2.1.4 it is enough to prove the theorem for non-directed graphs.

Denote the law of the limit graph (G, o) endowed with IID uniform labels $c(x)$ by μ. Fix T and let $\varepsilon_T > 0$ be such that if an event H satisfies $\mu(H) < \varepsilon_T$, then $\mu(H^{2T+1}) < 1/T$, as provided by Lemma 3.3.9. Let $M(T)$ be a matching as defined in Lemma 3.3.5.

We define the following events: let $\mathcal{X}_0 := \{\deg_{M(T)} o = 0\}$ and let $\mathcal{X}_{i,j}$ be the event that there is an edge $\{o, x\} \in M(T)$, such that x has the i^{th} largest label among the neighbors of o and o has the j^{th} largest label among the neighbors of x. Note that the above events are disjoint, $\mu\left(\mathcal{X}_0 \cup \left(\bigcup_{i,j} \mathcal{X}_{i,j}\right)\right) = 1$ and if $\{x, y\} \in M(T)$ then $(G, x) \in \mathcal{X}_{i,j}$ if and only if $(G, y) \in \mathcal{X}_{j,i}$. We can find constants $r = r(T)$ and $d = d(T)$ which satisfy the following: there are disjoint events $\mathcal{Y}_{i,j}, i, j \in [d]$ and $\mathcal{Y}_0 = \left(\cup_{i,j\in[d]}\mathcal{Y}_{i,j}\right)^c$ determined by the labeled neighborhood of radius r such that $\mu(H) < \varepsilon_T$ where $H := (\mathcal{Y}_0 \triangle \mathcal{X}_0) \cup \left(\bigcup_{i,j\leq d}(\mathcal{Y}_{i,j} \triangle \mathcal{X}_{i,j})\right) \cup \left(\bigcup_{\max\{i,j\}>d} \mathcal{X}_{i,j}\right)$, furthermore if $\deg_G o > d$, then $(G, o) \in \mathcal{Y}_0$. Denote by $\mathcal{B}(\mathcal{Y}_{i,j})$ the isomorphism types of neighborhoods of radius r which determine $\mathcal{Y}_{i,j}$.

Now we give all vertices of G_n uniform random [0,1] labels independently and denote the joint law of G_n and the labels by μ_n. We define the random matching $M_T(G_n)$ using the labels and the sets $\mathcal{B}(\mathcal{Y}_{i,j})$: let an edge $\{x, y\}$ be in $M_T(G_n)$ iff there is a pair (i, j) such that $B_{G_n}(x, r) \in \mathcal{B}(\mathcal{Y}_{i,j})$, y has the j^{th} largest label among the neighbors of x, and $B_{G_n}(y, r) \in \mathcal{B}(\mathcal{Y}_{j,i})$, x has the i^{th} largest label among the neighbors of y. The edge set $M_T(G_n)$ is a matching, because the events $\mathcal{B}(\mathcal{Y}_{i,j})$ are disjoint. We can define a matching $M_T(G)$ of G in the same way. Note, that $M_T(G)$ does not necessarily coincide with $M(T)$ but it satisfies $|\mu(o \in V(M(T))) - \mu(o \in V(M_T(G)))| < 2\varepsilon_T$ by the definition of $M_T(G)$. It follows by Lemma 3.3.7 that $\lim_{T\to\infty} \mu\big(o \in V(M_T(G))\big) = \lim_{T\to\infty} \mu\big(o \in V(M(T))\big) = m_E(G, o)$.

Denote by $\mathcal{Q}_T$ the event that there is an augmenting path for M_T of length less than T starting from the root. Let $Q_T(G_n)$ be the random set of vertices v of G_n such that $(G_n, v) \in \mathcal{Q}_T$ and let $q_T(G_n) := \frac{|Q_T(G_n)|}{|V(G_n)|}$. The event $(G_n, x) \in Q_T$ depends on $B_{G_n}(x, r+2T+1)$ by the definition of M_T. Furthermore, in the limiting graph G, an augmenting path of length less than T can start from o only if there is a vertex x on that path with $(G, x) \in H$, hence we have $\mathcal{Q}_T(G, o) \subseteq H^{2T+1}$. It follows from the convergence $G_n \to (G, o)$ that

$$\lim_{n\to\infty} \mathbb{E}(q_T(G_n)) = \lim_{n\to\infty} \mu_n(\mathcal{Q}_T(G_n, o)) \leq \mu(H^{2T+1}) < \frac{1}{T},$$

hence $\mathbb{E}(q_T(G_n)) < 2/T$ for n large enough. We have by [40, Lemma 2.1], that

$$\frac{|M_T(G_n)|}{|V(G_n)|} \leq m(G_n) \leq \frac{T+1}{T}\frac{|M_T(G_n)|}{|V(G_n)|} + q_T(G_n). \tag{3.3.2}$$

Taking expectation in (3.3.2) with respect to μ_n, we have for n large enough that

$$\begin{aligned}\mu_n\left(o \in V(M_T(G_n))\right) =& \mathbb{E}\left(\frac{|M_T(G_n)|}{|V(G_n)|}\right) \\ \leq \mathbb{E}(m(G_n)) \leq & \frac{T+1}{T}\mu_n\left(o \in V(M_T(G_n))\right) + \frac{2}{T},\end{aligned}$$

where o is a uniform random vertex of G_n. Since the event $\{o \in V(M_T(G_n))\}$ depends only on the $(r(T)+1)$-neighborhood of x, the convergence of the graph sequence implies $\lim_{n\to\infty}\mu_n(o \in V(M_T(G_n))) = \mu(o \in V(M_T(G)))$. It follows by letting $T \to \infty$ that $\mathbb{E}(m(G_n))$ converge to $\lim_{T\to\infty}\mu(o \in M_T(G)) = m_E(G, o)$. □

3.3.2 Almost sure convergence of the directed matching ratio

In this section, we examine the most widely used network models and the oriented versions of them: random d-regular graphs, Erdős–Rényi random graphs, the random configuration model and preferential attachment graphs. As referred in Section 2.2, each model has a local weak limit, hence Theorem 3.3.3 shows that the expected values of the directed matching ratios converge. The results of Section 2.3 imply that for these graph models, the almost sure convergence of the matching ratio holds as well. In this section, we list these consequences.

Theorem 3.3.10 (Almost sure convergence of the matching ratio)**.** *1) Let G_n be a sequence of undirected finite graphs defined on a common probability space that converges almost surely in the local weak sense and let $\overrightarrow{G_n}$ be a sequence of random directed graphs obtained from G_n by giving each edge a random orientation independently. Then $m(\overrightarrow{G_n})$ converges almost surely to the constant* $\lim_{n\to\infty}\mathbb{E}(m(\overrightarrow{G_n}))$.

2) Let G_n be the sequence of random directed graphs given by the preferential attachment rule. Then $m(G_n)$ converges almost surely to the constant $\lim_{n\to\infty}\mathbb{E}(m(G_n))$.

Proof. The result of Theorem 3.3.3 on the convergence of the expected value of the directed matching ratio and Lemma 2.3.6 on the almost sure convergence of an almost sure convergent undirected sequence with independently oriented edges imply Part 1).

The second part of the theorem follows from Theorem 3.3.3 and Theorem 2.3.8 on the almost sure local weak convergence of preferential attachment graphs. □

The directed versions of the sequences of random d-regular graphs, Erdős–Rényi random graphs or sequences given by the random configuration model are obtained by orienting the edges of the non-directed versions independently. Since these graph sequences are known to converge almost surely in the undirected case (see Examples 2.3.5 and 2.3.4), it follows by Part 1) of Theorem 3.3.10 that their directed matching ratios converge almost surely. By our Proposition 2.1.4 and [26, Theorem 2] on the limit of the matching ratio of convergent graph sequences, one can compute the value of the limit of the directed matching ratio when the limit is a unimodular Galton–Watson tree. In Corollaries 3.3.12 and 3.3.13 we also present the results given by this argument.

Corollary 3.3.11 (Almost sure convergence of the directed matching ratio of the random configuration model)**.** *Let $\vec{G}_n$ be a sequence of random directed graphs given by the random configuration model with degree distribution ξ satisfying $\mathbb{E}(\xi^p) < \infty$ for some $p > 2$. Then $\vec{G}_n$ converge almost surely in the local weak sense to $\overrightarrow{UGW}(\xi)$ and $m(\vec{G}_n)$ converges almost surely to $m_E(\overrightarrow{UGW}(\xi))$.*

The sequence of random directed d-regular graphs is a special case of the random configuration model (with degree distribution ξ being constant d). The connected component of the root $\bar{\bar{o}}$ of the bipartite representation $\bar{\bar{\mathbb{T}}}_d$ has law $UGW(Binom(d, 1/2))$, hence we have the following:

Corollary 3.3.12 (Almost sure convergence of the directed matching ratios of directed random regular graphs)**.** *Let $\vec{G}_n$ be the sequence of random d-regular graphs on n vertices with randomly oriented edges. Then the matching ratios converge almost surely to the constant*

$$\lim_{n\to\infty} m(\vec{G}_n) = m_E\big(UGW(Binom(d, 1/2))\big).$$

For directed Erdős–Rényi graphs one can compute the exact value of the almost sure limit of the matching ratio, using the results of [57] or [26, Theorem 2].

Corollary 3.3.13. *Let $\vec{\mathcal{G}}_{n,2c/n}$ be a sequence of directed Erdős–Rényi graphs with parameter $2c$. Then almost surely*

$$\lim_{n\to\infty} m(\mathcal{G}_{n,2c/n}) = 1 - \frac{t_c + e^{-ct_c} + ct_c e^{-ct_c}}{2} \tag{3.3.3}$$

where $t_c \in (0, 1)$ is the smallest root of $t = e^{-ce^{-ct}}$.

Proof. According to Example 2.3.4 and Lemma 2.3.6, the sequence of directed Erdős–Rényi random graphs converge almost surely in the local weak sense to $\overrightarrow{UGW}(Poi(2c))$, and hence $\lim_{n\to\infty} m(\vec{\mathcal{G}}_{n,2c/n}) = m_E(\overrightarrow{UGW}(Poi(2c)))$, where $Poi(2c)$ denotes the Poisson distribution with parameter $2c$. The connected component of the root in the bipartite representation of $\overrightarrow{UGW}(Poi(2c))$ has law $UGW(Poi(c))$, which is the almost sure local weak limit of the non-directed Erdős–Rényi random graphs $\mathcal{G}_{n,c/n}$ with parameter c. It is known (see [57] or [26, Theorem 2]), that for this graph sequence $\lim_{n\to\infty} m(\mathcal{G}_{n,c/n})$ equals the right hand side of (3.3.3) almost surely. By Remark 3.3.2 we have

$$\lim_{n\to\infty} m(\mathcal{G}_{n,c/n}) = m_E(UGW(Poi(c))) = m_E(\overrightarrow{UGW}(Poi(2c))).$$

This proves (3.3.3). □

Chapter 4

Percolation critical probabilities and unimodular random graphs

4.1 Introduction

The notion of local weak convergence was introduced for sequences of *finite* graphs. However, the definition also applies for sequences of *infinite* random rooted graphs and the same question arises naturally: do certain parameters of infinite graphs converge along local weak convergent sequences? In this chapter, we examine critical parameters related to percolation, originally defined for deterministic infinite graphs. The results of this chapter have been published in Beringer, Pete and Timár [20].

4.1.1 Motivation and results

There are several definitions of the critical probability for percolation on the lattices $\mathbb{Z}^d$, which have turned out to be equivalent not only on $\mathbb{Z}^d$, but also in the more general context of arbitrary transitive graphs [70, 2, 47, 10, 35]. One of our goals is to investigate the relationship between these different definitions when the graph G is an extremal unimodular random graph [17, 4], which is the natural extension of transitivity to the disordered setting. We examine the generalizations of $p_c = \sup\{p : \mathbb{P}_p(\text{there is an infinite cluster}) = 0\}$, $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$ and $\tilde{p}_c$ defined by Duminil-Copin and Tassion in [35]. The last quantity was in fact designed to give a simple new proof of $p_c = p_T$ for transitive graphs, and to address the question of locality of critical percolation: whether the value of p_c depends only on the local structure of the graph.

More precisely, Schramm's "locality conjecture", stated first explicitly in [16], says the continuity of percolation critical probability in the class of transitive graphs:

Conjecture 4.1.1 (Schramm)**.** *If G_n is a sequence of vertex-transitive infinite graphs such that G_n converges locally to G and $\sup_n p_c(G_n) < 1$ then $p_c(G_n) \to p_c(G)$ holds.*

Typically, however, the natural setting for such locality statements is not the class of transitive graphs, but the class of unimodular random graphs. Indeed, there are several interesting probabilistic quantities, most often related in some way to random walks, which have turned out to possess locality, mostly in the generality of unimodular random graphs: see [17, 65, 67, 31, 12, 49] for specific examples, and [75, Chapter 14] for a partial overview. Therefore, it is natural to investigate Schramm's conjecture in the setup of unimodular random graphs and see what the proper notion of critical probability may be from the point of view of locality.

The conjecture has been proven for some special transitive graphs: Grimmett and Marstrand [46] proved that $p_c\left(\mathbb{Z}^2 \times \{-n, \dots, n\}^{d-2}\right) \xrightarrow{n\to\infty} p_c(\mathbb{Z}^d)$. Benjamini, Nachmias and Peres [16] verified that the convergence holds if (G_n) is a sequence of d-regular graphs with large girth and Cheeger constants uniformly bounded away from 0. Martineau and Tassion [69] proved that the convergence holds if (G_n) is a sequence of Cayley graphs of Abelian groups converging to a Cayley graph G of an Abelian group. Hutchcroft [55] showed that the statement is true for graph sequences with a uniform exponential lower bound on their volume growth.

The lower semicontinuity of the critical probability, i.e., the inequality

$$\liminf_{n\to\infty} p_c(G_n) \geq p_c(G)$$

is known for any convergent sequence of transitive graphs; see [75, Section 14.2], and [35].

In Subsection 4.1.2, we define the generalized critical probabilities p_c, p_T, $\tilde{p}_c$, p_T^a, and $\tilde{p}_c^a$ for unimodular random graphs; somewhat simplistically saying, the first three will be quenched versions of the quantities mentioned above, while the last two will be annealed versions. In Section 4.2, we examine the relationship between these different generalizations. Our results are summarized in Table 4.1. The one sentence summary is that $p_c = \tilde{p}_c$ always holds, but otherwise almost anything can happen, unless the random graph satisfies some very strong uniformity conditions; one that we call "uniformly good" suffices for most purposes.

In Section 4.3, we investigate the extension of Schramm's conjecture for unimodular random graphs:

Question 4.1.2. *Does $p_c(G_n)$ converge to $p_c(G)$ if G_n is a sequence of unimodular random graphs, $G_n \to G$ in the local weak sense and* $\sup p_c(G_n) < 1$*?*

First we note (Example 4.3.2) that locality holds for unimodular Galton-Watson trees with bounded degrees, but not in general; this shows that it is natural to restrict one's attention to bounded degree unimodular random graphs. In Subsection 4.3.2, we give conditions which imply $\lim p_c(G_n) = p_c(G)$. In Subsection 4.3.3, we show by examples that there are sequences of unimodular random graphs such that $G_n \to G$ but $p_c(G) > \lim p_c(G_n)$ or $p_c(G) < \lim p_c(G_n) < 1$. These examples indicate a negative answer to Question 4.1.2, i.e., Schramm's conjecture does not hold in the generality of unimodular random graphs, although many such statements, formulated originally for transitive graphs, extend to this class. A recent result of Angel and Hutchcroft [8] provides counterexamples from the class of unimodular random graphs for two further classical conjectures related to percolation.

A corollary to our positive results is that if G is a transitive graph of sub-exponential volume growth, then there exists a sequence of invariant bi-Lipschitz spanning subgraphs G_n such that $p_c(G_n) \to 1$. As we will explain in Section 4.4, this is a strengthening of the simple fact that groups of sub-exponential growth have cost 1, as defined in [59], studied further in [43, 44]. We do not know if this strengthening holds for all groups of cost 1, which class includes, besides all amenable groups, direct products $\mathbb{G} \times \mathbb{Z}$ for any group $\mathbb{G}$, and $SL(d, \mathbb{Z})$ with $d \geq 3$. A related question is whether every amenable transitive graph has an invariant random Hamiltonian path. This is the invariant infinite version of what is known as Lovász' conjecture, namely, that every finite transitive graph has a Hamiltonian path, even though he has not conjectured a positive answer. The best general results seem to be [11] and [74].

4.1.2 Percolation and critical probabilities

In this section, we define the basic notions related to percolation theory that we will need in this chapter.

The study of percolation started with the work of Broadbent and Hammersley in 1957 [30] who introduced percolation as a probabilistic model for the flow of a fluid through a porus medium. Classical percolation theory examines percolation on transitive graph, especially in lattices, such as $\mathbb{Z}^d$; see the books of Kesten [58] and Grimmett [47] for an introduction to percolation theory. The most basic definition is the following:

Definition 4.1.3 (Percolation on a graph)**.** *Let G be a graph with countably many vertices and edges and fix a parameter $p \in [0, 1]$.*

Bernoulli bond percolation *with parameter p is a random subgraph ω of G with vertex set $V(\omega) = V(G)$ and a random edge set $E(\omega)$ given by the following distribution: each edge $e \in E(G)$ is present in $E(\omega)$ with probability p, and is in $E(G) \setminus E(\omega)$ with probability $1 - p$, independently for the edges.*

Bernoulli site percolation *with parameter p is a random subgraph ω of G induced by a random vertex set given by the following distribution: each vertex is in ω with probability p and is not in ω with probability $1 - p$, independently for the vertices.*

Site percolation is in some sense more general than bond percolation: bond percolation on a graph G can be modeled by site percolation of the *line graph* of G, which is the graph with vertex set $E(G)$ where two vertices are adjacent iff the corresponding edges share a vertex. The converse does not hold, there are graphs such that site percolation on them is not equivalent to any bond percolation. However, many statements for bond percolation hold also for site percolation.

The name Bernoulli indicates that the edges or vertices are independently removed. The definition can be generalized in many ways: the probability of keeping a vertex or edge can be different for some set of vertices or edges, or the condition of independence can be substituted with some kind of dependence. Such more general models are the oriented percolation introduced by Broadbent and Hammersley [30], the random cluster model [42, 48] and the Ising model.

For simplicity, we will consider only *bond* percolation processes on unimodular random graphs. We refer to Bernoulli bond percolation with parameter p as **Bernoulli(p) percolation** for short. For a fixed configuration ω of the random graph G let $\mathbb{P}_p^\omega$ be the probability measure obtained by the Bernoulli(p) bond percolation on ω and let $\mathbb{E}_p^\omega$ be the expectation with respect to $\mathbb{P}_p^\omega$. The percolation **cluster** (i.e., the connected component) of the root o will be $\mathcal{C}_o$.

A fundamental and long studied question in percolation theory is the value of the critical probabilities $p_c = \sup\{p : \mathbb{P}_p(|\mathcal{C}_o| = \infty) = 0\}$ first defined by Hammersley and $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$ introduced by Temperley. These quantities have natural generalizations to *extremal* unimodular random graphs. Let (G, o) be an extremal unimodular random graph. In this case, the critical probability $p_c(\omega)$ of an instance of (G, o) is almost surely a constant and the same holds for p_T (see [4], Section 6.). Hence one can define

$$\begin{aligned} p_c &= \inf\left\{p : \mu\left(\mathbb{P}_p^\omega\left(|\mathcal{C}_o| = \infty\right) > 0\right) = 1\right\} \\ &= \sup\left\{p : \mu\left(\mathbb{P}_p^\omega\left(|\mathcal{C}_o| = \infty\right) = 0\right) = 1\right\} \end{aligned}$$

and

$$\begin{aligned} p_T &= \sup\left\{p : \mu\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) < \infty\right) = 1\right\} \\ &= \inf\left\{p : \mu\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) = \infty\right) = 1\right\}. \end{aligned}$$

It may happen that although $\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) < \infty$ for μ-almost every ω, the expectation of these quantities with respect to μ is infinite. This provides a second natural extension of p_T to unimodular random graphs concerning the average size of $\mathcal{C}_o$:

$$\begin{aligned} p_T^a &= \sup\left\{p : \mathbb{E}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) < \infty\right\} \\ &= \inf\left\{p : \mathbb{E}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) = \infty\right\}. \end{aligned}$$

It follows from the definitions that $p_c \geq p_T \geq p_T^a$. It is known that $p_c = p_T$ in the case of transitive graphs; see [70, 2, 10, 35]. For unimodular random graphs (even with sub-exponential volume growth), the three critical probabilities can differ; we will present such graphs in Examples 4.2.10 and 4.2.11.

Duminil-Copin and Tassion [35] introduced the following local quantity for transitive graphs: let (G, o) be a rooted graph, $S \in \mathcal{S}(G)$ be a finite subgraph containing the root, and define

$$\phi_p(S) := \sum_{e \in \partial_E S} p\,\mathbb{P}_p(o \overset{S}{\longleftrightarrow} e^-)\,,$$

the expected number of open edges on the boundary of S such that there is an open path from o to e^- in S. Then, they defined the critical probability

$$\begin{aligned} \tilde{p}_c &:= \sup\{p : \text{there is an } S \in \mathcal{S}(G) \text{ s.t. } \phi_p(S) < 1\} \\ &= \inf\{p : \phi_p(S) \geq 1 \text{ for all } S \in \mathcal{S}(G)\}\,. \end{aligned} \tag{4.1.1}$$

They proved that transitive graphs satisfy $p_c = \tilde{p}_c$.

How to generalize this definition to unimodular random graphs is not a priori clear. The simplest way to define a similar critical probability seems to be a quenched version: find a suitable $S_\omega \in \mathcal{S}(\omega)$ for almost every configuration ω. For a subgraph $S \in \mathcal{S}(\omega)$ denote by

$$\phi_p^\omega(S) := \sum_{e \in \partial_E S} p\,\mathbb{P}_p^\omega\left(o \overset{\omega,p}{\underset{S}{\longleftrightarrow}} e^-\right)$$

the expected number of open edges on the boundary of S in ω such that there is an open path from o to e^- in the percolation on ω with parameter p. Then let

$$\tilde{p}_c := \sup\left\{p : \mu\left(\left\{\omega : \exists S_\omega \in \mathcal{S}(\omega) \text{ s.t. } \phi_p^\omega(S_\omega) < 1\right\}\right) = 1\right\}. \tag{4.1.2}$$

Remark 4.1.4. *Suppose p satisfies the following: for almost every ω there is an $S_\omega \in \mathcal{S}(\omega)$ with $\phi_p^\omega(S_\omega) < c$. Then unimodularity implies [4, Lemma 2.3.] that for almost every ω and every vertex x there is some finite connected set $S_{\omega,x} \ni x$ such that*

$$\phi_p^{\omega,x}(S_{\omega,x}) := p \sum_{e \in \partial_E S_{\omega,x}} \mathbb{P}\left(x \overset{\omega,p}{\underset{S_{\omega,x}}{\longleftrightarrow}} e^-\right) < c.$$

In the original definition of $\tilde{p}_c$ (equation (4.1.1)), there is no control on what the set S could be, which makes the definition rather ineffective. This becomes particularly problematic in the random graph case (equation (4.1.2)), where a bad neighborhood of o may force S_ω to be huge and hard to find. However, it will follow from our Lemma 4.2.3 that, for transitive graphs, the existence of an S with $\phi_p(S) < 1$ is equivalent to the existence of a positive integer r with $\phi_p(B(o,r)) < 1$. This provides a second natural extension of the definition of $\tilde{p}_c$ to the random case: we consider the ball of radius r in the random graph ω and we take the expectation of $\phi_p^\omega(B_\omega(o,r))$ with respect to μ. Then the following critical probability is another extension of the definition of $\tilde{p}_c$, an annealed version of $\tilde{p}_c$:

$$\tilde{p}_c^a := \sup\{p : \exists r \text{ such that } \mathbb{E}\left(\phi_p^\omega(B_\omega(o,r))\right) < 1\}.$$

4.2 Relationship of the critical probabilities of unimodular random graphs

We start by proving in Theorem 4.2.1 that all bounded degree unimodular graphs satisfy $p_c = \tilde{p}_c$. This will be useful in many of our later results.

In the transitive case, the quantity $\phi_p(S)$ in the definition of $\tilde{p}_c$ can be used to give a short proof (see [35]) of Menshikov's theorem [70]: if Γ is a transitive graph and $p < p_c(\Gamma)$, then there exist a $\varphi(p)$ such that

$$\mathbb{P}_p\left(o \leftrightarrow B(o,r)^c\right) \leq e^{-\varphi(p) r}. \tag{4.2.1}$$

If a graph satisfies this exponential decay for each $p < p_c$ and has sub-exponential volume growth, then it is easy to see that $p_T = p_c$. In Lemma 4.2.3, we give a condition for unimodular random graphs that implies (4.2.1), and we prove in Corollary 4.2.6 that this condition implies $p_c = p_T = p_T^a$ if the graph has uniform sub-exponential volume growth. However, in Examples 4.2.10 and 4.2.11 we present unimodular random graphs with uniform polynomial volume growth and $p_T < p_c$ and $p_T^a < p_T$, respectively. This shows that Menshikov's theorem is not true in the generality of unimodular graphs.

The results of this section are summarized in the following table:

$\tilde{p}_c = p_c$	bounded degree
$p_c \geq p_T \geq p_T^a$	always
$p_c = p_T^a$	bounded degree uniformly good with sub-exp. growth
$p_c > p_T$	Example 4.2.10, with polynomial growth
$p_T > p_T^a$	Example 4.2.11, with polynomial growth
$p_c \leq \tilde{p}_c^a$	bounded degree uniformly good
$p_c < \tilde{p}_c^a$	Example 4.2.7, bounded degree uniformly good
$p_c > \tilde{p}_c^a$	Example 4.2.8, not uniformly good

Table 4.1: Relationship of the critical probabilities

4.2.1 Positive results

Our first result is indispensable to the rest of the chapter. The second part of the proof is a slight modification of the proof in [35] for our setting, while the first part depends on new ideas. The main difficulty is that we cannot find isomorphic sets $S_{\omega,x}$ for different vertices x, and hence we cannot bound $\mathbb{P}_p\left(o \leftrightarrow B(o,r)^c\right)$ in terms of r. We build instead a tree T^ω using the sets $S_{\omega,x}$, and bound the probability that the subtree given by the percolation survives. The survival of that subtree is equivalent to the infinite size of the cluster of the root in the percolation on G.

Theorem 4.2.1. *If G is a bounded degree unimodular random rooted graph, then $p_c(G) = \tilde{p}_c(G)$.*

Proof. We prove first that $\tilde{p}_c \leq p_c$. Fixing $p < \tilde{p}_c$, we will show that $p \leq p_c$. We claim that there exists a constant $c = c(p) < 1$ such that we can find for almost every ω a set $S_\omega \in \mathcal{S}(\omega)$ that satisfies $\phi_p^\omega(S_\omega) \leq c$. Let $p' := \frac{p+\tilde{p}_c}{2} < \tilde{p}_c$. Let $S_\omega \in \mathcal{S}(\omega)$ be such that $\phi_{p'}^\omega(S_\omega) < 1$. The sets S_ω satisfy

$$\begin{aligned}\phi_p^\omega(S_\omega) = \sum_{e\in\partial_E S_\omega} p\mathbb{P}_p^\omega(o \leftrightarrow e^-) &\leq \frac{p}{p'} \sum_{e\in\partial_E S_\omega} p'\mathbb{P}_{p'}^\omega(o \leftrightarrow e^-)\\ &= \frac{p}{p'}\,\phi_{p'}^\omega(S_\omega) \leq \frac{p}{p'} =: c\,.\end{aligned}$$

Recall the definition of $\phi_p^{\omega,x}(S_{\omega,x})$ from Remark 4.1.4. Unimodularity implies that almost every ω satisfies the following: for each $x \in \omega$ there is a set $S_{\omega,x}$ containing x such that $\phi_p^{\omega,x}(S_{\omega,x}) \leq c$. Fix such an $S_{\omega,x}$ in an arbitrary measurable way.

Fix ω and denote by T^ω the following recursively defined tree: the vertices of the tree are finite sequences of vertices of ω. The root of the tree is (o). If $(x_0, x_1, \dots, x_k)$ is a vertex of T^ω, its children are the sequences $(x_0, x_1, \dots, x_k, x_{k+1})$ such that for all $j = 1, \dots, k+1$, we have $x_j \in \partial_V^{\text{out}} S_{\omega, x_{j-1}}$, and there exist vertices $x'_j \in \partial_V^{\text{in}} S_{\omega, x_{j-1}}$ such that $x'_j \sim x_j$, with paths from x_{j-1} to x'_j in $S_{\omega, x_{j-1}}$ that are disjoint from each other and from the edges $\{x'_j, x_j\}$, as $j = 1, \dots, k+1$. We say that the union of the above paths and edges is a *good path* through $x_0, x_1, \dots, x_k, x_{k+1}$. See Figure 4.1. Denote by $L_n := \{(x_0, x_1, \dots, x_n) \in T^\omega\}$ the vertex set of T^ω on the nth level.

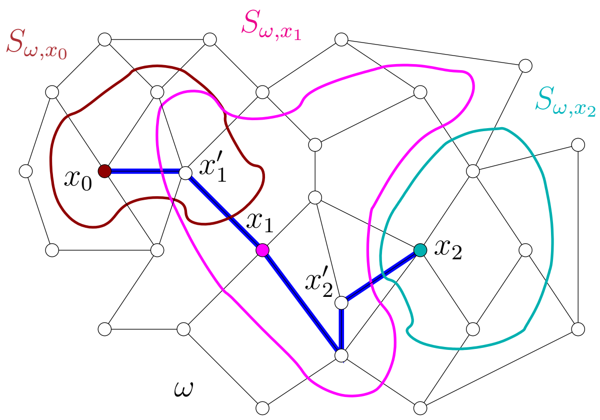

Figure 4.1: A "good path" that gives the vertex (x_0, x_1, x_2) of T^ω.

Let $T^\omega(p)$ be the random subtree of T^ω defined in a similar way using the same sets $S_{\omega,x}$ but allowing only good paths that are open in Bernoulli(p) percolation on ω. It is easy to check that in fact $T^\omega(p) \subseteq T^\omega$. Denote by $L_n(p)$ the set of vertices of $T^\omega(p)$ in the nth level. A self-avoiding infinite ray inside the p-percolation configuration gives rise to a growing sequence of good paths in the percolated ω, therefore if the cluster of the origin in the p-percolation on ω is infinite, then there is an infinite path in $T^\omega(p)$. Conversely, an infinite path in $T^\omega(p)$ corresponds to an infinite growing sequence of open good paths in the p-percolated ω, which are necessarily parts of an infinite component containing the origin.

We claim that for almost every ω the expected number of vertices in $L_n(p)$ converges to 0 as $n \to \infty$. More precisely, the expectation of the number of vertices in $L_n(p)$ decreases exponentially in n. In the first two inequalities we use the notation $\Box$ for the occurrence of events on disjoint edge sets and we apply the BK inequality ([47], Theorem 2.12). We denote the event $\left\{x_0 \overset{\omega,p}{\underset{B}{\longleftrightarrow}} x_k \text{ by a good path through } x_0, x_1, \dots, x_k\right\}$

by $\left\{x_0 \overset{\omega,p}{\underset{B,(x_0,x_1,\ldots,x_k)}{\longleftrightarrow}} x_k\right\}$.

$$\mathbb{E}^\omega\big(|L_n(p)|\big) = \sum_{(x_0,\ldots,x_n)\in L_n} \mathbb{P}^\omega\left(x_0 \overset{\omega,p}{\underset{B,(x_0,\ldots,x_n)}{\longleftrightarrow}} x_n\right)$$

$$\leq \sum_{\substack{(x_0,\ldots,x_{n-1})\in L_{n-1}\\ e\in\partial_E S_{\omega,x_{n-1}}}} \mathbb{P}^\omega\left(\left\{x_0 \overset{\omega,p}{\underset{B,(x_0,\ldots,x_{n-1})}{\longleftrightarrow}} x_{n-1}\right\}\Box\{e \text{ is open}\}\Box\left\{x_{n-1} \overset{\omega,p}{\underset{S_{\omega,x_{n-1}}}{\longleftrightarrow}} e^-\right\}\right)$$

$$\leq \sum_{\substack{(x_0,\ldots,x_{n-1})\in L_{n-1}\\ e\in\partial_E S_{\omega,x_{n-1}}}} \mathbb{P}^\omega\left(x_0 \overset{\omega,p}{\underset{B,(x_0,\ldots,x_{n-1})}{\longleftrightarrow}} x_{n-1}\right) p\,\mathbb{P}^\omega\left(x_{n-1} \overset{\omega,p}{\underset{S_{\omega,x_{n-1}}}{\longleftrightarrow}} e^-\right)$$

$$= \sum_{(x_0,\ldots,x_{n-1})\in L_{n-1}} \mathbb{P}^\omega\left(x_0 \overset{\omega,p}{\underset{B,(x_0,\ldots,x_{n-1})}{\longleftrightarrow}} x_{n-1}\right) \phi_p^{\omega,x_{n-1}}(S_{\omega,x_{n-1}}) \leq \mathbb{E}^\omega\left(|L_{n-1}(p)|\right) c\,.$$

It follows by induction that $\mathbb{E}^\omega\big(|L_n(p)|\big) \leq c^n$. Therefore,

$$\mathbb{P}^\omega(|\mathcal{C}_o| = \infty) = \mathbb{P}^\omega(T^\omega(p) \text{ survives}) = \lim_{n\to\infty} \mathbb{P}^\omega(|L_n(p)| \geq 1) \leq \lim_{n\to\infty} \mathbb{E}^\omega|L_n(p)| = 0\,,$$

hence $p \leq p_c$.

Next we prove that $\tilde{p}_c \geq p_c$. Let

$$q(p) := \mu(\{\omega : \phi_p^\omega(S) \geq 1 \text{ for all } S \in \mathcal{S}(\omega)\}),$$

Note that $q(p)$ is non-decreasing in p, and $q(p) > 0$ for every $p > \tilde{p}_c$ by the definition of $\tilde{p}_c$.

Fix ω and let $H \in \mathcal{S}(\omega)$ be fixed. We will use Lemma 1.4. of [35]:

$$\frac{d}{dp}\mathbb{P}_p^\omega\left(o \overset{\omega,p}{\longleftrightarrow} H^c\right) \geq \left(1 - \mathbb{P}_p^\omega\left(o \overset{H,p}{\longleftrightarrow} H^c\right)\right) \inf_{S:o\in S\subseteq H} \phi_p^H(S) \geq C(p) \inf_{S:o\in S\subseteq H} \phi_p^H(S)\,,$$

where $C(p) = (1-p)^D \leq 1 - \mathbb{P}^\omega\left(o \overset{\omega,p}{\longleftrightarrow} H^c\right)$ for every ω and H, with D being the almost sure bound on the degree of the graph G. The probabilities above depend only on the structure of ω in $K = H \cup \partial_V^{\text{out}} H$, hence we can use the above inequality to estimate the derivative of the probability $\mu\left(o \overset{\omega,p}{\longleftrightarrow} B^\omega(o,r)^c\right)$, as follows. Consider the following sets of finite rooted graphs: let $\mathcal{H}_r$ be the set of possible $(r+1)$-neighborhoods of the graphs with degree at most D, i.e.

$$\mathcal{H}_r := \{(K,o) : \text{dist}_K(o,x) \leq r+1 \text{ and } \deg_K(x) \leq D\,,\ \text{for all } x \in V(K)\}\,,$$

and let

$$\mathcal{H}_r(p) := \left\{(K,o) \in \mathcal{H}_r : \phi_p^K(S) \geq 1\,,\ \text{for all } S \in \mathcal{S}(B_K(o,r))\right\}.$$

Note that

$$\sum_{K\in\mathcal{H}_r(p)} \mu\left(\{\omega : B_\omega(o,r+1) = K\}\right) =$$

$$\mu(\{\omega : \phi_p^\omega(S) \geq 1 \text{ for all } S \in \mathcal{S}(B_\omega(o,r))\}) \geq q(p),$$

hence we have

$$\begin{aligned}\frac{d}{dp}\mu\left(o \overset{\omega,p}{\longleftrightarrow} B(o,r)^c\right) &= \sum_{(K,o)\in\mathcal{H}_r} \mu\big(B_\omega(o,r+1)=K\big)\frac{d}{dp}\mathbb{P}_p\left(o \overset{K,p}{\longleftrightarrow} B_K(o,r)^c\right)\\ &\geq \sum_{(K,o)\in\mathcal{H}_r(p)} \mu\big(B_\omega(o,r+1)=K\big)C(p)\inf_{S:o\in S\subseteq B_K(o,r)}\phi_p^K(S)\\ &\geq q(p)\,C(p).\end{aligned}$$

Integrate the above inequality on the interval $\left[\frac{p+\tilde{p}_c}{2},p\right]$. Using the monotonicity of $q(p)$ and $C(p)$, we get

$$\mu\left(o \overset{\omega,p}{\longleftrightarrow} B(o,r)^c\right) \geq \frac{p-\tilde{p}_c}{2}\,q\left(\frac{p+\tilde{p}_c}{2}\right)C(p).$$

This gives a positive lower bound that is uniform in r. Thus $\mu\left(o \overset{\omega,p}{\longleftrightarrow} \infty\right) > 0$, and $p \geq p_c$. □

One advantage of the definition of $\tilde{p}_c$ for transitive graphs is that it enables one to check whether a certain p is under $\tilde{p}_c$ using a finite witness. This characteristic makes the next definition natural.

Definition 4.2.2. *We say that a bounded degree unimodular random graph G is* ***uniformly good*** *if for any $p < p_c$ there exists a positive integer $r(p)$ such that $\mu_G(\{\omega : \exists S_\omega \subseteq B_\omega(o,r(p)), o\in S_\omega \text{ s.t. } \phi_p^\omega(S_\omega) < 1\}) = 1$.*

This class of graphs includes unimodular quasi-transitive graphs (obvious) and unimodular random trees of uniform sub-exponential growth (see Definition 4.2.5 and the proof of Proposition 4.3.6). Furthermore, uniformly good unimodular graphs satisfy the following exponential decay of $\phi_p(B_\omega(o,r))$ in r.

Lemma 4.2.3. *Let G be a bounded degree unimodular random graph. G is uniformly good if and only if for all $p < p_c$ there are constants $c = c(p) < 1$ and $R(p)$ such that if $r \geq R(p)$, then $\phi_p^\omega(B) \leq c^r$ for almost every ω and every finite $B \supseteq B_\omega(o,r)$.*

For the proof of Lemma 4.2.3 we use the same tree T^ω as in the proof of Theorem 4.2.1. The uniformly good property implies a uniform linear lower bound in r on the distance of the root from any vertex of T^ω that corresponds to a boundary point of B (namely the points of the set π defined in the proof). This property and the boundedness of the size of the sets $S_{\omega,x}$ allows us to prove the estimate of the lemma.

Proof. If the constants $c(p)$ and $R(p)$ exist, then the sets $S_\omega := B_\omega(o,R(p))$ indicate that G is uniformly good.

To prove the other direction, assume that G is uniformly good, and fix $p < p_c$. We can show as in the proof of Theorem 4.2.1 that there exists a constant $c_0 < 1$ and a positive integer r_0 such that for almost every ω and every $x \in \omega$ there exists a finite connected set $S_{\omega,x} \subseteq B_\omega(x, r_0)$ containing x that satisfies $\phi_p^{\omega,x}(S_{\omega,x}) \leq c_0$. Fix an ω and the sets $S_{\omega,x}$ as above, a positive integer r and a finite set $B \supseteq B_\omega(o,r)$. We define the trees T^ω and $T^\omega(p)$ as in the proof of Theorem 4.2.1. On every directed path in T^ω from o to infinity there is a first vertex $(x_0, \ldots, x_k)$ such that $x_k \notin B$. Let π be the set of these vertices, i.e.

$$\pi := \{(x_0, \ldots, x_k) \in T^\omega : x_0, \ldots, x_{k-1} \in B, x_k \notin B\}.$$

Note that π is a minimal set in T^ω that separates o from infinity, hence every non-backtracking infinite path from o has exactly one vertex in π. An argument as in the first part of the proof of Theorem 4.2.1 shows that

$$\begin{aligned}\mathbb{E}^\omega\left(|\pi \cap T^\omega(p)|\right) &= \sum_{(x_0,\ldots,x_k)\in\pi} \mathbb{P}^\omega\Big(x_0 \xleftrightarrow[B,(x_0,\ldots,x_k)]{\omega,p} x_k\Big) \\ &\leq \sum_{(x_0,\ldots,x_k)\in\pi} \sum_{(x'_1,\ldots,x'_k)} \prod_{j=1}^{k} \mathbb{P}^\omega\Big(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\Big)p =: F(\pi,p),\end{aligned}$$

where $(x'_1, \ldots, x'_k)$ denotes a sequence of vertices in ω such that $x'_j \in S_{\omega,x_{j-1}}$ and $x'_j \sim x_j$ for any $j = 1, \ldots, k$. First we bound $\phi_p^\omega(B)$ in terms of $F(\pi,p)$ using the uniform bound on the size of the sets $S_{\omega,x}$, then we prove a geometric bound on $F(\pi,p)$ using a linear bound in r on the distance of o and π in T^ω. These two estimates will imply the statement of the lemma.

Denote by $\bar\pi$ the set of the parents of the vertices in π, i.e.

$$\bar\pi := \{(x_0, \ldots, x_k) \in T^\omega : x_0, \ldots, x_k \in B, \exists x_{k+1} \notin B, (x_0, \ldots, x_{k+1}) \in T^\omega\}.$$

If for some $e \in \partial_E B$ the event $\big\{o \xleftrightarrow[B]{\omega,p} e^-\big\}$ occurs, then there is some $(x_0, \ldots, x_k) \in \bar\pi$ such that there is a good path through $x_0, \ldots, x_k$ in the percolation and a disjoint path from x_k to e^- in S_{ω,x_k}. For any fixed $(x_0, \ldots, x_k)$ the number of edges in $\partial_E B \cap (E(S_{\omega,x_k}) \cup \partial_E S_{\omega,x_k})$ is bounded above by $|E(S_{\omega,x_k}) \cup \partial_E S_{\omega,x_k}| \leq D^{r_0+1}$ where D is the almost sure bound on the degree of the graph G. We have

$$\begin{aligned}\phi_p^\omega(B) &= p \sum_{e\in\partial_E B} \mathbb{P}^\omega\Big(o \xleftrightarrow[B]{\omega,p} e^-\Big) \\ &\leq \sum_{e\in\partial_E B} \sum_{\substack{(x_0,\ldots,x_k)\in\bar\pi \\ (x'_0,\ldots,x'_k)}} \mathbb{P}_p^\omega\left(\Big\{x_0 \xleftrightarrow[B,(x_0,\ldots,x_k)]{\omega,p} x_k\Big\}\Box\Big\{x_k \xleftrightarrow[S_{\omega,x_k}]{\omega,p} e^-\Big\}\right) \\ &\leq \sum_{\substack{(x_0,\ldots,x_k)\in\bar\pi \\ (x'_0,\ldots,x'_k)}} \left(\prod_{j=1}^{k} \mathbb{P}^\omega\Big(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\Big)p\right) \sum_{e\in\partial_E B\cap\big(E(S_{\omega,x_k})\cup\partial_E S_{\omega,x_k}\big)} \mathbb{P}_p^\omega\Big(x_k \xleftrightarrow[S_{\omega,x_k}]{\omega,p} e^-\Big) \\ &\leq \sum_{\substack{(x_0,\ldots,x_k)\in\bar\pi \\ (x'_0,\ldots,x'_k)}} \left(\prod_{j=1}^{k} \mathbb{P}^\omega\Big(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\Big)p\right) D^{r_0+1} = F(\bar\pi,p)D^{r_0+1}. \qquad (4.2.2)\end{aligned}$$

To estimate (4.2.2), note that $F(\pi, p)$ equals

$$\sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x'_0,\dots,x'_k)}} \left(\prod_{j=1}^{k} \mathbb{P}^{\omega}\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\right)p\right) \sum_{\substack{x_{k+1}:(x_0,\dots,x_{k+1})\in\pi \\ x'_{k+1}\in S_{\omega,x_k}, x'_{k+1}\sim x_{k+1}}} \mathbb{P}^{\omega}\left(x_k \xleftrightarrow[S_{\omega,x_k}]{\omega,p} x'_{k+1}\right)p$$

$$\geq \sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x'_0,\dots,x'_k)}} \left(\prod_{j=1}^{k} \mathbb{P}^{\omega}\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\right)p\right)p^{r_0+1} = F(\bar\pi, p)p^{r_0+1}$$

by the assumption that the graph is uniformly good. Combined this with (4.2.2) gives

$$\phi_p^{\omega}(B) \leq \frac{D^{r_0+1}}{p^{r_0+1}} F(\pi, p). \tag{4.2.3}$$

Now we show that $F(\pi, p) \leq c_0^{\frac{r}{r_0}}$, which combined with (4.2.3) proves the lemma. Let $\pi_n := \bigcup_{m\leq n}(\pi\cap L_m)\cup\{v\in L_n : v \text{ has a descendant in } \pi\}$, which is a minimal vertex set that separates the root from infinity. Let $R := \max\{n : L_n\cap\pi\neq\emptyset\} < \infty$, thus $\pi = \pi_R$. Note that each π_n is the disjoint union of $\pi_{n+1}\setminus L_{n+1}\subseteq\pi$ and $\pi_n\setminus\pi_{n+1}\subseteq L_n$. We estimate $F(\pi, p)$ by summing over a larger set: the union of $\pi_R\setminus L_R$ and $\{(x_0,\dots,x_R) : (x_0,\dots,x_{R-1})\in\pi_{R-1}\setminus\pi_R, x_R\in\partial_V^{\text{out}}S_{\omega,x_{R-1}}\}\supseteq\pi_R\cap L_R$. That is, using the bound

$$\sum_{e\in\partial_E S_{\omega,x_{R-1}}} \mathbb{P}^{\omega}\left(x_{R-1} \xleftrightarrow[S_{\omega,x_{R-1}}]{\omega,p} e^-\right)p = \phi_p^{\omega,x_{R-1}}(S_{\omega,x_{R-1}}) \leq 1$$

for the second term in the following estimation, we have that

$$F(\pi, p) \leq \sum_{(x_0,\dots,x_k)\in\pi_R\setminus L_R} \sum_{(x'_1,\dots,x'_k)} \prod_{j=1}^{k} \mathbb{P}^{\omega}\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\right)p$$

$$+ \sum_{\substack{(x_0,\dots,x_{R-1})\in\pi_{R-1}\setminus\pi_R \\ (x'_1,\dots,x'_{R-1})}} \left(\prod_{j=1}^{R-1} \mathbb{P}^{\omega}\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\right)p\right) \sum_{e\in\partial_E S_{\omega,x_{R-1}}} \mathbb{P}^{\omega}\left(x_{R-1} \xleftrightarrow[S_{\omega,x_{R-1}}]{\omega,p} e^-\right)p$$

$$\leq \sum_{(x_0,\dots,x_k)\in\pi_{R-1}} \sum_{(x'_1,\dots,x'_k)} \prod_{j=1}^{k} \mathbb{P}^{\omega}\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x'_j\right)p = F(\pi_{R-1}, p).$$

A similar argument shows that $F(\pi, p) \leq F(\pi_n, p)$ for any $n \leq R$. If $(x_0,\dots,x_k)\in\pi$, then $\text{dist}_{\omega}(o, x_k) \geq r$, hence the distance between o and π in T^{ω} is at least $\frac{r}{r_0}$, thus $\pi_n = L_n$ for any $n \leq \frac{r}{r_0}$. If we apply the above argument for $F(\pi_n, p)$ with $n \leq \frac{r}{r_0}$, then the first term disappear, and the inequality $\phi_p^{\omega,x_{n-1}}(S_{\omega,x_{n-1}}) \leq c_0$ gives

$$F(\pi, p) \leq F(\pi_{\frac{r}{r_0}}, p) \leq F(\pi_{\frac{r}{r_0}-1}, p)c_0 \leq \cdots \leq c_0^{\frac{r}{r_0}}.$$

This combined with (4.2.3) proves the lemma. □

Corollary 4.2.4. *If G is a uniformly good unimodular graph, then $p_c \leq \tilde{p}_c^a$.*

Proof. Let $p < p_c$, and let c and $R(p)$ be as in Lemma 4.2.3. We have $\mathbb{E}\left(\phi_p^\omega\left(B_\omega(o, R(p))\right)\right) \le c^R < 1$, thus $p \le \tilde{p}_c^a$. □

We will see in Example 4.2.8 that, without the assumption of uniform goodness, the inequality $p_c \le \tilde{p}_c^a$ does not necessarily hold. Also, we will show in Example 4.2.7 that there are uniformly good graphs with $p_c < \tilde{p}_c^a$. However, if a graph has uniform sub-exponential growth, then the critical probabilities coincide, as in the transitive case. Sub-exponential volume growth will also appear in Example 4.3.8 and Proposition 4.4.1.

Definition 4.2.5. *We say that a unimodular graph G has* ***uniform sub-exponential volume growth*** *if for any $c < 1$ and $\varepsilon > 0$ there is an R such that $\mathbb{P}_G\left(\omega : |B_\omega(o,r)|c^r < \varepsilon\right) = 1$ for any $r > R$.*

Corollary 4.2.6. *If G is a uniformly good unimodular graph with uniform sub-exponential volume growth, then $p_c = p_T = p_T^a$.*

Proof. Let $p < p_c = \tilde{p}_c$ and let c and $R(p)$ be as in Lemma 4.2.3. Denote by D the maximum degree of G. Let $R > R(p)$ such that $\mu\left(\{\omega : |B_\omega(o,r)|c^{r/2} < 1\}\right) = 1$ for any $r > R$ and let ω satisfy this event for all $r > R$ simultaneously. Then we have

$$\begin{aligned}
\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) &= \sum_{n=1}^{\infty} \mathbb{P}_p^\omega\left(|\mathcal{C}_o| \ge n\right) = \sum_{r=1}^{\infty} \sum_{n=|B_\omega(o,r)|+1}^{|B_\omega(o,r+1)|} \mathbb{P}_p^\omega\left(|\mathcal{C}_o| \ge n\right) \\
&\le \sum_{r=1}^{\infty} \sum_{n=|B_\omega(o,r)|+1}^{|B_\omega(o,r+1)|} \mathbb{P}_p^\omega\left(o \overset{p,\omega}{\longleftrightarrow} B_\omega(o,r)^c\right) \\
&\le \sum_{r=1}^{\infty} |B_\omega(o,r+1)| \min\{\phi_p^\omega\left(B_\omega(o,r)\right), 1\} \\
&\le \sum_{r=2}^{R+1} |B_\omega(o,r)| + \sum_{r=R+1}^{\infty} |B_\omega(o,r+1)|c^r \\
&\le \sum_{r=2}^{R+1} D^r + \sum_{r=R+1}^{\infty} c^{r/2} < \infty
\end{aligned}$$

This gives a uniform upper bound on $\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)$ thus $\mathbb{E}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) < \infty$. It follows that $p \le p_T^a$, hence $p_T^a \ge p_c$. The other direction follows from the definition of p_T^a. □

4.2.2 Counterexamples

In this section, we give examples of graphs with $\tilde{p}_c^a > p_c$ and $\tilde{p}_c^a < p_c$; see Examples 4.2.7 and 4.2.8, respectively. These examples show that the inequality in Corollary 4.2.4 can be strict and that without the uniformly good assumption, even the reverse inequality can hold.

We show in Examples 4.2.10 and 4.2.11 that there are unimodular random graphs of uniform sub-exponential (in fact, quadratic) volume growth, but $p_T < p_c$ and $p_T^a < p_T$. Both constructions will use Bernoulli percolation on $\mathbb{Z}^2$ as an ingredient; moreover, although we define the graph in the second example as a vertex replacement of $\mathbb{Z}^2$, it could be defined even as an invariant random subgraph of $\mathbb{Z}^2$.

Our first example shows that the inequality $\tilde{p}_c^a \geq p_c$ in Corollary 4.2.4 can be strict even for a quasi-transitive graph.

Example 4.2.7. *There is a quasi-transitive graph with $\tilde{p}_c^a > p_c$.*

Proof. Let $H_{k,l}$ be the following finite directed multigraph: the vertex set is $\{x_0, x_1, \dots, x_k\}$, and we have l loops at x_0, one edge from x_0 to each x_j, $j = 1, \dots, k$, and one from each x_j back to x_0. Let $T_{k,l}$ be the directed cover of $H_{k,l}$ based at x_0. Consider two copies of $T_{k,l}$ and connect the roots of them by an edge to get the infinite quasi-transitive graph $G_{k,l}$, which has vertices of degree 2 and $k + l + 1$. One can easily compute that to get a unimodular random graph one has to choose the root according to $\mu(\deg o = 2) = 1 - \mu(\deg o = k + l + 1) = \frac{k}{k+2}$. Hence $\mathbb{E}_{G_{k,l}}(\deg o) = \frac{4k+2l+2}{k+2}$. The equality $\mathbb{E}_{G_{k,l}}\left(\phi_p^\omega(B_\omega(o,0))\right) = p\,\mathbb{E}_{G_{k,l}}(\deg o)$ implies that $\tilde{p}_c^a \geq \left(\mathbb{E}_{G_{k,l}}(\deg o)\right)^{-1} = \frac{k+2}{4k+2l+2}$. On the other hand, the critical probability of a directed cover of a finite graph is $p_c(T_{k,l}) = \left(\mathrm{br}(T_{k,l})\right)^{-1} = \left(\mathrm{growth}(T_{k,l})\right)^{-1} = \left(\lambda_*(H_{k,l})\right)^{-1}$, where $\lambda_*(H)$ is the largest positive eigenvalue of the directed adjacency matrix of $H_{k,l}$; see [68, Section 3.3] and [64]. One can thus compute that $p_c(G_{k,l}) = p_c(T_{k,l}) = \frac{2}{l+\sqrt{l^2+4k}}$. If we set, e.g., $k = 3, l = 5$, then we have $p_c(G_{3,5}) = \frac{2}{5+\sqrt{37}} < \frac{5}{24} = \left(\mathbb{E}_{G_{3,5}}(\deg o)\right)^{-1} \leq \tilde{p}_c^a(G_{3,5})$. □

Our next example is the canopy tree, which is not uniformly good and has exponential volume growth. This unimodular graph fails to satisfy the statements of both Corollaries 4.2.4 and 4.2.6.

Example 4.2.8. *The canopy tree Λ (see Definition 2.2.6) satisfies $p_c = 1$ and $p_T = \tilde{p}_c^a = \frac{1}{\sqrt{2}}$, thus this is an example of a not uniformly good unimodular graph with $p_T = p_c > \tilde{p}_c^a$.*

Proof. It is easy to check that $p_T = \frac{1}{\sqrt{2}}$ and $\mathbb{E}(\phi_p(B(o,r)))$ equals $2p(\sqrt{2}p)^r$ if r is even, and equals $3(\sqrt{2}p)^{r+1}/2$ if r is odd. Thus it converges to 0 for $p < 1/\sqrt{2}$, while remains above 1 for $p < 1/\sqrt{2}$, which implies the claim. □

Before presenting Examples 4.2.10 and 4.2.11, we prove a lemma which will be useful in our examples.

Lemma 4.2.9. *Let Q_n be the subgraph of $\mathbb{Z}^2$ spanned by the box $[-n,n]^2$. For any $\varepsilon > 0$ there is a probability $p_1 < 1$ such that for n large enough, the vertices $(0,-n)$, $(0,n)$, $(-n,0)$, $(n,0)$ are in the same cluster in Bernoulli(p_1) percolation on Q_n with probability at least $1 - \varepsilon$.*

Proof. The occurrence of the events in the following two claims implies the occurrence of the event in the statement of the lemma, hence we will be done by a union bound.

Claim 1: For any $p > 1/2$ and $n > n_0(p, \varepsilon)$ large enough, in Bernoulli(p) percolation on Q_n, with probability at least $1 - \varepsilon/2$, there is a *giant cluster* with the following properties: it joins all the sides of Q_n, while every other cluster in Q_n has diameter at most $n/5$. This was proved in [9, Proposition 2.1].

Claim 2: There exists $p_1 < 1$ such that for all n and all $p > p_1$,

$$\mathbb{P}_p \left(\text{diam}(\mathcal{C}_{(0,n)}) \geq n \right) \geq 1 - \varepsilon/8 .$$

Similarly for $(0, -n)$, $(-n, 0)$, and $(n, 0)$, instead of $(0, n)$.

Proof of Claim 2: If there is no open path in the dual percolation joining a dual vertex in $[-n+\frac{1}{2}, -\frac{1}{2}] \times \{n+\frac{1}{2}\}$ to a dual vertex in $[\frac{1}{2}, n-\frac{1}{2}] \times \{n+\frac{1}{2}\}$, then there is a primal open path from $(0, n)$ to $(\{-n\} \times [0, n]) \cup ([-n, n] \times \{0\}) \cup (\{n\} \times [0, n])$, and hence $\text{diam}(\mathcal{C}_{(0,n)}) \geq n$.

On the other hand, for any pair of dual vertices, $x \in [-n+\frac{1}{2}, -\frac{1}{2}] \times \{n+\frac{1}{2}\}$ and $y \in [\frac{1}{2}, n-\frac{1}{2}] \times \{n+\frac{1}{2}\}$, we have

$$\mathbb{P}_p \left(x \overset{Q_n}{\longleftrightarrow} y \text{ by a dual-open path of length } k \right) \leq (3(1-p))^k .$$

Moreover, if the distance of x and y is larger than k, then this probability is of course 0, hence for each k there are at most k^2 relevant pairs (x, y). Therefore, for every n,

$$\mathbb{P}_p \left(\text{diam}(\mathcal{C}_{(0,n)}) < n \right) \leq \sum_{k=1}^{\infty} k^2 (3(1-p))^k =: f(p) < \infty ,$$

where $f(p)$ converges to 0 as $p \to 1$. □

The next example shows, that the classical sharpness of the phase transition of percolation fails in the class of unimodular graphs. The graph in Example 4.2.10 is not uniformly good.

Example 4.2.10. *There is a unimodular graph with uniform polynomial volume growth and $p_T < p_c$. In particular, the exponential decay of two-point connection probabilities fails for $p \in (p_T, p_c)$ on this graph.*

Proof. We define the graph G as an edge replacement (see [4], Example 9.8) of the canopy tree (see Definition 2.2.6): each $e \in L_E(n)$ is replaced by $(Q_{2^n}(e), (0, -2^n), (0, 2^n))$, where $Q_{2^n}(e)$ is isomorphic to Q_{2^n}. It is easy to see that the volume of $B_G(o, r)$, for any root o and radius r, is at most Cr^2, for some absolute constant $C < \infty$. Indeed, if the root is in $Q_{2^n}(e)$, then $B_G(o, r)$ intersects the cubes $Q_{2^l}(e')$ with $e' \in (\xi, e)$ only if $l \leq \log_2 r$ or $l = n$. Furthermore, each such $Q_{2^l}(e')$ has more vertices than the sum of the number of vertices of $Q_{2^k}(e'')$ with $e' \in (\xi, e'')$, which are the further cubes that may intersect $B_G(o, r)$. It follows that $|B_G(o, r)| \leq \max \left\{ r^2, \sum_{l=n}^{\log_2 r} 2^{2l+3} \right\} \leq Cr^2$.

We will now show that $p_T(G) < p_c(G) = 1$. Consider Bernoulli(p) percolation ω on G and, as a deterministic function of it, define the following percolation λ on Λ: an edge $e \in L_E(n)$ is open in λ if and only if the vertices $(0,-n)$ and $(0,n) \in Q_n(e)$ are connected by an open path in ω. Clearly, there exists an infinite cluster in ω if and only if there is an infinite cluster in λ. The law of λ is stochastically dominated by a Bernoulli($1-(1-p)^3$) percolation on Λ, because if $e \in L_E(n)$ is open, then at least one of the edges in $Q_n(e)$ adjacent to $(0,n)$ is open. The tree Λ has one end, hence, for any $p < 1$,

$$\mathbb{P}_p^G(\exists \text{ an infinite cluster}) \leq \mathbb{P}^\Lambda_{1-(1-p)^3}(\exists \text{ an infinite cluster}) = 0\,.$$

That is, $p_c(G) = 1$.

It can be easily computed that $p_T(\Lambda) = 1/\sqrt{2}$. Now let $0 < \varepsilon < 1 - 1/\sqrt{2}$. It follows from Lemma 4.2.9 that there exists $p_1 < 1$ and some large N such that $\mathbb{P}_{p_1}(e \in \lambda) \geq 1 - \varepsilon$ for all $e \in L_E(n)$ with $n \geq N$. Thus, for $o \in L(N)$, the cluster $\mathcal{C}_o$ in λ, restricted to the levels $n \geq N$, stochastically dominates Bernoulli($1-\varepsilon$) percolation on Λ. The latter has infinite expected size, hence the expected size of the cluster in ω of $(0,-N) \in Q_N(e)$ for $e \in L_E(N)$ is also infinite. That is, $p_T(G) \leq p_1 < 1$. □

The last example of this section shows that the critical probabilities p_T and p_T^a can differ, even for a unimodular random graph of polynomial volume growth. The graph in Example 4.2.11 can be viewed as a random invariant subgraph of $\mathbb{Z}^2$.

Example 4.2.11. *There is a unimodular graph with polynomial volume growth and* $p_T^a < p_T$.

Proof. Let X be a positive integer valued random variable such that $\mathbb{P}(X = k) = ck^{-5/2}$ for all $k \geq 1$. Then $\mathbb{E}X < \infty$ and $\mathbb{E}(X^2) = \infty$. We define the graph G as a vertex replacement (see Subsection 1.2.3) of $\mathbb{Z}^2$ with respect to the following labels as follow. Let $\{X_n, X'_n : n \in \mathbb{Z}\}$ be iid copies of X, and for each vertex $(m,n) \in \mathbb{Z}^2$, let $G_{(m,n)}$ be isomorphic to the subgraph of $\mathbb{Z}^2$ spanned by the vertices in $[0, 2X_m] \times [0, 2X'_n]$, and for the edges going from (m,n) to North, East, South, and West, let the image of $\varphi_{(m,n)}$ be the corresponding midpoint of the box $G_{(m,n)}$. We can also think of the resulting graph as an invariant random subgraph of $\mathbb{Z}^2$.

Denote by Y and Y' half the length of the sides of the box of o in G, i.e., the law of X_0 and X'_0 biased by $X_0 X'_0$. Then

$$\mathbb{P}(Y = k, Y' = l) = \frac{kl}{(\mathbb{E}X)^2}\mathbb{P}(X = k, X' = l),$$

hence Y and Y' are independent with distribution $\mathbb{P}(Y = k) = \frac{ck^{-3/2}}{\mathbb{E}X}$.

First we show that $p_T^a = \frac{1}{2}$. G is a subgraph of $\mathbb{Z}^2$, hence $p_T^a(G) \geq \frac{1}{2}$. Fix $p > \frac{1}{2}$ and let $\varepsilon > 0$. Denote by $M(Q_n)$ the largest cluster in percolation with parameter p in the box Q_n (the subgraph of $\mathbb{Z}^2$ spanned by the box $[-n,n]^2$), and let

$$\mathcal{A}(Q_n) := \big\{|M(Q_n)| \geq (1-\varepsilon)\theta(p)|Q_n|, \quad \operatorname{diam}(C) < \nu \log n \ \forall \text{ open cluster } C \neq M(Q_n)\big\},$$

where $\theta(p) = \mathbb{P}_p(|\mathcal{C}_o(\mathbb{Z}^2)| = \infty)$, and ν is chosen as follows: by [47, Theorem 7.61], there is an $N = N(p)$ and $\nu = \nu(p)$ such that, for any $n \geq N$,

$$\mathbb{P}_p(\mathcal{A}(Q_n)) > 1 - \varepsilon\,.$$

Let $Z := \min\{Y, Y'\}$, and consider the event $\mathcal{D}(G_{0,0}) := \big\{\mathrm{dist}(o, \partial_V^{\mathrm{in}} G_{0,0}) \geq \nu \log Z\big\}$. If Z is large enough, then $\mathbb{P}\big(\mathcal{D}(G_{0,0}) \,\big|\, Z\big) \geq 1 - \varepsilon$, since o is uniform in $G_{0,0}$. Assuming that $\mathcal{D}(G_{0,0})$ occurs, choose a box $Q_Z \subseteq G_{0,0}$ that contains o such that $\mathrm{dist}(o, \partial_V^{\mathrm{in}} Q_Z) \geq \nu \log Z$. Consider percolation on $\mathbb{Z}^2 \supset Q_Z$. If o is in the unique infinite cluster of this percolation on $\mathbb{Z}^2$, then the diameter of $\mathcal{C}_o(Q_Z)$ is at least $\nu \log Z$, hence

$$\mathbb{P}_p\Big(o \in M(Q_Z), \mathcal{A}(Q_Z) \,\Big|\, Z = n, \mathcal{D}(G_{0,0})\Big) > \theta(p) - \varepsilon$$

for n large enough. It follows that there is an N' such that

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\big) &\geq \sum_{n=N'}^{\infty} \mathbb{P}_p\Big(o \in M(Q_Z), \mathcal{A}(Q_Z), \mathcal{D}(G_{0,0}) \Big| Z = n\Big)\, \mathbb{P}(Z = n)\, (1-\varepsilon)\theta(p) n^2 \\
&\geq \sum_{n=N'}^{\infty} (\theta(p) - \varepsilon)(1-\varepsilon)\, \mathbb{P}(Z = n)\, (1-\varepsilon)\theta(p) n^2 = \infty\,,
\end{aligned}$$

as desired.

To show that $p_T > \frac{1}{2}$ let e be an edge in $\mathbb{Z}^2$, and let G_{e^-} and G_{e^+} be the subgraphs of G that correspond to the endpoints of the edge. Let $x := \varphi_{e^-}(e)$ and $y := \varphi_{e^+}(e)$, i.e. let $\{x, y\}$ be the edge in G that joins G_{e^-} and G_{e^+}. If there is an open path in $G(p)$ through the edge $\{x, y\}$, that joins two vertices in $G_{e^-} \setminus \{x\}$ and in $G_{e^+} \setminus \{y\}$, then the event $J(\{x, y\}) := \{\exists e' \in E(G_{e^-}) : e' \sim x, e' \text{ open}\} \cap \{\exists e' \in E(G_{e^+}) : e' \sim y, e' \text{ open}\} \cap \{\{x, y\} \text{ open}\}$ occurs. For a fixed configuration of G the events $J(\{\varphi_{e^-}(e), \varphi_{e^+}(e)\})$ are independent for different edges, and

$$\mathbb{P}_p(J(\{\varphi_{e^-}(e), \varphi_{e^+}(e)\})) = p(1 - (1-p)^3)^2.$$

This probability is strictly increasing in p and there is a $p_0 > \frac{1}{2}$ such that $p(1-(1-p)^3)^2 > \frac{1}{2}$ iff $p > p_0$. We consider a random subset $H = H\left(G(p)\right) \subseteq E(\mathbb{Z}^2)$ obtained from the percolation $G(p)$: let $e \in H$ if and only if the event $J(\{\varphi_{e^-}(e), \varphi_{e^+}(e)\})$ occurs in $G(p)$. The law of H is the same as the law of Bernoulli($p(1-(1-p)^3)^2$) bond percolation. We want to estimate the expected size of $\mathcal{C}_o(G)$ conditioned on the size of $G_{0,0}$. If $\mathcal{C}_o(G)$ intersects a box G_v, then the connected component of o in H contains v. Therefore

$$\begin{aligned}
\mathbb{E}_G\Big(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) \,\Big|\, Y, Y'\Big) &\leq \mathbb{E}_G\left(\mathbb{E}_p^\omega\left(\sum_{v \in \mathbb{Z}^2 : v \in \mathcal{C}_o(H)} |G_v|\right) \middle|\, Y, Y'\right) \\
&\leq \mathbb{E}_G\big(\mathbb{E}_p^\omega(|\mathcal{C}_o(H)|)\big) \max\big\{Y^2, (Y')^2, (\mathbb{E}X)^2\big\}\,,
\end{aligned}$$

which is finite if $p < p_0$. It follows that for almost every configuration (ω, o) of (G, o) the expected size $\mathbb{E}_p^\omega(\mathcal{C}_o)$ is finite if $p < p_0$, hence $p_T \geq p_0$. $\square$

4.3 Locality of the critical probability

In this section, we examine Question 4.1.2, the question of Schramm's locality conjecture for transitive graphs: does $p_c(G_n)$ converge to $p_c(G)$ if $G_n \to G$ in the local weak sense? The original question in [16] (Conjecture 4.1.1) was phrased for sequences of transitive graphs that converge to a transitive graph in the local sense and satisfy $\sup p_c(G_n) < 1$. First we provide some simple examples of unimodular graphs where the conjecture holds. In Example 4.3.1, we note that if G_n and G are infinite clusters of an independent percolation with appropriate parameters, then the convergence holds. In Example 4.3.2, we discuss unimodular Galton–Watson trees, and give sufficient and necessary conditions on the offspring distribution to satisfy locality of p_c. Then we investigate the inequality $\liminf p_c(G_n) \geq p_c(G)$, which is known for transitive graphs; see [35] for a simple proof. In Proposition 4.3.3 we show by a similar argument that the critical probability $\tilde{p}_c^a$ satisfies this inequality for unimodular random graphs. We show in Propositions 4.3.4 and 4.3.6 that under certain restrictions on the graphs G and G_n the convergence $\lim p_c(G_n) = p_c(G)$ is true for unimodular random graphs. Examples 4.3.7 and 4.3.8 provide graph sequences with $\lim p_c(G_n) < p_c(G)$. These indicate that unimodular graphs do not satisfy Schramm's conjecture in general and show that both of the conditions in Proposition 4.3.4 are necessary. We show in Example 4.3.9 a sequence with $p_c(G) < \lim p_c(G_n) < 1$. In this example, G and each G_n satisfy the conditions of Corollaries 4.2.4 and 4.2.6, thus $p_c = p_T = p_T^a$ and also $\tilde{p}_c^a(G) < \lim \tilde{p}_c^a(G_n) < 1$. This shows that none of the generalizations of the critical probabilities satisfies the extension of Schramm's conjecture for unimodular graphs in general.

4.3.1 Basic examples

We present now two natural classes of unimodular random graphs that satisfy Schramm's conjecture.

Example 4.3.1. *Let G be a transitive unimodular graph and let $p_n, p_0 \in (p_c(G), 1]$, such that $\lim_{n\to\infty} p_n = p_0$. Let G_n be the connected component of the root in the Bernoulli(p_n) percolation on G conditioned to be infinite. Then $p_c(G_n) \to p_c(G_0) < 1$.*

Proof. The graph G_n is an extremal unimodular graph for every n by [4, Theorem 6.15]. It follows that $p_c(G_n)$ is an almost sure constant with respect to the distribution of G_n. We have

$$\mathbb{P}_p^{G_n}(|\mathcal{C}_o| = \infty) = \frac{\mathbb{P}_{pp_n}^{G}(|\mathcal{C}_o| = \infty)}{\mathbb{P}_{p_n}^{G}(|\mathcal{C}_o| = \infty)}.$$

This probability is 0 iff $\mathbb{P}_{pp_n}^{G}(|\mathcal{C}_o| = \infty) = 0$, which holds if $p < p_c(G)/p_n$ and does not hold if $p > p_c(G)/p_n$. Since $p_c(G_n)$ needs to be a constant, it equals $p_c(G)/p_n$ almost surely, which converges to $p_c(G)/p_n = p_c(G_0)$.

□

Our second example, the class of unimodular Galton–Watson trees, is less trivial. Recall Definition 1.2.10, $\hat{X}$ from Section 2.2.1 and Proposition 2.2.11 that states that a sequence of infinite unimodular Galton–Watson trees $UGW_\infty(\hat{X}_n)$ converges to $UGW_\infty(\hat{X})$ in the local weak sense iff $X_n \to X$ in distribution. The random graphs UGW_∞ are extremal unimodular random graphs [4].

Example 4.3.2. *Let $UGW_\infty(\hat{X})$ be the unimodular Galton–Watson tree with degree distribution $\hat{X}$, conditioned to be infinite. If the offspring distributions X_n and X are non-negative integer valued random variables with $\mathbb{E}X_n \geq 1$ and $\mathbb{E}X \geq 1$, then $p_c(UGW_\infty(\hat{X}_n)) \to p_c(UGW_\infty(\hat{X}))$ iff $\mathbb{E}X_n \to \mathbb{E}X$.*

Proof. The critical probability $p_c(UGW_\infty(\hat{X}))$ equals $\frac{1}{\mathbb{E}X}$ [68, Proposition 5.9], therefore $p_c(UGW_\infty(\hat{X}_n)) \to p_c(UGW_\infty(\hat{X}))$ iff $\mathbb{E}X_n \to \mathbb{E}X$. $\square$

Note that this example shows that p_c is a continuous function of $UGW_\infty(\hat{X})$ when the trees have a uniform bound on their degrees (by the Dominated Convergence Theorem), but not necessarily otherwise: if $X_n \to X$ in distribution, with $\mathbb{E}X_n \geq 1$ and $\mathbb{E}X \geq 1$, but $\mathbb{E}X_n \not\to \mathbb{E}X$, then the critical probabilities $p_c(UGW_\infty(\hat{X}_n))$ do not converge to $p_c(UGW_\infty(\hat{X}))$. Nevertheless, Fatou's lemma implies that the inequality $\limsup p_c(UGW_\infty(\hat{X}_n)) \leq p_c(UGW_\infty(\hat{X}))$ does hold without any assumptions. That is, if the trees do not satisfy the locality of p_c, then they also fail to satisfy the lower semicontinuity discussed in the next subsection, proved to hold in many cases, including transitive graphs. This suggests that a uniform bound on the degrees is a natural condition when we investigate the locality of p_c for unimodular graphs.

4.3.2 Lower semicontinuity and continuity

The quantity $\phi_p(S)$ can be used to give a short proof that $p_c(G)$ is lower semicontinuous in the local topology of transitive graphs: that is, $\liminf p_c(G_n) \geq p_c(G)$ holds; see [35, Section 1.2]. It can be proven for transitive graphs as follows: let $p < p_c(G)$, let $S \subset G$ be a set with $\phi_p^G(S) < 1$ and let r be such that $S \subset B_G(o,r)$. For n large enough $B_{G_n}(o,r) \simeq B_G(o,r)$, hence $\phi_p^{G_n}(S) < 1$, which implies $p \leq p_c(G_n)$. For bounded degree unimodular graphs, we will now show in a similar way that this inequality also holds for $\tilde{p}_c^a$; however, it fails for $\tilde{p}_c = p_c$, in general.

Proposition 4.3.3. *Let G_n and G be unimodular random graphs with uniformly bounded degrees. If G_n converges to G then $\liminf_{n\to\infty} \tilde{p}_c^a(G_n) \geq \tilde{p}_c^a(G)$.*

Proof. Let $p < \tilde{p}_c^a(G)$ and let r be such that $\mathbb{E}_G\left(\phi_p^\omega(B_\omega(o,r))\right) < 1 - \varepsilon$ with some $\varepsilon > 0$. Let n be large enough to satisfy

$$\sum_{H \in \mathcal{H}_{r+1}} \left|\mu_{G_n}\left(B_\omega(o,r+1) = H\right) - \mu_G\left(B_\omega(o,r+1) = H\right)\right| < \frac{\varepsilon}{2D^{r+1}},$$

where D is a uniform bound on the degrees of G_n and G and $\mathcal{H}_r$ is the set of possible r-neighborhoods of the root in graphs with maximum degree D. Any $H \in \mathcal{H}_{r+1}$

satisfies $\phi_p^H(B_\omega(o,r)) \le D^{r+1}$. We obtain

$$\begin{aligned}
\mathbb{E}_{G_n}\left(\phi_p^\omega(B_\omega(o,r))\right) &= \sum_{H\in\mathcal{H}_{r+1}} \mu_{G_n}\left(B_\omega(o,r+1)=H\right)\phi_p^H(B_\omega(o,r)) \\
&\le \sum_{H\in\mathcal{H}_{r+1}} \big[\mu_G\left(B_\omega(o,r+1)=H\right)\phi_p^H(B_\omega(o,r)) \\
&+ \left|\mu_{G_n}\left(B_\omega(o,r+1)=H\right)-\mu_G\left(B_\omega(o,r+1)=H\right)\right||\partial_E B_H(o,r)|\big] \\
&\le \mathbb{E}_G\left(\phi_p^\omega(S)\right)+\frac{\varepsilon}{2} < 1.
\end{aligned}$$

It follows that $\tilde{p}_c^a(G_n) \ge p$ thus $\liminf \tilde{p}_c^a(G_n) \ge \tilde{p}_c^a(G)$. □

Our next proposition states that if G_n converges to a uniformly good unimodular graph G in a uniformly sparse way, then $p_c(G_n) \to p_c(G)$, i.e., the assumptions of Proposition 4.3.4 imply a positive answer to Question 4.1.2. After the proof, we present an example that shows how this proposition can be applied. Another application of the proposition appears in Example 4.4.2.

Proposition 4.3.4. *Let G be a uniformly good unimodular random graph. Furthermore, let G_n be unimodular random graphs with uniformly bounded degrees that converge to G in the local weak sense, in a* uniformly sparse *way: there is a positive integer k such that for each n there is a coupling ν_n of μ_G and μ_{G_n} such that $G \subseteq G_n$ and there is a sequence of positive integers $r_n \to \infty$ that satisfies $|(E(G_n)\setminus E(G)) \cap B_{G_n}(o,r_n)| \le k$ ν_n-almost surely. Then*

$$\lim_{n\to\infty} p_c(G_n) = p_c(G).$$

Proof. First, $G \subseteq G_n$ implies that $p_c(G) \ge p_c(G_n)$ for all n. For the sake of simplicity, we prove the inequality $\lim p_c(G_n) \ge p_c(G)$ for $k=1$. It can be proved for general k in a similar way. Let $p < p_c(G)$. Our aim is to find a subset $B_n \in \mathcal{S}(G_n)$ for n large enough with $\phi_p^{G_n}(B_n) < 1$. Let n be sufficiently large to satisfy $r_n/2 > R(p)$ and $c^{r_n/2} < \frac{1}{3}$. Fix a pair (ω,ω_n) that satisfies the sparseness condition for r_n. Then, in the smaller ball $B_{\omega_n}(o,r_n/2)$, there is at most one edge $\{x,y\} \in \omega_n \setminus \omega$. If this edge exists, let $B_n := B_{\omega_n}(o,r_n/2) \cup B_{\omega_n}(x,r_n/2) \cup B_{\omega_n}(y,r_n/2)$; otherwise, just let $B_n := B_{\omega_n}(o,r_n/2)$. Note that $B_n \subset B_{\omega_n}(o,r_n)$. Similarly, let $B := B_\omega(o,r_n/2) \cup B_\omega(x,r_n/2) \cup B_\omega(y,r_n/2)$, omitting those terms in the union that do not exists in ω. (Note that it may happen that x or y does not exist in ω, but not both, since $B_{\omega_n}(o,r_n/2)$ is connected.) The sets B_n and B satisfy $\partial_E B_n = \partial_E B$. We claim that we have $\phi_p^{\omega_n}(B_n) < 1$. There are three possibilities in terms of the edge $\{x,y\}$ for an open path connecting o and a vertex e^- in B_n: it connects o and e^- in B or it connects x or y to e^- in B. It follows that

$$\begin{aligned}
\phi_p^{\omega_n}(B_n) &= p \sum_{e\in\partial_E B_n} \mathbb{P}^{\omega_n}\left(o \xleftrightarrow[B_n]{\omega_n,p} e^- \right) \\
&= p \sum_{e\in\partial_E B_n} \Bigg[\mathbb{P}^{\omega_n}\left(o \xleftrightarrow[B_n]{\omega_n\setminus\{x,y\},p} e^- \right) + \mathbb{P}^{\omega_n}\left(\{o \xleftrightarrow[B_n]{\omega_n,p} x\} \Box \big\{\{x,y\} \text{ open}\big\} \Box \{y \xleftrightarrow[B_n]{\omega_n,p} e^-\} \right) \\
&\qquad + \mathbb{P}^{\omega_n}\left(\{o \xleftrightarrow[B_n]{\omega_n,p} y\} \Box \big\{\{x,y\} \text{ open}\big\} \Box \{x \xleftrightarrow[B_n]{\omega_n,p} e^-\} \right) \Bigg] \\
&\le p \sum_{e\in\partial_E B} \Bigg[\mathbb{P}^{\omega}\left(o \xleftrightarrow[B]{\omega,p} e^- \right) + p\,\mathbb{P}^{\omega}\left(\{o \xleftrightarrow[B]{\omega,p} x\} \Box \{y \xleftrightarrow[B]{\omega,p} e^-\} \right) \\
&\qquad + p\,\mathbb{P}^{\omega}\left(\{o \xleftrightarrow[B]{\omega,p} y\} \Box \{x \xleftrightarrow[B]{\omega,p} e^-\} \right) \Bigg] \\
&\le p \sum_{e\in\partial_E B} \left[\mathbb{P}^{\omega}\left(o \xleftrightarrow[B]{\omega,p} e^- \right) + \mathbb{P}^{\omega}\left(y \xleftrightarrow[B]{\omega,p} e^- \right) + \mathbb{P}^{\omega}\left(x \xleftrightarrow[B]{\omega,p} e^- \right) \right] \\
&= \phi_p^{\omega}(B) + \phi_p^{\omega,y}(B) + \phi_p^{\omega,x}(B) \ < \ 1
\end{aligned}$$

by Lemma 4.2.3. If x or y does not exist in ω, all its appearances in the above formulas involving ω can be replaced by the other vertex, and the inequalities remain true. It follows that $p \le \tilde{p}_c(G_n) = p_c(G_n)$. □

The following example is a graph sequence G_n where Proposition 4.3.4 applies.

Example 4.3.5. *There is a uniformly good unimodular graph G and a sequence of unimodular graphs G_n that satisfy the assumptions of Proposition 4.3.4.*

Proof. Let G be a uniformly good unimodular graph of bounded degree; e.g., a unimodular quasi-transitive graph. Let $H_n \subset V(G)$ be an invariant subset (i.e., given by a unimodular labeling) such that $\min\{\mathrm{dist}_G(x,y) : x, y \in H_n\} \ge n$ almost surely. Such a subset can be produced as a factor of iid process: let $\{\xi_x : x \in V(G)\}$ be iid uniform random variables on $[0,1]$ and let $H_n := \{x : \xi_x = \min\{\xi_y : y \in B_G(x,n)\}\}$. Consider now an invariant perfect matching of the points of H_n (that is, an invariant partition of H_n into pairs) and let G_n be the union of that matching and G. An example of such a perfect matching can be constructed as follows. Let $\{\zeta_e : e \in V(G)\}$ be iid uniform random variables on $[0,1]$ and consider the distance function d on $V(G)$ defined as $d(x,y) = \inf \sum_{e\in P} \zeta_e$, where P ranges over all paths connecting x and y. It is easy to check that the infimum exists and is in fact a minimum; also, one can show that with the resulting metric the set H_n is discrete, non-equidistant, and has no descending chains (see [53] for the definitions). By a method similar to the proof of Proposition 9 in [53], one can show that the stable matching on H_n is a perfect matching, just as desired.

For quasi-transitive graphs G, we have $p_T = p_c$. Then it is not surprising that, for any $p < p_c$, once n is large enough, adding the sparse perfect matching cannot glue too many of the rather small finite clusters of G together, and hence we still have $p < p_c(G_n)$. That is, one expects $p_c(G_n) \to p_c(G)$. This indeed holds by our general proposition, while an actual direct proof would need to handle some non-trivial technicalities. □

In the quite special setting of unimodular trees of uniform sub-exponential growth (see Definition 4.2.5), the assumption of uniformly sparse convergence from Proposition 4.3.4 can be relaxed. This proposition gives further examples of uniformly good unimodular graphs (see Definition 4.2.2), while the convergence part will be used in Section 4.4.

Proposition 4.3.6. *If G is a bounded degree unimodular random tree with uniformly sub-exponential volume growth (see Definition 4.2.5), then all five critical percolation densities equal 1, and G is uniformly good.*

If G_n is a sequence of bounded degree unimodular random graphs with uniformly sub-exponential volume growth and girth tending to infinity, then $p_c(G_n)$, $\tilde{p}_c(G_n)$, $\tilde{p}_c^a(G_n)$ all tend to 1.

Proof. We start by proving the statement about the sequence G_n with girth tending to infinity. By the uniform sub-exponential growth, for each $p < 1$ there are positive integers $r = r(p)$ and $n_0(p)$ such that

$$|B_{G_n}(o_n, r)|\, p^r < 1 \tag{4.3.1}$$

for every $n \geq n_0(p)$, almost surely. Now, by the girth tending to infinity, there exists $n_1(p) \geq n_0(p)$ such that, for every $n \geq n_1(p)$, the ball $B_{G_n}(o_n, r)$ is a tree, and therefore

$$\phi_p^{G_n}(B_{G_n}(o_n, r)) \leq |B_{G_n}(o_n, r)|\, p^r\,. \tag{4.3.2}$$

Combining (4.3.1) and (4.3.2), and taking $p \to 1$, the balls $B_{G_n}(o_n, r)$ show that $\tilde{p}_c(G_n)$ and $\tilde{p}_c^a(G_n)$ tend to 1. By Theorem 4.2.1, we also have $p_c(G_n) \to 1$.

Now, if G is a unimodular tree of sub-exponential growth, then (4.3.2) holds for every r, hence $\tilde{p}_c^a(G) = \tilde{p}_c(G) = p_c(G) = 1$, and uniform goodness is also clear from the definition. Then Corollary 4.2.6 implies $p_T(G) = p_T^a(G) = 1$, as well. □

4.3.3 Counterexamples

Our first example will show that even if we keep the condition of uniformly sparse convergence of G_n to G of Proposition 4.3.4, without G being uniformly good, the conclusion may not hold. Next, Example 4.3.8 will show that keeping the limit uniformly good but removing the condition of uniform sparseness will make the conclusion false. Finally, Example 4.3.9 will show that the inequality of the lower semicontinuity may be strict even when invariant subgraphs G_n of $\mathbb{Z}^2$ converge to $\mathbb{Z}^2$.

Example 4.3.7. *There exists a sequence (G_n) of invariant random subgraphs of a Cayley graph, converging to an invariant subgraph G in a uniformly sparse way, such that* $\lim p_c(G_n) < p_c(\lim G_n)$.

Proof. The first step is to construct an invariant percolation on a Cayley graph of the lamplighter group all whose clusters are isomorphic to the canopy tree Λ (Definition 2.2.6). In more detail:

Consider the generators $\{Rs, R, sL, L\}$ of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z} = \oplus_{\mathbb{Z}}\mathbb{Z}_2 \rtimes \mathbb{Z}$, where $R := (0,1), L := (0,-1)$, and $s := (e_0, 0) \in \mathbb{Z}_2 \wr \mathbb{Z}$ with $e_0 \in \{0,1\}^{\mathbb{Z}}$, $(e_0)_j = \delta_{0,j}$. It is well-known (see, e.g., [83]) that the Cayley graph with respect to these generators is the Diestel–Leader graph DL(2,2). This graph can be defined using two trees $\mathbb{T}^1$ and $\mathbb{T}^2$ which both are 3-regular infinite rooted trees with a distinguished end and Busemann functions $\mathfrak{h}_i : \mathbb{T}^i \to \mathbb{Z}, i = 1, 2$, as in Example 4.2.10. Each vertex $x \in \mathbb{T}^i$ has exactly one neighbor $\bar{x}$ with $\mathfrak{h}_i(\bar{x}) = \mathfrak{h}_i(x) - 1$, called the parent of x. We call the other two neighbors the children of x. Now consider the following percolation on $\mathbb{T}^1$: for each vertex x we delete the edge connecting x to one of its two children, independently with equal probabilities. We get a random subgraph of $\mathbb{T}^1$ consisting of infinite simple paths. We then delete the edges in the graph DL(2,2) whose first coordinate is a deleted edge in $\mathbb{T}^1$. The resulting random subgraph $\mathcal{F} \subset$ DL(2,2) is invariant under the action of the lamplighter group and it consists of infinitely many components which are all isomorphic to the canopy tree $\Lambda \subset \mathbb{T}$. The probability that the root is in the n^{th} level of its component in $\mathcal{F}$ is clearly 2^{-n-1}. The canopy tree with a random root chosen according to this distribution is a unimodular random graph, as it also must be the case by Proposition 1.2.6.

The significance of the canopy tree for this construction (as in Example 4.2.10) will be that it has one end, thus $p_c(\Lambda) = 1$, while one can easily compute that $p_T(\Lambda) = 1/\sqrt{2}$.

Now let $\mathbb{G}$ be the free product of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ and the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. Let Γ be the left Cayley graph of $\mathbb{G}$ with respect to the generators $\{a, Rs, R, sL, L\}$ where a is the generator of the free factor $\mathbb{Z}_2$. Let $\beta : \mathbb{G} \longrightarrow \mathbb{Z}_2 \wr \mathbb{Z}$ be the natural projection homomorphism: if $w = a_1 b_1 \dots a_k b_k$ is a word in $\mathbb{G}$ such that $a_j \in \mathbb{Z}_2, b_j \in \mathbb{Z}_2 \wr \mathbb{Z}, j = 1, \dots, k$, then $\beta(w) := b_1 \dots b_k \in \mathbb{Z}_2 \wr \mathbb{Z}$. We now define G to be the following random spanning subgraph of Γ: let e be in $E(G)$ iff $\beta(e^-)$ and $\beta(e^+)$ are connected by an edge in $\mathcal{F}$. The distribution of G is invariant under the action of $\mathbb{G}$ and each component of G is a canopy tree, hence $p_c(G) = 1$.

We define a sequence (G_n) of random subgraphs of Γ converging to G. We choose an element $b \in \{0, 1, \dots n-1\}$ uniformly at random. For each vertex in $L_{\mathbb{T}^1}(b + kn), k \in \mathbb{Z}$ we choose one of its descendants in $L_{\mathbb{T}^1}(b + (k+1)n)$ uniformly at random and we choose all vertices in $L_{\mathbb{T}^2}(-b + kn)$. Let S_n be the set of edges $e \in E(\Gamma)$ such that e is labeled by the generator a and both coordinates of $\beta(e^-) = \beta(e^+)$ are chosen vertices in the above procedure. Let $G_n := G \cup S_n$.

We show that $p_c(G_n) \leq \frac{1}{\sqrt{2}}$ for all n. Let $p > \frac{1}{\sqrt{2}} = p_T(\Lambda)$, let n be a positive integer and consider Bernoulli(p) percolation on G_n. Denote by $T(v)$ the component of the vertex v in G and by $\mathcal{C}_v$ the component of the vertex v in the percolation on G_n. Let $s(v) := \min\{l : L_{T(v)}(l) \cap S_n \neq \emptyset\}$. We define a branching process depending on the percolation on G_n. For each vertex v of Γ let $N_v := \{ax : x \in T(v) \cap \mathcal{C}_v \cap S_n \setminus \{v\}, \{x, ax\}$ is open$\}$. Let $Z_1 := N_o$ and let $Z_{k+1} := \bigcup_{v \in Z_k} N_v$. Note that $Z_i \neq Z_j, i \neq j$ and $Z_j \subset \mathcal{C}_o$. The distribution of $|N_v|$ depends only on the level of v in $T(v)$ and on $s(v)$. The distribution of $|N_v|$ conditioned on

$\{o \in L_{T(o)}(l), s(v) = s\}$ with any l and s stochastically dominates the distribution of $|N_v|$ conditioned on the event $\{v \in L_{T(v)}(0), s(v) = n-1\}$. Therefore the distribution of $|Z_k|$ stochastically dominates the distribution of the k^{th} generation of the Galton–Watson process with offspring distribution $|N_v|$ conditioned on $\{v \in L_{T(v)}(0), s(v) = n-1\}$, which has infinite expectation. Hence $\mu(\liminf |Z_k| > 0) > 0$ which implies $\mu(|\mathcal{C}_o| = \infty) > 0$. □

Example 4.3.8. *There exists a sequence (G_n) of invariant random subgraphs of a Cayley graph such that $\lim p_c(G_n) < p_c(\lim G_n)$ and $\lim G_n$ is uniformly good.*

Proof. Let Γ be a Cayley graph of a finitely generated group such that there exists a random subgraph $\bar{G}$ which satisfies the following: the distribution of $\bar{G}$ is invariant under the action of the group, it consists of infinitely many infinite components and each component has critical percolation probability $\bar{p} < 1$. (A very simple example is that Γ is $\mathbb{Z}^d$ and $\bar{G}$ is a lamination by copies of $\mathbb{Z}^{d-1}$, with $d \geq 3$.) Let G' be an invariant random connected subgraph of Γ such that $p_c(G') > \bar{p}$. For example, if Γ is amenable, then one can choose G' to be an invariant spanning tree of Γ, which always exists and has at most two ends, and hence $p_c(G') = 1$; see [14], Theorem 5.3. Moreover, if Γ has sub-exponential volume growth (see Definition 4.2.5), then so does the spanning tree G', and it is uniformly good by Proposition 4.3.6.

Now let $\varepsilon_n \to 0$ be a sequence of positive numbers and let G_n be the following random subgraph of Γ: we remove each component of $\bar{G}$ with probability $1 - \varepsilon_n$ and keep it with probability ε_n independently for each component. Let G_n be the union of G' and the remaining components of $\bar{G}$. It follows from Proposition 1.2.6 that G_n is unimodular. The sequence (G_n) converges to G', but $p_c(G_n) \leq \bar{p} < p_c(G')$ for each n. The sequence $p_c(G_n)$ has a convergent subsequence, hence we can choose the corresponding subsequence $\varepsilon_{k(n)}$, and get $\lim p_c(G_{k(n)}) \leq \bar{p} < p_c(G')$.

We get a similar example that is uniformly good if we set $\Gamma := \mathbb{Z}^5$, $\bar{G} := \bigcup_{y \in \mathbb{Z}^2} \{y\} \times \mathbb{Z}^3$ and $G' := \bigcup_{x \in \mathbb{Z}^3} \mathbb{Z}^2 \times \{x\}$. In this example, G' is not connected, but each G_n is connected almost surely, and $p_c(G_n) \leq p_c(\mathbb{Z}^3) < p_c(\lim G_n) = p_c(\mathbb{Z}^2) < 1$ for each n. □

Example 4.3.9. *There exists a sequence (G_n) of invariant random subgraphs of a Cayley graph such that $1 > \lim p_c(G_n) > p_c(\lim G_n)$.*

Proof. We define G_n as a vertex and edge replacement (see Subsection 1.2.3 and [4, Example 9.8]) of $\mathbb{Z}^2$ where we replace each vertex x by the graph Q_x isomorphic to Q_n (the subgraph of $\mathbb{Z}^2$ spanned by the box $[-n, n]^2$) and we replace each edge by a path of length two that joins the middle points of the neighboring sides of the boxes corresponding to the endpoints of the edge. The graphs G_n can be considered as deterministic subgraphs of $\mathbb{Z}^2$ with a randomly chosen root. The sequence G_n converges to $\mathbb{Z}^2$.

We show that $\frac{1}{2} < \lim p_c(G_n) < 1$. Denote by $G_n(p)$ the subgraph obtained by the Bernoulli(p) percolation on G_n, and let $H_n(p)$ be the following percolation on $\mathbb{Z}^2$: let an edge $\{x, y\}$ open, iff both edges are open in the path that joins the boxes

Q_x and Q_y in G_n. The existence of an infinite cluster in $G_n(p)$ implies the existence of an infinite cluster in $H_n(p)$. The law of H_n equals the law of the Bernoulli(p^2) percolation on $\mathbb{Z}^2$, hence $p_c(G_n) \geq \frac{1}{\sqrt{2}}$ for each n.

To show that $\limsup p_c(G_n) < 1$, we define the percolation $\bar{H}_n(p)$ on $\mathbb{Z}^2$. Denote by $\mathcal{A}_x(n)$ the event that the vertices $(0,-n)$, $(0,n)$, $(-n,0)$, $(n,0)$ are in the same cluster in Bernoulli(p) percolation on the box $Q_x \subset G_n$. Let an edge $\{x,y\} \in \bar{H}_n(p)$, iff $\{x,y\} \in H_n(p)$, and both of the events $\mathcal{A}_x(n)$ and $\mathcal{A}_y(n)$ occurs. The existence of an infinite cluster in $\bar{H}_n(p)$ implies the existence of an infinite cluster in $G_n(p)$. Let $1 > p_0 > \frac{1}{2}$ be arbitrary. There is an $\varepsilon > 0$ such that if the marginals of a 2-dependent percolation on $\mathbb{Z}^2$ are at least $(1-\varepsilon)^4$, then this percolation stochastically dominates Bernoulli(p_0) percolation; see [60, Theorem 0.0]. Lemma 4.2.9 implies, that we can find constants $1-\varepsilon < p_1 < 1$ and N such that for any $p > p_1$, $n \geq N$ and for any vertex $x \in V(\mathbb{Z}^2)$ the event $\mathcal{A}_x(n)$ occurs with probability at least $1-\varepsilon$, thus $\mathbb{P}(e \in \bar{H}_n(p)) \geq p_1^2(1-\varepsilon)^2 \geq (1-\varepsilon)^4$ for any edge $e \in E(\mathbb{Z}^2)$. The events $\{e_1 \in \bar{H}_n\}$ and $\{e_2 \in \bar{H}_n\}$ are independent if the distance of e_1 and e_2 is at least 2, hence $\bar{H}_n(p)$ stochastically dominates Bernoulli(p_0) percolation. It follows that $\limsup p_c(G_n) \leq p_1 < 1$. □

4.4 On transitive graphs of cost 1

As proved in [14, Theorem 5.3], a transitive graph G is amenable if and only if it has an invariant spanning tree $\mathcal{T}$ with at most two ends, hence with expected degree 2 and $p_c(\mathcal{T}) = 1$. Briefly: for the existence of $\mathcal{T}$ for an amenable G, see the proof of Proposition 4.4.1 below, while from an invariant connected spanning graph $\mathcal{T}$ with $p_c(\mathcal{T}) = 1$ it is not hard to construct an invariant mean on G, and thus deduce amenability.

Proposition 4.3.6 tells us that, under the stronger condition of sub-exponential growth, we get a spanning tree $\mathcal{T}$ with the stronger property $p_T(\mathcal{T}) = p_T^a(\mathcal{T}) = 1$. Moreover, we can achieve approximately 1-dimensional percolation behavior $p_c(G_k) \to 1$ via connected spanning subgraphs that have the same large-scale geometry as G.

Proposition 4.4.1. *If G is a transitive amenable graph, then there is a sequence of invariant random subgraphs G_k which satisfies the following: each G_k is a bi-Lipschitz (in particular, connected) spanning subgraph of G, the girth of G_k tends to infinity and G_k locally converges to an invariant random spanning tree $\mathcal{T}$ with at most two ends.*

If G is a transitive graph with sub-exponential volume growth, then $\lim p_c(G_k) = 1$.

Proof. We construct $\mathcal{T}$ as in [14], Theorem 5.3: let F_n be a sequence of Følner sets such that $\sum_{n=1}^{\infty} \frac{|\partial_E F_n|}{|F_n|} < 1$. For each n and $x \in V(G)$ choose a random $g_{x,n} \in Aut(G)$ that takes o to x, and a random bit $Z_{x,n}$ that equals 1 with probability $\frac{1}{|F_n|}$. Choose all $g_{x,n}$ and $Z_{x,n}$ independently. Let $\omega_n := E(G) \setminus \bigcup_{x \in V(G), Z_{x,n}=1} \partial_E(g_{x,n} F_n)$; i.e., we remove all edges in the boundaries of the translates of F_n with $Z_{x,n} = 1$. Let $\bar{\omega}_n = \bigcap_{k \geq n} \omega_k$. Each $\bar{\omega}_n$ has only finite components.

To construct $\mathcal{T}$ and G_k, choose uniform labels L_e in [0,1] independently for each $e \in E(G)$. For each finite component of $\bar{\omega}_1$ take the minimal spanning tree of the component with respect to the labels. Denote by T_1 the union of these trees. Let T_2 be the union of T_1 and the edges in $\bar{\omega}_2 \setminus \bar{\omega}_1$ with minimal labels such that the components of T_2 are spanning trees of the components of $\bar{\omega}_2$. Continue inductively, and let $\mathcal{T} := \bigcup T_n$. This is an invariant random spanning tree, which has at most 2 ends (otherwise it would have infinitely many ends, which is impossible, since G is amenable).

To construct G_k we define a color for each edge. Let all edges in $\mathcal{T}$ be green. In each component of $\bar{\omega}_1$ do the following: consider the edge with the smallest label which has no color. If there is a path of length at most k between its endpoints consisting of green edges, then color it red, otherwise color it green. Continue inductively for the edges in the component. This procedure defines a color for each edge of $\bar{\omega}_1$. If all edges in $\bar{\omega}_n$ have a color, then continue coloring the edges of $\bar{\omega}_{n+1} \setminus \bar{\omega}_n$ in the same way. Let G_k be the union of the green edges. It follows from the construction that G_k is invariant, its girth is at least $k+2$ and for each edge of G there is a path in G_k between its endpoints with length at most k. The sequence G_k converges to $\mathcal{T}$.

If G has sub-exponential volume growth, then so does $\mathcal{T}$ and each G_k, and all of them are unimodular (by [77, Corollary 1] and Proposition 1.2.6 above). Thus $p_c(G_k) \to 1$ follows from Proposition 4.3.6. □

It might be surprising at first sight that, as opposed to having a spanning subgraph with $p_c = 1$, the existence of a sequence G_k as in the proposition does not imply amenability: if we chose the graph G in the next example to be any non-amenable Cayley graph, then $G \times \mathbb{Z}$ is non-amenable as well. Our originial example was the non-amenable special case when G is the 3-regular infinite tree, but it was simplified and generalized by Yuval Peres, as follows.

Example 4.4.2. *Let G be the Cayley graph of a finitely generated group. Then $G \times \mathbb{Z}$ has a sequence of invariant bi-Lipschitz subgraphs G_k with $p_c(G_k) \to 1$.*

Proof. Let $S = \{s_1, \ldots, s_n\}$ be the generating set of the group that defines the Cayley graph. Consider the following subgraphs $G_k \subseteq G \times \mathbb{Z}$: we keep all the edges in the subgraphs $\{v\} \times \mathbb{Z}$ and the edges $\{e\} \times \{nkj + ik\}$ where $j \in \mathbb{Z}$ and e is an edge with label s_i. We choose a uniform random integer $b \in \{0, \ldots nk - 1\}$ and translate this subgraph by (id, b) to get the invariant subgraph G_k of $G \times \mathbb{Z}$. Each G_k is clearly bi-Lipschitz equivalent to $G \times \mathbb{Z}$. On the other hand, we have $p_c(G_k) \to 1$: either from Proposition 4.3.4, or more directly, by observing that the universal cover T_k of G_k can be obtained from the $n+1$-regular infinite tree by replacing "$\frac{n}{n+1}$ proportion" of the edges by a path of length at least k; for this tree, it is easy to see that $p_c(T_k) \to 1$, while $p_c(T_k) \leq p_c(G_k)$ holds by [68, Theorem 6.47]. □

So, what is the class of transitive graphs for which the existence of such a sequence G_k may be expected? The answer seems to have something to do with the notion of cost from measurable group theory. The *cost of a group* $\mathcal{G}$ is defined as half of the infimum of the expected degrees of its invariant connected spanning graphs. The $\mathcal{G}$*-cost of a transitive graph* G may be defined similarly, over $\mathcal{G}$-invariant random connected spanning *sub*graphs of G, where $\mathcal{G} \le Aut(G)$ is a vertex-transitive subgroup of graph-automorphisms. It is not known in general that, if we first fix a Cayley graph G of $\mathcal{G}$, then the $\mathcal{G}$-cost of G is always as small as the cost of $\mathcal{G}$ (which is the cost of the complete graph on $\mathcal{G}$). Nevertheless, we have seen that cost 1 can be achieved inside any Cayley graph of any amenable group (since the expected degree of an infinite unimodular tree with at most two ends is 2).

We will now show that a sequence of invariant spanning subgraphs G_k with $p_c(G_k) \to 1$ implies that the cost is 1. The bi-Lipschitz condition does not appear here, but it is quite possible that once we have a sequence with $p_c(G_k) \to 1$, it can always be modified to fulfill the bi-Lipschitz property, as well. Note that the bi-Lipschitz condition is also natural from the point of view of Elek's combinatorial cost for sequences of finite graphs [37].

Lemma 4.4.3. *If* Γ *is a Cayley graph of* $\mathcal{G}$*, and there exists a sequence of* $\mathcal{G}$*-invariant connected spanning subgraphs* $G_k \subset \Gamma$ *with* $p_c(G_k) \to 1$*, then the cost of* Γ*, hence of* $\mathcal{G}$*, is 1.*

Proof. Take $\varepsilon_k \to 0$ such that $p_c(G_k) > 1-\varepsilon_k$. Then, all clusters of Bernoulli$(1-\varepsilon_k)$ percolation on G_k are finite almost surely. Let the set of closed edges be denoted by $\eta_k \subset G_k \subset \Gamma$, an invariant percolation itself. In each finite cluster, take a uniform random spanning tree, a subtree of G_k. The union of all these finite spanning trees and η_k will be ω_k. One the one hand, it is clear that ω_k is a connected spanning subgraph of G_k, hence of Γ. On the other hand, the expected degree of o in ω_k is at most $\mathbb{E}\deg_{\eta_k}(o) + 2 \le d\varepsilon_k + 2$, where $\deg_\Gamma(o) = d$. As $k \to \infty$, we obtain that the cost of Γ is 1. □

We do not know if the converse of Lemma 4.4.3 holds:

Question 4.4.4. *Does there exist, for any Cayley graph* G *of any group* $\mathcal{G}$ *of cost 1, a sequence of* $\mathcal{G}$*-invariant bi-Lipschitz spanning subgraphs* $G_k \subset G$ *with* $p_c(G_k) \to 1$*? At least for amenable* G*?*

For amenable Cayley graphs G, a first step of independent interest could be a positive answer to the following question, mentioned in Subsection 4.1.1:

Question 4.4.5. *For any amenable Cayley graph, is there an invariant random spanning subtree of sub-exponential growth? More boldly, does there always exist an invariant random Hamiltonian path?*

Köszönetnyilvánítás

Szeretném megköszönni a témavezetőmnek, Timár Ádámnak az érdekes problémákat, hasznos tanácsokat, amikkel bevezetett a matematikai kutatások világába. Köszönöm továbbá Pete Gábornak a tanulságos közös munkát.

Hálás vagyok a Bolyai Intézetnek a támogató légkörért, amiben elkezdhettem a doktori tanulmányaimat, továbbá Abért Miklósnak és csoportjának az inspiráló közegért, amiben dolgozhattam.

Köszönöm a sok biztatást a barátaimnak, a családomnak és különösen a páromnak, Czank Tamásnak.

Summary

The aim of the thesis is to examine a fundamental question related to local weak convergence: are certain parameters determined by the local structure of the graph? After introducing random graphs and local weak convergence, we examine the continuity of two graph parameters. In Chapter 3 we investigate the concentration and limiting properties of the (directed) matching ratio of finite random graphs. In Chapter 4 we generalize parameters related to Bernoulli percolation to infinite unimodular random graphs and study the convergence of them along local weak convergent sequences.

In Chapter 1 we introduce the basic notions and the families of random graphs that we study in the further chapters. In the thesis we consider finite random graphs and (possibly infinite) unimodular random rooted graphs. For random rooted graphs, unimodularity is a natural symmetry assumption that generalizes the features of the most studied families of random graphs, such as invariant random subgraphs of Cayley graphs and local weak limits of finite graphs. Unimodular random rooted graphs obeys the *Mass Transport Principle* which will be a useful tool in our proofs in Chapters 3 and 4.

In Chapter 2, we present the concept of *local weak convergence* introduced by Benjamini and Schramm [17]. A sequence of random rooted graphs (G_n, o_n) converges to the random rooted graph (G, o) in the local weak sense if for every $r > 0$, the distribution of the ball of radius r around the root in G_n converges to the distribution of the ball around o with radius r in G. We illustrate the definition by several examples which will be used also later in the thesis. We present a non-trivial example of convergence of infinite random graphs without a uniform bound on the degrees: the sequence of infinite unimodular Galton–Watson trees converges if and only if their offspring distributions converge. We further examine a stronger notion, the almost sure local weak convergence of sequences of finite graphs and present a few examples which will be important in Chapter 3. We prove that the sequence of finite random graphs obtained by the preferential attachment rule converges almost surely in the local weak sense.

In Chapter 3, we examine the concentration and limiting properties of the (directed) matching ratio of finite random graphs, i.e., the relative size of the maximum size (directed) matching. This parameter is closely related to an important parameter in control theory, as shown by Liu, Slotine and Barabási [61]. The results of this chapter were motivated by the observations of [61], that suggested that the

matching ratio of directed graphs is in some cases essentially determined by the degrees of the graphs and converges along certain local weak convergent sequences. We formulate and prove rigorously this two main statements of [61], which were based on numerical results and heuristics from statistical physics.

Our results on the concentration of the matching ratio are presented in Section 3.2. In Theorem 3.2.1, we give a bound on the probability that the difference between the directed matching ratio of a directed random graph given by the random configuration model with a fixed sequence of degrees and its expected value is larger than ε. This probability is exponentially small in $|V(G)|^2/|E(G)|$ which implies a strong concentration for large graphs with a degree sequence given by a random variable with finite expected value. The theorem holds with fixed in- and out-degrees and also when only the total degrees are fixed. In Theorem 3.2.3, we prove that preferential attachment graphs satisfy a similar strong concentration phenomenon.

In Section 3.3, we generalize the result of Elek and Lippner [40] on the convergence of the matching ratio, originally proved for deterministic graph sequences with a uniform bound on the degrees. We show in Theorem 3.3.3 that the mean of the directed matching ratio of local weak convergent sequences of random directed graphs with a bound only on the expected value of the degrees converges to the properly defined matching ratio parameter of the limiting graph. This extended theorem allows us to apply it for the most widely used families of scale-free networks, which were the main motivation of [61]. Our results imply that the matching ratio of Erdős–Rényi random graphs, the random configuration model and the preferential attachment graphs converges in a strong sense: it converges almost surely.

In Chapter 4, we investigate the continuity of parameters of infinite unimodular random graphs related to Bernoulli percolation. We examine the generalizations of the classical percolation critical probabilities $p_c = \sup\{p : \mathbb{P}_p(\text{there is an infinite cluster}) = 0\}$, $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$ and $\tilde{p}_c$ defined by Duminil-Copin and Tassion in [35]. The last quantity was in fact designed to give a simple new proof of $p_c = p_T$ for transitive graphs, and to address the question of locality of critical percolation: whether the value of p_c depends only on the local structure of the graph.

In Section 4.2, we examine the relationship between the generalizations of the critical probabilities $p_c, p_T, \tilde{p}_c$, originally introduced for transitive graphs, to *extremal unimodular random graphs*. We also investigate two further natural generalizations, that are annealed versions of the above critical probabilities and are denoted by p_T^a and $\tilde{p}_c^a$. We show in Theorem 4.2.1 that the equation $p_c = \tilde{p}_c$ also holds for bounded degree unimodular random graphs. The inequality $p_c \geq p_T \geq p_T^a$ follows from the definitions, but besides this, anything can occur in the class of unimodular graphs. As our examples in Section 4.2.2 show, the classical sharpness of the phase transition can fail even for invariant random subgraphs of $\mathbb{Z}^2$, i.e. for unimodular graphs the inequality $p_T < p_c$ can hold.

In Section 4.3, we examine Schramm's conjecture [16] in the case of unimodular random graphs: does $p_c(G_n)$ converge to $p_c(G)$ if G_n converge to G in the local weak sense and $\sup p_c(G_n) < 1$? In Propositios 4.3.4 and 4.3.6 we give conditions which imply $\lim p_c(G_n) = p_c(\lim G_n)$. However, our examples in Section 4.3.3 show that the locality conjecture does not hold in the generality of bounded degree unimodular random graphs: there are sequences of unimodular graphs such that G_n converges to G but $p_c(G) > \lim p_c(G_n)$ or $p_c(G) < \lim p_c(G_n) < 1$ (Examples 4.3.7 and 4.3.9). In fact, Example 4.3.9 shows that none of the generalized critical probabilities are continous in the class of unimodular random graphs.

As a corollary to our positive results, we show in Proposition 4.4.1 that for any transitive graph with sub-exponential volume growth there is a sequence $\mathcal{T}_n$ of large girth bi-Lipschitz invariant subgraphs such that $p_c(\mathcal{T}_n) \to 1$. It remains open whether this holds whenever the transitive graph has cost 1.

Összefoglalás

A disszertáció célja a lokális gyenge konvergenciához kapcsolódó egyik alapvető kérdés vizsgálata: meghatározza a gráf lokális struktúrája bizonyos paraméterek értékét? A véletlen gráfok és a lokális gyenge konvergencia bevezetése után két paraméter folytonosságát vizsgáljuk. A 3. fejezetben véges véletlen (irányított) gráfok párosítási arányának koncentrálódásával és konvergenciájával foglalkozunk. A 4. fejezetben Bernoulli perkolációhoz kapcsolódó paramétereket általánosítunk végtelen unimoduláris véletlen gráfokra, és ezek konvergenciáját vizsgáljuk lokálisan gyengén konvergens sorozatok mentén.

Az 1. fejezetben bevezetjük az alapvető fogalmakat és a további fejezetekben vizsgált véletlen gráfosztályokat. A disszertációban véges véletlen gráfokkal és (végtelen) unimoduláris gyökeres gráfokkal foglalkozunk. Véletlen gyökeres gráfok esetén az unimodularitás egy természetes szimmetria feltevés, ami a leggyakrabban vizsgált véletlen gráfok, mint például Cayley gráfok invariáns részgráfjainak vagy véges gráfok lokálisan gyenge limeszeinek tulajdonságait általánosítja. Az unimoduláris gráfok teljesítik az úgynevezett *tömeg transzportációs elvet*, ami egy hasznos eszköz lesz számunkra a 3. és 4. fejezet bizonyításaiban.

A 2. fejezetben bemutatjuk a Benjamini és Schramm [17] által bevezetett *lokális gyenge konvergencia* fogalmát. Azt mondjuk, hogy gyökeres véletlen gráfok (G_n, o_n) sorozata lokálisan gyengén konvergál a (G, o) gyökeres gráfhoz, ha minden $r > 0$ esetén a G_n gráf gyökere körüli r sugarú gömb eloszlása tart a G gráf gyökere körüli r sugarú gömb eloszlásához. A definíciót számos példával illusztráljuk, amelyek eredményeit a későbbiekben is felhasználjuk. Bemutatunk egy nem triviális példát végtelen gráfok egy olyan sorozatának konvergenciájára, ahol nincs egyenletes korlát a fokszámokra: az unimoduláris Galton–Watson fák sorozata pontosan akkor konvergál egy unimoduláris Galton–Watson fához, ha az utódeloszlások sorozata is konvergál. Vizsgálunk továbbá egy erősebb fogalmat, a majdnem biztos lokális gyenge konvergenciát, amire mutatunk néhány, a 3. fejezetben fontossá váló példát. Bebizonyítjuk, hogy a preferential attachment gráfok sorozata majdnem biztosan lokálisan konvergens.

A 3. fejezetben véges (irányított) véletlen gráfok párosítási arányát, azaz a maximális méretű párosítás relatív méretét vizsgáljuk. Liu, Slotine és Barabási [61] megmutatták, hogy a párosítási arány szorosan kapcsolódik a kontrollelmélet egy fontos paraméterhez. A fejezet eredményeit ennek a cikknek az észrevételei motiválták, amelyek arra utaltak, hogy az irányított gráfok párosítási arányát bizonyos esetekben lényegében meghatározza a fokszámeloszlás, továbbá ez a paraméter konvergál egyes lokálisan gyengén konvergáló sorozatok mentén. Ezeket a numerikus eredményeken és statisztikus fizikából származó heurisztikus módszereken alapuló eredményeket fogalmazzuk meg precízen és bizonyítjuk be a fejezetben.

A 3.2. részben bemutatjuk a koncentrációval kapcsolatos eredményeinket. A 3.2.1. tételben becslést adunk rögzített fokszámsorozat mellett a véletlen konfigurációs modellel megadott véletlen gráf esetén annak a valószínűségére, hogy a párosítási arány ε-nál jobban eltér a várható értékétől. Ez a valószínűség exponenciálisan kicsi $|V(G)|^2/|E(G)|$-ben, amiből nagy gráfok esetén egy erős koncentráció következik, ha a fokszámsorozatot egy véges várható értékű valószínűségi változó adja meg. A tétel teljesül rögzített ki- és be-fokok esetén, továbbá akkor is, ha csak a teljes fokszámot rögzítjük. Hasonló koncentráció eredményt bizonyítunk preferential attachment gráfok esetén a 3.2.3. tételben.

A 3.3. részben általánosítjuk Elek és Lippner tételét a párosítási arány konvergenciájáról [40], amelyet eredetileg egyenletesen korlátos fokszámú determinisztikus gráfsorozatokra bizonyítottak. A 3.3.3. tételben bebizonyítjuk, hogy véletlen irányított véges gráfok lokálisan gyengén konvergens sorozata esetén, melynél csak a fokszám várható értékéről teszünk fel korlátosságot, a párosítási arány várható értéke konvergál a limeszgráfon definiált megfelelő paraméterhez. Ez az általánosabb tétel lehetővé teszi, hogy a leggyakrabban vizsgált skálafüggetlen gráfmodellekre alkalmazzuk, amelyek a fő motivációt jelentették Liu, Slotine és Barabási [61] számára. Az eredményeinkből következik az alábbi gráfok párosítási arányának erős értelemben vett (majdnem biztos) konvergenciája: Erdős–Rényi gráfok, véletlen konfigurációs modell, preferential attachment gráfok.

A 4. fejezetben végtelen unimoduláris véletlen gráfok Bernoulli perkolációhoz kapcsolódó paramétereinek folytonosságával foglalkozunk. Az alábbi klasszikus kritikus valószínűségeket vizsgáljuk: $p_c = \sup\{p : \mathbb{P}_p(\text{van végtelen fürt}) = 0\}$, $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$, továbbá a Duminil-Copin és Tassion [35] által definiált $\tilde{p}_c$. Ezen utóbbi mennyiség bevezetésének segítségével egy egyszerű új bizonyítást adtak a $p_c = p_T$ egyenlőségre tranzitív gráfok esetén, továbbá egy lehetséges eszköznek bizonyult a perkolációs kritikus valószínűségek lokalitásának vizsgálatában, azaz annak eldöntésében, hogy a p_c kritikus valószínűség csak a gráf lokális struktúrájától függ-e.

A 4.2. részben az eredetileg tranzitív gráfokra bevezetett $p_c, p_T, \tilde{p}_c$ kritikus valószínűségek *extremális unimoduláris véletlen gráfokra* való általánosításait vizsgáljuk. További két természetes általánosítást is vizsgálunk, amelyek a fenti kritikus valószínűségek átlagoláson alapuló változatai, ezek p_T^a és $\tilde{p}_c^a$. A 4.2.1. tételben megmutatjuk, hogy a $p_c = \tilde{p}_c$ egyenlőség minden korlátos fokú unimoduláris gráfra fennáll. A $p_c \geq p_T \geq p_T^a$ egyenlőtlenség következik a definícióból, de ezen kívül bármi előfordulhat az unimoduláris gráfok osztályában. Ahogy a 4.2.2 részben leírt példáink mutatják, a klasszikus éles fázisátmenet nem teljesül még $\mathbb{Z}^2$ invariáns részgráfjaira se, azaz unimoduláris véletlen gráfokra fennállhat a $p_T < p_c$ egyenlőtlenség.

A 4.3. részben Schramm sejtését [16] vizsgáljuk unimoduláris gráfok esetén: tart a kritikus valószínűségek $p_c(G_n)$ sorozata $p_c(G)$-hez, amennyiben G_n lokálisan gyengén konvergál G-hez és $\sup p_c < 1$? A 4.3.4. és 4.3.6. állításokban feltételeket adunk, amelyekből következik a $\lim p_c(G_n) = p_c(\lim G_n)$ egyenlőség. Azonban a 4.3.3. részben leírt példáink mutatják, hogy a lokalitásról szóló sejtés nem teljesül általánosan korlátos fokú unimoduláris gráfokra: léteznek unimoduláris gráfoknak olyan sorozatai, amelyekre a $p_c(\lim G_n) > \lim p_c(G_n)$ vagy a $p_c(\lim G_n) < \lim p_c(G_n) < 1$ egyenlőtlenség áll fenn (4.3.7. és 4.3.9. példák). A 4.3.9. példa azt is mutatja, hogy semelyik általánosított kritikus valószínűség sem folytonos az unimoduláris gráfok osztályán.

A pozitív eredményeink következményeként megmutatjuk a 4.4.1. állításban, hogy minden szubexponenciális növekedésű tranzitív gráfnak vannak olyan $\mathcal{T}_n$ nagykörű bi-Lipschitz invariáns részgráfjai, amelyekre $p_c(\mathcal{T}_n) \to 1$ teljesül. Nyitott kérdés, hogy ez teljesül-e tranzitív gráfokra, melyek költsége 1.

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