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The Chooser-Picker games

PHD Dissertation

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Contents

Abstract

In the first part of this thesis we make an attempt to review some of the element of the theory of Combinatorial Games that have relevance to our work. Out of this vast area we tried to list those that underline the deep connections in Mathematics, reveal the difficulty of the process in giving exact proofs or even the right notions. We also wanted to stress from the very beginning the most important speciality of that subject: we have a compass to sail these troubled waters; the are the varying heuristics.

The theory started with the mathematical examination of such games that tic-tac-toe or hex, where a player need to achieve certain position in order to win. For those both the *Achievement game* and the *Positional game* terms are used. In the original forms the goal of both players was the same, reaching a winning position; this version is called now in retrospective as the strong or “Maker-Maker” version. The weak or “Maker-Breaker” is already a heuristic to understand the strong games; here one player (Maker) goes for the goal as before, while the other (Breaker) wins by preventing Maker to achieve this goal.

This games have an intimate relation to the Probabilistic Method. Many tools that has been developed to handle random events have counterpart in games, although the transition not always apparent, and frequently far from being trivial. In this process other heuristics, the “accelerated” and “biased” games help us. In the first more steps can be done at each turn of the game, while in the second the accelerations of the players might differ from each other.

This setup allows the development of several powerful methods for evaluating the possible outcome of games. Among those we describe strategy stealing arguments, pairing strategies, weight functions, auxiliary games etc. We also introduce the Reader to the recent development of a new variants of positional games, which are proved to be very nice and useful tools for analysing complex games. These are the *Picker-Chooser* (P-C) and *Chooser-Picker* (C-P) games.

The main goal of this work is to understand Picker-Chooser (or Chooser-Picker) games as deeply as possible. Undoubtedly, the central problem is here if these games are really good heuristics for the Maker-Breaker games? Our efforts are concentrated on this so-called Beck’s conjecture. We confirm the conjecture for a number of special cases, and this path also leads us to other natural problems. This quest has three main parts.

At first we examine the complexity of Picker-Chooser and Chooser-Picker games. Here we found that it is NP-hard to decide the winner for both P-C and C-P games [24]. Then we discuss the Picker-Chooser version of well-known games, to explore the differences and similarities among the various types. The examined games are the C-P 4×4 *Tic-Tac-Toe*, the Picker-Chooser version of generalized Shannon switching game, the Chooser-Picker version of the k-in-a-row and some of the Chooser-Picker, Maker-Breakr and Picker-Chooser Torus games. We also generalize the idea of pairing strategies,

and give a computer free proof for their non-existence in the case of the most notorious Harary game, the Snaky [24, 21]. We improve a little on the “Erdős-Selfridge” theorem for Chooser-Picker games, although a gap still remains here [21].

Secondly, we solve with the Chooser-Picker 7-in-a-row game. This game is quite interesting because the last really valuable result for the 8-in-a-row game was made more than 30 years ago. (Playing on the infinite board, the second player can achieve a draw in the 8-in-a-row game.) Since then, many people has tried to prove a draw for the 7-in-a-row, so far unsuccessfully. In light of Beck’s conjecture the Chooser-Picker version of that game should be a Picker win, that we prove. The proof is a bit lengthy and a non-trivial case study. After that, we sketch some ideas how might attack the original (Maker-Breaker) version of this game [22].

Finally we will discuss the Picker-Chooser diameter games. The diameter of random graphs and the outcome of various Maker-Breaker games are both notoriously hard to decide. Here we have found a very interesting result that how different are the outcomes of the Maker-Breaker versions and from that of the Picker-Chooser versions [2, 23]. Unlike the Maker-Breaker case, here the upper and lower bounds are of the same order.

Chapter 1

Introduction

1.1 All games bright and beautiful

What could be more appropriate than borrowing the title of John H. Conway's paper from 1977 for a section intended to go through on the whole history of Games? Games and Mathematics do not blend easily: for long centuries the games, especially the games of chance, had been treated with even more despise than Mathematics.[1] Worse, playing games were not only illegal sometimes, but dismissed as childish behavior, and a serious scholar definitively cannot risk to endeavor such a frivolous activity, although the subtle paradox of Zeno about the hare and turtle could have been a warning sign. Still, the old habits die hard.

Then, in 1654, the ideas of Blaise Pascal and Pierre Fermat gave birth, and, more importantly, *respect*, to a new field of Mathematics, Probability theory. After that it took less than 300 years that the notion of randomness became one of the most important ones in science, somehow rehabilitating the suspicious dice.

The deterministic games had to wait longer, although it started earlier. Another Frenchman, Bachet de Méziriac, published the first mathematical book that was devoted to recreational mathematics completely in 1612, see [20]. Bachet's famous *subtraction game*[2] became a prototype for the *Combinatorial Games*; it is a two person game with a discrete (and finite) set of states, it has rules governing the transition among those, and the player who is unable to move according to these rules, loses the game. It contains the basic ingredients one might expect from a mathematical game: simple rules, clean but nontrivial solution; in this case by the notion of divisibility.

The noble chess has also inspired a lots of efforts that led to Zermelo's great theorem and it became a benchmark problem for Computer Science, and later for testing the strength of hardware configuration. One might wonder, why the similarly adored game of Go did not give more to the development of the theory? While the main reason of this is the difficulty of the game, the elusiveness of the rules also play some role. Elwyn Berlekamp spent considerable effort to compare the "dialects" of Go that result in differ-

[1] While the mathematician had been banned from Rome for a certain period, laws against games were constructed all the time [72].

[2] In fact, Bachet defined it by addition. Two players tell positive number by taking turns. Starting at zero, they add an arbitrary integer between one and ten to the previous number, and the player, who announces one hundred is the winner.

ent outcomes in subtle situations. Since it makes mathematical approach impossible, he introduced his own variation “Mathematical Go” instead, and solved hard endgames, see [13]. Note, that the rules of chess contained loopholes, too.

The golden era of certain games started with John von Neumann[3] and Emile Borel. Neumann proved the famous *minimax theorem* in full generality that is a milestone in the theory of matrix games. His followers (David Gale, Harold Kuhn and Alan Tucker) connected Neumann’s theory to the newly formed subject of Linear Programming, that is still one of the best understood and computable part of game theory [31]. Note that von Neumann proposed the notion and investigation of LP duality and conjectured the equivalence of the so-called Strong Duality theorem and the minimax theorem.

In to give a solid foundation to theoretical economy, Neumann, together with Oscar Morgenstern, introduced the notion and study of *Cooperative games* in their classical book, see [50]. The impact of that book was tremendous, it has not only reshaped economy, but its language and its point of view still dominates the field. However, one aspect of the competition is missing from their approach, the case of non-cooperative players. John Forbes Nash took this important new step in his seminal paper [47]. It turned out that for general games the players’ strategies have distinguished distributions, now we called those *Nash equilibriums* such that a player does not win (in expectation) from deviating from it by alone. Thereafter the theory of cooperative and non-cooperative games swallowed the theoretical economy; while there is no mathematical Nobel prize, the Nobel Memorial prize for Economy was given several times for game theoretical work.[4]

Better or worse, the entities have started to multiply. Nash himself made attempts to include the cooperative games to his theory, that is to find non-cooperative *mechanisms* such that the arising Nash equilibrium(s) would be a solution of the cooperative version of the game.[5] The Nash equilibrium has been specialized (Kuhn, sub-game perfect equilibrium), generalized (Aumann, correlated equilibrium), led to new notions (Shapley value), and sometimes it has been even applied (Maynard-Smith, Evolutionary Stable Strategies).

In the shadow of these theories there were two other lines of games that are just special classes of the “already solved” matrix games. One of those was started with Charles Bouton’s NIM, continued with the Grundy-Sprague theory and culminated in Conway’s games, see [13, 18]. Although Conway’s theory is a mixture of set theory and arithmetic, it was coined as *Combinatorial Game Theory*, since it usually deals with finite objects.

The other field that we discuss in detail also of combinatorial nature, and its origin is even more humble than the previous one: those games come from Tic-Tac-Toe, 5-in-a-row or Nine-Man-Morris. The common in those games is that in order to win, the player needs to achieve a prescribed pattern; this explains the name *Positional Games*. No matter what the name is, these games have numerous links to diverse parts of mathematics. Without completeness we might mention Ramsey theory, on-line algorithms, Random Graphs, Topology, Complexity etc, for an excellent guide see the recent book of József Beck, [10]. It is not a great surprise that the field inherits all the beauty and difficulty experienced in the previous subjects. Therefore it is an appropriate language to express and study well-

[3]Of course his “real” name is Neumann János, or “Jancsi” to the initiated, but he used mainly the von Neumann form in his publications.

[4]Among those were Harsányi János, who received this prize together with John Nash and Reinhardt Selten in 1994.

[5]This is the so-called *Nash program.* Its success or even its possibility is debated; one thing is sure: a great number of works have been published on that.

known old problems, and to get a fresh view on those.

Finally, let us say a few words about Mathematics, and the mathematics of Positional Games. Mathematics is hard. Finite mathematics is even harder, since most of its laws have no apparent cause, and the geometric intuition cannot capture the essence of the phenomena. Indeed, we do not know the outcome of seemingly innocent small games, not to mention how to play those, and we have not much hope to find out these recently. Still, there are some beacons in the dark, making parts of the field tractable; these are what this work all about. The most important of those are the random heuristics, the acceleration of games and the Chooser-Picker versions. Here our main goal is to understand the last one as much as possible.

1.2 Combinatorial games

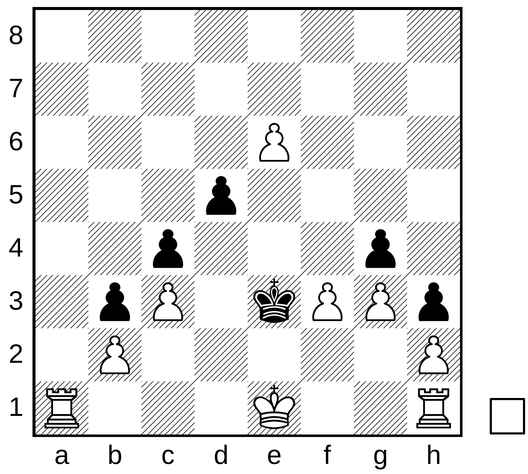
The archetype of combinatorial games is the game of Chess. Among other lessons it also shows that one can never really trust in rules written natural languages. Before 1972 the official rules for castling by FIDE were:

Castling consists of moving the king two squares toward a rook, then placing the rook on the other side of the king, adjacent to it.[2] Castling is only permissible if all of the following conditions hold:

1. The king and rook involved in castling must not have previously moved;
2. There must be no pieces between the king and the rook;
3. The king may not currently be in check, nor may the king pass through or end up in a square that is under attack by an enemy piece (though the rook is permitted to be under attack and to pass over an attacked square).

However, Tim Krabbé has composed the following piece that illustrated the incompleteness or unwanted ambiguity of the rule:

Tim Krabbé, 1972



Mate in 3

The winning line is:

1. e6-e7, Ke3$\times$f3
2. e7-e8R, Kf3-g2
3. 0-0-0-0-0-0 mate!

Here the unusual notation "0-0-0-0-0-0" means the "long castle" of the white King goes from e1 to e3 and the Rook goes from e8 to e2. Clearly, this move conforms to the rules 1. 2. and 3. To eliminate such bizarre possibilities an additional assumption had been added to the rule:

4. The king and the rook must be on the same rank.

From the Chess and thousands of others games the following scenario can be distilled, that more or less formalize what we call *Combinatorial Game*.

- There are two players, I (White) and II (Black), and I starts the game.
- There are finitely many positions and a starting position is given.
- The feasible (legal) moves of the players are given in every positions.
- The players take turns.
- Every sequence of legal moves are finite.
- A sequence of legal moves beginning with the starting position and ending with an end position is a game.
- The outcome in every end position is determined; one of the players win or it is a draw.
- Both players have all information; they know the rules and they legal moves, remember the moves they had already took, see all his/her and opponent moves etc.
- No moves or rules that depend on some randomness.

Zermelo used more or less the same definition to spell out and prove[6] his famous theorem:

Theorem 1.1. *A combinatorial game is either win by one of the players, or both players have a strategy resulting in a draw.*

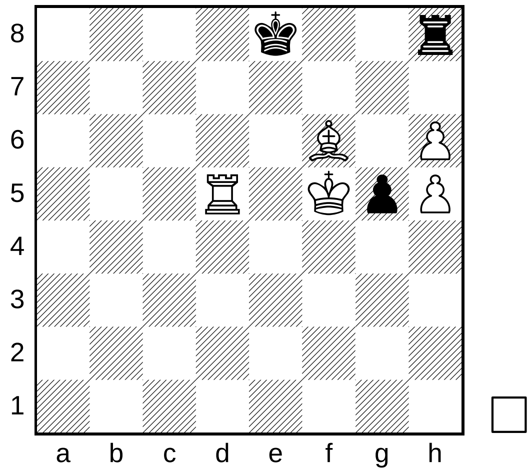
Sketch of the proof. One can built up a tree T such that the vertices are pairs sub games[7]. For the (possible) sub games x, y the directed edge (x, y) exists if the player, who is about to move at x, has a move that leads to the sub game y. Let r be the sub game consisting of the a starting position with empty history and White is about to move. With this T is the rooted tree that can be reached from r along the edges defined before. Since all plays are finite, the directed paths in T end up in leaves. The outcome of the sub game in each leaf can be decided, and one may compute the outcome in each inner vertex of T by *backward labeling.* $\square$

[6]The original proof of the theorem was wrong. It was fixed and the statement was considerably strengthened by Kőnig, Kalmár and Neumann, see [19, 76]. The result allows the use of *indirect proofs* for finite games, so sometimes it is referred as *game theoretic tertium non datur*.

[7]That is not only positions, but a description about the actual play leading to that position.

Remarks. In some sense the algorithm used in the proof of Theorem 1.1 solves *all* combinatorial games, but this is just the theory. The tree T is usually too big, although sometimes symmetries may help, or one can use the method for solving important sub games. We also have to mention that notions of position, play and strategies are more subtle then one may think for the first glance. The position itself is not enough to determine the game as we can see from the following piece that is a nice example for *existence arguments*.

W. Langstaff *Chess Amateur* 1922.



Mate in 2

The computer chess programs usually get into trouble finding the solution. After some deliberation one realizes that there are two different lines that the game has reached this position.[8] If King or Rook of Black has already moved, then 1. Kf5-e6 leads to the inevitable mate 2. Rd5-d8. On the other hand, if neither of these pieces has moved before, that is Black still has the right for castling, and fending off the Rd8 mate, another pattern arises. In that case the only possible last move of Black was 0. -, g7-g5, since a pawn on the square g6 would have attacked White's King. So in that case White has the right to capture the pawn on g5 *en passant*, that is 1. h5×g6, and for 1. -, 0-0 White mates with 2. h6-h7, for any other steps with 2. Bd5-d8.

So the general rule is that history cannot be erased; we shall see further example for this later. Still another issue is the need to make the notion of strategies more precise. Centuries ago there persisted a funny belief there might be a "winning formula" for Chess, and if a player applies it, wins the game regardless the color. Certainly the better minds dismissed this, asking what would happen if *both* players follow that magic formula?

The strict way to think about strategies is to call them functions that map the sub games to the set of legal moves that are possible in the given sub game. However, this approach also has paradoxical consequences. There are infinite games in which one of the player surely wins at the end, but neither of them have winning strategies! The first game of that nature was the Banach-Mazur game, then Gale and Steward, later McKenzie and Paris gave interesting examples, see [9].

[8] A rule for chess compositions requires that the position must come from the standard starting position by taking legal steps. In *fairy chess* problems this rule does not apply.

Finally, there are some games for which the labels of the sub games can be computed without searching for the whole game-tree T. The best known example for that is the game of Nim.

1.2.1 Nim

Nim is a two-player mathematical game of strategy in which players take turns removing objects from distinct heaps. On each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap. The person who makes the last move (i.e., who takes the last object) wins.

Variants of Nim have been played since ancient times. The game is said to have originated in China (it closely resembles the Chinese game of “Jianshizi”, or “picking stones”), but the origin is uncertain; the earliest European references to Nim are from the beginning of the 16th century. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1901, but the origins of the name were never fully explained. The name is probably derived from German nimm meaning “take”, or the obsolete English verb nim of the same meaning. It should also be noted that rotating the word NIM by 180 degrees results in WIN [75].

Nim (or more precisely the system of nimbers) is fundamental to the Sprague-Grundy theorem, which essentially says that in normal play every impartial game is equivalent to a Nim heap that yields the same outcome when played in parallel with other normal play impartial games (see disjunctive sum).

Since the number of objects are finite, so the game can not be a draw. So if a player could avoid the loss then he/she wins. This is the base of the strategy of Bouton. Let suppose that we can define a property $\mathcal{P}$ of a NIM game in that way:

(i) If all of the heaps are empty, then P fulfilled.

(ii) If property $\mathcal{P}$ does not fulfilled, then it is possible to move such a way that after it P is exists.

(iii) If P exists in a stage, then it won’t be in the next stage.

If at the beginning P does not exist, then the first player (because of (ii)) chooses a step to satisfy P. Therefore after the second player’s turn (if there is any) the first is also in such a situation that P does not exists (because of (ii) again). It means that the first has a legal move again, which drives to a stage with property $\mathcal{P}$. Sooner or later the objects/stones will be diminished, $\mathcal{P}$ holds at the end and the player who has to move loose. That looser is the second player. (If at the beginning P exists then the result is a second player win).

Some words about the strategy:

Nim has been mathematically solved for any number of initial heaps and objects; that is, there is an easily calculated way to determine which player will win and what winning moves are open to that player. In a game that starts with heaps of 3, 4, and 5, the first player will win with optimal play.

The key to the theory of the game is the binary digital sum of the heap sizes, that is, the sum (in binary) neglecting all carries from one digit to another. This operation is also known as “exclusive or" (xor) or “vector addition over $GF(2)$". Within combinatorial game theory it is usually called the nim-sum, as will be done here. The nim-sum of x and y is written $x \oplus y$ to distinguish it from the ordinary sum, $x + y$. In normal play, the winning strategy is to finish every move with a Nim-sum of 0. For example of the sum of heaps with size 3, 4, and 5 is the following: $X = 3 \oplus 4 \oplus 5 = 2$.

1.2.2 Conway theory

Here we just give some hint about this great theory, since it has little relevance to our work, but it would be unfair to ignore it completely. For a detailed introduction see [18, 13]. Conway melted the constructions of Dedekind, Cantor and Neumann into a very general notion of games, that includes practically *everything* [9] that can be considered as Mathematics. He builds up games recursively from small games that are listed as left (L) and right (R). That is the form of all games is $\{L|R\}$, where L and R are lists of already defined games. The general element, or *option* of L (R) is denoted by x^L (x^R). The players are also called L and R, when it does not cause ambiguity. They take turns, and a legal step for L (R) is to pick a game from the list L (R). The player, who cannot make a legal step, loses the game; this is the connection to Nim.

Of course, the whole journey starts with the game $\{|\}$ in which both lists are empty. The next games are the $\{\{|\}|\}$, $\{|\{|\}\}$ and $\{\{|\}|\{|\}\}$. The player who starts the game $\{|\}$ loses, since there is nothing to pick. The same argument shows that L (R) wins $\{\{|\}|\}$ ($\{|\{|\}\}$), while the player who moves wins $\{\{|\}|\{|\}\}$.

As more and more games appear in that process, one needs to order those and make equivalence classes containing those that differ only formally. Let $x \geq y$ if $x^R \leq y$ and $x \leq y^L$ holds for no x^R, x^L. An $x = \{L|R\}$ is a *number*, if for all x^R, x^L are numbers and $x^R \leq x^L$ never holds.

The games x and y are *identical*, $x \equiv y$, if their left and right sets are identical, while those are *equal*, $x = y$, if $x \leq y$ and $y \leq x$.

The sum and product of games are defined such a way that conform to the one Dedekind used in the case of *cuts*. If $x = \{L|R\}$, then $x \not\leq x^L$ and $x^R \not\leq x$. The sum of the games x and y should be the game where the player on move decides to take a game from either x or y, and the game continues on that game. That is $x + y := \{x^L+y, x+y^L|x^R+y, x+y^R\}$, $-x := \{-x^R|-x^L\}$ and $xy := \{x^Ly+xy^L-x^Ly^L, x^Ry+xy^R-x^Ry^R|x^Ly+xy^R-x^Ly^R, x^Ry+xy^L-x^Ry^L\}$.

One possibility here is to develop the arithmetic of games. The games form a field in which the numbers are an ordered subfield. The $0 := \{|\}$, $1 := \{0|\}$ and $-1 := \{|0\}$ are numbers, while $\{0|0\}$ is not a number and cannot be compared by them. Therefore the relations “$\equiv$" and “$=$" differ from each other. A number $x = \{L|R\}$ is determined by the largest element of L and the smallest element of R, but it may very well happen that $xz \neq yz$ even though $x = y$, if x, y, z are not numbers. Some more numbers are:

$2 := \{1|\} = \{-1,1|\} = \{0,1|\} = \{-1,0,1|\}$, $-2 := \{|-1\} = \{|-1,0\} = \{|-1,1\} = \{|-1,0,1\}$, $1/2 := \{0|1\} = \{-1,0|1\}$, $-1/2 := \{-1|0\} = \{-1|0,1\}$.

To evaluate a complicated number the so-called *simplicity theorem* is very handy:

[9] If one stays within the *cumulative hierarchy.*

Theorem 1.2. *Suppose for $x = \{L|R\}$ that some number z satisfies $x^L \not\geq z \not\geq x^R$ for all x^L, x^R-re, but no options of z satisfies the same condition.*[10] *Then $x = z$.*

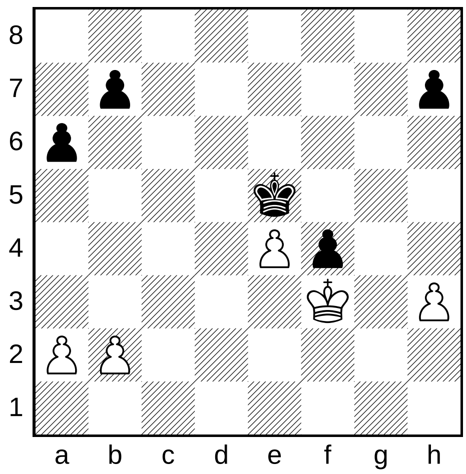
We have $x \geq z$ unless some $x^R \leq z$ or $x \leq z^L$. But from $x \leq z^L$ we can deduce $x^L \not\geq x \leq z^L < z \not\geq x^L$ for all x^R, x^L, from which we have $x^L \not\geq z^L \not\geq x^R$, contradiction the assumption about z. That is $x \geq z$, similarly $z \geq x$, so $x = z$. □

Going on that line one can get the dyadic rationals and all real numbers embedded in games. There are more exotic objects, too. Such are like the infinity $\omega := \{0, 1, 2, \dots\}$, Leibniz *infinitesimals*, e. g. $1/\omega = \{0|1, 1/2, /1/4, \dots\}$, and not only the successor of ω, $\omega + 1 = \{0, 1, 2, \dots, \omega|\}$ but its *predecessor* $\omega - 1 = \{0, 1, 2, \dots|\omega\}$, which has no equivalent among the ordinals.

The other way is to get use of the defined games. Every combinatorial games can be written in the form $\{L|R\}$. The games has their value that, roughly, says that a player is how much ahead to the opponent. If a game x is the some of games, that is $x = \sum_{i\in} y_i$, and the value of each y_i is known, then it is straightforward to compute the value (and the outcome) of x. [11]

The following example came up in a real chess game, and Conway's theory is quite appropriate to explain the subtleties of the resulting position.

Schweda-Sika, Brno, 1929



□

White starts and wins

In fact Euwe and Hooper noticed that the position is win by the player who moves, and Elkies found an elegant proof for this claim [53]. One can see right away the player who is forced to move his King first loses the game.[12] On the other hand, they will run out of run pawn movements, and breakthrough (and consequently pawn promotion) is not possible. The moves on the Queen side and King side are independent, that is a game decomposes to the *disjunctive sum* of the two sub games. In order to compute this sum, we need some notations. Let $* := \{0|0\}$ (*star*), $\uparrow := \{0|*\}$ (*up*), $\uparrow * := \{0, *|0\} = \uparrow + *$

[10] That is, either $z^R \leq x^L$ or $z^L \geq x^R$ hold for some choice.
[11] Indeed, the special case of this addition is the Nim addition.
[12] With other words, the central part of the game is $\{|\} =: 0$.

(*up-star*), $\Uparrow := \{0 | \uparrow *\} = \uparrow + \uparrow$ (*double up*), and $\Uparrow * := \{0 | \uparrow\} = \uparrow + \uparrow + *$ (*double up-star*). Furthermore $\downarrow := - \uparrow$, $\Downarrow := - \Uparrow$ and $\Downarrow * := - \Uparrow *$.

With this, the sub-game on the the h-file is $\Downarrow *$. The sub-game on Queen side needs some case checking; it is $\uparrow$, see [53]. Now $\uparrow + \Downarrow * = \downarrow *$. $\downarrow * || 0$, that is the game is a first player win. White can start with 1. h4 giving Black the game $\uparrow + \downarrow = 0$, while Black wins with 1. - a5.

However, the games we shall discuss are rarely of this nature; until the last few steps those cannot be broken into sum of simpler games. So we will need other, mainly combinatorial tools to attack them.

1.3 Methods, issues and paradigms

We make an attempt to collect the most important stuff about combinatorial games. This venture is far from being complete, the size of the subject prevents us from achieving this. The volumes of Berlekamp, Conway and Guy [13], Beck [9], Nowakowski et al. [53] and the collection of Fraenkel [30] give good references for that. Here we want to mention only those parts of the theory that have close relation, or even continuation in our work. Even in that case, we try to do it briefly.

First we go through on methods that can give the outcome of a game *without* exploring the whole game-tree and then introduce heuristics that help to understand to essence of some games. Note that these methods and ideas are not independent, their elements come up together in the examples and the applications.

1.3.1 Pairing

A really old chestnut is the infinite *placing coins* (to the table) game. In the easiest case a round table is given, onto which the players have to take round coins in turns such that (1) two coins must not overlap (2) the weight center of each coin must be supported by the table. The player, who cannot make a legal move, loses the game.

White has a winning *pairing strategy*: place the first coin to the center, and for each step of Black, put down the next coin in a centrally symmetric place. This way White restores the central symmetry of the leftover table[13], so whenever Black has a legal move White also has. But the game is finite in length, since the area of the table is finite, so one of the player must lose, and from the previous argument it must be Black.

There are many relatives of that result. In a similar one, first appeared in lecture note than in print [62], the unavoidable defeat of a player comes from a combinatorial fact. A beautiful old result of Harper and Chvátalova, see e. g. [43] says that if one labels the vertices of the $n \times n$ grid with the numbers $1, \dots, n^2$, then there will be two neighboring vertices where the difference of the labels at least n. Moreover there is a labeling in which the differences are not more than that.[14]

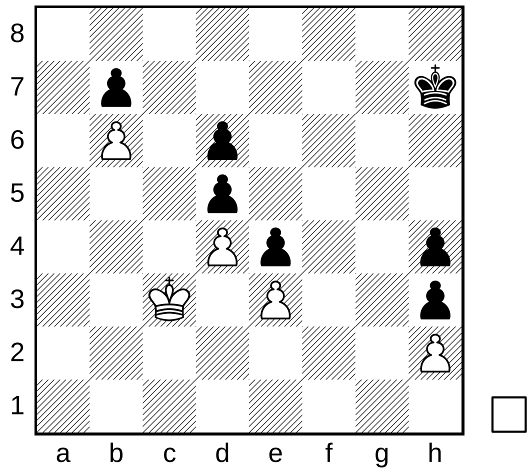
[13] If the table is not centrally symmetric at the beginning, this strategy breaks down, and for the general case nothing is known.

[14] That is the *bandwidth of the $n \times n$ grid is n*. In general to decide the bandwidth of a graph is NP-complete.

We make the two players labeling the vertices of the grid with the numbers $1, \ldots, n^2$, using one number only once, and the player whose move first causes a difference at least n loses the game. Obviously the game is finite and it cannot be a draw, so from Theorem 1.1 one of the player has winning strategy. In fact, it is almost the same as in the Placing coins game, just one pair up not only the vertices of the grid but the numbers also. With the pairing $\mathcal{M} = \{(i, n^2 - i + 1) : 1 \leq i \leq \lfloor n^2/2 \rfloor\}$ White wins if $n = 2k + 1$, while Black wins if $n = 2k$. (Again, if the graph is not central symmetric, not much is known. Note that one can make out a game from a combinatorial impossibility theorem ofttimes.)

The well-known notion of different types of *opposition* in Chess endgames or *bamboo* in Go are pairings used every day. The *corresponding* or *conjugate* squares in the famous piece of Kornél Éberszt is a pinnacle of this direction, see [63].

K. Éberszt



White moves and draw

Here White has to protect the squares f3 and c6. In means that White's King must move to a5 (e2) if Black's King moves to c6 (g4).[15] From here one can pair up some other squares recursively: b4-d7, a4-d8, c3-e6, b3-e7, a3-e8, d2-f5(h5), c2-f6(h6), b2-f7(h7), a2-f8(h8), d1-g5, c1-g6, b1-g7, a1-g8. The squares a8, b8, c8 have no specific pairs, just White has to make sure to occupy b4 or a4 when Black leaves these ones. (E.g. a5-c8, b5-b8, a5-a8 is OK.)

1.3.2 Potential functions

The potential function appeared in special combinatorial games, in which Black does nothing, just gives back the position to White.[16] The best known examples are Conway's frogs [13] pp. 715–717, and the classification of the Peg game positions by de Bruijn [15]. The board is the infinite two-dimensional lattice (or rather an infinite board with

[15]The pairing b5-d7 is not good, since from that position White needs 4 moves to reach e2, while Black needs only 3 moves to get to g4.

[16]To put it differently, such a game is a one person game, or *Solitaire*, like the Rubik cube, or a formal proof of a theorem.

squares if we consider the dual) in the first and a finite subset of it in the second. There are pieces (men, frogs, pegs) on the board, and the legal steps are jumps. A piece x can jump a neighboring piece y, if the square s on the other of y is empty. Alas, the jump costs the life of y, so the number of pieces is decreasing.

Conway asked, if all frogs are placed on one side of a horizontal line ℓ, how far the frog may go to the other side of ℓ. The goal of the Peg game is to remove all of the pegs but one. Accordingly, de Bruijn was interested in describing the equivalence classes of peg sets, that is those configuration that might be reached from each other with the legal step or its reverse.

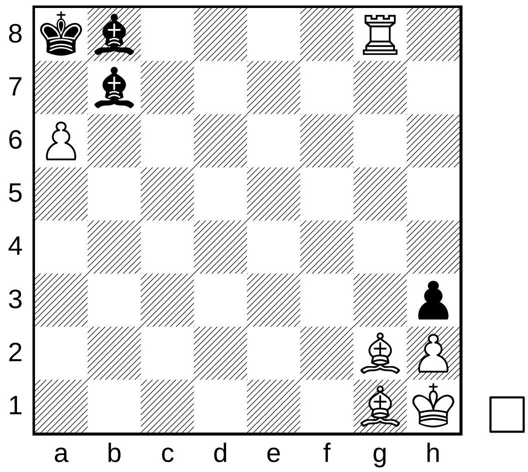
To answer these questions potential functions are developed. First one labels the cells with some numbers, and compute the potential by adding up the labels on the occupied squares.[17]

However Conway used real labels that were decreasing exponentially in both ways from a starting point, while de Bruijn appropriate pairs from $GF(2)$. Conway showed that his potential function is non-increasing, and with this he deduced that the frogs would never reach the fifth row.

The potential function of de Bruijn's one is, and together with other considerations are enough to classify the reachability in the mentioned sense.

In one of the greatest examples the pairing and potential function approach is mixing: the potential function helps in describing the pairing among positions of the game.

G. Broecker, London Chess Fortnightly, 1892.



Selfmate[18] in 9

While the solution was known, no one really understood the game (except perhaps the composer). Then Gyula Neukomm [51] found a brilliant description for the winning strategy.

[17]The function, parity of a permutation, in Samuel Lloyd famous "15-game" is rather a clever algebraic invariant then a potential function.

[18]That is White forces Black to win the game. One can always construct the so-called *misére* version of a game declaring that the winner is, who would lose with the original rules.

It is obvious, that the only way to fulfill the requirement Black has to be forced to capture White's white square Bishop in such a way that the Rook cannot protect the checkmate on the main diagonal. Neukomm gave values to the squares what the Rook and the white squared Bishops, were about to use. These are (for the Rook) h8=1, c8=2, d8=3, e8=4, f8=5 and g8=6. For the Bishops: g2=2, f3=3, e4=4, d5=5, c6=6 and b7=7. Let the value of a piece the label of the square the piece occupies. Let the value of White be the sum of the values of the Rook and the white colored white square Bishop, while the value of Black be the value of the black colored white squared Bishop.

Now White's winning strategy is to move with the Rook or the Bishop that equalizes his/her value with the value of Black. Furthermore White should increase the value of the Bishop if it is possible. With this the play become finite, and the longest variation takes nine steps.

Let us give a less known, simple game, the "Breeding stones" game. The board is the first quarter of the infinite board, see higher. One refers to a cell like (i, j) if it is on the i^{th} column and the j^{th} row. The game starts with one stone on $(1, 1)$, and a legal step is to duplicate a stone on (i, j) and to move its two successors to the squares $(i + 1, j)$ and $(i, j + 1)$, provided those are empty. Our garden is

$$G = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}.$$

The goal is to rid of G from stones. Let us label the square (i, j) with 2^{2-i-j}. This results in a potential constant one. One the other hand, there are two other invariants: the number of stones in the first row and column. It mean that to reach the goal we should put stones to all available squares which is clearly impossible in finite steps.

1.3.3 Acceleration

When a game (or its researcher) is exhausted, it is always an option to consider the *acceleration* of it. It means that in one turn a player might take more then one legal steps. The role of the two players are not necessarily the same, the (a, b) affix mean that White takes a, while Black takes b steps in turn, if it has sense at all, see [9, 59, 61].

There are several nice, playable games created this way, the $(2, 2)$-six-in-a-row[19], $(2, 2)$-Connect four (with the additional rule that White takes only one in the first round), $(1, 2)$-Chess, where Black has only the pawn and the King etc.

For an unsettled game the motivation is the following: we hope that the outcome of an accelerated game similar to the original one (with some natural changes) and *easier* to compute it. It is not always the case, accelerations can bring surprises [2].

A great frustration for the mathematicians that the outcome of the Chess is still not known. The only thing that we know from Theorem1.1 is that either some of the players win, or it is a draw. Alas, no matter how extremely unlikely that Black win it, we cannot even prove that White has a draw. What about some acceleration? White wins the (a, a)-chess if $a \geq 4$ in the first round, and for $a = 3$ in the second round.[20]

The case of $(2, 2)$-chess is not solved completely, but at least we know that Black cannot win. The usual argument goes like this. Suppose for contrary that Black has a

[19] It is possible to put $a \neq b$, and set up different goals for Black and White.

[20] After 1. e2-e3, Qd1-f3, Bf1-c4, it is impossible to protect the square f7, and hence the King.

winning strategy. Let White starts with a pair of reversible moves, e. g. 1. Nb1-c3, Nc3-b1, and then just pretend he/she is Black and *steal* the opponent strategy.

In fact, this proof has a hole that was probably discovered several time, maybe the first time by A. N. Kolmogorov see [33]. The problem is with the argument above that White has already made a pair of steps, and according to the rules of Chess the player, who could achieve a position third times may apply for a draw. So if the assumed winning strategy of Black would lead the starting position two times, then Black can have a draw that is not available for White![21]

A way to fix the argument is to use two boards in it. On the first board start as before, on the second play what was Black answer on the first board. Then play on the first board as Black answered on the second and so on. This results in two games that are "mirror images" of each other, on the other hand Black wins in both, because of the assumed winning strategy, that is a contradiction.

1.3.4 Cake cuts

One of the oldest problem in life is how to achieve fair division? Having a two persons and goods that are easy to separate, like a cake or gold-dust one can rely on the *gold-digger* algorithm: one cuts the other chooses. Of course if there are more persons and they might get envy, we have more trouble, see [70, 71].

One might use the gold-digger method in games, such that one player makes a selection (picks up something), while the other decides which part he/she likes, and takes (chooses among the alternatives).[22] For example if we have some money, we can divide it by the classical gold-digger algorithm, or one player (call Picker) picks up two coins/banknotes, and the other (Chooser) takes one, the other goes to Picker. One can see right away, that playing greedily, Chooser gets at least half of the money. If the distributed "goods" have some interactions, the game became more complex. If two captains want to select their soccer teams from a bunch of kids, Picker may pair up a the best defender with the best midfield player. Chooser faces problems, since giving up one of them seriously restricts the later choices. (So they do not do this; the captains take turns to pick they favorite players.)

In [69] Spencer studied a nice game, that illustrates both this cake-cutting approach and randomness. In the *Tenured game* the Dean and the Chair(man) play against each other with the faculty of a department. The Chair makes two groups from the faculty, the Dean can fire one group but has to promote every person in the other group. The Dean's goal to get rid of the whole faculty, while the Chair would like to save at least one person. The Chair wins if after some promotions a person reaches a tenured position. Let a_i be the number of faculty members, who need i promotion to be secure.

Theorem 1.3. *The Dean wins iff* $\sum_{i=0}^{\infty} a_i 2^{-i} < 1$.

The "if" part only. The Dean just flips a fair coin, and chooses randomly. The expected number of people reaching tenure is exactly $\sum_{i=0}^{\infty} a_i 2^{-i} < 1$. But if the Chair had a winning strategy, there would be at least one tenured guy at the end. Since there is no draw in this game, the Dean must have a winning strategy. $\square$

[21] True, it is a quite weird scenario, but this is what Mathematics is all about. One have to be careful, it is very easy to err dealing with obvious looking statements on games.

[22] This sequential selection is the opposite of the acceleration, it *slows down* the gold-digger algorithm.

1.3.5 Randomness

From the solution of the *Moriarty paradox*, or more broadly speaking from the invention of *mixed strategies* it has been clear that coin tossing can be extremely useful in playing matrix games. These games have a hidden[23] element, while the players know their opponent strategies and the (expected) payoffs, they do not know the actual strategy that the opponent are going to use.

One of the mysterious facts of life that randomness is so useful in completely deterministic situations. There is no general theory for explanation; in the case of games we have the following picture: instead of two perfect players imagine two perfectly mindless ones. They explore the game tree randomly, and get to a leaf (and the corresponding outcome) accordingly. If one player tends to win in that random game, then perhaps the same person can also win anyway.

Of course it is not a big deal to construct a game for which this intuition fails. Let T be a binary tree of hight n and the leaves are labeled by the binary numbers from $00\ldots0$ to $11\ldots1$, meaning that the ith digit describes that the player on move goes to left (zero) or right (one). White wins iff the game ends up in a leaf with all odd digits are one, loses otherwise. Of course, White loses the random game with probability $1-2^{\lfloor -n/2\rfloor}$, and still wins the normal game.

Even when the random intuition predicts a win for a player, the actual winning strategy might have little to do with randomness. However, this little is not nothing, as we can see from the algorithmic approach of the Tenured game. One just have to recognize that the condition $\sum_{i=0}^{\infty} a_i 2^{-i} < 1$ should be maintained by the choice of the Dean, and apply induction. When the Chair makes two groups out of the faculty, for at least one of those the sum taken only on the members of group is less than one half. Let us promote this group and fire the other one. The condition still holds but the number of faculty is less; we are ready by the (implicit) induction hypothesis. □

With this we have completed the cycle: we started with a deterministic game, used a random heuristic to conjecture the outcome (even gave a probabilistic proof), and finally we *derandomized* the heuristic by a *potential function* and got explicit winning strategy in *polynomial time*.

1.3.6 Complexity

We assume that the Reader are familiar the basic concepts of Complexity theory as it is presented for example in [55]. The approach of defining the various complexity measures of certain languages can be very fruitful, and sometimes completely misleading.[24] Indeed, it is easier to argue, why *not* to expect much from that tools.

First of all, for a concrete game, like chess, go or any given finite game of form $x = \{L|R\}$ has a constant complexity. Alas, this does not mean that that one could tell the outcome of small games readily. It is quite amusing (and rather frustrating at the same

[23] Some authors call these *incomplete information games*, others reserve this name for those games which really have unknown element. Note, that in matrix games the information provided is enough if one plays the game several times.

[24] The sheer statements about e.g. the NP-completeness of a problem or the worst case behavior of an algorithm are usually dismissed in the practice, where real problems are to be solved.

time) that no matter of the great theories and fast computers, the solution for tiny, innocent looking games are still hopeless.

One also gets into a dead end considering families of games instead of concrete ones. The proof of Theorem 1.1 shows that the outcome of a game can be computed by backward labeling the corresponding game tree in *linear* time. Too bad, this linearity is measured by the size of the game tree and this tree might be a big one. (Going through on the pairs of *all* possible strategies of the players might need double exponential time in the size of the input.) For certain games, where the history of a play does not matter (i. e. a sub game can be identified with a position) the search reduces to a *Game graph*, and the size of that graph is still exponential in the input.

Some of the (infinite) games are simply unsolvable in the sense that neither player has winning or drawing strategies. The first example of those were the Banach-Mazur game, later Gale and Stewart constructed simpler ones. McKenzie and Paris showed that this phenomenon occurs even among the *Positional games*[25], see [9].

One may think that this strange behavior comes from the fact the sets of possible strategies are too big. (In the analysis of these games some forms of the Axiom of Choice is used). So let us consider only the *computable strategies*. The game as follows. Let U be an universal Turing machine, and White starts the game by giving a word x. Then Black wins iff he can tell how many (possibly infinite) steps U will take on input x. A sure win for either players requires the solution of the *Halting problem.*[26] It is amusing that instead of the abstract machine U, finitely many marked node on the infinite grid and some natural rules might also results in the same. We can make out a two person game from zero person Conway's Game of Life: White places a finite number of cells C and Black has to tell, if the descendants of C will live forever or die out. Black wins iff his answer is good. But the computation of the universal Turing machine given any input x can be coded with appropriate initial cells in the Life game, see [13].

Even simple looking, finite games might become tricky if we restrict the computational power of the players. Jones in [44] proposed the following game: given the polynomial

$$Q(x_1,\ldots,x_5) = x_1^2 + x_2^2 + 2x_1x_2 - x_3x_5 - 2x_3 - 2x_5 - 3,$$

White and Black alternate in assigning nonnegative integer values to the variables in order $x_1,\ldots,x_5$. White wins if, with the substitution, $Q = 0$, otherwise Black wins. The outcome of the game is depend on a yet unknown number theoretic problem. To see this let us write Q as

$$Q(x_1,\ldots,x_5) = (x_1 + x_2)^2 + 1 - (x_3 + 2)(x_5).$$

Since Black picks x_2, he has a winning strategy if and only if there are infinitely many primes of form $n^2 + 1$.

In a similar game, see [29], the computing powers restricted even more. Now our polynomial is

$$Q(x_1,\ldots,x_4) = x_1 - x_2x_4 - x_2 - x_4 - 1,$$

and the players alternately assign values to x_1, x_2, x_3, x_4 in this order. White has to select x_1 as a *composite* integer, $x_1 > 1$, x_3 any positive integer, and Black selects any

[25] We shall discuss these games in detail in the following Chapters.

[26] More precisely, White should exhibit an x for which the halting is undecidable.

positive integers. Black wins if $Q(x_1, \dots, x_4) = 0$; otherwise White wins. Clearly Black has a winning strategy, since

$$Q(x_1, \dots, x_4) = x_1 - (x_2 + 1)(x_4 + 1),$$

and x_1 is composite. But the computation of Black's winning strategy requires significantly greater resources than putting up a tough resistance by White. White can find two large primes p and q about the same size, and setting $x_1 = pq$. Then, in order to win, Black has to factor x_1 that might be too hard problem if the time is limited.

Going through on all these irregularities we might try to measure the hardness of a game not by itself, but by the computational complexity of an *infinite family*. Perhaps the best known example is *Geography*, see in [55]. A directed graph G and a distinguished vertex $x \in V(G)$ are given. The players alternately move a token along an edge, starting from x, and a vertex may be visited only once. A player who cannot take a legal move, looses the game. Let GEOGRAPHY be the language of all graphs for which White wins.

Theorem 1.4. *GEOGRAPHY is a PSPACE-complete language.*

The proof is quite standard, one takes an arbitrary instance of QSAT[27] and reduces it to an word of GEOGRAPHY. QSAT is the language of true words of form

$$\Phi = \exists x_1 \forall x_2 x_3 \dots x_\ell \forall x_{\ell+1} \dots \forall x_{n-1} x_n \phi,$$

where ϕ is a Boolean formula in conjunctive normal form of variables $x_1, \dots, x_n$.

Some problems (or rather languages) concerning games turn out to be NP-complete. In these cases we might say that the game is hard, although these complexity issues involve several possible inputs, not only the usual starting position of a (family of) game(s). On the other hand, and it is more justifiable, a game is easy, if the outcome (winning strategy etc.) can be given in polynomial time as a function of the input.

[27] Of course QSAT is a PSPACE-complete language.

Chapter 2

Positional games

In the main part of the dissertation we will deal with games what we call *Positional games*. In the positional games there is usually two player, they are moving alternately and the aim of the game is to occupy (or prevent to occupy by the other) a winning set by possessing all of the element of that set. Note, that misére versions of this games are also possible; in those games the goal is the *avoidance* of the winning sets.

2.1 Tic-Tac-Toe type of games

In this section we introduce a family of combinatorial games that have very deep connections with other parts of Combinatorics. Before specific definitions, let us see some examples.

2.1.1 Tic-Tac-Toe

The most well-known hypergraph game is the *Tic-Tac-Toe* game. Here the aim of the game is to occupy a row, or a column or a diagonal on a $N \times N$ board.

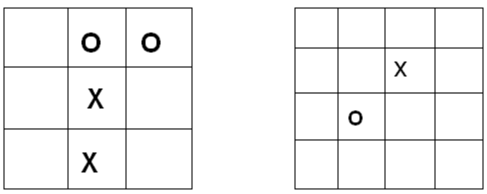


Figure 2.1: The 3×3 and the 4×4 *Tic-Tac-Toe* games. Usually X is the first player, O is the second player

One remark on how complex these small games are: The *Tic-Toc-Tac-Toe* is similar to the *Tic-Tac-Toe*, but here the board is $4 \times 4 \times 4$. By playing a little, it is easy to see that *Tic-Tac-Toe* can be won by the first player. The first player also can win the *Tic-Toc-Tac-Toe* game but there was needed 1500 hours calculation time for Oren Patashnik at 1979. In the same year Allis *strongly* solved the game, that is he computed the outcome of *all* possible positions. Still, there is no easily describable winning strategy, see [56]. The

$5 \times 5 \times 5$ and $6 \times 6 \times 6$ the versions[1] are still open, although probably these are draw games.

To give a common frame for these games, we introduce the notion of *hypergraph games*.

2.1.2 Hypergraph games

Given an arbitrary hypergraph $\mathcal{F} = (V, E)$ (or $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$) the first and second players take elements of V in turns. We call this games *hypergraph games*. When we think about hypergraph games, we usually think on a *board game* where the fields are the vertices and certain subsets of the board are the winning sets.

We call these games *Maker-Maker games* (or *strong games*) where the player, who takes all elements of an edge $A \in E$ first wins the game [7]. The Maker-Maker game are associated to a (false) natural fairness; that is why so many played version of those. In fact, these games are *hot*[2] by Conway's terminology:

Theorem 2.1 (Nash, Hales-Jewett)**.** *In the strong version of a game, the first player wins or the game is draw.*

Proof. Assume for contradiction that the second player has a winning strategy S. The application of this strategy by the first player is as follows. The first player starts the game arbitrarily and then forget about his first step and plays the second player assumed winning strategy S. If S would require a field that the first player have already marked, then the first player declares he take this move and make an arbitrarily move that he forgets. Since to have an "extra field" on the board never cause harm to any player, the first player wins, contradiction.[3]. $\square$

This technique is called *"strategy stealing."* The invention of it is attributed to John Nash, who first applied it to the game called *Hex*. For positional games it was formally proved by Alfred Hales and Robert Jewett in 1963, see [35].

2.1.3 Hex game

This game was invented independently by Piet Hein (1942) and John Nash. The players are placing stones into a rhombus board with $n \times n$ hexagonal grid. The goal is to form a connected path of one's stones linking the opposing sides of the board marked by the players color, before the opponent connects his or her sides in a similar fashion. Note, that Hex is *not* a hypergraph game. The first player who completes his or her connection, wins the game. The Hex, unlike some of the games which are only interesting on mathematical point of view, is exciting and addictive game. People make puzzles, competitions of it; here $n = 10$ or $n = 11$. (These sizes are unexplored, the best result was achieved by Kohei Noshita who gave explicit winning strategy for $n \leq 8$ [52].)

[1] The boards are three dimensional cubes consisting $5^3 = 125$, and $6^3 = 216$ smaller cubes, and the winning sets are the lines and diagonals.

[2] Stricly speaking this means that the player, who moves, wins the game. In generally hot means that the right of a move always an advantage in the game.

[3] Note that the proof is existential and does not provides information about how should play to win.

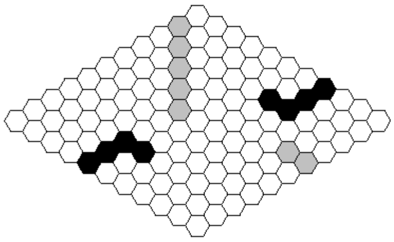


Figure 2.2: The Hex game. The players alternately color the hexagons. The aim of the game is to make a one color path between the two opposite sides.

Theorem 2.2 (Nash - Gale)**.** *If all cells of the Hex board are colored by the two players then there exist a N-E, S-W path.*

Proof. This proof is a sketch of the proof given by David Gale in his article on Hex and the Brouwer Fixed-Point Theorem, [32]. It is uses only some well-known facts of graph theory.

Let G be a planar 2-connected graph in which all vertices have degree two or three such each face of G corresponds to a cell on a Hex board. We add four more edges and pendant vertices; one to each corner of the board. We can assume that all cells are occupied because a game of Hex cannot end before a player wins or no cell is available. Now make a subgraph G' of G created by keeping the edges that separate two faces of different colors on the Hex board and the additional edges at the corners.

It is easy to see that G' will mostly consist of vertices of degree zero or two. Exactly four vertices will have degree one, namely the pendant vertices connecting to the four corners: a, b, c and d. It is an exercise that graphs with vertices of degree less than three consist of isolated vertices, simple cycles and simple paths. Since G' has exactly four vertices of degree one there must be exactly two of these paths.

These two paths must connect $a - b$ and $c - d$ or $a - d$ and $b - c$. In the first case white will have won, in the second black. □

Now we can decide the outcome of the game Hex:

Theorem 2.3 (John Nash, 1949)**.** *The first player wins the Hex.*

Proof. Follows immediately from Theorem 2.2 and Theorem 2.1. □

2.2 Heuristics

Of course one has greatest motivation in solving Maker-Maker games, or in general those games that proved to be worthy, like Chess or Go. But as we mentioned in the introduction, it is very hard even to have clues for what type of results can be proved at all. In most part of Mathematics there are good examples and (relatively) simple exercises that help to shape the consequent theories. In natural sciences the experiments play the same role. Alas, one cannot rely on rich source of good examples of well understood hypergraph games. All but the smallest concrete games are still unsolved, and even when the outcome is known, the existence or exhaustive search proofs are not really illuminating.

We believe this problem was one of the main factors that slowed down the development of the field for so long.

The breakthrough was the invention of new games that (1) preserve some characteristics of the original game (2) hopefully easier to handle (3) the outcome can be related to the original.

Some of these games are interesting or having significance of thier own, while others, like the random heuristics, are more diagnostic tools than games. The examination of games needs both analysis and synthesis involving smaller games and combinatorial arguments. This process leads to making tools for tools as it quite often happens elsewhere. The most important heuristics are the *Maker-Breaker games, the acceleration, the Picker-Chooser (and Chooser-Picker) games* and *biased games.* The last approach is closely related to the *random heuristic* [9] or *"Erdős paradigm"* [45] that might be considered as zero person game played on some board. Let us see some details.

2.2.1 The weak version of the games

It is quite natural to define the so-called *weak version* of the positional games [6], where the the first player win by completing a winning set any time, while the second player wins if he/she can prevent the first player's win. It means that the first player do not have to be afraid of that the second player occupies a winning set. This version is also called as *Maker-Breaker game* from obvious reason. Of course the first player is Maker, and the second is Breaker. It is easy to see the following statement, see [7].

Statement 2.4. *If the Breaker wins in the weak version of a game, then the strong version is draw.*

Remarks. If the first player wins in the weak version of the game, it does not follows that the first player wins the strong game. A simple example for this phenomenon is the 3×3 Tic-Tac-Toe: In the picture 2.1 it can be seen, that in the weak version the Maker can place to the lower right square and win; while in the strong game the first player should defend, hence the chance of winning disappear. Since there is no tie at the game Hex, so the two versions coincide. Generally the languages of both the strong and weak games are PSPACE-complete languages, see [65, 16].[4]

2.2.2 Random heuristic

We have already sketched a idea of two players who randomly move in the game tree in subsection 1.3.5. For a Maker-Breaker hypergraph game $(V, \mathcal{F})$ this kind of randomness can be approximated by a more comfortable probability space. Let us take the vertices one by one, toss a (fair) coin; if it shows head, then give the vertex to Maker, otherwise to Breaker. This way the hypergraph is *2-colored*, all vertices get the color of either Maker or Breaker. A 2-coloring is *good*, if all edges contains both colors.[5] The expected number of one-colored edges is $\sum_{A \in \mathcal{F}} 2^{-|A|+1}$, which immediately gives the following theorem.

[4] One has to be careful with these results. A concrete game is always of constant complexity. For the PSPACE-completeness one need no only an infinite family, e. g. the Hex or Go for $n \in \mathbb{N}$, but an infinite family of starting positions.

[5] For the more formal general definition see subsection 2.3.1.

Theorem 2.5 (Erdős, 1963)**.** *There is a good 2-coloring of the hypergraph $\mathcal{F} = (V, E)$ if*

$$\sum_{A \in E} 2^{-|A|+1} < 1.$$

As a heuristic, one may think that the Maker-Breaker game on the hypergraph (V, E) is a draw, that turned out to be true:

Theorem 2.6 (Erdős-Selfridge, 1973)**.** *If Maker starts the weak version of the $(\mathcal{F} = V, E)$ hypergraph game and*

$$\sum_{A \in E} 2^{-|A|+1} < 1,$$

then Breaker wins[6].

We wait with the proof of Theorem 2.15 until subsection 2.3.2. Though, we have to stress that one cannot overestimate the impact of the method developed for that. It gave the Probabilistic method a new meaning and meant a breakthrough in the research of Positional games.

2.2.3 Accelerated and biased games

If both of the players claim $k > 1$ vertices in each turn, then we talk about an *accelerated* games. In general, if the players claim a and b vertices, then we call it the *biased* version of the game.[7]

We shall touch the accelerated k-in-a-row in the next chapter, and discuss a biased game diameter game deeply in section 5.2. Note that these are examples where the heuristic fails for the Maker-Breaker game, while be in great accordance with other heuristics, e.g. with the random heuristic. A nice, and very useful result is an Erdős-Selfridge type of theorem by Beck [4].

Theorem 2.7. *[4] If E is the family of winning sets of a positional game, then Breaker has a winning strategy in the $(a : b)$ game when*

$$\sum_{A \in E} (1 + b)^{1-|A|/a} < 1.$$

Another striking line connects the theory of random graphs and the biased games [17, 2, 5, 45, 12]. Here the theory of random graphs [14] and Positional Games blend nicely with the following setup. As defined earlier, Maker wins if a monotone property $\mathcal{P}$ holds for the subgraph of his edges. Our purpose is to find the smallest b_0, for which Breaker wins the $(1 : b_0)$-game. While to get the exact value of b_0 is almost impossible, one may shows asymptotic upper and lower bounds on it. Some of the best examples are: for Hamiltonicity and maximum degree, see Beck and Hefetz et al. [8, 5, 40], for planarity, colorability and graph minor games, Hefetz et al. [39], for building a specific graph G or creating a large component, Bednarska and Łuczak [11, 12].

[6]The condition for the uniform case is sometimes spelled out as $|E| + \Delta(\mathcal{F}) < 2^n$, where $\Delta(\mathcal{F})$ is the maximum degree of the hypergraph.

[7]The biased games arise quite often and naturally when one builds a global strategy out of strategies of auxiliary games defined on non disjoined sub boards.

2.2.4 The Chooser-Picker and the Picker-Chooser games

Studying the very hard clique games, Beck [6] introduced a new type of heuristic, that proved to be a great success. He defined the *Picker-Chooser* or shortly P-C and the *Chooser-Picker* (C-P) versions of a Maker-Breaker game that resembles fair division, (see [70, 71]).

In these versions Picker takes an unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. When $|V|$ is odd, the last element goes to Chooser. Beck obtained that conditions for winning a Maker-Breaker game by Maker and winning the Picker-Chooser version of that game by Picker *coincide* in several cases. Furthermore, Breaker's win in the Maker-Breaker and Picker's win in the Chooser-Picker version seem to occur together. That is the Picker-Chooser (Chooser-Picker) games are themselves heuristics for the Maker-Breaker games.

The probabilistic intuition also helps in studying Picker-Chooser (Chooser-Picker) games. Let $a = b = 1$ and $||\mathcal{F}|| = \max_{A \in E(\mathcal{F})} |A|$ be the *rank of the hypergraph* $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$. In that case, there is an almost perfect analogue of Theorem 2.7 as follows:

Theorem 2.8. *[6, 21] 3.6 If*

$$T(\mathcal{F}) := \sum_{A \in E(\mathcal{F})} 2^{-|A|} < \frac{1}{3\sqrt{||\mathcal{F}|| + 0.5}},$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $\mathcal{F}$. If $T(\mathcal{F}) < 1$, then Chooser wins the Picker-Chooser game on $\mathcal{F}$.

2.2.5 Beck's conjecture

Beck [6] has another interesting remark, namely that Picker may win easily the Picker-Chooser game if Maker wins the corresponding Maker-Breaker game. He formulates this as follows:

"Note that Picker has much more control in the Picker-Chooser version than Chooser does in the Chooser-Picker version, or Maker does in the Maker-Breaker version so the Picker-Chooser game is far the simplest case. This relative simplicity explains why we start with the Picker-Chooser game instead of the perhaps more interesting Maker-Breaker game."

The study of these games gives invaluable insight to the Maker-Breaker version. For some hypergraphs the outcome of the Maker-Breaker and Chooser-Picker versions is the same [6, 21]. In all cases it seems that Picker's position is at least as good as Breaker's. It was formalized in the following conjecture.

Conjecture 2.9. *[21] If Maker (as the second player) wins the Maker-Breaker game, then Picker wins the corresponding Picker-Chooser game. If Breaker (as the second player) wins the Maker-Breaker game, then also Picker wins the Chooser-Picker game.*

The definition of these games suggests, that the Chooser-Picker game is easier for Picker, then the Maker-Breaker version for Breaker (the first player can be kept under strong control by Picker), although the other direction of this implication should stand in the most cases. We have found only one non-trivial game, where the outcome is not the same. [22]

One can ask what is the use of such a conjecture? Usually it is easier to analyze a Chooser-Picker game than the corresponding Maker-Breaker game. So if we think that Maker wins a weak game, then to confirm it we first check the Picker-Chooser version, and we must see that Picker wins. Again, if Breaker's win is expected, then the Chooser-Picker version should be a Picker's win.

So not just the Beck's conjecture gives the importance of this games, but that paradigm, that these games are very close to the weak version. (See more about Beck's results on clique-games in [6]). For example if we can prove that Picker loses the 6-in-a-row, it would be a strong argument in addition to that the same could happen in the normal version of that game. Therefore the following rule is not just an arbitrary trick for generalization, but a useful and elegant tool for understand better the Maker-Breaker games.

It is therefore necessary for the Chooser-Picker Games infinite version the following restriction: At the beginning Chooser can select a bounded subset of the board, where they will play. Because if they play on the infinite board, then Picker could select points far from each other, and it is a trivially winning strategy for Picker.

2.3 Tools

2.3.1 Pairing strategy

The *pairing strategies* has been extended to language of game theory. Here the aim is to coloring the $(V, \mathcal{F})$ hypergraph's vertices by two colors so that none of the winning sets are monochrome. Here the game can be a draw.

The pairing strategies of hypergraph games are from [35]. Alfred Hales and Robert Jewett introduced the games $HJ(n, d)$, where n and d are natural numbers. The board of the $HJ(n, d)$ is a d dimensional cube, which is assembled by n^d little cubes (in all edges lies n tiny cubes). Formally the basic set of the hypergraph are the d length serials, where each coordinates are integers between 1 and n. It means that $V(HJ(n, d)) = 1, ..., n^d$. The edges of the hypergraph are such n triples, which elements can be arranged on that way that in a fixed coordinate the serials are $1, 2, ..., n$, $n, n-1, ..., 1$ or constant. The $HJ(3, 2)$ is the *Tic-Tac-Toe*, and the $HJ(4, 3)$ is the *Tic-Toc-Tac-Toe*.

Definition 2.10. *An assignment $\chi : V \to 1, \ldots, k$ of the $\mathcal{F} = (V, E)$ hypergraph is a good coloring, if all $A \in \mathcal{F}$ subset has at least 2 elements. The minimal k, which has good coloring is the chromatic number of $\mathcal{H}$. We mark it by $\chi(\mathcal{H})$.*

If for a hypergraph $\mathcal{F}$ the $\chi(\mathcal{F}) > 2$, then the game on it cannot be a draw. A good example for this the game $HJ(3, 3)$ or *Tic-Tac-Toe* game. On the other hand $HJ(4, 3)$ (the *Tic-Toc-Tac-Toe*) is an example for that the first player could have a winning strategy even if $\chi(\mathcal{F}) = 2$.

Theorem 2.11 (Hales-Jewett)**.** *For all n natural number exist such $d > 0$ integer, that the hypergraph game $HJ(n, d)$ has chromatic number greater than 2.*

It means that if d is enough large, then there is no tie at Hales-Jewett games, and due to the strategy-steeling argument the first player wins. But this theorem also gives only an existential proof, and moreover for a given n the boulder for d is extremely high. The best result is Shaharon Shelah's, where $d(n)$ is in $\mathcal{E}^5$ Grzegorczyk hyerarchy (see [66]).

The following theorem is one of the most basic one in Combinatorics:

Theorem 2.12 (Kőnig D.-Ph. Hall)**.** *The $\{A_i\}_{i=1}^m$ system of finite sets has system of distinct representatives, iff for all $I \subseteq 1, \ldots, m$ stands $| \cup_{i \in I} A_i| \geq |I|$.*

The game-theoretic application of this is the following theorem:

Theorem 2.13 (Hales-Jewett)**.** *If in a finite $(V, \mathcal{F})$ hypergraph game for all $\mathcal{G} \subset \mathcal{F}$ the*

$$| \bigcup_{A \in \mathcal{G}} A| \geqslant 2|G|,$$

stands, then the game is draw.

– It means, that if all subsets of the hypergraph has two times more vertices then edges, then the game is draw. The proof of this statement is following from Kőnig Dénes and Philip Hall's theorem. By using this theorem we can see that the 5×5 is a draw. (At the beginning there is 25 point and 12 edges, and for all k length winning sets have at least $2k$ vertices. And also easy to see that the smallest vertices/edge rate is when we get the whole board.)

Proof. If $H = \{A_1, ..., A_m\}$, then be $H^* = \{A_1, A_1^*, A_2, A_2^*, ..., A_m, A_m^*\}$, where $A_i = A_i^*$ for all $i = 1, ..., m$-re. Easy to see, that from $| \cup_{A \in G} A| \geq 2|G|$ follows that: for all $\mathcal{G}^* \subset H^*$ choosing $| \cup_{A \in G^*} A| \geq |\mathcal{G}|$. It means that the theorem above can be applied for the system of H^* . Be $S = \{a_1, a_1^*, a_2, a_2^*, ..., a_m, a_m^*\}$ a system of distinct representatives. The second player should follow this strategy: For anytime when the first player chooses an element from S (these element can be either a_i-t or a_i^*), then the second should choose an element with the same index (a_i^* or a_i), otherwise step freely. The first can not get an A_i for $i = 1, ..., m$, because from $a_i, a_i^* \in A_i$ at least one is owned by the second. $\square$

Note that at the hypergraph $HJ(n, d)$ where n is odd then all vertices are member of $\frac{1}{2}(3^d - 1)$ winning sets. If n is even, then this number is $2^d - 1$. By using this we can get the following theorem:

Theorem 2.14 (Hales-Jewett, 1963)**.** *The game $HJ(n, d)$ is draw, if $n \geq 3^d - 1$ and $n = 2l + 1$, or if $n \geq 2^{d+1} - 2$ and $n = 2l$.*

For example the 6×6- Tic-Tac-Toe (=$HJ(6, 2)$) is a draw. (We had seen before that the 5×5 is also a draw. Of course, a similar case study gives the same for 6×6, that is the game is a draw, while Theorem 2.14 would not induce this result. It is quite common in Combinatorics that a general theorems yield weaker results than special case studies.)

2.3.2 Weight functions

An another approach is by the usage of *weight functions*. Here the dangerousness of a position is represented by the weight of the game. If one player occupies many vertices from an edge (and none by the other) then this edge has a "heavy" weight.

The so called Erdős-Selfridge weights are frequently used. An edge $A \in \mathcal{F}$ weight's is $2^{-|A|}$ and doubles each time when Maker occupies a new vertex from it. Be m the number of occupied vertices in the edge A by Maker after its ith step. At this time the weight of A is:

$$w_i(A) = 2^{-|A|+m},$$

if Breaker does not have vertices in A, otherwise $w_i(A) = 0$. The weight of a vertex is the sum of that edge-weights where the vertex is in the edge:

$$w_i(x) = \sum_{x \in A} w_i(A)$$

It can be shown, that if Breaker always gets the largest weight of the graph, then the function $w_i = \sum_{A \in \mathcal{F}} w_i(A)$ is monotonously decreasing in i. Considering that $w_1 \leq \sum_{A \in \mathcal{F}} 2^{-|A|+1}$ and that if Maker wins at the kth step than $w_k \geq 1$, derives the following theorem:

Theorem 2.15 (Erdős-Selfridge, 1973)**.** *If Maker starts the weak version of the (V, E) hypergraph game and*

$$\sum_{A \in E} 2^{-|A|+1} < 1,$$

then Breaker wins.

By using this, we can prove that the 5×5 is a draw: there are 12 edges and all of them are 5 length. Plugging in these to the condition of Theorem 2.15, we get: $12 * 1/32 < 1/2$.

We should remark, that the Erdős-Selfridge theorem is sharp: it means that there are such hypergraph families, where $\sum_{A \in E} 2^{-|A|+1} = 1$ and Maker wins.

The theorem above is a derandomization of Erdős' previous theorem. Note that if we coloring randomly the vertices of the graph V by two color, then $\sum_{A \in E} 2^{-|A|+1}$ is the expected number if monochrome edges. It means that there exists a good two coloring of (V, E), what we found constructively.

There is an analogue for biased games too, see the theorem below 2.16. It is also sharp for all $p, q \in \mathbb{N}$, and for $p = q = 1$ it gives back Theorem 2.15.

Theorem 2.16 (Beck)**.** *The $(\mathcal{F}, p, q)$ biased Maker-Breaker game is a draw, if*

$$\sum_{A \in E(\mathcal{F})} (1+q)^{-|A|/p} < \frac{1}{1+q},$$

where Maker moves p, Breaker moves q-t steps afterward.

Another application for weights is that it might helps to find a good possible move of a game: Maker and Breaker should try the most dangerous steps at first, where the dangerousness is represented by E-S weights (see before): $2^{|A|-m}$, where m is the number of occupied vertices.

2.3.3 Back-tracking

Suppose that the board consists indexed cells (for example 1,2,3,4,5,....,32), and our duty is to get through all of the possible cases by checking on which position who is the winner, and who wins from the initial position.

The initial stance is: $M = 1$, $B = 2$, $level = 1$

1. Maker steps to the smallest unoccupied field, Breaker too.
2. If Maker does not win, then $level = level + 1$, goto 1. (next steps)
3. If Maker occupies a winning-set: $level = level - 1$ and Breaker steps to the next cell. (here Maker wins)
4. If there is no winning set without B: $level = level - 1$ and Maker steps to the next cell. (here Maker cant win, it is good for Breaker)
5. Maker wins if $level = 0$; and Breaker steps to the next cell (e.g to 3); Breaker wins if $level = 0$; and Maker steps to the next cell (e.g to 3);

Typically back-tracking goes together with *branch and bound* techniques, where are other tricks which fastens stage 2, 3, 4.

We use that algorithm at the section 4.6, when we calculate the result of a specific sub-game (an auxiliary game).

2.3.4 Auxiliary games

It usually help if we split the game to smaller *auxiliary games*. In fact the pairing is also can be considered as an auxiliary game (where the sub-game's hypergraph has only two vertices and one edges). The aim is to win the game by playing independently on these subgraphs (sub-boards). Some applications are the proof of the 9-in-a-row is a draw, 8-in-a-row is a draw see below 4.1. We will also use it in the next chapter.

Here we list some simple facts from [21] that are very useful in analyzing concrete games. For the sake of completeness we give the proofs, too.

2.3.5 Pairing lemma

Lemma 2.17. *[21] If in the course of the (Chooser- Picker) game (or just already at the beginning) there is a two element winning set $\{x, y\}$ then Picker has an optimal strategy starting with $\{x, y\}$.*

Proof. It is enough to see that if Picker has a winning strategy p, then there exists a starting with $\{x, y\}$ – call it p^* which is also Picker win. If the strategy p asks later $\{x, y\}$: Assume that playing p^* Chooser can win on given distribution of $\{x, y\}$, than Chooser could pretend that this distribution already happened before. In this way playing strategy

p also could use the same strategy. If in strategy p Picker compelled to ask not at once x and y:

Then Chooser could both ask x and y when they are separately turns up with other elements (or one of these is the remaining one for Chooser) in strategy p. And therefore Chooser wins. □

For the better adaptability of the C-P games, we should prove the following lemma.

2.3.6 The monotonicity lemma

It looks very desirable to extend such a successful heuristic to games played on infinite hypergraphs. However, one has to be careful since in that case Picker might offer a set of vertices $A \subset V$ such that every edge contain at most one element from A, which is a trivial winning strategy for Picker. A possible remedy is add a step at the beginning: Chooser selects a finite set $X \in V$, and they play on the *induced sub-hypergraph* that is keep only those edges $A \in \mathcal{F}$ for which $A \subset X$. More formally, given the hypergraph $(V, \mathcal{F})$ let $(V \setminus X, \mathcal{F}(X))$ denote the hypergraph where $\mathcal{F}(X) = \{A \in \mathcal{F}, A \cap X = \emptyset\}$.

Lemma 2.18. *[21] If Picker wins the Chooser-Picker game on* $(V, \mathcal{F})$*, then Picker also wins it on* $(V \setminus X, \mathcal{F}(X))$.

Proof. By induction it is enough to prove the statement for $X = \{x\}$, i. e., $|X| = 1$. Assume that p is a winning strategy for Picker in the game on $(V, \mathcal{F})$. That is in a certain position of the game the value of the function p is a pair of unselected elements that Picker is to give to Chooser. We can modify p in order to get a winning strategy p^* for the Chooser-Picker game on $(V \setminus \{x\}, \mathcal{F}(\{x\}))$. Let us follow p while it does not give a pair $\{x, y\}$. Getting a pair $\{x, y\}$, we ignore it, and pretend we are playing the game on $(V, \mathcal{F})$, where Chooser has taken y and has returned x to us. If $|V|$ is odd, there is a $z \in V$ at the end of the game that would go to Chooser. Here Picker's last move is the pair $\{y, z\}$. Picker wins, since Chooser could not win from this position even getting the whole pair $\{y, z\}$. If $|V|$ is even, p^* leads to a position in which y is the last element, and it goes to Chooser. But the outcome is then the same as the outcome of the game on $(V, \mathcal{F})$, that is a Picker's win. □

This lemma is useful tool at the next chapters, because if a bounded set S cannot be partitioned into uniform sub-games, then it can be increased to S', which can be split into such sub-games. And if Picker wins on S', then also can win on S.

Chapter 3

Some results on Chooser-Picker games

3.1 Introduction

Studying the very hard clique games, Beck [6] introduced a different type of heuristic, that proved to be a great success. He defined the *Picker-Chooser* or shortly P-C and the *Chooser-Picker* (C-P) versions of a Maker-Breaker game that resembles fair division, (see [70, 71]).

In these versions Picker takes an unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. When $|V|$ is odd, the last element goes to Chooser.

Beck obtained that conditions for winning a Maker-Breaker game by Maker and winning the Picker-Chooser version of that game by Picker coincide in several cases. Furthermore, Breaker's win in the Maker-Breaker and Picker's win in the Chooser-Picker version seem to occur together.

However, one has to be careful to spell out a good conjecture, since it is easy to check that Chooser wins the 2×2 hex.

The precise form of Beck's conjecture was stated before 2.9.

Remarks. It is enough to prove Conjecture 2.9 for Picker-Chooser games since the Chooser-Picker case would follow. To see this one just considers $(V, \mathcal{F}^*)$, the *transversal hypergraph* of $(V, \mathcal{F})$. That is $\mathcal{F}^*$ contains those minimal sets $B \subset V$ such that for all $A \in \mathcal{F}$, $A \cap B \neq \emptyset$. Note that Breaker as a first (second) player wins the Maker-Breaker $(V, \mathcal{F})$ iff Maker as a first (second) player wins the Maker-Breaker $(V, \mathcal{F}^*)$.

The decision problem that if Picker wins a Picker-Chooser (or Chooser-Picker) game is at least NP-hard [62], but probably it is PSPACE-complete as that of the Maker-Breaker games, shown by Schaefer [65]. Still, for concrete games it can be easier to decide the outcome of the Picker-Chooser (Chooser-Picker) version than the Maker-Maker version. That is if Conjecture 2.9 is proved for a class of hypergraphs then the easier Picker-Chooser (Chooser-Picker) games can be used in an alpha-beta pruning algorithm for the harder Maker-Breaker game. A natural class for that us the otherwise hopeless, Hales-Jewett or torus games for low dimension (see [7, 35]). We discuss some examples and useful tools for that direction in Section 4.2. Here we would emphasize the extension of Picker-Chooser games to infinite hypergraphs and the role of Lemma 2.18 in this case. These might be used in solving Harary-type of polyomino problems for Chooser-Picker games for which the Maker-Breaker versions were studied by Harary, Blass, Pluhár and Sieben [13, 59, 67].

3.2 On the complexity of Chooser-Picker positional games

Since the Maker-Breaker (and the Maker-Maker) games are PSPACE-complete (see [65]) it would support both Conjecture 2.9, and the above coincidence with Chooser-Picker games to see that the Chooser-Picker or Picker-Chooser games are not easy as well. To prove PSPACE-completeness for positional games is more or less standard, see [64, 16]. Here we can prove something weaker because of the asymmetric nature of these games.

Theorem 3.1. *It is NP-hard to decide the winner in a Picker-Chooser game.*

Theorem 3.2. *It is NP-hard to decide the winner in a Chooser-Picker game.*

In Section 3.6 we generalize the pairing strategies first formalized by Hales and Jewett [35]. As an application, we show there is no pairing strategy for the game "Snaky," see [37, 38, 68]. Finally, we compare the actual complexity of these games on a specific hypergraph, the 4×4 torus in Section 3.7.

3.3 Proofs of Theorems 3.1 and 3.2

Both proofs are based on the usual reduction method. We reduce $3 - \mathrm{SAT}$ to Chooser-Picker or Picker-Chooser games.

Proof of Theorem 3.1. Consider an arbitrary CNF formula $\phi(x_1, \ldots, x_n) \in 3 - \mathrm{SAT}$. We denote $\phi = C_1 \wedge \cdots \wedge C_k$, where $C_i = \ell_{i_1} \vee \ell_{i_2} \vee \ell_{i_3}$ and ℓ_{i_j} is a literal for $i \in \{1, \ldots, k\}$ and $j = 1, 2, 3$. With a slight abuse of notation, we use C_i also to denote the set of literals in it. That is, if there exists a clause $C_i = x_2 \vee \bar{x}_5 \vee x_6$, then we also denote the set $C_i = \{x_2, \bar{x}_5, x_6\}$.

We will exhibit a hypergraph $\mathcal{H}_\phi = (V, E)$ such that the Picker-Chooser game is a win for Chooser if and only if ϕ is satisfiable.

The vertex set will be $V = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$. Let $\mathcal{B} \subset 2^V$ have the property that $B \in \mathcal{B}$ if, for all $i \in \{1, \ldots, n\}$, B contains either x_i or $\bar{x}_i$ but not both. The edge set E consists of the sets A such that $A = C_i \cup B$ for some i and some $B \in \mathcal{B}$.

Note that $\mathcal{B}$, and consequently E, has a short (polynomial in ϕ) description even though $|E| \geq |\mathcal{B}| = 2^n$.

Claim 1 allows us to restrict our attention to games in which Picker has a specific kind of strategy.

Claim 1: If Picker fails to select pairs of the form $\{x_i, \bar{x}_i\}$ in each round, then Chooser has a winning strategy.

Proof: We assume to the contrary: Let $\{x, y\}$ be the first pair selected by Picker such that $\{x, y\} \neq \{x_i, \bar{x}_i\}$ for any $i \in \{1, \ldots, n\}$. In that case, Chooser keeps, say, x, and waits until Picker offers up $\bar{x}$ in a pair. In that round, Chooser takes $\bar{x}$, and wins the game, since Picker cannot take any $B \in \mathcal{B}$. This proves Claim 1. $\square$

First we show that if Picker-Chooser on $\mathcal{H}_\phi$ is a win for Chooser, then ϕ is satisfiable. According to Claim 1, we may assume that Picker's strategy is to select pairs of the form $\{x_i, \bar{x}_i\}$ resulting in the fact that such pairs are shared among Picker and Chooser for all i. Assume that Chooser wins the game on $\mathcal{H}_\phi$, and set $\hat{x}_i = 1$ if Chooser holds x_i, and $\hat{x}_i = 0$ otherwise. Picker holds all elements of some $B \in \mathcal{B}$, so the assumption means that Chooser has an element in each of the C_i's. That is, $\phi(\hat{x}_1, \ldots, \hat{x}_n) = 1$.

Next we show that if ϕ is satisfiable, then Picker-Chooser on $\mathcal{H}_\phi$ is a win for Chooser. Since ϕ is satisfiable, there exist $\hat{x}_1, \ldots, \hat{x}_n$, such that $\phi(\hat{x}_1, \ldots, \hat{x}_n) = 1$. Consider the Picker-Chooser game on $\mathcal{H}_\phi$. By Claim 1, we may assume that, in each round, Picker offers a pair of the form $\{x_i, \bar{x}_i\}$. In that case, Chooser takes x_i if and only if $\hat{x}_i = 1$, and wins the game. This proves Theorem 3.1. $\square$

Proof of Theorem 3.2. Let us use the same set-up and notation for the CNF formula ϕ as in the proof of Theorem 3.1. We want to define a hypergraph $\mathcal{H}_\phi = (V, E)$ such that the Chooser-Picker game on $\mathcal{H}_\phi = (V, E)$ is a Picker's win if and only if ϕ is satisfiable.

Let the vertex set be $V = \{a_i, b_i, c_i, d_i\}_{i=1}^n$. The edge set, E, consists of all edges A such that

- $A \subset \{a_i, b_i, c_i, d_i\}$ and $|A| = 3$ for some $i \in \{1, \ldots, n\}$,
- $A = \{a_i, a_j, a_k, b_i, b_j, b_k\}$ for a clause $C = x_i \vee x_j \vee x_k$,
- $A = \{a_i, a_j, a_k, b_i, b_j, c_k\}$ for a clause $C = x_i \vee x_j \vee \bar{x}_k$,
- $A = \{a_i, a_j, a_k, b_i, c_j, c_k\}$ for a clause $C = x_i \vee \bar{x}_j \vee \bar{x}_k$,
- $A = \{a_i, a_j, a_k, c_i, c_j, c_k\}$ for a clause $C = \bar{x}_i \vee \bar{x}_j \vee \bar{x}_k$.

Claim 2 allows us to restrict our attention to games in which Chooser has a specific kind of strategy.

Claim 2:

- If Picker picks a pair (x, y) such that $\{x, y\} \not\subset \{a_i, b_i, c_i, d_i\}$ for some $i \in \{1, \ldots, n\}$, then Chooser has a winning strategy.
- Chooser has an optimal strategy that results in always choosing a_i and always giving d_i to Picker.

In particular, this means that we may assume that for all i, Picker either picks $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$. Moreover, Chooser will get a_i and Picker will get d_i and each player will get exactly one of (b_i, c_i).

Proof: Suppose Picker offers a pair (x, y) for which $x \in \{a_i, b_i, c_i, d_i\}$ but $y \notin \{a_i, b_i, c_i, d_i\}$. Consider the first such instance. In that case, Chooser chooses x, and ultimately wins by choosing at least two more elements from $\{a_i, b_i, c_i, d_i\} \setminus \{x\}$, giving Chooser every element of some A of size 3. So, for all i, Picker will pick either $\{(a_i, d_i), (b_i, c_i)\}$ or $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$. Hence, Chooser and Picker will have at least one member of each set of size 3.

However, no d_i appears in any of the sets of size 6 and so if Chooser wins by choosing d_i, then he must also win by not choosing d_i. Finally, suppose Picker picks the pair (a_i, b_i)

or (a_i, c_i). Chooser will choose a_i in either case because every A of size 6 that contains either b_i or c_i will also contain a_i. So, once again, Chooser can only benefit by choosing a_i over b_i or c_i. Summarizing, if Picker plays optimally; i.e., always taking pairs with the same subscript, then for every winning strategy in which Chooser chooses d_i, there exists a winning strategy in which he does not and for every winning strategy in which Chooser does not choose a_i, there exists a winning strategy in which he does.

So, we may assume that Picker picks either $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$ for all i because if Picker picks $\{(a_i, d_i), (b_i, c_i)\}$, then the outcome is the same except that he cannot control which of $\{b_i, c_i\}$ he will be given by Chooser. This proves Claim 2. □

Now let Picker's $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$ moves correspond to setting the value of $x_i = 1$ or $x_i = 0$, respectively.

First we show that if Chooser-Picker on $\mathcal{H}_\phi$ is a win for Picker, then ϕ is satisfiable. We may assume that Chooser plays according to the restrictions imposed by Claim 2. At the end of the game, Picker has exactly one of $\{b_i, c_i\}$. Chooser has a_i for all $i \in \{1, \ldots, n\}$. Let $\hat{x}_i = 1$ if Picker has b_i and $\hat{x}_i = 0$ otherwise. By the construction of $\mathcal{H}_\phi$, this means that $\phi(\hat{x}_1, \ldots, \hat{x}_n) = 1$.

Next we show that if ϕ is satisfiable, then Picker-Chooser on $\mathcal{H}_\phi$ is a win for Picker. Suppose that there is some assignment that $\phi = (\hat{x}_1, \ldots, \hat{x}_n)$. Picker makes sure to get b_i (i.e., Picker picks $\{(a_i, b_i), (c_i, d_i)\}$) if $\hat{x}_i = 1$, and makes sure to get c_i (i.e., Picker picks $\{(a_i, c_i), (b_i, d_i)\}$) if $\hat{x}_i = 0$. Because of Claim 2, we may assume that Chooser will always choose a_i for all $i \in \{1, \ldots, n\}$. As a result, Picker will get at least one element from every $A \in E$, and wins the game. This proves Theorem 3.2. □

Note that this theorem implies that Chooser-Picker games are NP-hard, even in the case of hypergraphs (V, E), for which $|A| \leq 6$ for all $A \in E$.

3.3.1 4×4 tic-tac-toe

Before proving the harder Theorems 3.4 and 3.6 we demonstrate Picker's strategies for the Chooser-Picker version of well-known games. The 4×4 tic-tac-toe is a draw game, and Breaker wins it as a second player. The later statement can be proved by a little strengthening of Theorem 2.7, see in [7].

One may rightfully expects that Picker wins the Chooser-Picker version of the 4×4 tic-tac-toe, and indeed this is the case.

Proposition 3.3. *Picker wins the Chooser-Picker version of the* 4×4 *tic-tac-toe.*

Proof. Picker takes the two endpoints of the main diagonal, and then the two "middle points" of the other diagonal first. Considering the symmetries, we get the picture on Figure 1. (Chooser's pieces are in white, Picker's are in black.) Then Picker selects the pairs indicated by the thin lines. □

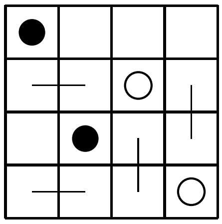


Figure 1

3.4 Picker-Chooser version of the generalized Shannon switching game

Now we prove Conjecture 2.9 for the Picker-Chooser version of Shannon switching game in the generalized version as Lehman did in [46]. Let $(V, \mathcal{F})$ be a matroid, where $\mathcal{F}$ is the set of bases, and Picker wins by taking an $A \in \mathcal{F}$. Note, that this is equivalent with the Chooser-Picker game on $(V, \mathcal{C})$, where $\mathcal{C}$ is the collection of *cutsets* of the matroid $(V, \mathcal{F})$, that is for all $A \in \mathcal{F}$ and $B \in \mathcal{C}$, $A \cap B \neq \emptyset$.

Theorem 3.4. *Let $\mathcal{F}$ be collection of bases of a matroid on V. Picker wins the Picker-Chooser $(V, \mathcal{F})$ game, if and only if there are $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$.*

The notation and the proof closely follow the ones given in [54] for the Maker-Breaker case.

Proof. First we show that if there are no two disjoint $A, B \in \mathcal{F}$ then Chooser wins. Let $\mathcal{M}_1 = (V, \mathcal{F})$ and $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_1$ be the union matroid of $\mathcal{M}_1$ with itself. The rank function $r_{\mathcal{M}}$ of the union matroid of $\mathcal{M} = M_1 \vee \cdots \vee M_k$ is the following,

$$r_{\mathcal{M}}(S) = \min_{T \subset S} \left\{ |S \setminus T| + \sum_{i=1}^{k} r_i(T) \right\},$$

where the matroids are defined on the same ground set S, and the matroid $\mathcal{M}_i$ has the rank function r_i. We have $\min_{T \subset V} \{|V \setminus T| + 2r_1(T)\} = r_{\mathcal{M}}(V) < 2r_1(V)$, since $\mathcal{M}_1$ does not have two disjoint bases. Equivalently, $|V \setminus T| < 2(r_1(V) - r_1(T))$. Receiving a pair (x, y), Chooser keeps an element of $V \setminus T$ if possible. At the end of the game Chooser owns at least $\lceil |V \setminus T|/2 \rceil$ elements of $V \setminus T$.

That is Picker may own at most $\lfloor |V \setminus T|/2 \rfloor < r_1(V) - r_1(T)$ elements of $V \setminus T$ at the end of the game.

Let Y be the elements of Picker at the end of the game. Clearly,

$$r_1(Y) \leq r_1(Y \cap (V \setminus T)) + r_1(T) < r_1(V) - r_1(T) + r_1(T) = r_1(V),$$

that is Picker has lost the game.

For the other direction, we assume that $A, B \in \mathcal{F}$, $A \cap B = \emptyset$, and use induction. We consider the matroid $\mathcal{M}/y \setminus x$ given a pair (x, y) taken by Chooser and Picker, respectively. Clearly Picker wins the game for $\mathcal{M}$ if he can win it for $\mathcal{M}/y \setminus x$. (The dimension of $\mathcal{M}/y \setminus x$ is one less than that of $\mathcal{M}$, and if A' is a base of $\mathcal{M}/y \setminus x$, then $A' \cup \{y\}$ is a base of $\mathcal{M}$.)

All we need here is the *strong base exchange axiom* (or rather theorem), that says if A and B are bases of a matroid $\mathcal{M}$, then there exist $x \in A$, $y \in B$ such that both $\{A \setminus \{x\}\} \cup \{y\}$ and $\{B \setminus \{y\}\} \cup \{x\}$ are also bases of $\mathcal{M}$. Picker selects the pair (x, y) such that the above applies, and reduces the game to either $\mathcal{M}/y \setminus x$ or $\mathcal{M}/x \setminus y$. Since $A \setminus \{x\}$ and $B \setminus \{y\}$ are disjoint bases both in $\mathcal{M}/y \setminus x$ and $\mathcal{M}/x \setminus y$, we can proceed. □

3.5 Erdős-Selfridge type theorems for P-C and C-P games

The Erdős-Selfridge theorem [27] gives a very useful condition for Breaker's win in a Maker-Breaker $(V, \mathcal{F})$ game. Note, that here we use the simpler notion, and the set of edges are $\mathcal{F}$.

Using a stronger condition, Beck [6] proves Picker's win in a Chooser-Picker $(V, \mathcal{F})$ game. (For the P-C version he proved a sharp result that we include here.) Let $||\mathcal{F}|| = \max_{A \in \mathcal{F}} |A|$ be the rank of the hypergraph $(V, \mathcal{F})$.

Theorem 3.5. *[6] If*

$$T(\mathcal{F}) := \sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{8(||\mathcal{F}|| + 1)}, \tag{3.1}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $(V, \mathcal{F})$. If $T(\mathcal{F}) < 1$, then Chooser wins the Picker-Chooser game on $(V, \mathcal{F})$.

We improve on his result by showing:

Theorem 3.6. *If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{3\sqrt{||\mathcal{F}|| + \frac{1}{2}}}, \tag{3.2}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $(V, \mathcal{F})$.

It is worthwhile to spell out a special case of Conjecture 2.9 for this case, that would be the sharp extension of Erdős-Selfridge theorem to Chooser-Picker games.

Conjecture 3.7. *If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Picker wins the Chooser-Picker game on the hypergraph $(V, \mathcal{F})$.

Proof. We shall modify the proof of Theorem 3.5 appropriately. The idea of the proof is to associate a weight function $T(\mathcal{F})$ to a hypergraph $(V, \mathcal{F})$ that measures the danger for Picker. The value of T becomes 1 iff Chooser wins the game, so Picker tries to keep T down. In Maker-Breaker games the greedy selection works, see the classical Erdős-Selfridge theorem in [27] or in [7]. Let

$$T(\mathcal{F}) = \sum_{A\in\mathcal{F}} 2^{-|A|},\ T(\mathcal{F};v) = \sum_{v\in A\in\mathcal{F}} 2^{-|A|} \ \text{ and } \ T(\mathcal{F};v,w) = \sum_{\{v,w\}\subset A\in\mathcal{F}} 2^{-|A|}$$

for an arbitrary hypergraph $(V, \mathcal{F})$.

Assume that after the ith turn Chooser already has the elements $x_1, x_2, \ldots, x_i$ and Picker has the elements $y_1, y_2, \ldots, y_i$. Now Picker picks a 2-element set $\{v, w\}$, from which Chooser will choose x_{i+1}, and the other one (i. e. y_{i+1}) will go back to Picker. Let $X_i = \{x_1, x_2, \ldots, x_i\}$ and $Y_i = \{y_1, y_2, \ldots, y_i\}$. Let $V_i = V \setminus (X_i \cup Y_i)$. Clearly $|V_i| = |V| - 2i$. Let $\mathcal{F}(i)$ be the truncated subfamily of $\mathcal{F}$ which consists of the unoccupied parts of the still dangerous winning sets:

$$\mathcal{F}(i) = \{A \setminus X_i : A \in \mathcal{F},\ |A \setminus X_i| \le \lceil |V_i|/2 \rceil,\ A \cap Y_i = \emptyset\}.$$

Here we will deviate a little from Beck's proof, since he includes all sets $A \in \mathcal{F}$, $|A \setminus X_i| \le |V_i|$ in $\mathcal{F}(i)$ if $A \cap Y_i = \emptyset$. But if $|A \setminus X_i| > \lceil |V_i|/2 \rceil$, then Picker *automatically* gets an element of A, so deleting these sets from $\mathcal{F}(i)$ does not change the outcome of the game.

Let $\mathcal{F}(end) = \mathcal{F}(\lceil |V|/2 \rceil)$, i. e., these are the unoccupied parts of the still dangerous sets at the end of the play. Chooser wins iff $T(\mathcal{F}(end)) \ge 1$, so to guarantee Picker's win it is enough to show that $T(\mathcal{F}(end)) < 1$. Let x_{i+1} and y_{i+1} denote the $(i+1)$th elements of Chooser and Picker, respectively.

Then we have

$$T(\mathcal{F}(i+1)) = T(\mathcal{F}(i)) + T(\mathcal{F}(i);x_{i+1}) - T(\mathcal{F}(i);y_{i+1}) - T(\mathcal{F}(i);x_{i+1},y_{i+1}).$$

It follows that

$$T(\mathcal{F}(i+1)) \le T(\mathcal{F}(i)) + |T(\mathcal{F}(i);x_{i+1}) - T(\mathcal{F}(i);y_{i+1})|.$$

Introduce the function

$$g(v,w) = g(w,v) = |T(\mathcal{F}(i);v) - T(\mathcal{F}(i);w)|$$

which is defined for any 2-element subset $\{v, w\}$ of V_i. Picker's next move is that 2-element subset $\{v_0, w_0\}$ of V_i for which the function $g(v, w)$ achieves its minimum. Since $\{v_0, w_0\} = \{x_{i+1}, y_{i+1}\}$, we have

$$T(\mathcal{F}(i+1)) \le T(\mathcal{F}(i)) + g(i), \tag{3.3}$$

where

$$g(i) = \min_{v,w:v\ne w,v,w\subset V_i} |T(\mathcal{F}(i);v) - T(\mathcal{F}(i);w)|. \tag{3.4}$$

To estimate $g(i)$ we take a lemma from [6]. It is an easy exercise for the reader.

Lemma 3.8. *If $t_1, t_2, \ldots, t_m$ are non-negative real numbers and $t_1 + t_2 + \ldots + t_m \le s$, then*

$$\min_{1\le j<\ell\le m} |t_j - t_\ell| \le \frac{s}{\binom{m}{2}}.$$

We distinguish two phases of the play.

Phase 1: $|V_i| = |V| - 2i > 2||\mathcal{F}||$. (Note that Beck uses $|V_i| > ||\mathcal{F}||$.) Simple counting shows that

$$\sum_{v \in V_i} T(\mathcal{F}(i); v) \leq ||\mathcal{F}|| T(\mathcal{F}(i)).$$

By Lemma 3.8 and (3.4),

$$g(i) \leq \frac{||\mathcal{F}||}{\binom{|V_i|}{2}} T(\mathcal{F}(i)),$$

so by (3.3),

$$T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i)) \left\{ 1 + \frac{||\mathcal{F}||}{\binom{|V_i|}{2}} \right\}.$$

Since $1 + x \leq e^x = \exp(x)$, we have

$$T(\mathcal{F}(i+1)) \leq T(\mathcal{F}) \exp \left\{ ||\mathcal{F}|| \sum_{j=0}^{i} \frac{1}{\binom{|V_j|}{2}} \right\}.$$

It is easy to see that

$$\sum_{i:|V_i|>2||\mathcal{F}||} \frac{1}{\binom{|V_i|}{2}} < \frac{1}{2||\mathcal{F}||},$$

so if i_0 denotes the last index of the first phase then

$$T(\mathcal{F}(i_0+1)) < \sqrt{e} T(\mathcal{F}). \tag{3.5}$$

Phase 2: $|V_i| = |V| - 2i \leq 2||\mathcal{F}||$. Then a similar counting as in *Phase 1* gives

$$\sum_{v \in V_i} T(\mathcal{F}(i); v) \leq \left\lceil \frac{|V_i|}{2} \right\rceil T(\mathcal{F}(i)).$$

One checks that $T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i))$ when $2 \leq |V_i| \leq 4$. If $|V_i| \geq 4$, then by Lemma 3.8 and (3.4),

$$g(i) \leq \frac{1}{|V_i| - 1} T(\mathcal{F}(i)),$$

so by (3.3),

$$T(\mathcal{F}(i+1)) \leq \frac{|V_i|}{|V_i| - 1} T(\mathcal{F}(i)). \tag{3.6}$$

Let us recall the well-known Wallis's formula, $\lim_{n\to\infty} \frac{1}{2n+1} \prod_{i=1}^{n} \frac{(2i)^2}{(2i-1)^2} = \frac{\pi}{2}$. Since $\frac{(2n+2)^2}{(2n+1)(2n+3)} > 1$ for all $n \in \mathbb{N}$, we have the inequality for all $n \in \mathbb{N}$

$$\prod_{i=1}^{n} \frac{2i}{2i-1} < \sqrt{\frac{\pi}{2}(2n+1)}. \tag{3.7}$$

By repeated application of (3.6) we have

$$T(\mathcal{F}(end)) \leq T(\mathcal{F}(i_0+1)) 2 \prod_{i:2 \leq |V_i| \leq 2||\mathcal{F}||} \frac{|V_i|}{|V_i| - 1} \leq T(\mathcal{F}(i_0+1)) 2 \prod_{j=2}^{||\mathcal{F}||} \frac{2j}{2j-1}.$$

Now using (3.7), (3.5) and (3.2), we have

$$T(\mathcal{F}(end)) < T(\mathcal{F}(i_0+1))\sqrt{\pi(||\mathcal{F}||+\frac{1}{2})} \leq \sqrt{e\pi}T(\mathcal{F})\sqrt{||\mathcal{F}||+\frac{1}{2}} < 1.$$

Since Chooser cannot completely occupy a winning set, Theorem 3.6 follows. □

To further explore this direction, a generalization of Theorem 3.6 for biased games is needed. No attempt is made here to get the best possible form, for our needs the following lemma will be sufficient and useful. See the proof of it in Chapter 5

Lemma 3.9. *Picker wins the Chooser-Picker* $(1 : b)$ *biased game on the hypergraph* $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$ *if*

$$\frac{v}{b+1}\sum_{A\in E(\mathcal{F})} 2^{-|A|/b} < 1,$$

where $v = |V(\mathcal{F})|$.

3.6 Pairing strategies revisited

3.6.1 Pairing strategies in general

Pairing strategies appear in a plethora of games, see [13]. Certain kind of pairing strategies were introduced to the theory of Positional Games by Hales and Jewett in [35]. Based on these pairing strategies they proved the following theorem.

Theorem 3.10. *[35] Breaker wins a Maker-Breaker game on the hypergraph* (V, E) *if* $|\cup_{A\in\mathcal{G}} A| \geq 2|\mathcal{G}|$ *for all* $\mathcal{G} \subset E$.

The idea is to use the celebrated Kőnig-Hall theorem[1], and exhibit a "double" *system of distinct representatives (SDR)*, in the hypergraph (V, E). A set $X \subset V$ is an SDR if $|X| = |E|$, and there is a bijection $\phi : X \to E$ such that for all $x \in X$, $x \in \phi(x)$. If X and Y are SDR's of (V, E) with the bijections ϕ and ψ where $X \cap Y = \emptyset$, then $\rho = \psi^{-1}\phi$ is a bijection $\rho : X \to Y$. Breaker, even as a second player, wins by using ρ. That is, Breaker takes $\rho(x)$ [takes $\rho^{-1}(y)$] if Maker takes an $x \in X$ [a $y \in Y$], and an arbitrary untaken element $v \in V$ if Maker takes a $w \in V \setminus (X \cup Y)$.

While Theorem 3.10 works fine for some games, it has its drawbacks. It rarely gives sharp results, which is not surprising considering the PSPACE-completeness of those games. Another problem is that the Kőnig-Hall theorem (and consequently Theorem 3.10) applies only to finite hypergraphs. A remedy for this is a lesser known theorem of Marshall Hall Jr., that requires only the local finiteness of the hypergraph (V, E). We say that (V, E) is *locally finite* if $\deg(x) := |\{A : x \in A \in E\}| < \infty$ for all $x \in V$.

Theorem 3.11. *[36] There is a SDR in a locally finite hypergraph* (V, E) *iff* $|\cup_{A\in\mathcal{G}} A| \geq |\mathcal{G}|$ *for all* $\mathcal{G} \subset E$.

Still, Theorem 3.10 does not apply directly if $|V| < 2|E|$, for instance, one must use other ideas to tackle the k-in-a-row games in two or in higher dimensions, see [60].

[1] A generalized form of this theorem will be spelled out in the next paragraph as Theorem 3.11.

Definition 3.12. *The bijection $\rho : X \to Y$, where $X \cap Y = \emptyset$ and $X, Y \subset V$, is a* **winning pairing strategy** *for Breaker in the Maker-Breaker game on hypergraph (V, E) if for all $A \in E$ there is an $x \in X$ such that $\{x, \rho(x)\} \subset A$.*

Of course, we assume that both the function ρ and the decision problem that determining whether any set $Y \subset V$ has the property that $Y \subset A \in E$ are computable in polynomial time in the size of description of (V, E). (For the sake of simplicity we consider only the case when both V and E are finite.) Having the bijection ρ, Breaker wins by taking $\rho(x)$ [taking $\rho^{-1}(y)$] if Maker's last move was $x \in X$ [was $y \in Y$]. To decide the existence of ρ is not easy in general. Let us denote the class of hypergraphs for which Breaker has a winning pairing strategy by $\mathcal{B}$.

Theorem 3.13. *Determining whether a hypergraph is in $\mathcal{B}$ is NP-complete.*

Proof. Given a bijection ρ that witnesses a winning pairing strategy, one checks for an $A \in E$ if there is an $x \in X$ such that $\{x, \rho(x)\} \subset A$. For any pair (A, x) it can be done in polynomial time, and $|E||V|$ is an upper bound on the number of such pairs. Consequently, $\mathcal{B} \in \mathrm{NP}$.

To show that $\mathcal{B}$ is NP-hard one can use basically the same argument as in the proof of Theorem 3.2. There is, however, a simpler reduction. Let ϕ be an arbitrary CNF in 3-SAT. We construct a hypergraph $\mathcal{H}_\phi = (V, E)$ such that $V = \{r_i, b_i, p_i\}_{i=1}^n$ and the edge set, E, consists of all edges A such that

- A is $\{r_i, b_i, p_i\}$ for all $i \in \{1, \ldots, n\}$,
- $A = \{p_i, r_i, p_j, r_j, p_k, r_k\}$ for a clause $C = x_i \vee x_j \vee x_k$,
- $A = \{p_i, r_i, p_j, r_j, p_k, b_k\}$ for a clause $C = x_i \vee x_j \vee \bar{x}_k$,
- $A = \{p_i, r_i, p_j, b_j, p_k, b_k\}$ for a clause $C = x_i \vee \bar{x}_j \vee \bar{x}_k$,
- $A = \{p_i, b_i, p_j, b_j, p_k, b_k\}$ for a clause $C = \bar{x}_i \vee \bar{x}_j \vee \bar{x}_k$.

A winning pairing strategy for Breaker cannot contain both $\{p_i, r_i\}$ or $\{p_i, b_i\}$ for $1 \le i \le n$, because the strategy is a bijection. But such a strategy must contain one of $\{p_i, r_i\}$ or $\{p_i, b_i\}$ in order to have at least one pair of the form $\{x, \rho(x)\}$ in each of the edges of size 3. Let $x_i = 1$ if $\{p_i, r_i\}$ is present, while $x_i = 0$ otherwise. As a result, a clause C associated to its corresponding set A of size 6 is satisfied if and only if A contains a pair. $\square$

Remarks. If the hypergraph (V, E) is almost disjoint, then Breaker has a winning pairing strategy iff $|\cup_{A \in \mathcal{G}} A| \ge 2|\mathcal{G}|$ for all $\mathcal{G} \subset E$, that is one gets back the assumption of Theorem 3.10. This case can be decided in polynomial time in the description of (V, E). As in Theorem 3.2, $\mathcal{B}$ is NP-complete for hypergraphs (V, E), where $|A| \le 6$ for $A \in E$. A result of Hegyháti [41] implies that the existence of a winning pairing strategy can be decided in polynomial time if $|A| \le 3$ for $A \in E$. The cases when $|A| \le 4$ or $|A| \le 5$, to the best of our knowledge, are open.

3.6.2 Applications for k-in-a-row and Snaky

Let d_2 be the maximum pair degree in (V, E), that is $d_2 = \max_{x \neq y} d_2(x, y)$, where $d_2(x, y) = |\{A : \{x, y\} \subset A \in E\}|$.

Proposition 3.14. *If Breaker has a winning pairing strategy then $d_2|X|/2 \geq |\mathcal{G}|$ must hold for all $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

Proof. Simply locate the pairs in the winning pairing strategy. There are at most $|X|/2$ such pairs, which are disjoint. Each pair will be a subset of at most d_2 edges. Since each edge of $\mathcal{G}$ must have a pair as a subset, the number of edges must be at most $d_2|X|/2$. □

Now we can explain why pairing strategies can work for the game k-in-a-row for sufficiently large n only if $k \geq 9$, see [13]. In the k-in-a-row game, $d_2 = k - 1$, and if X is an $n \times n$ board, then $|\mathcal{G}| = 4n^2 + O(kn)$. By Proposition 3.14, we have $(k-1)n^2/2 \geq 4n^2 + O(kn)$; that is, $k \geq 9 + o(n)$.

Another example in which we can use this ideas is the polyomino game Snaky, which were examined by Harary [37], Harborth and Seeman [38], and Sieben [68]. This game is a Maker-Breaker game in which the board consists of the cells of the infinite grid and Maker's goal is to occupy all of the cells in an isomorphic copy of the polyomino Snaky, shown in Figure 3.1.

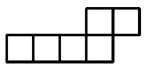


Figure 3.1: The polyomino Snaky. The "head" is the pair of cells in the upper row. The "body" is the set of four consecutive cells in the lower row.

Using a computer search, Harborth and Seeman [38] showed that there is no pairing strategy for Breaker in this game. We give a computer-free proof for their statement:

Theorem 3.15. *[38] Breaker has no pairing strategy to avoid the isomorphic copies of the polyomino "Snaky."*

Proof. Assume to the contrary that there is a winning pairing ρ for Breaker. Let P_ℓ be the polyomino which consists of ℓ consecutive squares of the table.

First we show that ρ cannot be a pairing for the polyomino P_4. Let us assume that ρ is such a pairing, and consider an $n \times n$ board X such that the edges of $\mathcal{G}$ consist of the P_4's on X. Since $d_2 = 3$, Proposition 3.14 gives that $3n^2/2 \geq 2n^2 + O(n)$, which is a contradiction if n is sufficiently large.

On the other hand, if ρ is a pairing for Snaky, then we will show that it must be a pairing for P_5. To see this, we assign labels to the cells such that cells receive the same label iff they are paired by ρ. Let us take the longest set of consecutive cells R such that no labels are repeated on R. We may assume that either those labels are $1, \ldots, \ell$ for some $\ell \geq 5$, or R is infinite.

We first consider the case $\ell = 5$, and in doing so let us refer to a cell of the grid by its lower left lattice point. If ρ is not a pairing for P_5, then we may assume, without loss of generality, that the set of cells $L = \{(1, 0), \ldots, (5, 0)\}$ contains no pairs. These cells are labeled by $1, \ldots, 5$ on the left-hand side of Figure 3.2. Since $\ell = 5$, the both the cells

					$?$	$?$	$?$
L	$\diamond$	1	2	3	4	5	$\diamond$

	E	F	C				
A	A	$\diamond$	C		$\diamond$		
$\diamond$	1	2	3	4	5	6	$\diamond$
B	B	$\diamond$	D		$\diamond$		
	F	E	D				

Figure 3.2: The cases $\ell = 5$ and $\ell = 6$.

$(0, 0)$ and $(6, 0)$ are in a pair with some cell of L. (We indicate the cells that have indices which matching with an element of L by a diamond, otherwise by capital letters.) This leaves only three elements of L that can be matched with a cell the rows above and below of L.

Consider the Snakys that have four cells in L. The head of the snake will have two cells in one of 4 disjoint sets of three consecutive cells in the row above or the row below L. Without loss of generality, we may assume that the three consecutive cells $\{(4, 1), (5, 1), (6, 1)\}$. That is, no cell of L is matched by the cells $\{(4, 1), (5, 1), (6, 1)\}$, labeled by "?" in Figure 3.2. But in that case ρ should contain, as pairs, both $\{(4, 1), (5, 1)\}$ and $\{(5, 1), (6, 1)\}$, which is impossible. So we may assume that $\ell > 5$.

Remark. In the case that $\ell > 5$, or ℓ is infinite, we again have a set L containing no pairs such that $|L| = \ell$. Every three consecutive cells in the rows above and below L must contain at least one cell whose label is matched to a cell of L, otherwise we finish the argument as in case $\ell = 5$. Here by "the rows above and below L" we mean sets that extend one cell longer than the end of L if L is finite or if L terminates in one direction.

Second is the case of $\ell = 6$ and we may assume that $\{(1, 0), \ldots, (6, 0)\}$ receive distinct labels. We will show that the only possible pattern is shown in the right-hand side of Figure 3.2. There are diamonds in the cells $(0, 0)$ and $(7, 0)$. Four diamonds remain to be placed and each set of three consecutive cells above and below L. The only possible locations do to so are $(2, \pm 1)$ and $(5, \pm 1)$. This ensures that $\{(0, 1), (1, 1)\}$ and $\{(0, -1), (1, -1)\}$ form pairs, which we label with "A" and "B", respectively.

Note that neither diamonds above and below the cell "2" can also be labeled by "2", otherwise the diamond, its right neighbor, and the cells $3, 4, 5, 6$ would be a pairing-free Snaky. The cells above and below the cell "3" are labeled "C" and "D", respectively. At this moment C could be equal to D. However, if we consider a standing Snaky on the cells $\{(1, 2), (1, 1), (2, 1), (2, 0), (2, -1), (2, -2)\}$, the only unpaired cells are those that are labeled with "E". If we consider a standing Snaky with the same body and the head towards the upper right, the only unpaired cells are those labeled "C" in the right-hand side of Figure 3.2. Symmetrically, we may assign labels "D" and "F" as shown in the figure. This, however, leads to a contradiction, since there would be a pairing-free Snaky again. In particular, the upper E and F cells make the head, and the body consists of the diamond above the cell "2", the cell of the lower C, the empty cell above "4" and the diamond above the cell "5". So, we may assume that $\ell > 6$.

The third case, where $\ell = 7$, is impossible since the rows above and below L should contain three diamonds each to avoid the snakes and two are needed to the right and left of L. This totals at least 8, more than the 7 that are available.

				$\diamond$	A	A	$\diamond$			
L	$\diamond$	1	2	3	4	5	6	7	8	$\diamond$

					$\diamond$	$\diamond$	$\diamond$				
L		1	2	3	4	5	6	7	8	9	
					$\diamond$	$\diamond$	$\diamond$				

Figure 3.3: The cases $\ell = 8$ and $\ell \geq 9$.

In the fourth case, where $\ell = 8$, we have at most eight diamonds around L, two of those at the ends, and every three consecutive cells above and below L containing at least one diamond. So, there are ten cells above L and ten cells below L to receive the remaining 6 diamonds. There must be one in the three leftmost cells above L, in the three rightmost cells above L, in the three leftmost cells below L and in the three rightmost cells below L. Only two diamonds remain. One must be above one of the cells labelled "3", "4", "5" or "6". A diamond cannot be above the cell labelled "4" or "5" because for the two Snakys with heads equal to $\{(4,1),(5,1)\}$ and bodies in L, the diamond either represents one of $\{1,2,3,4\}$ or one of $\{5,6,7,8\}$. Hence, one of these Snakys must be pairing-free. As a result, the cells $\{(4,1),(5,1)\}$ must be paired with each other and so we label them with "A". See the diagram in the left-hand side of Figure 3.3. Because every three consecutive cells must contain at least one diamond, the cells above the cells labeled "3" and "6" are labeled with a diamond. This is a contradiction to the fact that only one diamond can be above these cells. So, we may assume that $\ell > 8$.

In the fifth case, where $\ell \geq 9$ and is finite, all cells above and below the cells $4, \ldots, \ell - 3$, the "critical region", must be diamonds. It is the same idea as in the previous case: If, say the cell above "4", is A, then so is the cell above "5". But the same is true for the cells above "5" and "6". Not only must the cells in the critical region be diamonds, there must be a total of at least 4 more above at below L to cover all of the triples of consecutive cells. With the additional two on the endpoints, there must be at least $2(\ell - 6) + 4 + 2$ diamonds, that is impossible, given that the total number of diamonds is at most ℓ, which is at least 9.

Finally, suppose L is infinite. Take 13 consecutive cells of L, call it L'. In the critical region of L' there must be $2(13 - 6) = 14$ cells with diamonds, but they must repeat the labels in the cells of L', a contradiction. This concludes the proof of the fact that a pairing for Snaky must be a pairing for P_5.

We exhibit two pairings for P_5. The pairing T_1 is like a chessboard, where the fields are 2×2, and alternately packed by a standing and lying pairs of dominoes as in the left-hand side of Figure 3.4. The pairing T_2 is like an infinite zipper, repeated in both directions, see the right-hand side of Figure 3.4.

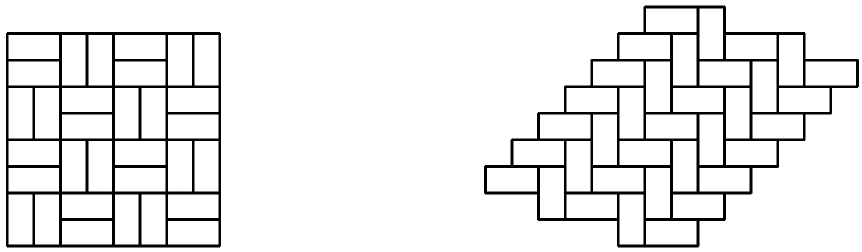


Figure 3.4: The parings T_1 and T_2.

Lemma 3.16. *A pairing for P_5 is either the translated and rotated copy of T_1 or T_2.*

Proof. Let us consider a pairing, ρ, for P_5. A pair $\{x, \rho(x)\}$ is *good* if x and $\rho(x)$ are neighbouring cells. If $\{x, \rho(x)\}$ is good, then $d_2(x, \rho(x)) = 4$, otherwise it is smaller. The number of P_5's are $2n^2 + O(n)$ on an $n \times n$ sub-board X, so Proposition 3.14 implies that at all but $O(n)$ pairs on X are good. It follows that, if n is sufficiently large, then there is a $Y \subset X$, $k \times k$ square sub-board that contains only good pairs. I. e. this $k \times k$ sub-board is paired by dominoes.

There are either two dominoes meeting at their longer sides, or the two long sides meet but are offset by one unit. In these cases the immediate neighbouring dominoes are forced to be in the pattern of T_1 or T_2, respectively.

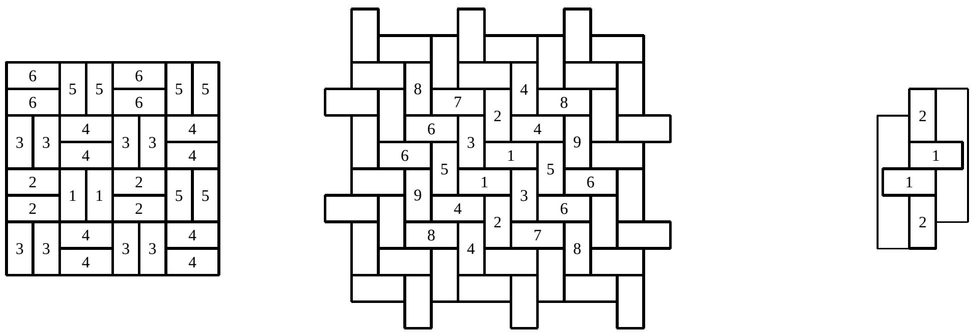


Figure 3.5: The forcing for pairs and filling.

We will show that if we have a large enough pattern of dominoes, then the pairs in the neighbouring cells are forced to be in either T_1 or T_2. First suppose that, within the pattern tiled by dominoes that two dominoes share a long edge, as in the dominoes labelled with “1” in the left-hand side of Figure 3.5. Since the pairs can only occur as dominoes, we can use horizontal P_5's to ensure the pairing is oriented as in the dominoes labelled with “2”. Vertical P_5's ensure the orientations of the dominoes labelled “3”. We can continue in this fashion, getting the 8×8 pattern in the left-hand side of Figure 3.5. Once this is determined, one can extend the pattern to a larger rectangle, forcing not just the domino condition, but the T_1 pattern itself. This can be seen by first taking horizontal P_5's in rows 1,2,5,6 that have two cells outside of the pattern. Then taking vertical P_5's in columns 9,10, the pattern can be extended to an 8×10 rectangle. This can be continued ad infinitum, showing that the entire $n \times n$ board must be in the pattern T_1.

Next, suppose that whenever two dominoes meet at their long edge in the sub-board, that they are offset by one unit, since two dominoes meeting at their long edge will force the pattern T_1. See dominoes labelled "1" in the diagrams in the center or the right-hand side of Figure 3.5. The pairs must occur as dominoes and so vertical P_5's ensure that the dominoes labelled with "2" are placed in that location. Now, consider the right-hand side of Figure 3.5. Two P_5's are indicated by thin lines. Since the dominoes cannot share a long side, this forces the placement of the dominoes labelled with "3".

In fact, if we know that a sub-board is tiled with dominoes that do not share a long edge, then the configuration must be that of T_2. It remains to show that if we have a large enough fragment of T_2 in a sub-board, then, even if the board is not guaranteed to be tiled with dominoes, it must be completed to a T_2 pattern. The other pairs are forced even without the assumption that those are in dominoes, since the otherwise a P_5 containing no pair would arise.

To see how we can use this sub-board to extend T_2 to the whole board, we first show in the center of Figure 3.5 how enough pairs can be formed under the assumption that every pair forms a domino and no pair of dominoes can share a long edge. The numbers show the order in which dominoes can be taken. Then, in Figure 3.6 we show how, under no assumptions that the pairs occur as dominoes, that the dominoes that cover the 7×7 board can be extended to cover a 9×9 board. Again, the numbers show the order in which dominoes can be taken.

The general approach is that one can force new horizontal dominoes in every third row that touch the left and right border of the small square and vertical dominoes in every third column that touch the top and bottom border. From there, the rest of the larger square is easy to complete. This can continue *ad infinitum* until the board is filled. This concludes the proof of Lemma 3.16. ☐

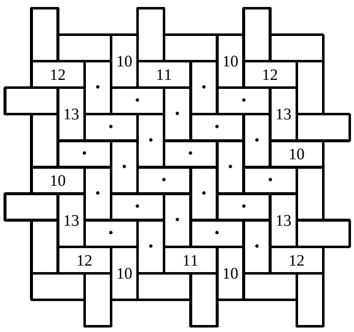


Figure 3.6: Expanding a 7×7 square to a 9×9 square. The dominoes given by the 7×7 square are marked with "·".

By Lemma 3.16, the pairs of ρ are either in the pattern T_1 or the pattern T_2, but none of those are pairings for Snaky. This concludes the proof of Theorem 3.15. ☐

3.7 Torus games

To test Beck's paradigm from Conjecture 2.9that Chooser-Picker and Picker-Chooser games are similar to Maker-Breaker games, we check the status of concrete games defined on the 4×4 torus. That is, we identify the opposite sides of the grid, and consider all lines of slopes 0 and ± 1 and size 4 to be winning sets. We denote the torus, along with those winning sets with the notation 4^2. For the general definition of torus games, see [7]. We use a chess-like notation to refer to the elements of the board. We note that the hypergraph of winning sets on 4^2 is not almost disjoint, see e. g. the two winning sets $\{a2, b1, c4, d3\}$ and $\{a4, b1, c2, d3\}$. See Figure 3.7. We consider four possible games on 4^2: Maker-Maker, Maker-Breaker, Chooser-Picker and Picker-Chooser. According to [7], the Maker-Maker version of 4^2 is a draw, and, according to [21], Picker wins the Chooser-Picker version. Here, we investigate the Maker-Breaker and the Picker-Chooser versions. In fact, the statement of the Maker-Breaker version implies the result for the Maker-Maker version, while the proof of it contains the proof of the Chooser-Picker version.

Proposition 3.17. *Breaker wins the Maker-Breaker version of the 4^2 torus game.*

Proof. Using the symmetry of 4^2, we may assume, without loss of generality, that Maker takes $a4$. Breaker's move will then be to take $d1$. Up to isomorphism, there are eight cases depending on the next move of Maker. The first element of the pair is Maker's move, while the second is Breaker's answer: 1. $(c3, b2)$, 2. $(b3, b2)$, 3. $(c2, b2)$, 4. $(b4, c3)$, 5. $(c4, b4)$, 6. $(d4, c3)$, 7. $(d2, a3)$ and 8. $(d3, b1)$.

In the first seven cases Breaker has winning pairing strategies. All eight cases are shown in the first two rows of Figure 3.7 and the pairs appear under the labels A, B, C, D, and E. We leave it to the reader to check that the pairs block all 16 winning sets.

In the eighth case Breaker does not have pairing strategy, but the game reduces to one of the seven prior cases unless Maker plays $a3$, $a2$ or $a1$ in the third step of the game. In that case, Breaker plays $b4$, $a3$ or $b2$, respectively, and wins by the pairing strategy shown in the third row of Figure 3.7. $\square$

Note that in the Chooser-Picker version of the game 4^2, Picker can achieve a position isomorphic to Case 1. That is, Picker wins.

If Conjecture 2.9 were true, then Breaker has an easier job in the Maker-Breaker version than Chooser has in the Picker-Chooser game. For the 4×4 torus the outcome of these games are the same, although this is much harder to prove.

Proposition 3.18. *Chooser wins the Picker-Chooser version of 4^2, the 4×4 torus game.*

Sketch of the proof. The full proof needs a lengthy exhaustive case analysis. However, some branches of the game tree may be cut by the following result of Beck [6]: Chooser wins the Picker-Chooser game on $\mathcal{H}$ if $T(\mathcal{H}) := \sum_{A \in E(\mathcal{H})} 2^{-|A|} < 1$.

In our case, $T(\mathcal{H}) = 16 \times 2^{-4} = 1$, which just falls short. Instead we use a similar method using so-called *potential functions*. We assign weights to each edge at the i^{th} stage such that $w_i(A) = 0$ if Chooser has taken an element of A, otherwise it is $2^{-f(A)}$, where $f(A)$ is the number of untaken elements of A. The weight of a vertex x is $w_i(x) = \sum_{x \in A} w_i(A)$, while the total weight is $w_i := \sum_{A \in E(\mathcal{H})} w_i(A)$.

Note that Picker wins if and only if both $w_8 \geq 1$ and $w_0 = T(\mathcal{H}) = 1$. When a pair (x, y) is offered, Chooser can always take the one with larger weight, which results in a non-increasing total weight. In fact, if the weights of x and y differ or both x and y are elements of an A of positive weight, then the total weight strictly decreases.

In order to have any possibility of winning, Picker has to select x and y of equal weights and no edge of positive weight containing both. By the symmetries of the board, we may assume Picker gets $a4$ and Chooser gets $c3$ in the first round. After that, Picker has

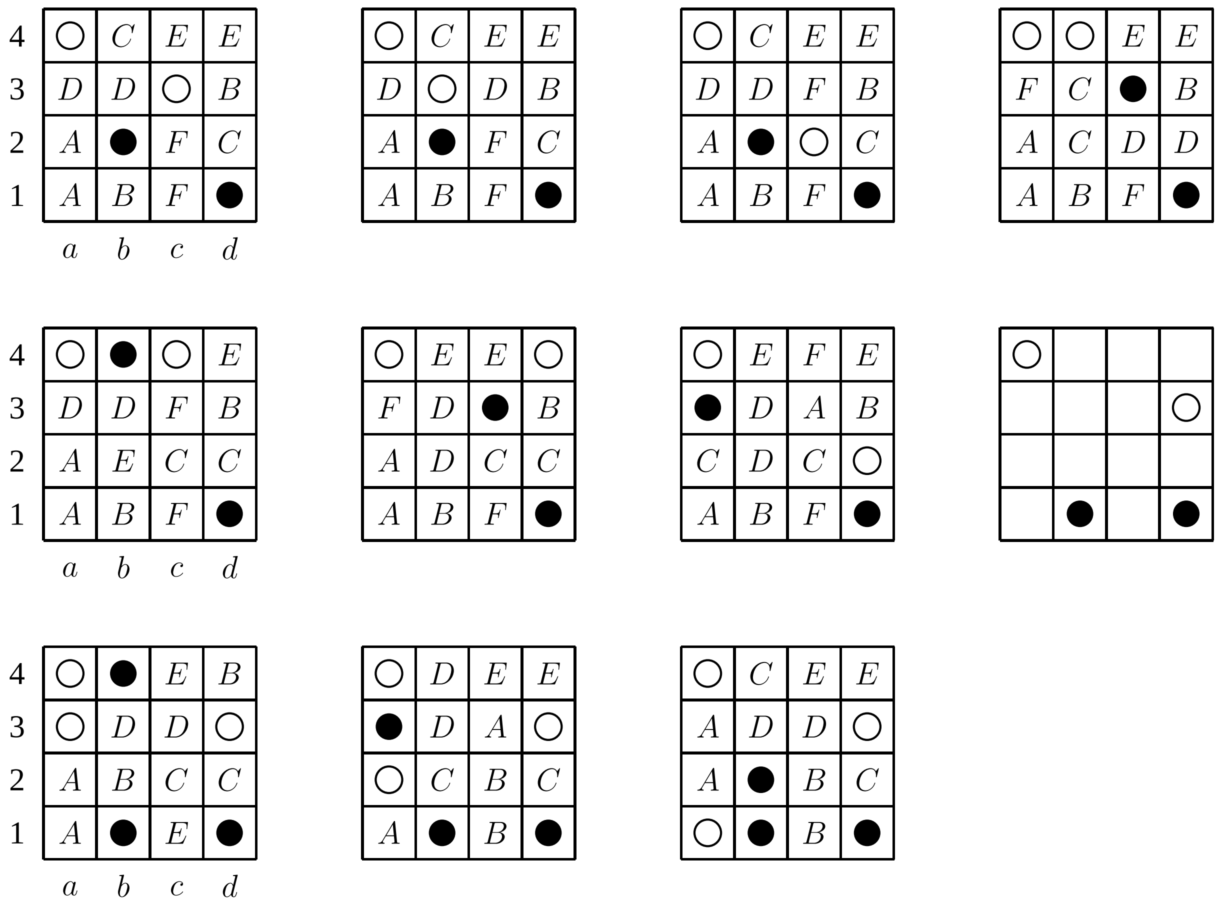


Figure 3.7: The pairings used by Picker in the game 4^2.

only pairs (x, y) that do not result in a loss for Picker: $(b4, d3)$, $(a3, c4)$, $(b3, d4)$, $(a3, b3)$, $(a3, d3)$, $(b3, d3)$, $(a1, b2)$ and $(a1, d2)$, see Figure 3.8. The letter P [C] designates the vertex taken by Picker [Chooser] in the first step, the numbers are the weights of the vertices.

4	P	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$
3	$\frac{4}{16}$	$\frac{4}{16}$	C	$\frac{4}{16}$
2	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{3}{16}$
1	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{5}{16}$
	a	b	c	d

Figure 3.8: The beginning of the Picker-Chooser 4^2 game.

The rest of the proof is similar to that of the prior step: one needs to check that Chooser has winning strategy for each of the eight nontrivial responses of Picker. We omit the details. $\square$

Chapter 4

The Chooser-Picker 7-in-a-row game

4.1 The k-in-a-row game

The k-in-a-row game is that hypergraph game, where the vertices of the graphs are the fields of an infinite graph paper ($\mathbb{Z}^2$), and the winning sets are the consecutive cells (horizontal, vertical or diagonal) of length k. If one of the players gets a length k line, then he wins otherwise the game is draw. Note the assuming perfect play, the winner is always the first player, or it is a draw by the strategy stealing argument of John Nash, [13]. More details about k-in-a-row games in [61, 62].

The board of the classical 5-in-a-row game is a graph paper or the 19×19 Go board[1], and the players' goal is to get five squares in a row vertically, horizontally or diagonally first.

It is easy to see that the first player wins if $k \leq 4$, and a delicate case study by Allis [1] shows that the first player wins for $k = 5$ on the 19×19 or even in the 15×15 board. From this result it does *not* follow that the same is true for the infinite board, as it sometimes claimed, [25]. Theoretically it can be occurred, that by placing there an other winning set - the new game is a draw. This phenomenon is called *Extra Set paradox*. The simplest example for it is the following: In the figure 4.1 there is a hypergraph with 8 branches (these are the winning sets). The players marks the vertices one-by-one. This first player can easily win this game, but if we add that extra 3 length wining set, then the game is draw.

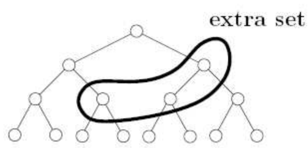


Figure 4.1: If the first player chooses one-one vertices for all of the branchings, then he/she wins. But if the first player cares for the second player not to occupy the extra winning set, then the first can not win.

So the case $k = 5$ is still open on the infinite board, but Allis' result implies that Maker wins for $k = 5$ in the Maker-Breaker version.

[1]The Go-Moku rules differs from the 5-in-a-row, see e. g. [13].

It is much less known that the Kőnig-Hall theorem can be extended to the infinite case, at least if the hypergraph is *locally finite*. This is due to Marshall Hall Jr., see [36].

Theorem 4.1 (M. Hall Jr.)**.** *The system of finite sets $\{A_i\}_{i=1}^{\infty}$ has system of distinct representatives, if for all finite $I \subset \mathbb{N}$ $|\cup_{i\in I} A_i| \geq |I|$.*

We can use this theorem for the k-in-a-row game played in the infinite board. It gives a draw for $k \geq 15$. If we play on a d dimensional board this number is $k \geq 2(3^d - 1) - 1$ [60]. If $d = 2$ then we get $k \geq 15$, if $d = 3$ then $k \geq 53$, if $d = 4$ then $k \geq 159$ and so on...

The game is a *blocking draw*, i. e. Breaker wins the Maker-Breaker version a if $k \geq 9$, proved first by Shannon and Pollak by using H letter shape, auxiliary sub-boards. The Breaker wins all of the sub-game, and it means that the game is draw (even the weak and the strong case). Later even a pairing strategy was given, [7, 13].

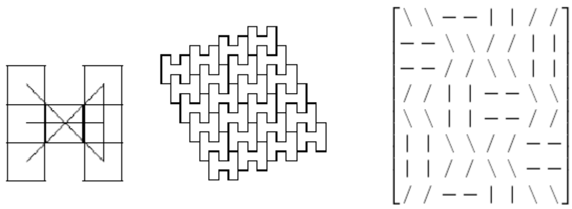
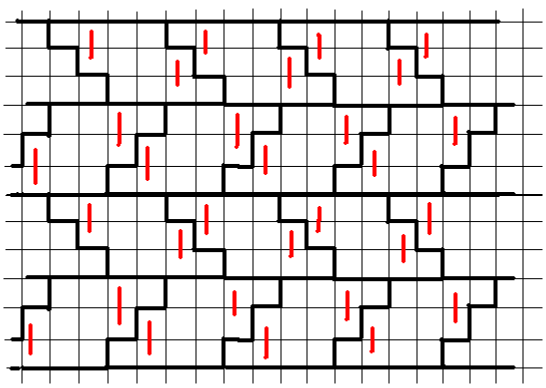


Figure 4.2: The 9-in-a-row draw: proof by two different tillings.

Finally Breaker's win was published by A. Brouwer under the pseudo name T. G. L. Zetters[2] for $k = 8$, [34]. The proof is also uses sub-games. The authors made two serious mistakes when they sent their solution to the *American Monthly* for R. K. Guy and J. L. Selfridge problem. One was that they believed that the 7-in-a-row also can be handled such "easy" way. The other was the alias, because since 30 years nobody could improve this result. [3]



[2]This pun certainly refers to the proof that Shannon and Pollak gave for the case $k = 9$. They used "H" shaped sub-boards, here more sophisticated sub-boards are needed. Note that "zetter" means "typesetter" in Dutch.

[3]A. E. Brouwer's homepage, see http://www.win.tue.nl/ aeb/publications.html#pubpapers refers to that he was one on the authors.

Figure 4.3: The original 8-in-a-row is draw [34] and the C-P version is also a draw [21]. Both uses this tilling (the winning sets are 3-4 length diagonal and straight lines and two additional pairs). It can be seen, that neither direction can be marked 8 consecutive cells, it means that none of them has a whole hyper-edge.

Both the Maker-Maker and the Maker-Breaker versions of the k-in-a-row for $k = 6, 7$ are open. These are wisely believed to be draws (Breaker's win) but, despite of the efforts spent on those, not much progress has been achieved.

4.1.1 Accelerated, and biased k-in-a-row games

Now here comes an example for accelerated games: the accelerated k-in-a-row. The theorem below (see [61]) shows the length of the winning set in formula $p + f(p)$ where p is the speed of the acceleration.

Theorem 4.2 (Pluhár, [61])**.** *If $f(p) \geq 80\log(p) + 160$ and $p \geq 1000$, them Breaker wins the $A_{p+f(p)}(p, p)$ accelerated game. If $f(n) = \frac{\log_2 p}{\log_2 \log_2 p} - 1$, then Maker wins the $A_{p+f(p)}(p, p)$ accelerated game.*

In more specific cases there can be proved more. For example Wu and Huang [74] found the following result :

Theorem 4.3 (Wu-Huang [74])**.** *The biased 6-in-a-row $(p = 2, q = 3)$ (e.g. $Connect(6, 2, 3)$) is a second player win.*

Before proving the C-P 7-in-a-row game, we prove the easier C-P 8-in-a-row game:

4.2 The Chooser-Picker 8-in-a-row game

Proposition 4.4. *Picker wins the Chooser-Picker version of the game 8-in-a-row on any $B \subseteq \mathbb{Z}^2$.*

Proof. First we need to use the lemma 2.18

We shall cut up the infinite board to sub-boards in the same way as it was in [34], see also Figure 3. The left tile and its mirror image are the bases of the tiling. The winning sets for the these sub-boards are the rows, the diagonals of slope one, and the two pairs indicated by the thin lines. The middle of the picture shows the tiling itself. We use one type of tile in an infinite strip, and its mirror image in the neighboring stripes. On the right side of Figure 3 the transformed tile is drawn, where the winning sets are the rows, columns and the indicated two pairs.

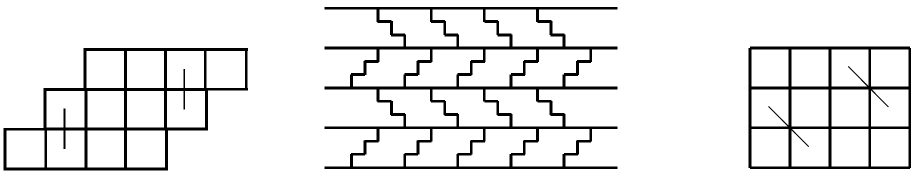


Figure 3

Let $\bar{B}$ be the union of those sub-boards meeting B. We show that Picker wins the Chooser-Picker 8-in-a-row game for the board $\bar{B}$. Note that $\bar{B}$ is a union of sub-boards. Picker plays auxiliary games on the sub-boards independently of each other with the goal of preventing Chooser from getting a winning set of a sub-board.

To achieve this goal, Picker selects the two pairs first on any sub-board, that give rise to the possible positions shown on Figure 4. Then Picker uses the appropriate winning pairing strategy indicated by the thin lines. One checks easily that if Picker wins all the auxiliary games then he wins the Chooser-Picker 8-in-a-row game on playing $\bar{B}$, too. Finally, by Lemma 2.18, Picker wins on B. $\square$

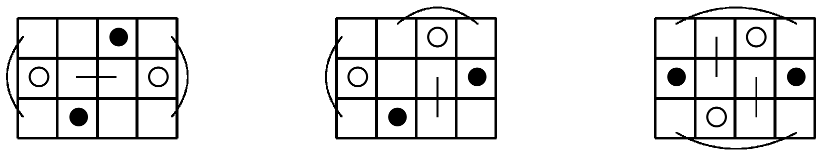


Figure 4

Now we prove the C-P 7-in-a-row game:

4.3 The Chooser-Picker 7-in-a-row game

Theorem 4.5. *Picker wins the Chooser-Picker 7-in-a-row game on every A subset of $\mathbb{Z}^2$.*

Let us start with the strategy of the proof. By applying the remedy mentioned before Lemma 2.18 at first Chooser determines the finite board S. We will consider a tiling of the entire plane, and play an auxiliary game on each tile (sub-hypergraph). It is easy to see, if Picker wins all of the sub-games, then Picker wins the game played on any K board which is the union of disjoint tiles. Let K be the union of those tiles which meet S. Since $S \subset K$, Lemma 2.18 gives that Picker also wins the game on S, too. Now we need to show a suitable tiling and to define and analyze the auxiliary games. The tiling guarantees that if Picker wins on in each sub-games then Chooser cannot occupy any seven consecutive squares on K.

Each tile is a 4×8 sized rectangle and the winning sets, for the sake of better understanding, are drawn on the following four board:

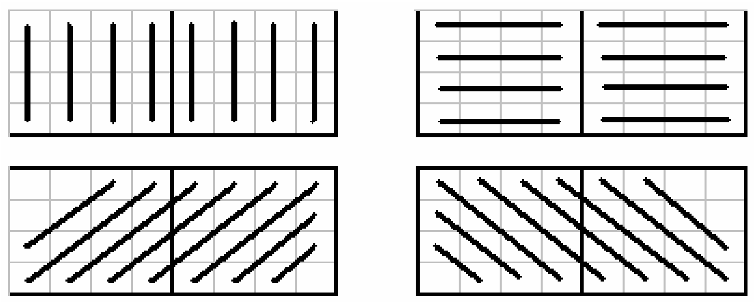


Figure 4.4: These are the winning-sets of the 4×8 rectangle. Easy to see, that there is exactly one symmetry (along the double line). Later we will make use of it.

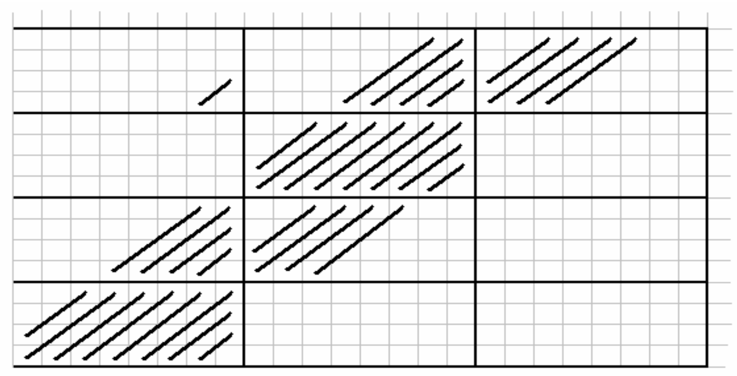


Figure 4.5: We can see, how to draw from playing on simple tile, the game played on the infinite chessboard: neither vertically, nor horizontally, nor diagonally (there is only one diagonal direction detailed) there are no seven consecutive squares without containing one winning set of a sub-game.

The key lemma for our proof is the following.

Lemma 4.6. *Picker wins the auxiliary game defined on the* 4×8 *rectangle.*

Before starting the proof of Lemma 4.6, let us estimate the actual complexity of the Maker-Breaker and Chooser-Picker versions of the auxiliary "4×8-game."

4.3.1 Some words about the Maker-Breaker case

Before of proving the theorem, let us says a few words about the weak version of the same subgame and the concept of its evaluation. The basic of the proof is a DFS algorithm for the double steps (first Maker, after Breaker). [4] An algorithm which tries all of the cases needs $32!$ steps. We use *branch and bound* techniques to reduce the cases. Regarding that fact that there is only one symmetry in the board it is not so easy task. So we has to use backtracking (see 2.3.3), weighting (see 2.3.2) and dominating techniques as well (see here).

Some useful techniques to fasten the game

- general:
  –It always helps if Breaker has a good defending heuristic, but it is "only" gets the square root of the $32!$
  –After 10-12 steps the board splits into pair-wise disjunct components. It is enough to study the separate components.
  –The players should play "balanced" in the two parts of the graph.
  – It is useful to make dictionaries for the often occurring stages (=final stages with some unoccupied cells at the end).

- weighting:
  –By using weighting techniques if Maker has a quick (one step) win then take it.
  –Breaker should occupy first such fields where are a lot of unoccupied winning sets.

- dominating:
  If Maker cannot win in the next step, then it is not worthwhile for Maker to step to a place where is at most one winning-set which is not marked by Breaker. It means that it is not worthwhile to step places where is just one winning set and more then two cells for it.

Remark. We checked with brute force computer search the M-B game on the same auxiliary board (see 4.3.1), but it is a Maker win!

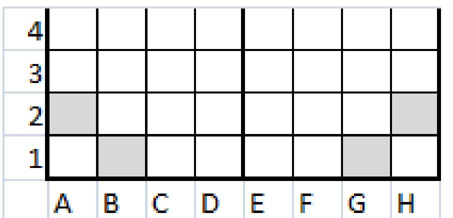


Figure 4.6: Encoding the table

[4] It is interesting that the proof is a BFS algorithm, but we use DFS, because we need to know the result of a stance before we start to examine an other one. So we use use DFS to examine the board.

Maker can force a win by the following moves (by using the encoding above). Not all of the Breakers move are directly forced, but if Breaker does not take that move sooner or later looses.

The begining:
A2-B1; G1-H2; C4-B3; E1-F1; A4-A3;
The trick:
(solution I) B2-B4; C3-C2; D4-A1; E2-D3
(solution II) B2-D4; E2-D3; D2-C2
Endgame:
G2-H3; G4-G3; F3-D1; E4-H1; F4 (Maker wins by double threat)

So we cannot use the same table again, to prove that the weak version (Maker-Breaker version) of this game is a Breaker win. One is tempted to look for other auxiliary games, which is not going to be easy. As a rule of thumb, it always good idea to check the C-P version of these games at first.

4.3.2 The Chooser-Picker case

In the Maker-Breaker version Maker has 32 possible moves, then Breaker has 31, so clearly the (unpruned) game-tree has size $32!$. Even worse, it may be hard to write down convincing evidence of the outcome after searching this tree. In the Chooser-Picker case, provided that Picker win, there is always a much shorter proof of the outcome. Picker exhibits two squares and depending on Chooser move, only two smaller games have to be searched. This leads to a game-tree of size 2^{16}, which is reasonable to search. (Note that if Chooser wins a Chooser-Picker game, the verification can be even harder than a proof for the corresponding Maker-Breaker version.)

With some consideration the length of the case-study of the Chooser-Picker version can be reduced, too. One tool of this is a classification on the partially filled tables. Let us denote the squares of a board T taken by Chooser or Picker by T_C and T_P, respectively. From Picker's point of view the table T is more dangerous than the table T' $(T > T')$ if $T'_C \subset T_C$ and $T_P \subset T'_P$. Thus if Picker has a p winning strategy on T, as a consequence of Lemma 2.18 playing the modified p Picker also wins on T'. See the application in 4.3.3.

An other gain is that Picker can ask an appearing two length winning set immediately by Lemma 2.17. (In the defined 4×8 auxiliary game there are two such pairs at the beginning already, and some appears later.)

Finally, we do not always have to go down to the leaves to the game-tree, since an appropriate pairing strategy may prove Picker's win in an inner vertex of that tree.

Our plan is for proving the key lemma is

I. Separating cases: A) and B) type cases.

II. Filling up one side of the auxiliary table using breath first search.

III. After a case classification filling up the other side.

4.3.3 The proof of the key lemma

The course of the proof is: We take the 2 piece of 2 length winning set. Picker picks them at the beginning (Picker can do this without any disadvantage thanks to the Lemma 2.18). Depending on Chooser selection, there are two cases:

A) Chooser gets the upper square at least one side.

B) Both side Chooser gets the lower ones.

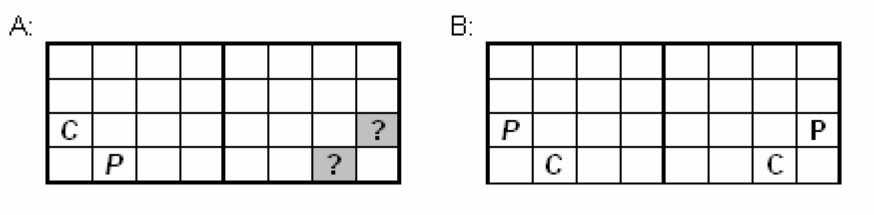


Figure 4.7: The two cases: case **A)**, and case **B)**.

From the characteristic of the game, the tree which describes the game is binary-tree, we should walk breadth first on the cases; thus the cases with the same parent are beside each other. (After Picker's move, these are the two possible choices of Chooser). According to this, the positions are indexed lexicographically: ***A, B, a, b, i, ii, I, II***. . .

Somehow remarkable, that we the use breath first search to write down the proof, but to find the value of the game we always use deep first search algorithms (because we have to know the outcome of the game, before we are looking to the next branch of the tree).

case A)

Without loss of generality, we may assume Chooser occupies the upper square on the left side (there might be the same on the right side). Now Picker's strategy is to fill up the left side and leave the least possible crossing winning sets are left alive (see more detailed at **Index_A**). On the pictures in the Appendix the special marks like =, *, +, etc. are the pairs to be asked by Picker. At that stage it makes no difference which squares are chosen by Chooser. And those marks also indicates the ending of a branch of the game-tree.

It is convenient to introduce to a new notation: It helps Chooser's game if we change one of Picker's square to a free square, and it is also advantageous for Chooser if he/she gets one of the free squares (P $\ll$ FREE $\ll$ C). It means that it is not necessary to prove a case if there exists a more dangerous one.

From both Picker and Chooser point of view a square (be occupied or not) is *uninteresting*, if each winning set which contains it is "dead." It does not change the outcome of the game if we give these squares to Picker.

Then after filling up the left side we can create equivalence classes using the relations and arrangements above. In this way one have to consider seven cases only:

The finish of these positions above (=filling up the right side) can be found detailed in the Appendix of the paper (see Index$_{Ra}$, Index$_{Rb}$, ...Index$_{Rg}$)

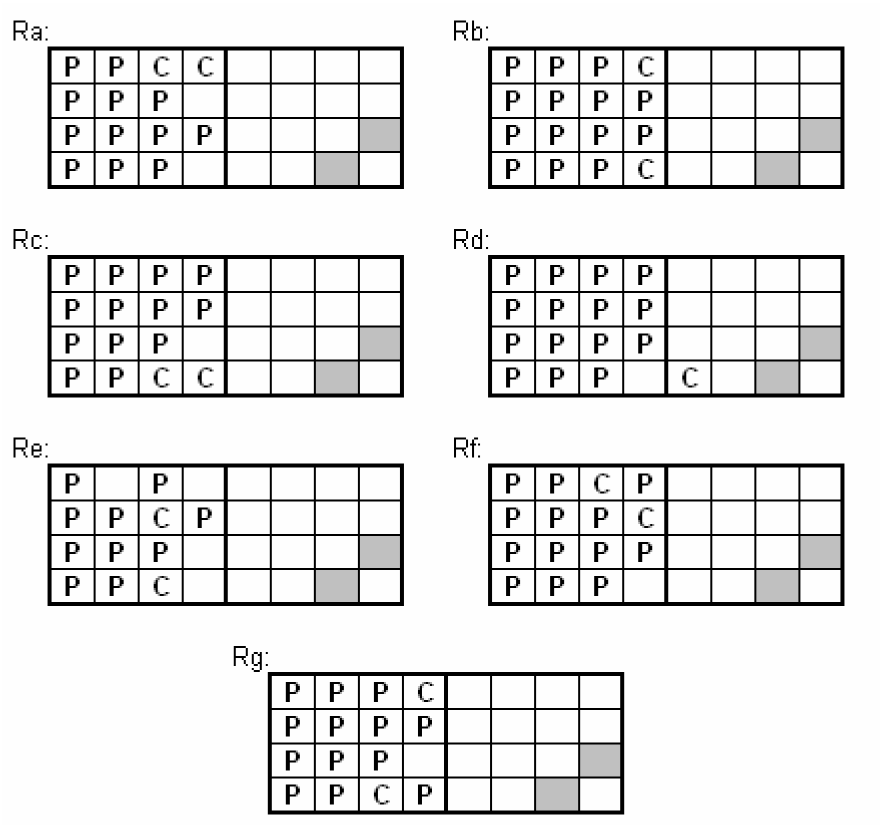


Figure 4.8: If on the “left side” matching Chooser occupies the “upper” square, than Picker can achieve one of this stages (or an equivalent or less perilous position, using arrangement above).

case B)

If case B) happens, then Picker asks the following two matching (see below), hence, using the symmetry, it is enough to examine the following three cases.

The results of the three cases are also detailed at the APPENDIX (see **Index**$_{Ba}$**,** **Index**$_{Bb}$**,** **Index**$_{Bc}$), that concludes the proof of Lemma 4.6 and consequently Theorem 4.5. □

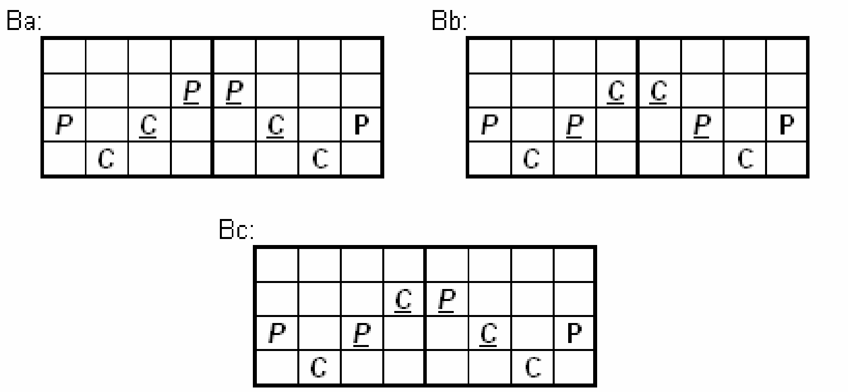


Figure 4.9: If at the beginning Chooser takes the “lower” squares on both side, then Picker asks the two colored pair of squares in the middle. It gives rise to three cases.

Chapter 5

The Picker-Chooser Diameter Game

5.1 Introduction to Graph Games

Large classes of Maker-Breaker games are defined on the complete graph on n vertices. The players take the edges of the graph in turns; Maker wins iff his subgraph has a given, usually monotone, property $\mathcal{P}$, see [8, 5, 12, 17, 40]. As we mentioned in subsection 2.2.3, in those cases a the random heuristic works very well. That is if p_0 is the *threshold*[1] for the property $\mathcal{P}$, than Maker should win the if $(1 : b)$-game if $b < 1/p_0$, while Breaker should win if $b > 1/p_0$.

This heuristic is so powerful, that sometimes even a random play confirm it [12].

After that the following results is quite surprising. Balogh et al. [2] introduced the $(a : b)$ d-diameter game, shortly $\mathcal{D}_d(a : b)$, which means that Maker wins iff the diameter of his subgraph is at most d. These games turned out to be very difficult and surprising; a detailed discussion will be given in Section 5.2. The main result of Balogh et al. was that Maker loses the game $\mathcal{D}_2(1 : 1)$ but Maker wins the game $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\log n)^{3/8})$.

That is the acceleration of a game may change the outcome dramatically. This phenomenon was first noted by Pluhár [61]. The outcome also changes a lot when one considers the Picker-Chooser version of the game $\mathcal{D}_2(1 : 1)$. Our main result is the following theorem.

Theorem 5.1. *In the Chooser-Picker game $\mathcal{D}_2(1 : b)$, Picker wins if $b < \sqrt{n/(16 \log_2 n)}$, while Chooser wins if $b > 3\sqrt{n}$, provided that n is large enough.*

5.2 Diameter and degree games

Let us repeat the definition from the introduction. The diameter d game $\mathcal{D}_d(a : b)$ is played on the edges of the complete graph K_n, Maker (Breaker) takes a (b) edges in each turn. If Maker's edges form a subgraph of diameter at most d at the end, then Maker wins the game, otherwise Breaker wins.

Balogh et al. [2] observed that the game $\mathcal{D}_2(1 : 1)$ defies the probabilistic intuition completely. Indeed, if one divides the edges of K_n among Maker and Breaker randomly, then Maker's subgraph will almost surely have diameter two. Still, Breaker has a simple

[1]It means roughly that $\mathcal{P}$ holds for $G(n, p)$ almost surely if $p - \epsilon > p_0$ and fails if $p < p_0 - \epsilon$, for any $\epsilon > 0$, n goes to infinity.

pairing winning strategy for $n > 3$, [2]. First taking an edge uv, such that neither ux nor vx has been taken by Maker for any vertex x. Then if Maker takes ux, taking vx follows, and if Maker takes vx, Breaker takes ux, otherwise an arbitrary edge is taken.

However, when playing the game $\mathcal{D}_2(2 : 2)$, this pairing strategy is not available for Breaker. Maker wins the game $\mathcal{D}_2(2 : 2)$, and even more, the game $\mathcal{D}_2(2 : b)$, where b grows polynomially in n, provided that n is large enough.

Theorem 5.2. *[2] Maker wins the game $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\ln n)^{3/8})$, and Breaker wins the game $\mathcal{D}_2(2 : (2+\epsilon)\sqrt{n/\ln n})$ for any $\epsilon > 0$, provided n is large enough.*

Note that the random graph $G(n, p)$ has diameter two with probability close to one if $p > n^{-0.5+\epsilon}$, while this probability is close to zero, if $p > n^{-0.5-\epsilon}$ and n is large enough. The breaking point b_0 of Theorem 5.1 is within that interval, so we may say the Picker-Chooser $\mathcal{D}_2(1 : b)$ game follows the probabilistic intuition.

To prove Theorem 5.1 we need to study the so-called *degree games*. Székely, Beck and Balogh et al. [73, 5, 2] showed that these games are interesting in their own right.

In such games one player tries to distribute his moves uniformly, while the other player's goal is to obtain as many edges incident to some vertex as possible. Given a graph G and a prescribed degree d, Maker and Breaker play an $(a : b)$ game on the edges of G. Maker wins by getting at least d edges incident to each vertex. For $G = K_n$ and $a = b = 1$ this game was investigated thoroughly in [73] and [5]. It was shown that Maker wins if $d < n/2 - \sqrt{n \ln n}$, and Breaker wins if $d > n/2 - \sqrt{n}/12$.

This is in agreement with the probabilistic intuition, since in $G_{n,1/2}$ the degrees of all vertices fall into the interval $[n/2 - \sqrt{n \log n}, n/2 + \sqrt{n \log n}]$ almost surely. We are interested only in the case of $G = K_n$.

Balogh et al. [2] proved the following lemma:

Lemma 5.3. *[2] Let $a \le n/(4 \ln n)$ and n be large enough. Then Maker wins the $(a : b)$ degree game on K_n if $d < \frac{a}{a+b}n - \frac{6ab}{(a+b)^{3/2}}\sqrt{n \ln n}$.*

As we do not wish to develop the complete theory of P-C (C-P) degree games, we state only a simple form that suffices our needs and furthermore provides an elegant proof.

Lemma 5.4. *Let $b < n/(8 \ln n)$ and n be large enough. Then Chooser wins the $(1 : b)$ Chooser-Picker degree game on K_n if $d < n - 1 - 3n/b$.*

5.3 Proofs

5.3.1 The case $\mathrm{a} = \mathrm{b} = 1$

Both directions of Theorem 5.1 rely heavily on the weight function method. It is worth noting that it is much easier to prove Picker's win in a special case. A brief discussion needs to follow so that we can introduce some of the notions used later.

Observation.

Picker wins the P-C game $\mathcal{D}_2(1 : 1)$ on the graph K_n, if $n > 22$.

Proof. Let us start with a definition. Playing the game, Picker *links* a set of vertices, if he achieves that all the distances among those are at most two.

At first Picker marks two non-incident edges (a, b) and (c, d). Chooser chooses one of them, for instance (c, d), while (a, b) goes back to Picker. Then Picker picks all pairs of edges $((p, a), (p, b))$ for $p \in V \setminus \{a, b\}$ one by one. We can partition $V \setminus \{a, b\} = A \cup B$, where A and B are the vertices connected directly to a and b, respectively. The vertices within A and B are linked together, and both sets are linked to both a and b.

Say, that $|A| \geq |B|$, which also means $|A| \geq 10$. We show that Picker can get a complete matching $\mathcal{M}$ for covering the vertices of A, if A is even. If A is odd, Picker can get a matching $\mathcal{M}$ and possibly a triangle T. Let the vertices of A be $1, \ldots, k$. Picker offers the edges $(1, 2)$ and $(1, 3)$ and gets back, for example, the edge $(1, 2)$. Then Picker offers the edges $(3, 4)$ and $(3, 5)$ and again we may assume that the edge $(3, 4)$ goes back to Picker and so on.

If A is even, Picker ends up with the almost perfect matching $\mathcal{M}'$ consisting of the edges $\{(i, i+1)\}$, for $i = 1, 3, 5, \ldots, k-2$. Then Picker offers the pair $(1, k-1), (2, k)$, and getting back, say, $(1, k-1)$. Finally, Picker offers the pair $(2, k), (k-1, k)$. Obviously, either $(k-1, k)$ or $(2, k)$ leads to a perfect matching $\mathcal{M}$, since either $\mathcal{M} = \mathcal{M}' \cup \{k-1, k\}$, or $\mathcal{M} = \{\mathcal{M}' \setminus \{(1, 2)\} \cup \{(1, k-1), (2, k)\}\}$.

If A is odd, $\mathcal{M}'$ is the same as before, exposing only the vertex k. Picker may ask for $(1, k), (3, k)$, then $(5, k), (7, k)$. He gets back two of these edges, say $(1, k)$ and $(5, k)$, and then asks for the pair $(2, k), (6, k)$. This result is a matching and a triangle, covering A.

Finally, Picker picks edges in pairs $(b', x), (b', y)$, where $(x, y) \in \mathcal{M}$, $b' \in B$. It links vertex b' to both x and y, or in case of a triangle $\{i, i+1, k\}$, to these vertices. $\square$

5.3.2 Proof of Lemma 5.4.

First, we transfer the degree game to a P-C game played on a hypergraph. The hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is such that $V(\mathcal{H})$ is the edges of K_n, while $A \in E(\mathcal{H})$ iff $|A| = \lceil 3n/b \rceil$ and all (graph) edges in A incident to a vertex x of K_n. To prove the lemma, it is enough to show that Chooser wins a P-C game on $\mathcal{H}$.

Let Chooser choose randomly and independently in each round, which means that Picker gets back any edge e with probability $1/(b+1)$. Hence, for any strategy of Picker, the probability that Picker gets every edges of an $A \in E(\mathcal{H})$ is not more than $(b+1)^{-\lceil 3n/b \rceil}$. By the Boole's inequality and since $\binom{n}{k} < (en/k)^k$, we have

$$\Pr(\text{Picker wins}) \leq \sum_{A \in E(\mathcal{H})} (b+1)^{-\lceil \frac{3n}{b} \rceil} = n \binom{n}{\lceil \frac{3n}{b} \rceil} (b+1)^{-\lceil \frac{3n}{b} \rceil} \leq n \left(\frac{e}{3}\right)^{\frac{3n}{b}} < 1,$$

if $b < n/(8 \ln n)$, and n is large enough. This means that Picker cannot have a winning strategy, and since the game has only two outcomes, Chooser must be the winner. $\square$

To prove Theorem 5.1, we prove Lemma 3.9 first.

5.3.3 Proof of Lemma 3.9.

We use weight functions, for more details see [27, 9]. Let λ be such that $\lambda^b = 2$. The *weight of an edge* A is $w_0(A) = \lambda^{-|A|}$ at the beginning. The weight of A at the i^{th} step is

$$w_i(A) = \begin{cases} \lambda^{-|A|+k} & \text{if Picker has no elements of } A \\ 0 & \text{otherwise,} \end{cases}$$

where k is the number of vertices in A occupied by Maker (Chooser). The *weight of a vertex* $x \in V(\mathcal{H})$ is $w_i(x) = \sum_{x \in A} w_i(A)$. The *total weight* at the i^{th} round is $w_i = \sum_{A \in \mathcal{H}} w_i(A)$.

Note that it is enough to show that Picker can guarantee $w_i < 1$ for all i. Indeed, if Chooser occupies an edge A at the i^{th} round for some i, then $w_i \geq w_i(A) = 1$. We will see how Picker keeps w_i small.

In each step Picker marks $b+1$ point and Chooser keeps b of those and one goes back to Picker. Thus, in each round i, the number of unoccupied vertices of the hypergraph v_i is decreased by $b+1$;

$$v_0 = v, v_1 = v - (b+1), \ldots, v_{\lfloor \frac{v}{b+1} \rfloor} = v_{last} < b+1.$$

Let $\widetilde{w}$ be the largest weight of a vertex in the i^{th} round. By the pigeon hole principle, there must be $b+1$ vertices, such that their weights are all in an interval I of length $D = \widetilde{w}(b+1)/v_i$. Picker picks those vertices. Let the endpoints of I be $\overline{w}$ and w^*, that is $I = [\overline{w}, w^*]$.

The biggest possible growth of the total weight function occurs if one vertex has weight $\overline{w}$, b vertices have weight w^* and Chooser keeps those. So, if Picker picks these vertices, than the total weight in the $(i+1)^{th}$ round can be bounded as follows:

$$w_{i+1} \leq w_i - \overline{w} + (\lambda^b - 1)w^* \leq w_i + (\lambda^b - 2)w^* + (w^* - \overline{w}) = w_i + (\lambda^b - 2)w^* + D.$$

Since $\lambda^b = 2$, we have that $w_{i+1} \leq w_i + D$. Now we plug in that $D = \widetilde{w}(b+1)/v_i$ and $\widetilde{w} \leq w_i$:

$$w_{i+1} \leq w_i + \frac{\widetilde{w}(b+1)}{v_i} \leq w_i + \frac{(b+1)w_i}{v_i} \leq \cdots \leq w_0 \prod_{k=0}^{\lfloor \frac{v}{b+1} \rfloor - 1} \left(1 + \frac{b+1}{v_k}\right).$$

To ease the notation let $B = b+1$, $\ell = \lfloor \frac{v}{b+1} \rfloor$, and we also use the inequality $1 + x < e^x$. We have that for all $i = 0, \ldots, \ell$

$$w_i \leq w_0 \prod_{k=0}^{\ell-1} \left(1 + \frac{B}{v - iB}\right) \leq w_0 \exp\left\{\sum_{k=0}^{\ell-1} \frac{B}{v - iB}\right\} \leq$$

$$w_0 \exp \ln \frac{v}{B} = w_0 \frac{v}{B} = w_0 \frac{v}{b+1} = \frac{v}{b+1} \sum_{A \in E(\mathcal{H})} 2^{-|A|/b} < 1,$$

by the assumption of the lemma. $\square$

5.3.4 Proof of Theorem 5.1

The second part of the theorem, i. e. Chooser wins if $b > 3\sqrt{n}$, comes from Lemma 5.4. Let Chooser play accordingly to that lemma, then Picker gets at most $(3n/b) - 1$ edges at any vertex $x \in K_n$, so the number of vertices that are linked to x is no more than $((3n/b) - 1)^2 < n - 1$.

To prove the first part of the theorem implies more work. We split the vertices of the graph into three approximately equal parts, X_1, X_2 and X_3. (Let X_i be $X_{i \mod 3}$ if $i > 3$.) The elements of X_i may be listed as $1, 2, \ldots, n/3$.[2] $E(X_i, X_j)$ denotes the edges between the sets X_i and X_j.

We will play two different games among and inside the parts. At the first game we link the points of X_i using $E(X_i, X_{i+1})$, for $i = 1, 2, 3$. At the second game we link the sets X_i with X_{i+1} playing on the edges of X_{i+1}.

Linking vertices within X_i.

The first game consists of $n/3$ auxiliary sub-games. At first, Picker links the vertices of X_i, for $i = 1, 2, 3$ playing on $E(X_i, X_{i+1})$.

The 1^{st} game: Picker asks for all the edges of the form $(1, x)$, where $1 \in X_i$ and $x \in X_{i+1}$ are in arbitrary order. About $\left\lfloor \frac{\lfloor n/3 \rfloor}{b+1} \right\rfloor$ of those edges go back to Picker. The set $A_1 = \{x : \text{Picker gets } (1, x), x \in X_{i+1}\}$.

The 2^{nd} game: Picker asks for all the edges of the form $(2, x)$, where $2 \in X_i$ and $x \in X_{i+1}$, paying attention to get at least one edge $(2, x)$ such that $x \in A_1$.
The set $A_2 = \{x : \text{Picker gets } (2, x), x \in X_{i+1}\}$.

In general:

The k^{th} game: Picker asks for all the edges of the form (k, x), where $k \in X_i$ and $x \in X_{i+1}$, paying attention to get at least one edge to every $A_1, \ldots, A_{k-1}$. Again, the set $A_k = \{x : \text{Picker gets } (k, x), x \in X_{i+1}\}$.

Clearly, if Picker wins all auxiliary games $1, \ldots, \lceil n/3 \rceil$, then he also links the vertices within X_i. Observe that Picker wins the k^{th} game iff Chooser cannot occupy completely any of the sets $C_j^k = \{(k, x) : k \in X_i, x \in A_j\}$, where $1 \leq j < k$.

Furthermore, if Picker can win the last game, then he wins the j^{th} game for $j < \lceil \frac{n}{3} \rceil$. So, we have to consider only the last game.

Picker applies Lemma 3.9. Here $v = \lfloor n/3 \rfloor$ and for all $j \in \{1, \ldots, \lceil n/3 \rceil\}$ $|A_j| = \lfloor \lfloor n/3 \rfloor / (b+1) \rfloor > n/(3(b+2))$. All we need to check is whether the inequality

$$\frac{v}{b+1} \sum_j 2^{-\frac{|A_j|}{b}} < \frac{n}{3(b+1)} \sum_{i=1}^{\lceil n/3 \rceil} 2^{-\frac{n}{3(b+1)^2}} < 1$$

[2]It can also be $\lfloor n/3 \rfloor$ and $\lceil n/3 \rceil$. In the proof we show that it works with $\lceil n/3 \rceil$, and the case $\lfloor n/3 \rfloor$ easily follows from that.

holds. Developing this formula, we get that the inequality holds if
$b \le \sqrt{\frac{n}{8\log_2 n}}$ and n is large enough.

Linking vertices of X_i to X_{i+1}.

Now we define a game where the players play within X_i to link the vertices of X_i to X_{i+1}, $i = 1, 2, 3$ using the edges Picker has already got in the first game.

For all $j \in X_{i+1}$ Picker wants to get an edge to every A_k, for $k = 1, \ldots, \lceil n/3 \rceil$. It obviously links j to all elements of X_i. As before, it is enough to show that Chooser cannot occupy completely any of the sets $F_{k,j} = \{(x, j) : x \in A_k \cap X_{i+1}, j \in X_{i+1}\}$.

The number of these sets is $\left(\lceil \frac{n}{3} \rceil\right)^2$, and there are $v = \binom{\frac{n}{3}}{2}$ edges within X_{i+1}. Plugging it into Lemma 3.9 we see Picker win if

$$\frac{\binom{\frac{n}{3}}{2}}{b+1}\left(\frac{n}{3}\right)^2 2^{-\frac{n}{3(b+1)^2}} < 1.$$

The inequality above clearly holds if $b \le \sqrt{n/(16\log_2 n)}$, which completes the proof of Theorem 5.1. $\square$

Conclusions

The main topics of the dissertation are the researches connected to Chooser-Picker (and Picker-Chooser) games and the examination of Beck's conjecture, Conjecture 2.9.

During the researches we have made the the following definitions and statements:

- We have redefined and specified the definition of Chooser-Picker and Picker-Chooser games Chapter 3.1;
- We have examined the complexity of these games and we found that it is NP-hard to decide the winner for both P-C and C-P games Theorem 3.2;
- We have formulated and proved the Pairing lemma, Lemma 2.17. We used this lemma for taking the trivial moves (without any disadvantages for Picker) in small board games, and with that we can increase the speed of finding who wins a game.
- We have proved the monotonicity lemma, Lemma 2.18. We applied this lemma for proving theorems for infinite boards by using finite auxiliary games and that lemma 4.2 and Theorem 4.5;
- We have improved Beck former Erdős-Selfridge type theorem for Chooser-Picker games, Theorem 3.6;
- We have found Picker's winning conditions in the Picker-Chooser version of the generalized Shannon switching game, Theorem 3.4;
- We have proved, that Picker wins the Chooser-Picker 4×4 TIC - TAC -TOE game 3.3.1;
- We have proved that Chooser wins the Picker-Chooser 4×4 torus game 3.18 it is in accordance with that Breaker wins the Maker-Breaker version;
- In the paper [24] we have also dealt with pairing strategies and we got the following results:
  1. Pairing strategies can work for the game k-in-a-row for sufficiently large n only if $k \geq 9$, Proposition 3.14;
  2. We gave a computer-free proof that Breaker has no pairing strategy to avoid the isomorphic copies of the polyomino Snaky, Theorem 3.15;
  3. We described Breaker's all pairing strategies that avoid P_5, Lemma 3.16.

- We have defined the Chooser-Picker k-in-a-row for infinite board and proved the following theorems:

  1. Picker wins the Chooser-Picker 8-in-a-row game Theorem 4.2;
  2. Picker also wins the Chooser-Picker 7-in-a-row game Theorem 4.5;
  3. For this we had proved that Picker wins a special auxiliary game played on a 4×8 board Lemma 4.6;
  4. We had examined the Maker-Breaker case for the same sub-game and we found that Breaker does not win there Subsection 4.3.1;

- We have defined the Chooser-Picker diameter game after the Maker-Breaker version.

  1. After Balogh et al [2] former results on Maker-Breaker diameter games [2] we also defined the winning conditions of Picker and Chooser playing on the complete graph with n vertices Theorem 5.1;
  2. For that we proved a lemma for biased (asymmetric) Picker-Chooser degree games, Lemma 5.4;
  3. We observed that Maker loses the diameter two game, but Picker wins the Picker-Chooser version of this, Observation 5.3.1.

Chapter 6

Appendix

6.1 Case study for the Picker-Chooser 4×4 torus game

Where the numbers are the weights of the vertices multiplied by 16.

	a	b	c	d
4	P	4	4	4
3	4	4	C	4
2	5	3	5	3
1	3	5	3	5

A1

	a	b	c	d
4	P	4	3	5
3	P	C	C	4
2	6	3	3	4
1	5	4	5	2

A2

	a	b	c	d
4	P	4	5	3
3	C	P	C	4
2	4	3	7	2
1	1	6	1	8

E1

	a	b	c	d
4	P	5	4	5
3	3	5	C	4
2	4	P	6	4
1	C	5	3	4

E2

	a	b	c	d
4	P	3	4	3
3	5	3	C	4
2	6	C	4	2
1	P	5	3	6

B1

	a	b	c	d
4	P	4	3	5
3	P	3	C	C
2	6	3	3	4
1	5	4	5	2

B2

	a	b	c	d
4	P	3	5	4
3	C	4	C	P
2	4	2	7	3
1	1	8	1	6

F1

	a	b	c	d
4	P	3	4	3
3	5	4	C	3
2	6	2	4	C
1	P	6	3	5

F2

	a	b	c	d
4	P	5	4	5
3	3	4	C	5
2	4	4	6	P
1	C	4	3	5

C1

	a	b	c	d
4	P	3	C	3
3	P	3	C	3
2	5	4	5	4
1	5	4	5	4

C2

	a	b	c	d
4	P	5	P	5
3	C	5	C	5
2	5	2	5	2
1	1	6	1	6

G1

	a	b	c	d
4	P	5	4	3
3	4	P	C	C
2	5	4	5	2
1	3	3	3	7

G2

	a	b	c	d
4	P	3	4	5
3	4	C	C	P
2	5	2	5	4
1	3	7	3	3

D1

	a	b	c	d
4	P	C	3	3
3	3	3	C	P
2	6	2	7	3
1	3	7	2	6

D2

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	4	4	3	3
1	3	3	4	4

H1

	a	b	c	d
4	P	3	3	C
3	3	P	C	3
2	6	3	7	2
1	3	6	2	7

H2

	a	b	c	d
4	P	5	5	P
3	5	C	C	5
2	4	3	3	4
1	3	4	4	3

A1

	a	b	c	d
4	P	4	3	5
3	P	C	C	4
2	6	3	3	4
1	5	4	5	2

A2

	a	b	c	d
4	P	4	5	3
3	C	P	C	4
2	4	3	7	2
1	1	6	1	8

A1a1

	a	b	c	d
4	P	P	4	6
3	P	C	C	C
2	5	3	1	5
1	5	1	7	1

A1a2

	a	b	c	d
4	P	C	2	4
3	P	C	C	P
2	7	3	5	3
1	5	7	3	3

A2a1

	a	b	c	d
4	P	P	4	5
3	C	P	C	3
2	C	4	6	1
1	1	7	1	6

A2a2

	a	b	c	d
4	P	C	6	1
3	C	P	C	5
2	P	2	8	3
1	1	5	1	10

A1b1

	a	b	c	d
4	P	6	P	5
3	P	C	C	5
2	6	C	2	3
1	5	5	3	2

A1b2

	a	b	c	d
4	P	2	C	5
3	P	C	C	3
2	6	P	4	5
1	5	3	7	2

A2b1

	a	b	c	d
4	P	P	6	4
3	C	P	C	C
2	3	5	5	1
1	1	5	1	7

A2b2

	a	b	c	d
4	P	C	4	2
3	C	P	C	P
2	5	1	9	3
1	1	7	1	9

A1c1

	a	b	c	d
4	P	P	4	7
3	P	C	C	1
2	5	3	1	6
1	4	C	6	1

A1c2

	a	b	c	d
4	P	C	2	3
3	P	C	C	7
2	7	3	5	2
1	6	P	4	3

A2c1

	a	b	c	d
4	P	4	3	C
3	C	P	C	3
2	5	P	8	2
1	1	8	1	7

A2c2

	a	b	c	d
4	P	4	7	P
3	C	P	C	5
2	3	C	6	2
1	1	4	1	9

A1d1

	a	b	c	d
4	P	6	2	6
3	P	C	C	2
2	6	4	2	P
1	4	C	6	2

A1d2

	a	b	c	d
4	P	2	4	4
3	P	C	C	6
2	6	2	4	C
1	6	P	4	2

B1

	a	b	c	d
4	P	4	3	5
3	P	3	C	C
2	6	3	3	4
1	5	4	5	2

B2

	a	b	c	d
4	P	3	5	4
3	C	4	C	P
2	4	2	7	3
1	1	8	1	6

B1a1

	a	b	c	d
4	P	5	P	6
3	P	5	C	C
2	6	3	2	C
1	5	2	3	5

B1a2

	a	b	c	d
4	P	5	C	2
3	P	3	C	C
2	6	5	4	P
1	5	2	7	3

B2a1

	a	b	c	d
4	P	P	7	4
3	C	5	C	P
2	3	2	6	C
1	1	9	1	4

B2a2

	a	b	c	d
4	P	C	3	4
3	C	3	C	P
2	5	2	8	P
1	1	7	1	8

B1b1

	a	b	c	d
4	P	7	P	6
3	P	3	C	C
2	6	3	C	2
1	5	2	5	3

B1b2

	a	b	c	d
4	P	3	C	2
3	P	5	C	C
2	6	5	P	4
1	5	2	5	5

B2b1

	a	b	c	d
4	P	4	6	P
3	C	C	C	P
2	3	1	5	5
1	1	7	1	5

B2b2

	a	b	c	d
4	P	2	4	C
3	C	P	C	P
2	5	3	9	1
1	1	9	1	7

B1c1

	a	b	c	d
4	P	6	4	P
3	P	C	C	C
2	5	5	1	3
1	5	1	7	1

B1c2

	a	b	c	d
4	P	4	2	C
3	P	P	C	C
2	7	3	5	3
1	5	3	3	7

B2c1

	a	b	c	d
4	P	5	4	P
3	C	3	C	P
2	C	1	6	4
1	1	6	1	7

B2c2

	a	b	c	d
4	P	1	6	C
3	C	5	C	P
2	P	3	8	2
1	1	10	1	5

B1d1

	a	b	c	d
4	P	7	4	P
3	P	1	C	C
2	5	6	1	3
1	4	1	6	C

B1d2

	a	b	c	d
4	P	3	2	C
3	P	7	C	C
2	7	2	5	3
1	6	3	4	P

B1e1

	a	b	c	d
4	P	6	2	6
3	P	2	C	C
2	6	P	2	4
1	4	2	6	C

B1e2

	a	b	c	d
4	P	4	4	2
3	P	6	C	C
2	6	C	4	2
1	6	2	4	P

B1f1

	a	b	c	d
4	P	P	5	6
3	P	5	C	C
2	2	5	3	5
1	C	2	6	3

B1f2

	a	b	c	d
4	P	C	1	2
3	P	3	C	C
2	10	3	3	1
1	P	2	4	5

C1

4	P	3	C	3
3	P	3	C	3
2	5	4	5	4
1	5	4	5	4
	a	b	c	d

C2

4	P	5	P	5
3	C	5	C	5
2	5	2	5	2
1	1	6	1	6
	a	b	c	d

C1a1

4	P	P	C	C
3	P	4	C	2
2	5	3	5	5
1	5	5	5	3
	a	b	c	d

C1a2

4	P	C	C	P
3	P	2	C	4
2	5	5	5	3
1	5	3	5	5
	a	b	c	d

C2a1

4	P	P	P	8
3	C	6	C	C
2	3	3	3	1
1	1	3	1	5
	a	b	c	d

C2a2

4	P	C	P	2
3	C	4	C	P
2	7	1	7	3
1	1	9	1	7
	a	b	c	d

C1b1

4	P	P	C	2
3	P	4	C	C
2	5	5	3	5
1	5	3	7	3
	a	b	c	d

C1b2

4	P	C	C	4
3	P	2	C	P
2	5	3	7	3
1	5	5	3	5
	a	b	c	d

C2b1

4	P	P	P	9
3	C	4	C	3
2	C	2	4	1
1	1	5	1	4
	a	b	c	d

C2b2

4	P	C	P	1
3	C	6	C	7
2	P	2	6	3
1	1	7	1	8
	a	b	c	d

C1c1

4	P	2	C	P
3	P	C	C	4
2	5	5	3	5
1	5	3	7	3
	a	b	c	d

C1c2

4	P	4	C	C
3	P	P	C	2
2	5	3	7	3
1	5	5	3	5
	a	b	c	d

C2c1

4	P	P	P	9
3	C	4	C	3
2	4	2	C	1
1	1	5	1	4
	a	b	c	d

C2c2

4	P	C	P	1
3	C	6	C	7
2	6	2	P	3
1	1	7	1	8
	a	b	c	d

C1d1

4	P	1	C	1
3	P	3	C	3
2	P	3	6	3
1	8	3	C	3
	a	b	c	d

C1d2

4	P	5	C	5
3	P	3	C	3
2	C	5	4	5
1	2	5	P	5
	a	b	c	d

C2d1

4	P	8	P	P
3	C	C	C	6
2	3	1	3	3
1	1	5	1	3
	a	b	c	d

C2d2

4	P	2	P	C
3	C	P	C	4
2	7	3	7	1
1	1	7	1	9
	a	b	c	d

C1e1

4	P	4	C	4
3	P	2	C	2
2	6	P	4	4
1	4	4	6	C
	a	b	c	d

C1e2

4	P	2	C	2
3	P	4	C	4
2	4	C	6	4
1	6	4	4	P
	a	b	c	d

C2e1

4	P	9	P	P
3	C	3	C	4
2	C	1	4	2
1	1	4	1	5
	a	b	c	d

C2e2

4	P	1	P	C
3	C	7	C	6
2	P	3	6	2
1	1	8	1	7
	a	b	c	d

C1f1

4	P	4	C	4
3	P	2	C	2
2	6	4	4	P
1	4	C	6	4
	a	b	c	d

C1f2

4	P	2	C	2
3	P	4	C	4
2	4	4	6	C
1	6	P	4	4
	a	b	c	d

C2f1

4	P	9	P	P
3	C	3	C	4
2	4	1	C	2
1	1	4	1	5
	a	b	c	d

C2f2

4	P	1	P	C
3	C	7	C	6
2	6	3	P	2
1	1	8	1	7
	a	b	c	d

C1g1

4	P	3	C	3
3	P	1	C	1
2	8	3	C	3
1	P	3	6	3
	a	b	c	d

C1g2

4	P	3	C	3
3	P	5	C	5
2	2	5	P	5
1	C	5	4	5
	a	b	c	d

C2g1

4	P	4	P	6
3	C	C	C	P
2	5	1	5	3
1	1	9	1	3
	a	b	c	d

C2g2

4	P	6	P	4
3	C	P	C	C
2	5	3	5	1
1	1	3	1	9
	a	b	c	d

D1

4	P	C	3	3
3	3	3	C	P
2	6	2	7	3
1	3	7	2	6
	a	b	c	d

D2

4	P	P	5	5
3	5	5	C	C
2	4	4	3	3
1	3	3	4	4
	a	b	c	d

D1a1

4	P	C	P	C
3	2	4	C	P
2	9	1	7	1
1	3	9	1	5
	a	b	c	d

D1a2

4	P	C	C	P
3	4	2	C	P
2	3	3	7	5
1	3	5	3	7
	a	b	c	d

D2a1

4	P	P	P	8
3	C	6	C	C
2	3	3	3	1
1	1	3	1	5
	a	b	c	d

D2a2

4	P	P	C	2
3	P	4	C	C
2	5	5	3	5
1	5	3	7	3
	a	b	c	d

D1b1

	a	b	c	d
4	P	C	P	2
3	C	4	C	P
2	7	1	7	3
1	1	9	1	7

D1b2

	a	b	c	d
4	P	C	C	4
3	P	2	C	P
2	5	3	7	3
1	5	5	3	5

D2b1

	a	b	c	d
4	P	P	8	P
3	6	C	C	C
2	3	3	1	3
1	3	1	5	1

D2b2

	a	b	c	d
4	P	P	2	C
3	4	P	C	C
2	5	5	5	3
1	3	5	3	7

D1c1

	a	b	c	d
4	P	C	P	1
3	3	4	C	P
2	8	1	6	C
1	3	9	2	5

D1c2

	a	b	c	d
4	P	C	C	5
3	3	2	C	P
2	4	3	8	P
1	3	5	2	7

D2c1

	a	b	c	d
4	P	P	6	4
3	4	6	C	C
2	P	4	4	2
1	4	2	C	4

D2c2

	a	b	c	d
4	P	P	4	6
3	6	4	C	C
2	C	4	2	4
1	2	4	P	4

D1d1

	a	b	c	d
4	P	C	P	3
3	1	4	C	P
2	7	2	7	3
1	C	8	1	6

D1d2

	a	b	c	d
4	P	C	C	3
3	5	2	C	P
2	5	2	7	3
1	P	6	3	6

D2d1

	a	b	c	d
4	P	P	4	6
3	6	4	C	C
2	4	P	2	4
1	2	4	4	C

D2d2

	a	b	c	d
4	P	P	6	4
3	4	6	C	C
2	4	C	4	2
1	4	2	4	P

D1e1

	a	b	c	d
4	P	C	2	P
3	4	C	C	P
2	5	3	5	5
1	3	7	3	5

D1e2

	a	b	c	d
4	P	C	4	C
3	2	P	C	P
2	7	1	9	1
1	3	7	1	7

D2e1

	a	b	c	d
4	P	P	5	5
3	3	7	C	C
2	3	5	P	4
1	C	2	3	5

D2e2

	a	b	c	d
4	P	P	5	5
3	7	3	C	C
2	5	3	C	2
1	P	4	5	3

D1f1

	a	b	c	d
4	P	C	3	P
3	2	3	C	P
2	4	3	7	5
1	C	6	2	7

D1f2

	a	b	c	d
4	P	C	3	C
3	4	3	C	P
2	8	1	7	1
1	P	8	2	5

D2f1

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	5	3	P	4
1	2	C	3	5

D2f2

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	3	5	C	2
1	4	P	5	3

D1g1

	a	b	c	d
4	P	C	2	4
3	P	C	C	P
2	7	3	5	3
1	5	7	3	3

D1g2

	a	b	c	d
4	P	C	4	2
3	C	P	C	P
2	5	1	9	3
1	1	7	1	9

D2g1

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	3	5	4	P
1	C	2	5	3

D2g2

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	5	3	2	C
1	P	4	3	5

D1h1

	a	b	c	d
4	P	C	3	2
3	P	3	C	P
2	7	2	6	C
1	5	7	3	4

D1h2

	a	b	c	d
4	P	C	3	4
3	C	3	C	P
2	5	2	8	P
1	1	7	1	8

D2h1

	a	b	c	d
4	P	P	5	5
3	7	3	C	C
2	5	3	4	P
1	2	C	5	3

D2h2

	a	b	c	d
4	P	P	5	5
3	3	7	C	C
2	3	5	2	C
1	4	P	3	5

D1i1

	a	b	c	d
4	P	C	4	1
3	3	P	C	P
2	6	1	8	C
1	3	7	2	7

D1i2

	a	b	c	d
4	P	C	2	5
3	3	C	C	P
2	6	3	6	P
1	3	7	2	5

D1j1

	a	b	c	d
4	P	C	4	3
3	1	P	C	P
2	5	2	9	3
1	C	6	1	8

D1j2

	a	b	c	d
4	P	C	2	3
3	5	C	C	P
2	7	2	5	3
1	P	8	3	4

D1k1

	a	b	c	d
4	P	C	3	5
3	1	3	C	P
2	5	3	8	P
1	C	6	1	7

D1k2

	a	b	c	d
4	P	C	3	1
3	5	3	C	P
2	7	1	6	C
1	P	8	3	5

E1

	a	b	c	d
4	P	5	4	5
3	3	5	C	4
2	4	P	6	4
1	C	5	3	4

E2

	a	b	c	d
4	P	3	4	3
3	5	3	C	4
2	6	C	4	2
1	P	5	3	6

E1a1

	a	b	c	d
4	P	5	5	P
3	5	3	C	2
2	3	P	4	5
1	C	C	5	5

E1a2

	a	b	c	d
4	P	5	3	C
3	1	7	C	6
2	5	P	8	3
1	C	P	1	3

E2a1

	a	b	c	d
4	P	P	5	5
3	6	C	C	4
2	5	C	2	3
1	P	5	4	3

E2a2

	a	b	c	d
4	P	C	3	1
3	4	P	C	4
2	7	C	6	1
1	P	5	2	9

E1b1

	a	b	c	d
4	P	5	5	P
3	5	C	C	5
2	3	P	4	5
1	C	3	5	2

E1b2

	a	b	c	d
4	P	5	3	C
3	1	P	C	3
2	5	P	8	3
1	C	7	1	6

E2b1

	a	b	c	d
4	P	5	P	5
3	5	2	C	3
2	8	C	C	2
1	P	4	3	5

E2b2

	a	b	c	d
4	P	1	C	1
3	5	4	C	5
2	4	C	P	2
1	P	6	3	7

E2c1

	a	b	c	d
4	P	5	5	P
3	5	C	C	5
2	5	C	2	3
1	P	5	3	4

E2c2

	a	b	c	d
4	P	1	3	C
3	5	P	C	3
2	7	C	6	1
1	P	5	3	8

E2d1

	a	b	c	d
4	P	4	6	P
3	4	3	C	5
2	6	C	4	2
1	P	3	C	5

E2d2

	a	b	c	d
4	P	2	2	C
3	6	3	C	3
2	6	C	4	2
1	P	7	P	7

E2e1

	a	b	c	d
4	P	4	3	3
3	P	3	C	1
2	9	C	2	3
1	P	C	2	4

E2e2

	a	b	c	d
4	P	2	5	3
3	C	3	C	7
2	3	C	6	1
1	P	P	4	8

E2f1

	a	b	c	d
4	P	2	5	3
3	4	P	C	4
2	7	C	6	1
1	P	3	C	7

E2f2

	a	b	c	d
4	P	4	3	3
3	6	C	C	4
2	5	C	2	3
1	P	7	P	5

F1

	a	b	c	d
4	P	3	4	3
3	5	4	C	3
2	6	2	4	C
1	P	6	3	5

F2

	a	b	c	d
4	P	5	4	5
3	3	4	C	5
2	4	4	6	P
1	C	4	3	5

F1a1

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	5	3	2	C
1	P	4	3	5

F1a2

	a	b	c	d
4	P	C	3	1
3	5	3	C	P
2	7	1	6	C
1	P	8	3	5

F2a1

	a	b	c	d
4	P	P	5	5
3	5	5	C	C
2	3	5	4	P
1	C	2	5	3

F1a2

	a	b	c	d
4	P	C	3	5
3	1	3	C	P
2	5	3	8	P
1	C	6	1	7

F1b1

	a	b	c	d
4	P	P	6	4
3	4	5	C	3
2	6	2	4	C
1	P	5	C	3

F1b2

	a	b	c	d
4	P	C	2	2
3	6	3	C	3
2	6	2	4	C
1	P	7	P	7

F2b1

	a	b	c	d
4	P	P	5	5
3	5	2	C	3
2	3	5	4	P
1	C	5	5	C

F2b2

	a	b	c	d
4	P	C	3	5
3	1	6	C	7
2	5	3	8	P
1	C	3	1	P

F1c1

	a	b	c	d
4	P	5	P	5
3	5	3	C	2
2	8	2	C	C
1	P	5	3	4

F1c2

	a	b	c	d
4	P	1	C	1
3	5	5	C	4
2	4	2	P	C
1	P	7	3	6

F2c1

	a	b	c	d
4	P	6	P	6
3	2	4	C	6
2	4	C	4	P
1	C	4	2	6

F2c2

	a	b	c	d
4	P	4	C	4
3	4	4	C	4
2	4	P	8	P
1	C	4	4	4

F1d1

	a	b	c	d
4	P	3	3	4
3	P	1	C	3
2	9	3	2	C
1	P	4	2	C

F1d2

	a	b	c	d
4	P	3	5	2
3	C	7	C	3
2	3	1	6	C
1	P	8	4	P

F1e1

	a	b	c	d
4	P	3	5	2
3	4	4	C	P
2	7	1	6	C
1	P	7	C	3

F1e2

	a	b	c	d
4	P	3	3	4
3	6	4	C	C
2	5	3	2	C
1	P	5	P	7

G1

	a	b	c	d
4	P	5	4	3
3	4	P	C	C
2	5	4	5	2
1	3	3	3	7

G2

	a	b	c	d
4	P	3	4	5
3	4	C	C	P
2	5	2	5	4
1	3	7	3	3

G1a1

	a	b	c	d
4	P	P	4	5
3	3	P	C	C
2	C	5	4	2
1	1	5	4	5

G1a2

	a	b	c	d
4	P	C	4	1
3	5	P	C	C
2	P	3	6	2
1	5	1	2	9

G2a1

	a	b	c	d
4	P	P	6	7
3	3	C	C	P
2	3	2	5	5
1	C	6	3	2

G2a2

	a	b	c	d
4	P	C	2	3
3	5	C	C	P
2	7	2	5	3
1	P	8	3	4

G1b1

	a	b	c	d
4	P	P	6	5
3	5	P	C	C
2	4	5	C	2
1	3	5	4	3

G1b2

	a	b	c	d
4	P	C	2	1
3	3	P	C	C
2	6	3	P	2
1	3	1	2	11

G2b1

	a	b	c	d
4	P	P	6	5
3	5	C	C	P
2	5	2	5	3
1	2	6	3	C

G2b2

	a	b	c	d
4	P	C	2	5
3	3	C	C	P
2	5	2	5	5
1	4	8	3	P

G1c1

	a	b	c	d
4	P	5	P	4
3	3	P	C	C
2	6	C	4	1
1	3	1	2	9

G1c2

	a	b	c	d
4	P	5	C	2
3	5	P	C	C
2	4	P	6	3
1	3	5	4	5

G2c1

	a	b	c	d
4	P	4	P	6
3	C	C	C	P
2	5	1	5	3
1	1	9	1	3

G2c2

	a	b	c	d
4	P	2	C	4
3	P	C	C	P
2	5	3	5	5
1	5	5	5	3

G1d1

	a	b	c	d
4	P	7	6	P
3	3	P	C	C
2	3	5	5	2
1	C	2	3	6

G1d2

	a	b	c	d
4	P	3	2	C
3	5	P	C	C
2	7	3	5	2
1	P	4	3	8

G2e1

	a	b	c	d
4	P	4	P	5
3	3	C	C	P
2	6	1	4	C
1	3	9	2	1

G2e2

	a	b	c	d
4	P	2	C	5
3	5	C	C	P
2	4	3	6	P
1	3	5	4	5

G1e1

	a	b	c	d
4	P	5	6	P
3	5	P	C	C
2	5	3	5	2
1	2	C	3	6

G1e2

	a	b	c	d
4	P	5	2	C
3	3	P	C	C
2	5	5	5	2
1	4	P	3	8

G2f1

	a	b	c	d
4	P	5	4	P
3	3	C	C	P
2	C	2	4	5
1	1	5	4	5

G2f2

	a	b	c	d
4	P	1	4	C
3	5	C	C	P
2	P	2	6	3
1	5	9	2	1

G1f1

	a	b	c	d
4	P	6	P	4
3	C	P	C	C
2	5	3	5	1
1	1	3	1	9

G1f2

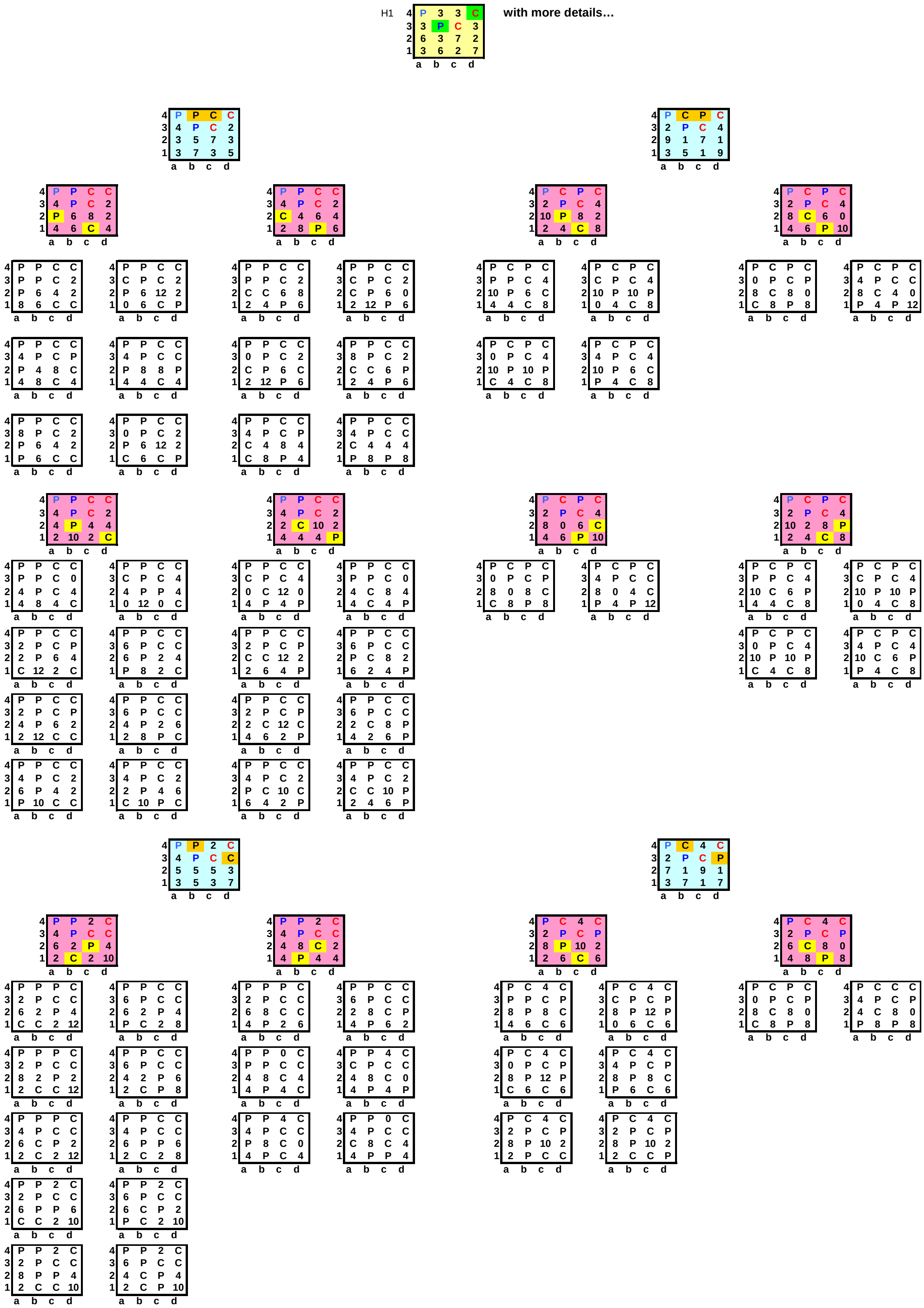
	a	b	c	d
4	P	4	C	2
3	P	P	C	C
2	5	5	5	3
1	5	3	5	5

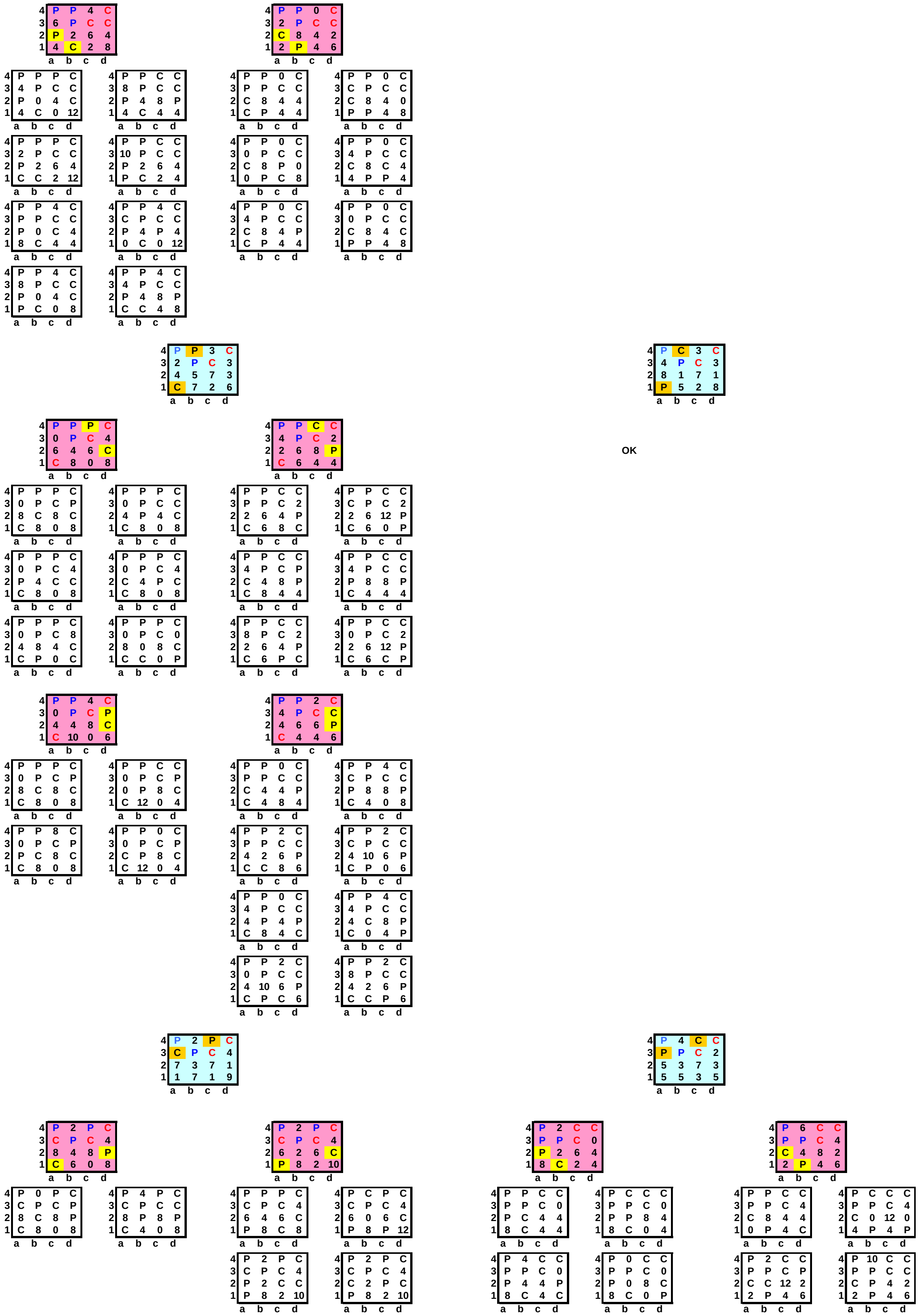
G2g1

	a	b	c	d
4	P	5	6	P
3	5	C	C	P
2	4	2	C	5
1	3	3	4	5

G2g2

	a	b	c	d
4	P	1	2	C
3	3	C	C	P
2	6	2	P	3
1	3	11	2	1


H1
with more details…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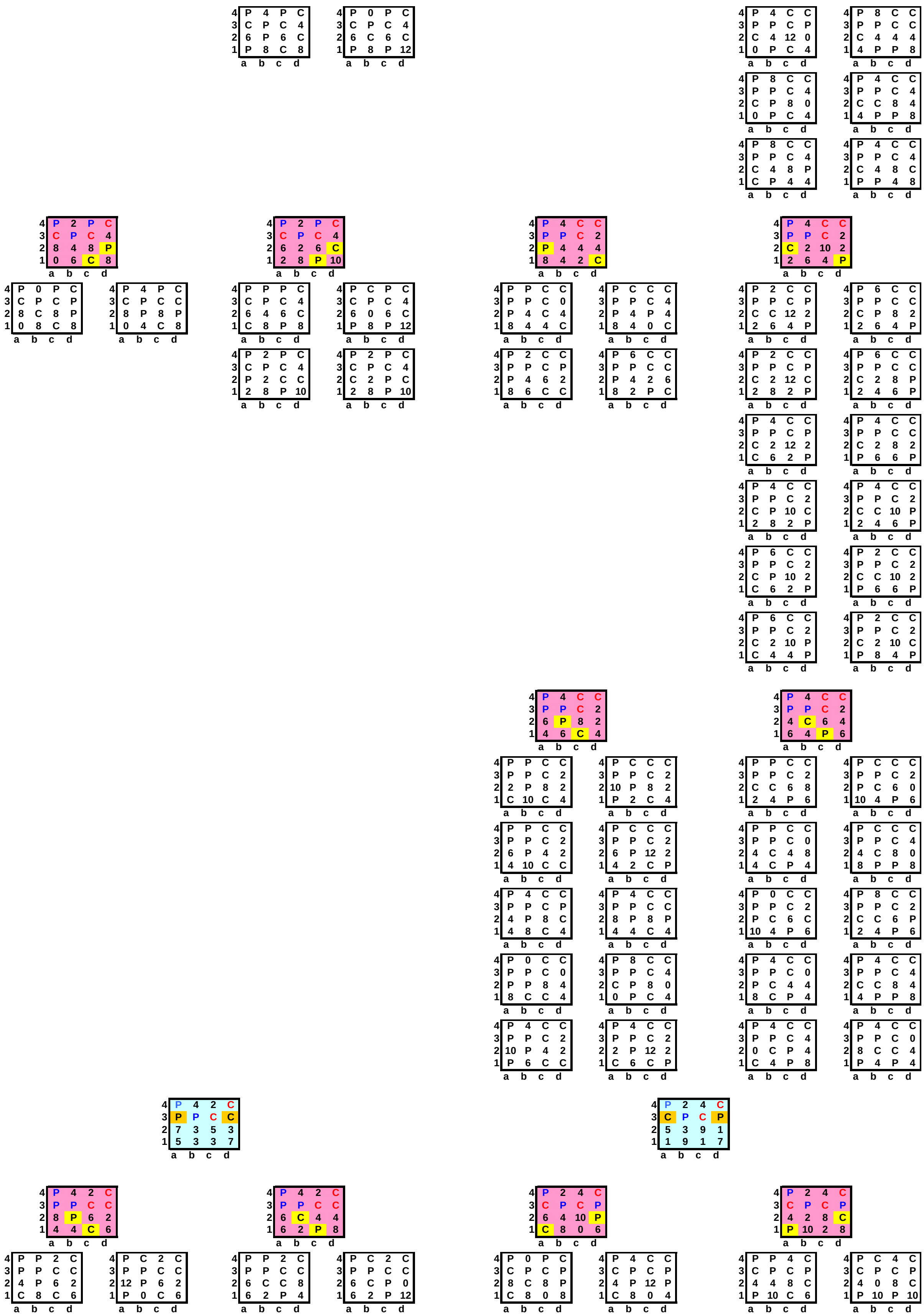
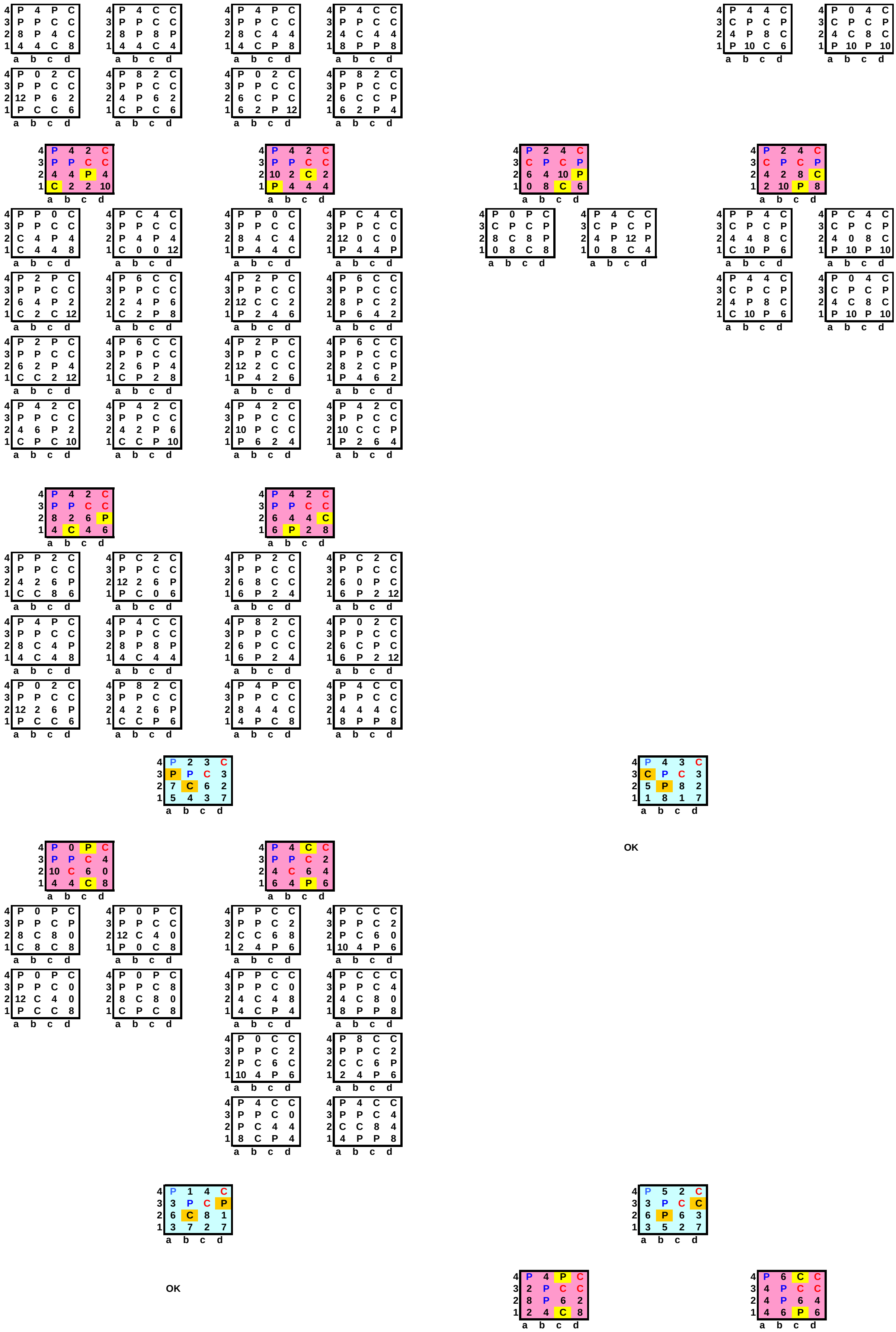
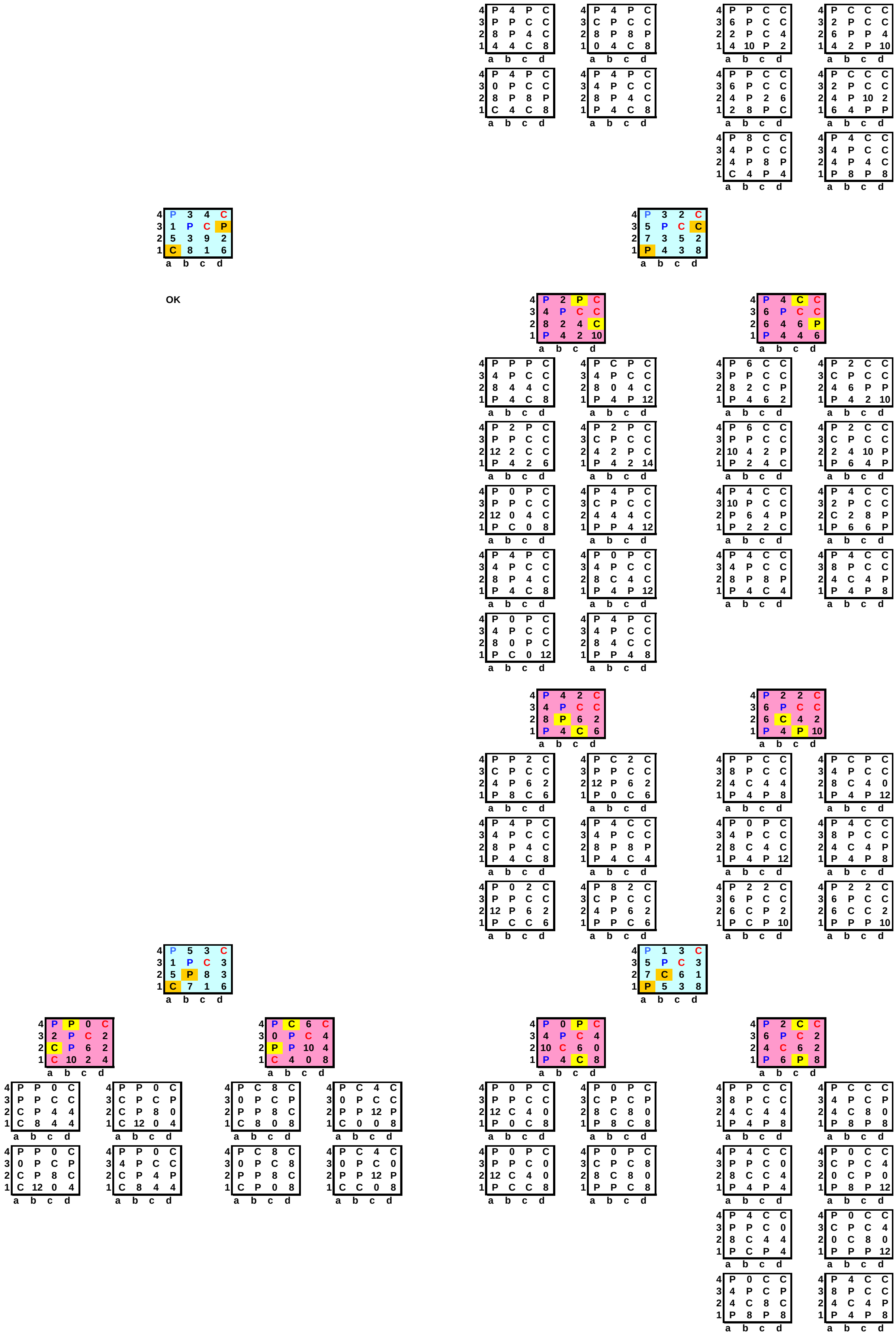



OK
OK


OK

	a	b	c	d
4	P	2	2	C
3	P	P	C	0
2	10	C	4	2
1	P	C	2	6

	a	b	c	d
4	P	P	C	C
3	P	P	C	0
2	8	C	4	4
1	P	C	4	4

	a	b	c	d
4	P	C	P	C
3	P	P	C	0
2	12	C	4	0
1	P	C	0	8

	a	b	c	d
4	P	0	P	C
3	P	P	C	0
2	12	C	4	C
1	P	C	0	8

	a	b	c	d
4	P	4	C	C
3	P	P	C	0
2	8	C	4	P
1	P	C	4	4

	a	b	c	d
4	P	0	4	C
3	C	P	C	6
2	4	C	8	0
1	P	P	4	10

	a	b	c	d
4	P	0	P	C
3	C	P	C	8
2	8	C	8	0
1	P	P	C	8

	a	b	c	d
4	P	0	C	C
3	C	P	C	4
2	0	C	8	0
1	P	P	P	12

	a	b	c	d
4	P	0	8	C
3	C	P	C	8
2	P	C	8	0
1	P	P	C	8

	a	b	c	d
4	P	0	0	C
3	C	P	C	4
2	C	C	8	0
1	P	P	P	12

H1

	a	b	c	d
4	P	3	3	C
3	3	P	C	3
2	6	3	7	2
1	3	6	2	7

H2

	a	b	c	d
4	P	5	5	P
3	5	C	C	5
2	4	3	3	4
1	3	4	4	3

see above…

H2a1

	a	b	c	d
4	P	P	8	P
3	6	C	C	C
2	3	3	1	3
1	3	1	5	1

H2a2

	a	b	c	d
4	P	C	2	P
3	4	C	C	P
2	5	3	5	5
1	3	7	3	5

H2b1

	a	b	c	d
4	P	8	P	P
3	C	C	C	6
2	3	1	3	3
1	1	5	1	3

H2b2

	a	b	c	d
4	P	2	C	P
3	P	C	C	4
2	5	5	3	5
1	5	3	7	3

H2c1

	a	b	c	d
4	P	6	4	P
3	P	C	C	C
2	5	5	1	3
1	5	1	7	1

H2c2

	a	b	c	d
4	P	4	6	P
3	C	C	C	P
2	3	1	5	5
1	1	7	1	5

H2d1

	a	b	c	d
4	P	4	6	P
3	4	C	C	6
2	P	2	4	4
1	4	4	C	2

H2d2

	a	b	c	d
4	P	6	4	P
3	6	C	C	4
2	C	4	2	4
1	2	4	P	4

H2e1

	a	b	c	d
4	P	5	5	P
3	5	C	C	5
2	3	P	4	5
1	C	3	5	2

H2e2

	a	b	c	d
4	P	5	5	P
3	5	C	C	5
2	5	C	2	3
1	P	5	3	4

H2f1

	a	b	c	d
4	P	5	5	P
3	7	C	C	3
2	5	P	4	3
1	2	3	5	C

H2f2

	a	b	c	d
4	P	5	5	P
3	3	C	C	7
2	3	C	2	5
1	4	5	3	P

H2g1

	a	b	c	d
4	P	5	5	P
3	3	C	C	7
2	3	4	P	5
1	C	5	3	2

H2g2

	a	b	c	d
4	P	5	5	P
3	7	C	C	3
2	5	2	C	3
1	P	3	5	4

H2h1

4	P	5	5	P
3	5	C	C	5
2	5	4	P	3
1	2	5	3	C
	a	b	c	d

H2h2

4	P	5	5	P
3	5	C	C	5
2	3	2	C	5
1	4	3	5	P
	a	b	c	d

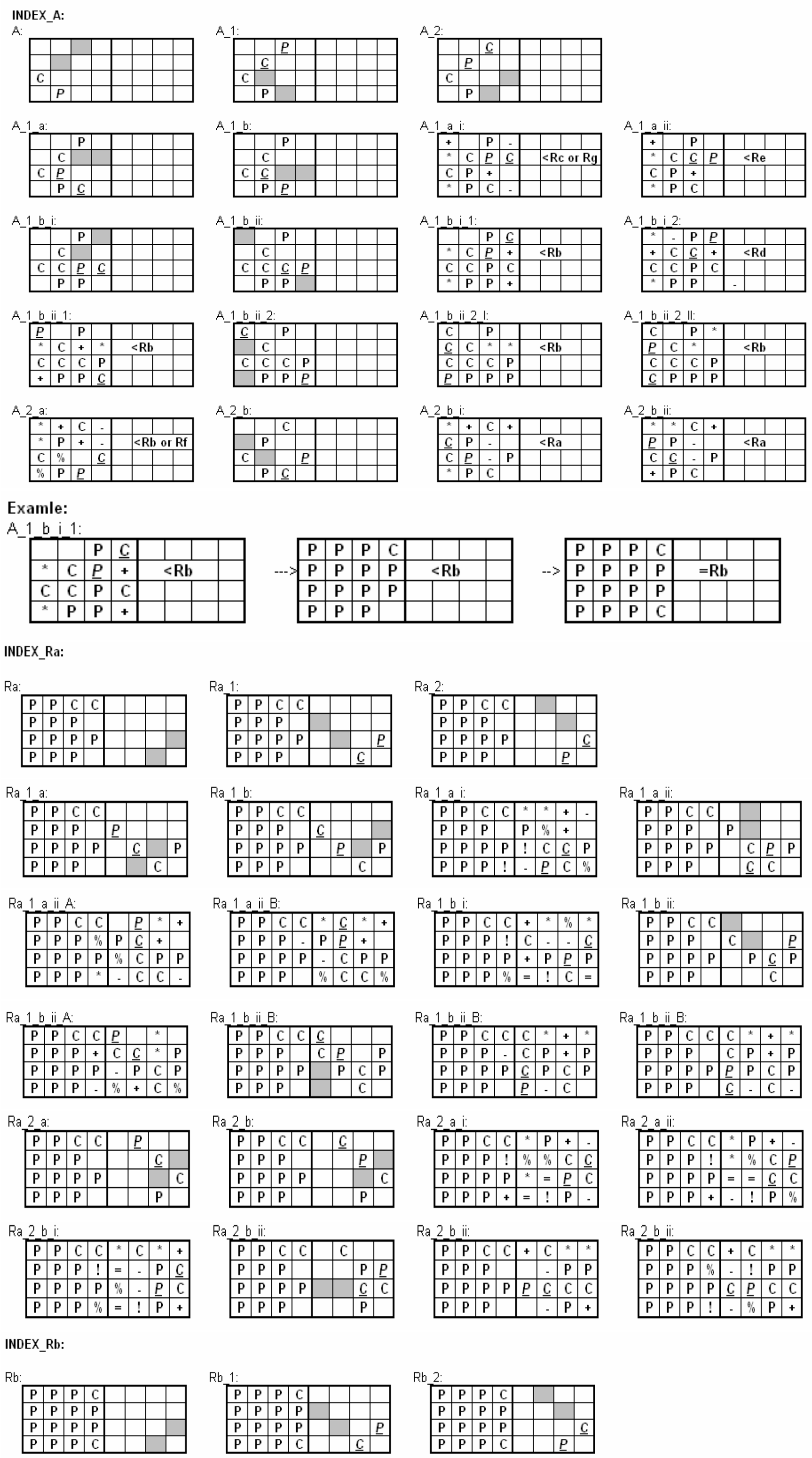
H2i1

4	P	6	4	P
3	6	C	C	4
2	4	4	2	P
1	2	C	4	4
	a	b	c	d

H2i2

4	P	4	6	P
3	4	C	C	6
2	4	2	4	C
1	4	P	4	2
	a	b	c	d

6.2 Case studies for the Chooser-Picker 7-in-a-row-game



Rb_1_a:

P	P	P	C				
P	P	P	P	*C*			
P	P	P	P		*P*		P
P	P	P	C			C	

Rb_1_b:

P	P	P	C				
P	P	P	P	*P*			
P	P	P	P		*C*		P
P	P	P	C			C	

Rb_1_a_i:

P	P	P	C	*	+	%	+
P	P	P	P	C	%	-	-
P	P	P	P	*	P	*P*	P
P	P	P	C	!	*C*	C	!

Rb_1_a_ii:

P	P	P	C				
P	P	P	P	C			
P	P	P	P		P	*C*	P
P	P	P	C		*P*	C	

Rb_1_a_ii_A:

P	P	P	C			*C*	
P	P	P	P	C		*P*	
P	P	P	P		P	C	P
P	P	P	C		P	C	

Rb_1_a_ii_B:

P	P	P	C	*		*P*	
P	P	P	P	C	+	*C*	+
P	P	P	P	-	P	C	P
P	P	P	C	-	P	C	*

Rb_1_a_ii_A_I:

P	P	P	C	*	+	*C*	+
P	P	P	P	C	*P*	*P*	
P	P	P	P	*C*	P	C	P
P	P	P	C	*	P	C	

Rb_1_a_ii_A_II:

P	P	P	C	*	+	*C*	+
P	P	P	P	C	*C*	*P*	
P	P	P	P	*P*	P	C	P
P	P	P	C		P	C	*

Rb_1_b_i:

P	P	P	C				
P	P	P	P	P			
P	P	P	P		C	*C*	P
P	P	P	C		*P*	C	

Rb_1_b_ii:

P	P	P	C				
P	P	P	P	P			
P	P	P	P		C	*P*	P
P	P	P	C		*C*	C	

Rb_1_b_i_A:

P	P	P	C	*		*P*	+
P	P	P	P	P	*	*C*	
P	P	P	P		C	C	P
P	P	P	C	+	P	C	

Rb_1_b_i_B:

P	P	P	C	*	%	*C*	%
P	P	P	P	P	+	*P*	
P	P	P	P	+	C	C	P
P	P	P	C		P	C	*

Rb_1_b_ii_A:

P	P	P	C		*P*	*	+
P	P	P	P	P	*C*	+	
P	P	P	P	*	C	P	P
P	P	P	C	-	C	C	-

Rb_1_b_ii_B:

P	P	P	C	*	*C*	*	+
P	P	P	P	P	*P*	+	
P	P	P	P		C	P	P
P	P	P	C	-	C	C	-

Rb_2_a:

P	P	P	C		*P*		
P	P	P	P			*C*	
P	P	P	P				C
P	P	P	C			P	

Rb_2_b:

P	P	P	C		*C*		
P	P	P	P			*P*	
P	P	P	P				C
P	P	P	C			P	

Rb_2_a_i:

P	P	P	C		P		
P	P	P	P			C	*P*
P	P	P	P			*C*	C
P	P	P	C			P	

Rb_2_a_ii:

P	P	P	C		P		
P	P	P	P			C	*C*
P	P	P	P			*P*	C
P	P	P	C			P	

Rb_2_a_i_A:

P	P	P	C		P		*
P	P	P	P		+	C	*P*
P	P	P	P	*P*	*C*	*C*	C
P	P	P	C	*		P	+

Rb_2_a_i_B:

P	P	P	C	*	P	-	
P	P	P	P	+	-	C	*P*
P	P	P	P	*C*	*P*	*C*	C
P	P	P	C	+		P	*

Rb_2_a_ii:

P	P	P	C		P	*	-
P	P	P	P	*P*	*C*	C	C
P	P	P	P	*	+	P	C
P	P	P	C	+		P	-

Rb_2_a_ii:

P	P	P	C	*	P		-
P	P	P	P	*C*	*P*	C	C
P	P	P	P	*	+	P	C
P	P	P	C	+		P	-

Rb_2_b_i:

P	P	P	C		C		
P	P	P	P			P	
P	P	P	P			*C*	C
P	P	P	C		*P*	P	

Rb_2_b_ii:

P	P	P	C	+	C	*	+
P	P	P	P	-	%	P	!
P	P	P	P	*	%	*P*	C
P	P	P	C	-	*C*	P	!

Rb_2_b_i:

P	P	P	C	*	C	*C*	*
P	P	P	P	+	*P*	P	%
P	P	P	P	-	-	C	C
P	P	P	C	+	P	P	%

Rb_2_b_i:

P	P	P	C	*	C	*P*	%
P	P	P	P	+	*C*	P	%
P	P	P	P	-	-	C	C
P	P	P	C	+	P	P	*

INDEX_Rc:

Rc:

P	P	P	P				
P	P	P	P				
P	P	P					
P	P	C	C				

Rc_1:

P	P	P	P				
P	P	P	P				
P	P	P					*C*
P	P	C	C			*P*	

Rc_2:

P	P	P	P				
P	P	P	P				
P	P	P					*P*
P	P	C	C			*C*	

Rc_1_a:

P	P	P	P		*C*		
P	P	P	P			*P*	
P	P	P					C
P	P	C	C			P	

Rc_1_b:

P	P	P	P		*P*		
P	P	P	P			*C*	
P	P	P					C
P	P	C	C			P	

Rc_1_a_i:

P	P	P	P		C	*P*	
P	P	P	P		*C*	P	
P	P	P					C
P	P	C	C			P	

Rc_1_a_ii:

P	P	P	P		C	*C*	
P	P	P	P		*P*	P	
P	P	P					C
P	P	C	C			P	

Rc_1_a_i_A:

P	P	P	P		C	P	
P	P	P	P		C	P	
P	P	P			*C*		C
P	P	C	C		*P*	P	

Rc_1_a_i_B:

P	P	P	P		C	P	
P	P	P	P		C	P	
P	P	P			*P*		C
P	P	C	C		*C*	P	

Rc_1_a_i_A_I:

P	P	P	P	+	C	P	-
P	P	P	P	*	C	P	-
P	P	P	*	*P*	C	*C*	C
P	P	C	C		P	P	+

Rc_1_a_i_A_II:

P	P	P	P	+	C	P	-
P	P	P	P	*	C	P	-
P	P	P	*	*C*	C	*P*	C
P	P	C	C	+	P	P	

Rc_1_a_i_B_I:

P	P	P	P		C	P	-
P	P	P	P	*	C	P	*C*
P	P	P	*	+	P	*P*	C
P	P	C	C	+	C	P	-

Rc_1_a_i_B_II:

P	P	P	P	-	C	P	
P	P	P	P	*	C	P	*P*
P	P	P	*	+	P	*C*	C
P	P	C	C	+	C	P	-

Rc_1_a_ii_A:

P	P	P	P	*P*	C	C	*C*
P	P	P	P	*	P	P	%
P	P	P	*	-	-	+	C
P	P	C	C		+	P	%

Rc_1_a_ii_B:

P	P	P	P	*C*	C	C	*P*
P	P	P	P	*	P	P	%
P	P	P	*	+	-	-	C
P	P	C	C	+	%	P	

Rc_1_b_i:

P	P	P	P		P		
P	P	P	P			C	*C*
P	P	P				*P*	C
P	P	C	C			P	

Rc_1_b_ii:

P	P	P	P		P		
P	P	P	P			C	*P*
P	P	P				*C*	C
P	P	C	C			P	

Rc_1_b_i_A:

P	P	P	P		P	*	-
P	P	P	P	*P*	*C*	C	C
P	P	P		*	+	P	C
P	P	C	C	+		P	-

Rc_1_b_i_B:

P	P	P	P	*	P		-
P	P	P	P	*C*	*P*	C	C
P	P	P		*	+	P	C
P	P	C	C	+		P	-

Rc_1_b_ii_A:

P	P	P	P		P		+
P	P	P	P		*	C	P
P	P	P		*P*	*C*	C	C
P	P	C	C	+		P	*

Rc_1_b_ii_B:

P	P	P	P	+	P	*	
P	P	P	P	-	*	C	P
P	P	P		*C*	*P*	C	C
P	P	C	C	-		P	+

Rc_2_a:

P	P	P	P				
P	P	P	P				
P	P	P				*P*	P
P	P	C	C		*C*	C	

Rc_2_b:

P	P	P	P				
P	P	P	P				
P	P	P				*C*	P
P	P	C	C		*P*	C	

Rc_2_a_i:

P	P	P	P				
P	P	P	P		*C*		
P	P	P			*P*	P	P
P	P	C	C		C	C	

Rc_2_a_ii:

P	P	P	P	-	+	+	%
P	P	P	P	*	*P*	%	
P	P	P	*	-	*C*	P	P
P	P	C	C	!	C	C	!

Rc_2_a_i_A:

P	P	P	P	*	+	*P*	
P	P	P	P	*	C	-	-
P	P	P	+	*C*	P	P	P
P	P	C	C	%	C	C	%

Rc_2_a_i_B:

P	P	P	P	+	+	*C*	
P	P	P	P	*	C	-	-
P	P	P	*	*P*	P	P	P
P	P	C	C	%	C	C	%

Rc_2_b_i:

P	P	P	P			C̲	
P	P	P	P			P̲	
P	P	P				C	P
P	P	C	C		P	C	

Rc_2_b_ii:

P	P	P	P		*	P̲	+
P	P	P	P	-	%	C̲	-
P	P	P	*	!	+	C	P
P	P	C	C	!	P	C	%

Rc_2_b_i_A:

P	P	P	P	-	+	C	+
P	P	P	P	*	C̲	P	
P	P	P	*	P̲		C	P
P	P	C	C		P	C	-

Rc_2_b_i_B:

P	P	P	P	-	+	C	+
P	P	P	P	*	P̲	P	
P	P	P	*	C̲		C	P
P	P	C	C	-	P	C	

INDEX_Rd:

Rd:

P	P	P	P				
P	P	P	P				
P	P	P	P				
P	P	P		C			

Rd_1:

P	P	P	P				
P	P	P	P				
P	P	P	P				C̲
P	P	P		C		P̲	

Rd_2:

P	P	P	P				
P	P	P	P				
P	P	P	P				P̲
P	P	P		C		C̲	

Rd_1_a:

P	P	P	P		P̲		
P	P	P	P			C̲	
P	P	P	P				C
P	P	P		C		P	

Rd_1_b:

P	P	P	P		C̲		
P	P	P	P			P̲	
P	P	P	P				C
P	P	P		C		P	

Rd_1_a_i:

P	P	P	P		P		P̲
P	P	P	P			C	
P	P	P	P		C̲		C
P	P	P		C		P	

Rd_1_a_ii:

P	P	P	P		P		C̲
P	P	P	P			C	
P	P	P	P		P̲		C
P	P	P		C		P	

Rd_1_a_i_A:

P	P	P	P	*	P		P
P	P	P	P	+	+	C	-
P	P	P	P	P̲	C	C̲	C
P	P	P		C	-	P	*

Rd_1_a_i_B:

P	P	P	P	*	P	+	P
P	P	P	P	*	-	C	-
P	P	P	P	C̲	C	P̲	C
P	P	P	+	C		P	

Rd_1_a_ii_A:

P	P	P	P	*	P		C
P	P	P	P	*	+	C	P̲
P	P	P	P	-	P	+	C
P	P	P	-	C		P	C̲

Rd_1_a_ii_B:

P	P	P	P	*	P	-	C
P	P	P	P	+	+	C	C̲
P	P	P	P	*	P	%	C
P	P	P	-	C	%	P	P̲

Rd_1_b_i:

P	P	P	P		C		
P	P	P	P			P	
P	P	P	P	P̲	C̲		C
P	P	P		C		P	

Rd_1_b_ii:

P	P	P	P		C		
P	P	P	P			P	
P	P	P	P	C̲	P̲		C
P	P	P		C		P	

Rd_1_b_i_A:

P	P	P	P	+	C	+	*
P	P	P	P		C̲	P	*
P	P	P	P	P	C	-	C
P	P	P		C	P̲	P	-

Rd_1_b_i_B:

P	P	P	P	+	C	+	*
P	P	P	P		P̲	P	-
P	P	P	P	P	C	-	C
P	P	P		C	C̲	P	*

Rd_1_b_ii_A:

P	P	P	P		C	P̲	C̲
P	P	P	P			P	
P	P	P	P	C	P		C
P	P	P		C		P	

Rd_1_b_ii_B:

P	P	P	P	*	C	C̲	P̲
P	P	P	P	*	+	P	-
P	P	P	P	C	P	%	C
P	P	P	+	C	-	P	%

Rd_1_b_ii_A_I:

P	P	P	P	*	C	P	C
P	P	P	P	*		P	C̲
P	P	P	P	C	P	+	C
P	P	P		C	+	P	P̲

Rd_1_b_ii_A_II:

P	P	P	P	*	C	P	C
P	P	P	P	*	-	P	P̲
P	P	P	P	C	P	-	C
P	P	P		C		P	C̲

Rd_2_a:

P	P	P	P				
P	P	P	P				
P	P	P	P				P
P	P	P		C	C̲	C	P̲

Rd_2_b:

P	P	P	P				
P	P	P	P				
P	P	P	P				P
P	P	P		C	P̲	C	C̲

Rd_2_a_i:

P	P	P	P				
P	P	P	P				P̲
P	P	P	P			C̲	P
P	P	P		C	C	C	P

Rd_2_a_ii:

P	P	P	P				
P	P	P	P				C̲
P	P	P	P			P̲	P
P	P	P		C	C	C	P

Rd_2_a_i_A:

P	P	P	P	*	+	P̲	-
P	P	P	P	*	+	C̲	P
P	P	P	P		-	C	P
P	P	P		C	C	C	P

Rd_2_a_i_B:

P	P	P	P	*	-	C̲	-
P	P	P	P	*	+	P̲	P
P	P	P	P	%	+	C	P
P	P	P	%	C	C	C	P

Rd_2_a_ii_A:

P	P	P	P	*	+	-	-
P	P	P	P	*	C̲	P̲	C
P	P	P	P	%	+	P	P
P	P	P	%	C	C	C	P

Rd_2_a_ii_B:

P	P	P	P	*	-	-	+
P	P	P	P	*	P̲	C̲	C
P	P	P	P		+	P	P
P	P	P		C	C	C	P

Rd_2_b_i:

P	P	P	P	*	*	-	%
P	P	P	P	+	P̲	C̲	
P	P	P	P	+	%	-	P
P	P	P		C	P	C	C

Rd_2_b_ii:

P	P	P	P	*	-	%	-
P	P	P	P	+	C̲	P̲	
P	P	P	P	+		*	P
P	P	P	%	C	P	C	C

INDEX_Re:

Re:

P		P					
P	P	C	P				
P	P	P					
P	P	C					

Re_1:

P		P					
P	P	C	P				
P	P	P					C̲
P	P	C				P̲	

Re_2:

P		P					
P	P	C	P				
P	P	P					P̲
P	P	C				C̲	

Re_1_a:

P		P			C̲		
P	P	C	P			P̲	
P	P	P					C
P	P	C				P	

Re_1_b:

P		P			P̲		
P	P	C	P			C̲	
P	P	P					C
P	P	C				P	

Re_1_a_i:

P		P			C		
P	P	C	P	P̲		P	
P	P	P	C̲				C
P	P	C				P	

Re_1_a_ii:

P		P			C		
P	P	C	P	C̲		P	
P	P	P	P̲				C
P	P	C				P	

Re_1_a_i_A:

P	!	P		*	C	*	+
P	P	C	P	P	P̲	P̲	-
P	P	P	C	%	%	C̲	C
P	P	C		!	-	P	+

Re_1_a_i_B:

P	!	P		*	C	*	+
P	P	C	P	P	C̲	P	+
P	P	P	C	%	-	P̲	C
P	P	C	%	!	-	P	

Re_1_a_ii_A:

P		P			C		
P	P	C	P	C	P̲	P	
P	P	P	P		C̲		C
P	P	C				P	

Re_1_a_ii_B:

P		P			C		
P	P	C	P	C	C̲	P	
P	P	P	P		P̲		C
P	P	C				P	

Re_1_a_ii_A_I:

P		P		*	C	*	-
P	P	C	P	C	P	P	+
P	P	P	P	P̲	C	C̲	C
P	P	C			+	P	-

Re_1_a_ii_A_II:

P		P		+	C	-	-
P	P	C	P	C	P	P	*
P	P	P	P	C̲	C	P̲	C
P	P	C		+		P	*

Re_1_a_ii_B_I:

P		P		-	C	%	%
P	P	C	P	C	C	P	+
P	P	P	P	*	P	C̲	C
P	P	C	*	-	+	P	P̲

Re_1_a_ii_B_II:

P		P			C		
P	P	C	P	C	C	P	
P	P	P	P		P	P̲	C
P	P	C				P	C̲

Re_1_a_ii_B_II_aa:

P		P		C̲	C	P̲	-
P	P	C	P	C	C	P	-
P	P	P	P	+	P	P	C
P	P	C		+		P	C

Re_1_a_ii_B_II_bb:

P		P		P̲	C	C̲	-
P	P	C	P	C	C	P	-
P	P	P	P	+	P	P	C
P	P	C	+			P	C

Re_1_b_i:

P	*	P		+	P	%	=
P	P	C	P	-	-	C	C̲
P	P	P	*	+	!	P̲	C
P	P	C	%	!		P	=

Re_1_b_ii:

P	*	P		=	P	%	-
P	P	C	P	=	!	C	P̲
P	P	P	*	+	+	C̲	C
P	P	C	%	-		P	!

Re_2_a:

P		P					
P	P	C	P				
P	P	P				*C*	P
P	P	C			*P*	C	

Re_2_b:

P		P					
P	P	C	P				
P	P	P				*P*	P
P	P	C			*C*	C	

Re_2_a_i:

P		P				*P*	
P	P	C	P			*C*	
P	P	P				C	P
P	P	C			P	C	

Re_2_a_ii:

P		P				*C*	
P	P	C	P			*P*	
P	P	P				C	P
P	P	C			P	C	

Re_2_a_i_A:

P	*	P		+		P	-
P	P	C	P	*P*	+	C	
P	P	P	*		-	C	P
P	P	C		*C*	P	C	

Re_2_a_i_B:

P		P	%	*	+	P	
P	P	C	P	*C*	-	C	-
P	P	P	+		%	C	P
P	P	C		*P*	P	C	*

Re_2_a_ii_A:

P	*	P	%	*P*	+	C	
P	P	C	P	*C*		P	
P	P	P	+	-	%	C	P
P	P	C	-	*	P	C	

Re_2_a_ii_B:

P	*	P		*C*	-	C	-
P	P	C	P	*P*	%	P	
P	P	P	*	+		C	P
P	P	C	+		P	C	%

Re_2_b_i:

P	*	P		%	*C*	%	!
P	P	C	P	*P*	+	!	
P	P	P	*	-	+	P	P
P	P	C	-	=	C	C	=

Re_2_b_ii:

P	*	P	+	-	*P*	:	=
P	P	C	P	*C*	!	=	!
P	P	P	*	-	+	P	P
P	P	C	:	%	C	C	%

INDEX_Rf:

Rf:

P	P	C	P				
P	P	P	C				
P	P	P	P				
P	P	P					

Rf_1:

P	P	C	P				
P	P	P	C				
P	P	P	P				*C*
P	P	P				*P*	

Rf_2:

P	P	C	P				
P	P	P	C				
P	P	P	P				*P*
P	P	P				*C*	

Rf_1_a:

P	P	C	P		*P*		
P	P	P	C			*C*	
P	P	P	P	*P*			C
P	P	P			*C*	P	

Rf_1_b:

P	P	C	P		*P*		
P	P	P	C			*C*	
P	P	P	P	*C*			C
P	P	P			*P*	P	

Rf_1_c:

P	P	C	P		*C*		
P	P	P	C			*P*	
P	P	P	P	*P*			C
P	P	P			*C*	P	

Rf_1_d:

P	P	C	P		*C*		
P	P	P	C			*P*	
P	P	P	P	*C*			C
P	P	P			*P*	P	

Rf_1_a_i:

P	P	C	P		P		-
P	P	P	C	*	*	C	*C*
P	P	P	P	P	+	*P*	C
P	P	P		+	C	P	-

Rf_1_a_ii:

P	P	C	P	*	P		
P	P	P	C		*	C	*P*
P	P	P	P	P	+	*C*	C
P	P	P		+	C	P	

Rf_1_b_i:

P	P	C	P		P		
P	P	P	C			C	
P	P	P	P	C	*P*	*C*	C
P	P	P			P	P	

Rf_1_b_ii:

P	P	C	P		P		
P	P	P	C			C	
P	P	P	P	C	*C*	*P*	C
P	P	P			P	P	

Rf_1_b_i_A:

P	P	C	P	*P*	P	*	+
P	P	P	C	-	*C*	C	-
P	P	P	P	C	P	C	C
P	P	P	*		P	P	+

Rf_1_b_i_B:

P	P	C	P	*C*	P		
P	P	P	C	*	*P*	C	+
P	P	P	P	C	P	C	C
P	P	P		*	P	P	+

Rf_1_b_ii_A:

P	P	C	P		P	+	*C*
P	P	P	C	*	*	C	-
P	P	P	P	C	C	P	C
P	P	P	+	*P*	P	P	-

Rf_1_b_ii_B:

P	P	C	P	*	P	+	*P*
P	P	P	C	*	-	C	-
P	P	P	P	C	C	P	C
P	P	P	+	*C*	P	P	

Rf_1_c_i:

P	P	C	P	*	C	-	-
P	P	P	C		+	P	*P*
P	P	P	P	P	+	*C*	C
P	P	P			C	P	*

Rf_1_c_ii:

P	P	C	P	*	C	*	-
P	P	P	C		+	P	*C*
P	P	P	P	P	+	*P*	C
P	P	P			C	P	-

Rf_1_d_i:

P	P	C	P	*P*	C	*	+
P	P	P	C		*C*	P	+
P	P	P	P	C	-	-	C
P	P	P	*		P	P	

Rf_1_d_ii:

P	P	C	P	*C*	C	+	+
P	P	P	C		*P*	P	*
P	P	P	P	C	-	-	C
P	P	P			P	P	*

Rf_2_a:

P	P	C	P				
P	P	P	C				
P	P	P	P	*C*			P
P	P	P			*P*	C	

Rf_2_b:

P	P	C	P				
P	P	P	C				
P	P	P	P	*P*			P
P	P	P			*C*	C	

Rf_2_a_i:

P	P	C	P	*		*P*	+
P	P	P	C	-	-	*C*	
P	P	P	P	C	+	%	P
P	P	P		*	P	C	%

Rf_2_a_ii:

P	P	C	P	-	-	*C*	
P	P	P	C	+	*	*P*	
P	P	P	P	C		%	P
P	P	P	*	+	P	C	%

Rf_2_b_i:

P	P	C	P	*	-	*	%
P	P	P	C	+	-	+	*C*
P	P	P	P	P	%	*P*	P
P	P	P		!	C	C	!

Rf_2_b_ii:

P	P	C	P				
P	P	P	C				*P*
P	P	P	P	P		*C*	P
P	P	P			C	C	

Rf_2_b_ii_A:

P	P	C	P				
P	P	P	C				P
P	P	P	P	P		C	P
P	P	P		*C*	C	C	*P*

Rf_2_b_ii_B:

P	P	C	P				
P	P	P	C				P
P	P	P	P	P		C	P
P	P	P		*P*	C	C	*C*

Rf_2_b_ii_A_I:

P	P	C	P		*	*P*	+
P	P	P	C		*	*C*	P
P	P	P	P	P	+	C	P
P	P	P		C	C	C	P

Rf_2_b_ii_A_II:

P	P	C	P	*	*	*C*	
P	P	P	C		+	*P*	P
P	P	P	P	P	+	C	P
P	P	P		C	C	C	P

Rf_2_b_ii_B_I:

P	P	C	P	*P*	*	+	
P	P	P	C		*C*	+	P
P	P	P	P	P	*	C	P
P	P	P		P	C	C	C

Rf_2_b_ii_B_II:

P	P	C	P	*C*	*	+	*
P	P	P	C		*P*	+	P
P	P	P	P	P		C	P
P	P	P		P	C	C	C

INDEX_Rg:

Rg:

P	P	P	C				
P	P	P	P				
P	P	P					
P	P	C	P				

Rg_1:

P	P	P	C				
P	P	P	P				
P	P	P					*C*
P	P	C	P			*P*	

Rg_2:

P	P	P	C				
P	P	P	P				
P	P	P					*P*
P	P	C	P			*C*	

Rg_1_a:

P	P	P	C		*P*		
P	P	P	P			*C*	
P	P	P					C
P	P	C	P			P	

Rg_1_b:

P	P	P	C		*C*		
P	P	P	P			*P*	
P	P	P					C
P	P	C	P			P	

Rg_1_a_i:

P	P	P	C	*	P		+
P	P	P	P	-	-	C	*C*
P	P	P		*	%	*P*	C
P	P	C	P	%		P	+

Rg_1_a_ii:

P	P	P	C	*	P		%
P	P	P	P	*	+	C	*P*
P	P	P		-	-	*C*	C
P	P	C	P	%		P	+

Rg_1_b_i:

P	P	P	C		C		
P	P	P	P			P	
P	P	P			*P*	*C*	C
P	P	C	P			P	

Rg_1_b_ii:

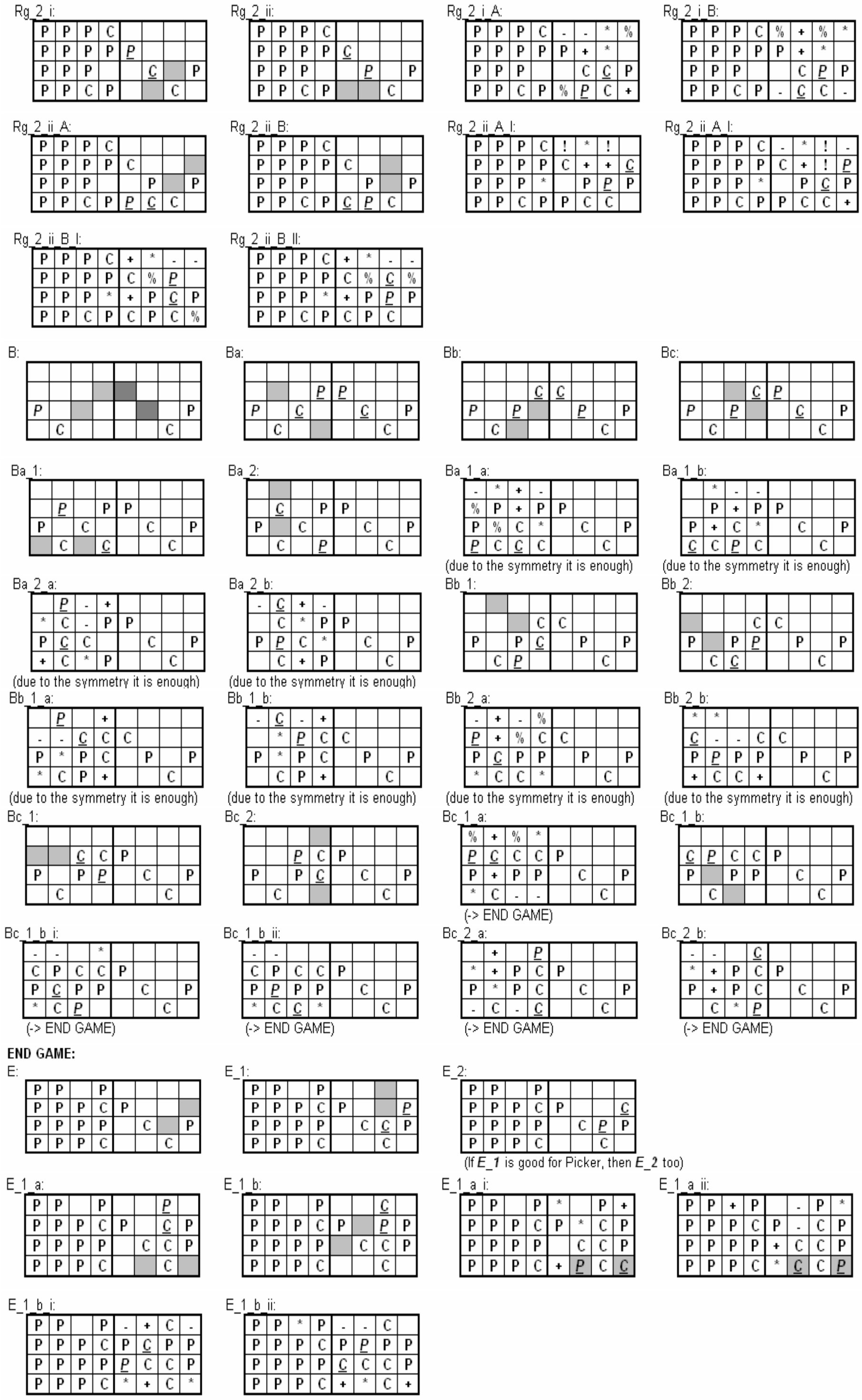
P	P	P	C		C	-	-
P	P	P	P	+	*	P	!
P	P	P	+	%	*C*	*P*	C
P	P	C	P	%	*	P	!

Rg_1_b_i_A:

P	P	P	C	+	C		*P*
P	P	P	P	-	+	P	*
P	P	P	-	%	P	C	C
P	P	C	P	%	*	P	*C*

Rg_1_b_i_B:

P	P	P	C	!	C	!	*C*
P	P	P	P	-		P	*
P	P	P	-	%	P	C	C
P	P	C	P	%	*	P	*P*


Rg_2_i:
Rg_2_ii:
Rg_2_i_A:
Rg_2_i_B:
Rg_2_ii_A:
Rg_2_ii_B:
Rg_2_ii_A_I:
Rg_2_ii_A_I:
Rg_2_ii_B_I:
Rg_2_ii_B_II:
B:
Ba:
Bb:
Bc:
Ba_1:
Ba_2:
Ba_1_a:
(due to the symmetry it is enough)
Ba_1_b:
(due to the symmetry it is enough)
Ba_2_a:
(due to the symmetry it is enough)
Ba_2_b:
(due to the symmetry it is enough)
Bb_1:
Bb_2:
Bb_1_a:
(due to the symmetry it is enough)
Bb_1_b:
(due to the symmetry it is enough)
Bb_2_a:
(due to the symmetry it is enough)
Bb_2_b:
(due to the symmetry it is enough)
Bc_1:
Bc_2:
Bc_1_a:
(-> END GAME)
Bc_1_b:
Bc_1_b_i:
(-> END GAME)
Bc_1_b_ii:
(-> END GAME)
Bc_2_a:
(-> END GAME)
Bc_2_b:
(-> END GAME)
END GAME:
E:
E_1:
E_2:
(If E_1 is good for Picker, then E_2 too)
E_1_a:
E_1_b:
E_1_a_i:
E_1_a_ii:
E_1_b_i:
E_1_b_ii:

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Chapter 7

Summary

Abstract

The main goal of this work is to understand Picker-Chooser (or Chooser-Picker) games and Beck's conjecture as deeply as possible. The text has three main parts.

At first we examine the complexity of Picker-Chooser(P-C) and Chooser-Picker(C-P) games. Here we found that it is NP-hard to decide the winner for both P-C and C-P games [24]. Then we discuss the Picker-Chooser version of well-known games, to explore the differences and similarities among the various types. The examined games are the C-P 4×4 *Tic-Tac-Toe*, the P-C version of generalized Shannon switching game, the C-P version of the k-in-a-row and some of the C-P, M-B (Maker-Breaker) and P-C Torus games. We improve a little on the "Erdős-Selfridge" theorem for C-P games, although a gap remains this and the conjectured form [21].

Secondly, we solve with the Chooser-Picker 7-in-a-row game. This game is quite interesting because the last really valuable result for the 8-in-a-row game (by playing on infinite board the 8-in-a-row game the second player can achieve a draw), was made more than 30 years ago. Since then all attempts to solve the 7-in-a-row was unsuccessful. The thesis deals with the Chooser-Picker version of the same problem. In that section we prove that the Chooser-Picker 8-in-a-row and the Chooser-Picker 7-in-a-row game is a Picker win. The proof is a bit lengthy and a non- trivial case study. After we sketch some idea how can we deal with the original (M-M or M-B) version of this game [22].

Finally we will discuss the P-C diameter games. Here we found a very interesting result that how different result is given by the Maker-Breaker version and the Picker-Chooser version [2, 23]. As we show the Picker-Chooser version restores the probabilistic intuition, just like the acceleration of the game.

7.1 Definitions, a conjecture and some new tools

7.1.1 The weak version of the games

There can be defined the weak version of the positional games [6], where the second player wins if he/she can achieve a draw. It means that the first player do not have to be afraid of (and defend against) that the second player occupies a winning set. Here the first player is called Maker, and the second is called Breaker. It is easy to see the following statement, see [7].

Statement 7.1. *If the Breaker wins in the weak version of the game, then the strong version is draw.*

7.1.2 Chooser-Picker and the Picker-Chooser games

Studying the very hard clique games, Beck [6] introduced a new type of heuristic, that proved to be a great success. He defined the *Picker-Chooser* or shortly P-C and the *Chooser-Picker* (C-P) versions of a Maker-Breaker game that resembles fair division, (see [71]). In these versions Picker takes an unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. When $|V|$ is odd, the last element goes to Chooser. Beck obtained that conditions for winning a Maker-Breaker game by Maker and winning the Picker-Chooser version of that game by Picker coincide in several cases. Furthermore, Breaker's win in the Maker-Breaker and Picker's win in the Chooser-Picker version seem to occur together.

The study of these games gives invaluable insight to the Maker-Breaker version. For some hypergraphs the outcome of the Maker-Breaker and Chooser-Picker versions is the same [6, 21]. In all cases it seems that Picker's position is at least as good as Breaker's. It was formalized in the following conjecture.

Conjecture 7.2. *If Maker (as the second player) wins the Maker-Breaker game, then Picker wins the corresponding Picker-Chooser game. If Breaker (as the second player) wins the Maker-Breaker game, then also Picker wins the Chooser-Picker game.[21]*

It is necessary for the Chooser-Picker Games infinite version the following restriction: At the beginning Chooser can select a bounded subset of the board, where they will play. Because if they play on the infinite board, then Picker could select points far from each other, and it is a trivially winning strategy for Picker.

7.1.3 Toolbar

Pairing lemma

Lemma 7.3 (Cs-P)**.** *If in the course of the (Chooser- Picker) game (or just already at the beginning) there is a two element winning set* $\{x, y\}$ *then Picker has an optimal strategy starting with* $\{x, y\}$ *.*

The monotonicity lemma

We mentioned that the at infinite version Chooser can select a bounded subset. In practice it means that Chooser selects a finite set $X \in V$, and they play on the *induced sub-hypergraph* that is keep only those edges $A \in \mathcal{F}$ for which $A \subset X$. More formally, given the hypergraph $(V, \mathcal{F})$ let $(V \setminus X, \mathcal{F}(X))$ denote the hypergraph where $\mathcal{F}(X) = \{A \in \mathcal{F}, A \cap X = \emptyset\}$.

Lemma 7.4. *[21] If Picker wins the Chooser-Picker game on $(V, \mathcal{F})$, then Picker also wins it on $(V \setminus X, \mathcal{F}(X))$.*

This lemma is useful tool at the next chapters, because if a bounded set S cant be partitioned into uniform sub-games, then it can be increased to S', which can be split into such sub-games. And if Picker wins on S', then also can win on S.

7.1.4 Some results on Chooser-Picker games

Complexity of Chooser-Picker positional games

Since the Maker-Breaker (and the Maker-Maker) games are PSPACE-complete, see [65], it would support both Conjecture 2.9, and the above heuristic to see that the Chooser-Picker or Picker-Chooser games are not easy, too. To prove PSPACE-completeness for positional games is more or less standard, see [65, 64, 16]. Here we can prove less because of the asymmetric nature of these games.

Theorem 7.5. *It is NP-hard to decide the winner in a Picker-Chooser game.*

Theorem 7.6. *It is NP-hard to decide the winner in a Chooser-Picker game.*

Both proofs are based on the usual reduction method. We reduce $3-\mathrm{SAT}$ to Chooser-Picker or Picker-Chooser games.

Note that Chooser-Picker games are NP-hard, even for hypergraphs (V, E), where $|A| \leq 6$ for $A \in E$.

4×4 **tic-tac-toe**

Proposition 7.7. *Picker wins the Chooser-Picker version of the 4×4 tic-tac-toe.*

Picker-Chooser version of the generalized Shannon switching game

We prove Conjecture 2.9 for the Picker-Chooser version of Shannon switching game in the generalized version as Lehman did in [46]. Let $(V, \mathcal{F})$ be a matroid, where $\mathcal{F}$ is the set of bases, and Picker wins by taking an $A \in \mathcal{F}$. Note, that this is equivalent with the Chooser-Picker game on $(V, \mathcal{C})$, where $\mathcal{C}$ is the collection of *cutsets* of the matroid $(V, \mathcal{F})$, that is for all $A \in \mathcal{F}$ and $B \in \mathcal{C}$, $A \cap B \neq \emptyset$.

Theorem 7.8. *Let $\mathcal{F}$ be collection of bases of a matroid on V. Picker wins the Picker-Chooser $(V, \mathcal{F})$ game, if and only if there are $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$.*

The proof closely follow the ones given by Oxley in [54] for the Maker-Breaker case.

Erdős-Selfridge type theorems for P-C and C-P games

The Erdős-Selfridge theorem [27] gives a very useful condition for Breaker's win in a Maker-Breaker $(V, \mathcal{F})$ game. Using a stronger condition, Beck [6] proves Picker's win in a Chooser-Picker $(V, \mathcal{F})$ game. (For the P-C version he proved a sharp result that we include here.) Let $||\mathcal{F}|| = \max_{A \in \mathcal{F}} |A|$ be the rank of the hypergraph $(V, \mathcal{F})$.

Theorem 7.9. *[6] If*

$$T(\mathcal{F}) := \sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{8(||\mathcal{F}|| + 1)}, \tag{7.1}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $(V, \mathcal{F})$. If $T(\mathcal{F}) < 1$, then Chooser wins the Picker-Chooser game on $(V, \mathcal{F})$.

We improved on his result by showing:

Theorem 7.10. *If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{3\sqrt{||\mathcal{F}|| + \frac{1}{2}}}, \tag{7.2}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $(V, \mathcal{F})$.

It is worthwhile to spell out a special case of Conjecture 2.9 for this case, that would be the sharp extension of Erdős-Selfridge theorem to Chooser-Picker games.

Conjecture 7.11. *If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Picker wins the Chooser-Picker game on $(V, \mathcal{F})$.

Torus games

To test Beck's paradigm we check the status of concrete games defined on the 4×4 torus, denoted by 4^2. That is we glue together the opposite sides of the grid, and consider all lines of slopes 0 and ± 1 as winning sets. For the general definition of torus games see [7]. We use a chess-like notation to refer to the elements of the board. The hypergraph of 4^2 is not almost disjoint, see e. g. the two winning sets $\{a2, b1, c4, d3\}$ and $\{a4, b1, c2, d3\}$. We can define four possible games on 4^2, those are the Maker-Maker, the Maker-Breaker, the Chooser-Picker and the Picker-Chooser versions. According to [7], the Maker-Maker version of 4^2 is a draw, and Picker wins the Chooser-Picker version, see [21]. In fact, the statement of the Maker-Breaker version implies the result for the Maker-Maker version, while the proof of it contains the proof of the Chooser-Picker version.

Proposition 7.12. *Breaker wins the Maker-Breaker version of the 4^2 torus game.*

According to Conjecture 2.9, Breaker has an easier job in the Maker-Breaker version than Chooser has in the Picker-Chooser game. For the 4×4 torus the outcome of these games are the same, although it is much harder to prove.

Proposition 7.13. *Chooser wins the Picker-Chooser version of the* 4×4 *torus game.*

Proof. **(sketch)** The full proof needs a lengthy exhaustive case analysis. However, some branches of the game tree may be cut by proof method of Beck's following result [6]: Chooser wins a Picker-Chooser game on $\mathcal{H}$ if $T(\mathcal{H}) := \sum_{A \in E(\mathcal{H})} 2^{-|A|} < 1$. □

It is important to remark that above we have seen an ordering due to its complexity: it is easier to get the result of the C-P case, then the M-B case, though it gives the same result. And it is far more hard to determine the P-C case then the Maker-Breaker case.

7.2 The Chooser-Picker 7-in-a-row game

7.2.1 The k-in-a-row game

The k-in-a-row game is that hypergraph game, where the vertices of the graphs are the fields of an infinite graph paper ($\mathbb{Z}^2$), and the winning sets are the consecutive cells (horizontal, vertical or diagonal) of length k. If one of the players gets a length k line, then he wins otherwise the game is draw. Note the assuming perfect play, the winner is always the first player, or it is a draw by the strategy stealing argument of John Nash, [13]. More details about k-in-a-row games in [61, 62].

Both the Maker-Maker and the Maker-Breaker versions of the k-in-a-row for $k = 6, 7$ are open. These are wisely believed to be draws (Breaker's win) but, despite of the efforts spent on those, not much progress has been achieved.

7.2.2 The C-P k-in-a-row game

Before proving the C-P 7-in-a-row game, we proved the easier C-P 8-in-a-row game (by playing auxiliary games in a "Z" shaped board, what used Zetters in [34]).

Proposition 7.14. *Picker wins the Chooser-Picker version of the game* 8*-in-a-row on any* $B \subseteq \mathbb{Z}^2$.

Theorem 7.15. *Picker wins the Chooser-Picker 7-in-a-row game on every A subset of* $\mathbb{Z}^2$.

By applying the remedy mentioned before Lemma 2.18 at first Chooser determines the finite board S. We will consider a tiling of the entire plane, and play an auxiliary game on each tile (sub-hypergraph). It is easy to see, if Picker wins all of the sub-games, then Picker wins the game played on any K board which is the union of disjoint tiles. Let K be the union of those tiles which meet S. Since $S \subset K$, Lemma 2.18 gives that Picker also wins the game on S, too. Now we need to show a suitable tiling and to define and analyze the auxiliary games. The tiling guarantees that if Picker wins on in each sub-games then Chooser cannot occupy any seven consecutive squares on K.

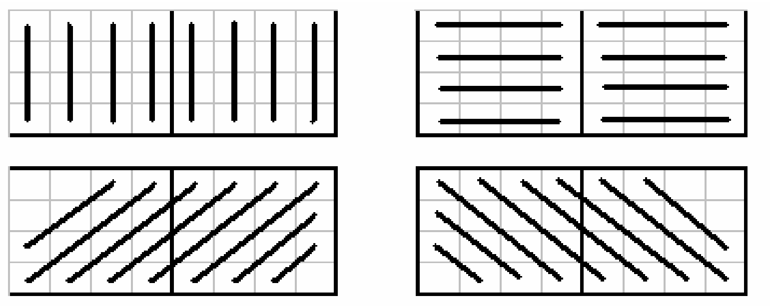

Figure 7.1: These are the winning-sets of the 4×8 rectangle. Easy to see, that there is exactly one symmetry (along the double line). Later we will make use of it.

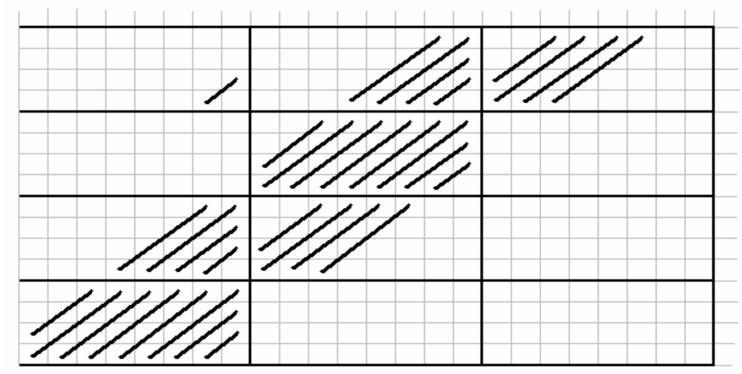

Figure 7.2: We can see, how to draw from playing on simple tile, the game played on the infinite chessboard: neither vertically, nor horizontally, nor diagonally (there is only one diagonal direction detailed) there are no seven consecutive squares without containing one winning set of a sub-game.

Each tile is a 4×8 sized rectangle and the winning sets, for the sake of better understanding, are drawn on the following four board:

The key lemma for our proof is the following.

Lemma 7.16. *Picker wins the auxiliary game defined on the 4×8 rectangle.*

Remark 7.17. *We checked with brute force computer search the M-B game on the same auxiliary board, but it is a Maker win! So we cannot use the same table again, to prove that the weak version (=the Maker-Breaker version) of this game is a Breaker win. One is tempted to look for other auxiliary games, which is not going to be easy. As a rule of thumb, it always good idea to check the C-P version of these games at first.*

7.3 The Picker-Chooser Diameter Game

7.3.1 Graph Games

Large classes of Maker-Breaker games are defined on the complete graph on n vertices. The players take the edges of the graph in turns; Maker wins iff his subgraph has a given, usually monotone, property $\mathcal{P}$, see [8, 5, 12, 17]. Balogh et al. [2] introduced the $(a : b)$

d-diameter game, shortly $\mathcal{D}_d(a : b)$, which means that Maker wins iff the diameter of his subgraph is at most d. These games turned out to be very difficult and surprising; a detailed discussion will be given in Section 5.2. The main result of Balogh et al. was that Maker loses the game $\mathcal{D}_2(1 : 1)$ but Maker wins the game $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\log n)^{3/8})$.

This means that the acceleration of a game may change the outcome dramatically, [61]. The outcome also changes a lot when one considers the Picker-Chooser version of the game $\mathcal{D}_2(1 : 1)$. Our main result is the following theorem.

Observation.

Picker wins the P-C game $\mathcal{D}_2(1 : 1)$ on the graph K_n, if $n > 22$.

Theorem 7.18. *In the Chooser-Picker game $\mathcal{D}_2(1 : b)$, Picker wins if $b < \sqrt{n/\log_2 n}/4$, while Chooser wins if $b > 3\sqrt{n}$, provided that n is large enough.*

The Picker-Chooser (Chooser-Picker) games are themselves heuristics for the Maker-Breaker games. As Theorem 3.6 shows, the conditions for winning a Maker-Breaker game by Breaker and winning the Chooser-Picker version of that game by Picker coincide in several cases. Furthermore, Breaker's win in the Maker-Breaker and Chooser's win in the Picker-Chooser version seem to occur together in some cases [6]. To further explore this connection, a generalization of Theorem 3.6 for biased games is needed. No attempt is made here to get the best possible form, for our needs the following lemma will be sufficient.

Lemma 7.19. *Picker wins the Chooser-Picker $(1 : b)$ biased game on the hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ if*

$$\frac{v}{b+1} \sum_{A \in E(\mathcal{H})} 2^{-|A|/b} < 1,$$

where $v = |V(\mathcal{H})|$.

Diameter and degree games

Balogh et al. [2] observed that the game $\mathcal{D}_2(1 : 1)$ defies the probabilistic intuition completely. Indeed, if one divides the edges of K_n among Maker and Breaker randomly, then Maker's subgraph will almost surely have diameter two. Still, Breaker has a simple pairing winning strategy for $n > 3$, [2]. First taking an edge uv, such that neither ux nor vx has been taken by Maker for any vertex x. Then if Maker takes ux, taking vx follows, and if Maker takes vx, Breaker takes ux, otherwise an arbitrary edge is taken.

However, when playing the game $\mathcal{D}_2(2 : 2)$, this pairing strategy is not available for Breaker. Maker wins the game $\mathcal{D}_2(2 : 2)$, and even more, the game $\mathcal{D}_2(2 : b)$, where b grows polynomially in n, provided that n is large enough.

Theorem 7.20. *[2] Maker wins the game $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\ln n)^{3/8})$, and Breaker wins the game $\mathcal{D}_2(2 : (2+\epsilon)\sqrt{n/\ln n})$ for any $\epsilon > 0$, provided n is large enough.*

To prove Theorem 5.1 we need to study the so-called *degree games*. Székely, Beck and Balogh et al. [73, 5, 2] showed that these games are interesting in their own right.

In such games one player tries to distribute his moves uniformly, while the other player's goal is to obtain as many edges incident to some vertex as possible. Given a graph G and a prescribed degree d, Maker and Breaker play an $(a:b)$ game on the edges of G. Maker wins by getting at least d edges incident to each vertex. We are interested only in the case of $G = K_n$. Balogh et al. [2] proved the following lemma:

Lemma 7.21. *[2] Let $a \leq n/(4\ln n)$ and n be large enough. Then Maker wins the $(a:b)$ degree game on K_n if $d < \frac{a}{a+b}n - \frac{6ab}{(a+b)^{3/2}}\sqrt{n\ln n}$.*

As we do not wish to develop the complete theory of P-C (C-P) degree games, we state only a simple form that suffices our needs

Lemma 7.22. *Let $b < n/(8\ln n)$ and n be large enough. Then Chooser wins the $(1:b)$ Chooser-Picker degree game on K_n if $d < n-1-3n/b$.*

To prove Theorem 5.1, we proved Lemma 3.9 first.

The second part of the theorem, i. e. Chooser wins if $b > 3\sqrt{n}$, comes from Lemma 5.4. Let Chooser play accordingly to that lemma, then Picker gets at most $(3n/b) - 1$ edges at any vertex $x \in K_n$, so the number of vertices that are linked to x is no more than $((3n/b)-1)^2 < n-1$.

To prove the first part of the theorem implies more work. We split the vertices of the graph into three approximately equal parts, X_1, X_2 and X_3. (Let X_i be $X_{i \bmod 3}$ if $i > 3$.) The elements of X_i may be listed as $1, 2, \ldots, n/3$.[1] $E(X_i, X_j)$ denotes the edges between the sets X_i and X_j.

We will play two different games among and inside the parts. At the first game we link the points of X_i using $E(X_i, X_{i+1})$, for $i = 1, 2, 3$. At the second game we link the sets X_i with X_{i+1} playing on the edges of X_{i+1}.

[1] It can also be $\lfloor n/3 \rfloor$ and $\lceil n/3 \rceil$. In the proof we show that it works with $\lceil n/3 \rceil$, and the case $\lfloor n/3 \rfloor$ easily follows from that.

Chapter 8

Összefoglaló

Absztrakt

Ennek a munkának az fő célja, hogy minél mélyebben megértsük a Picker-Chooser (vagy Chooser-Picker) játékokat és Beck sejtését. A dolgozatnak három fő részből áll:

Először a Picker-Chooser(P-C) és a Chooser-Picker(C-P) játékok komlexitását vizsgáltuk meg. Itt azt találtuk, hogy mind a P-C és a C-P játékok esetében NP nehéz eldönteni, hogy melyik játékos a nyerő.[24]. Ezután bemutattuk néhány ismert példán keresztül a Picker-Chooser játékokat, hogy felfedezzük az azonosságokat és eltéréseket a különböző játékok között. Megvizsgáltuk a C-P 4×4 *tic-tac-toe*-t, a P-C változatát az általánosított Shannon-féle kapcsolójátéknak, a C-P változatát a k-amőbának, valamint a C-P, M-B és P-C tórusz játékoknak. Egy kicsit javítottunk a C-P játékokra vonatkozó "Erdős-Selfridge" tételen is [21].

A második részben a Chooser-Picker 7-amőba játékot oldottuk meg. Ez a játék azért is nagyon érdekes, mert a legutolsó igazán értékes eredmény a 8-amőba játékra már több mint 30 évvel ezelőtti (a végtelen négyzetrácsos papíron a második játékos elérheti a döntetlent). A 7-amőba megoldására tett kísésletek mindeddig sikertelenek. A tézis ennek a játéknak a Chooser-Picker változatával foglalkozik. Ebben a fejezetben belátjuk, hogy a Chooser-Picker 8-amőbát és a Chooser-Picker 7-amőbát Picker nyeri. A bizonyítás egy kissé hosszadalmas, nem triviális esetvizsgálat. Eztán felvázolunk egy elképzelést, hogyan lehetne boldogulni az eredeti (M-M illetve M-B) változatával ennek a játéknak [22].

Az utolsó részben a P-C átmérő játékkal foglalkozunk. Itt nagyon érdekes megfigyelni az M-B és a P-C játékokra kapott eredmények különbözőségét [2, 23]. Megmutatjuk, hogy a valószínűségi intuiciónkhoz közel álló eredményt hoz a Picker-Chooser változat, csakúgy, mint a felgyorsítás..

8.1 Definíciók, egy sejtés és néhány eszköz

8.1.1 A játékok gyenge változata

A játékok gyenge változatának azt nevezzük, amikor a második játékos akkor nyer, ha döntetlent tud elérni. Ez azt jelenti, hogy a kezdő játékosnak nem kell félnie/védekeznie az ellen, hogy a második játékos elfoglalhat egy nyerőhalmazt. Itt a kezdő játékost Maker-nek (építő), a másodikat Breaker-nek (romboló) hívjuk. Könnyű belátni az alábbi állítást, lásd [7].

Állítás 8.1. Ha Breaker nyeri a a játék gyenge változatát, akkor az eredeti játék döntetlen.

8.1.2 Chooser-Picker és a Picker-Chooser játékok

Beck [6] az igen nehéz klikk játékok tanulmányozására bevezetett egy új típusú heurisztikát, mely igen sikeresnek bizonyult. Definiálta a *Picker-Chooser* vagy röviden P-C és a *Chooser-Picker* (C-P) változatait a Maker-Breaker játékoknak, mely igen hasonló a kétszemélyes torta felosztás problámához, (lásd [71]). Ezeknél a változatoknál Picker mindig kiválaszt két mezőt, majd Chooser választ közülük egyet, a másik Pickerhez kerül. A Picker-Chooser játékokban Picker felel meg Maker-nek és Chooser Breaker-nek, míg a Chooser-Picker játékoknál fordítva. Ha $|V|$ páratlan, akkor az utolsó elem Chooser-é. Beck azt tapasztalta, hogy Maker igen sok esetben pontosan akkor nyeri meg a Maker-Breaker játékot, amikor Picker a Picker-Chooser változatot. Ráadásul Breaker nyerései a M-B játékban, illetve Picker nyerései a C-P játékban úgy tűnik, hogy egybe esnek.

Ezen játékok tanulmányozása felbecsülhetetlen rátekintést enged a Maker-Breaker változatra. Néhány hipergráfra a végeredménye a Maker-Breaker és a Chooser-Picker változatnak ugyanaz [6, 21]. Általában úgy tűnik, hogy Picker helyzete legalább olyan jó, mint Breaker-é. Ezt az alábbi sejtésben mondható ki:

Sejtés 8.2. Ha a Maker-Breaker játékot Maker nyeri, akkor a Picker - Chooser játékot (mint második játékos) Picker nyeri; ha a Maker-Breaker játékot Breaker nyeri, akkor a Chooser-Picker játékot szintén (mint második játékos) Picker nyeri [21].

Szükséges a Chooser-Picker játékok végtelen változatának használhatóságához az alábbi megszorítás: Az elején Chooser kiválaszthatja egy korlátos részhalmazát a táblának, ahol majd játszanak. Erre azért van szükség, mert egy végtelen táblán Picker mindig kérhet egymástól távoleső pontokat és ez triviális nyerés Pickernek.

8.1.3 Eszköztár

Párosítási lemma

Lemma 8.3 (Cs-P)**.** Ha egy Chooser-Picker játék során (akár már a játék elején) van egy két elemű nyerőhalmaz $\{x, y\}$, akkor Picker-nek van olyan optimális nyerőstratégiája, amely $\{x, y\}$-nal kezdődik.

Monotonitási lemma

Korábban beláttuk, hogy végtelen tábla esetén Choosernek ki kell választania egy korlátos részhalmazát a tálának. Ez a gyakorlatban azt jelenti, hogy Chooser választ egy $X \in V$ részhalmazt, éa játszik az így *indukált rész-hipergráfon*, mely csak azokat az $A \in \mathcal{F}$ éleket tartalmazza, ahol $A \subset X$. Formálisabban: egy adott $(V, \mathcal{F})$ hipergráfra legyen $(V \setminus X, \mathcal{F}(X))$ az a rész-hipergráf, ahol $\mathcal{F}(X) = \{A \in \mathcal{F}, A \cap X = \emptyset\}$.

Lemma 8.4. [21] Ha Picker nyeri a Chooser-Picker játékot $(V, \mathcal{F})$-on, akkor Picker nyeri a $(V \setminus X, \mathcal{F}(X))$ hipergráfon is.

Ez a lemma hasznos lesz a következő fejezeteknél, ugyanis, ha egy korlátos S halmazt nem tudunk egyforma részekre feldarabolni, akkor megnövelhetjük S'-re, amit már fel lehet darabolni egyenlő részekre. És ha Picker nyer S'-n, akkor S-en is nyerni fog.

8.1.4 Néhány eredmény a Chosser-Picker játékokról

A Chooser-Picker játékok komplexitása

Miután a Maker-Breaker játékok (és a Maker-Maker) játékok PSPACE-teljesek, lásd [65], ezért mind a(z) 2.9 sejtés, mind a fenti heurisztika alapján a Picker-Chooser és a Chooser-Picker játékok sem ígérkeznek könyebbnek. Játékok PSPACE-teljességének beláttása többé-kevésbbé standard lásd [65, 64, 16]. Most mi ennél kevesebbet mutatunk be a vizsgált játékok asszimetrikus természete miatt.

Tétel 8.1. A Picker-Chooser játékoknál NP-nehéz eldönteni, hogy ki nyer.

Tétel 8.2. A Chooser-Picker játékoknál NP-nehéz eldönteni, hogy ki nyer.

Mindkét bizonyításban a $3 - \mathrm{SAT}$-ot vezetjük vissza Chooser-Picker, illetve Picker-Chooser játékokra.

Fontos megjegyezni, hogy a Chooser-Picker játékok NP-nehezek még azokra a (V, E) hipergráfokra is, ahol $|A| \leq 6$ minden $A \in E$.

4×4 **tic-tac-toe**

Állítás 8.5. Picker nyeri a Chooser-Picker 4×4 tic-tac-toe játékot.

Az általánosított Shannon-féle kapcsolójáték Picker-Chooser változata

Beláttuk a(z) 2.9 sejtést az általánosított Shannon-féle kapcsolójáték Picker-Chooser változatára, hasonlóan ahhoz, ahogy Lehman tette [46]. Legyen $(V, \mathcal{F})$ egy matroid, ahol $\mathcal{F}$ a bázisok halmaza, és Picker nyer ha elfoglal egy $A \in \mathcal{F}$ elemet. Jegyezzük meg, hogy ez ekvivalens egy $(V, \mathcal{C})$-on játszott Chooser-Picker játékkal a, ahol $\mathcal{C}$ a $(V, \mathcal{F})$ matroidból kivágot halmazok egy gyűjteménye minden $A \in \mathcal{F}$ és $B \in \mathcal{C}$, $A \cap B \neq \emptyset$-re.

Tétel 8.3. Legyen $\mathcal{F}$ a bázisok egy gyűjteménye a V csúcshalmazon értelmezett matroidnak. Picker akkor és csak akkor nyeri meg a játékot $(V, \mathcal{F})$-en, ha van olyan $A, B \in \mathcal{F}$, hogy $A \cap B = \emptyset$.

A bizonyítás Oxley [54] írásában található Maker-Breaker eset bizonyításához hasonló.

Erdős-Selfridge típusú tételek P-C és C-P játékokra

Maker-Breaker $(V, \mathcal{F})$ játék esetén az Erdős-Selfridge tétel [27] nagyon jól használható kritériumot fogalmaz meg Breaker nyerésére. A Chooser-Picker $(V, \mathcal{F})$ játék esetében Beck [6], jóval erősebb feltételt használva, bebizonyította Picker nyerését. (A P-C változatra éles eredményt bizonyított, melyet szintén belefoglaltunk az alábbi tételbe) Legyen $||\mathcal{F}|| = \max_{A\in\mathcal{F}} |A|$ a $(V, \mathcal{F})$ hipergráf rangja.

Tétel 8.4. [6] A $(V, \mathcal{F})$ hipergráfon játszott Chooser-Picker játékban, ha

$$T(\mathcal{F}) := \sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{8(||\mathcal{F}||+1)}, \tag{8.1}$$

akkor Pickernek van explicit nyerő stratégiája.

Ha $T(\mathcal{F}) < 1$, akkor Chooser nyeri a Picker-Chooser játékot a $(V, \mathcal{F})$ hipergráfon.

Ezt az eredmény megjavítottuk avval, hogy beláttuk a következőt:

Tétel 8.5. A $(V, \mathcal{F})$ hipergráfon játszott Chooser-Picker játékban, ha

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{3\sqrt{||\mathcal{F}|| + \frac{1}{2}}}, \tag{8.2}$$

akkor Pickernek van explicit nyerő stratégiája.

Érdemes kiemelni egy speciális esetét a(z) 2.9 sejtésnek és az Erdős-Selfridge tételnek a Chooser-Picker játékokra.

Sejtés 8.6. Ha

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

akkor Picker nyeri a Chooser-Picker játékot a $(V, \mathcal{F})$ hipergráfon.

Tórusz játékok

Beck paradigmáját leellenőriztük a 4×4-es tóruszon definiálható játékokon. A tóruszt a továbbiakban 4^2-nek jelöljük. Itt összeragasztjuk a négyzetháló szemközti oldalait és az 0 és ± 1 meredekségű vonalakból álló halmazokat tekintjük nyerőhalmazoknak. A tórusz játékok általános definícija megtalálható [7]-ben. A cellákra továbbiakban úgy hivatkozunk, mint ahogyan a sakkban szokták. A 4^2 tórusz hipergráfjában az élek többször is metszhetik egymást. Például a következő két nyerőhalmaznak két közös eleme is van: $\{a2, b1, c4, d3\}$ és $\{a4, b1, c2, d3\}$. Négy lehetséges játékot definiálunk a 4^2 hipergráfon. Ezek a Maker-Maker, a Maker-Breaker, a Chooser-Picker és a Picker-Chooser változatok. [7]-ől ismert, hogy a Maker-Maker változat döntetlen, a [21] cikkből, hogy Picker nyeri a Chooser-Picker játékot. Valójában, a Maker-Breaker változat eredményéből következik a Maker-Maker változaté is, valamint a Chooser-Picker bizonyítása is.

Állítás 8.7. Breaker nyeri a Maker-Breaker változatás a 4^2 tórusz játéknak.

Beck sejtésével (2.9) összhangban, Breakernek könnyebb dolga van a Maker-Breaker változatban, mint Choosernek a Picker-Chooser változatban. Bár a 4×4 tóruszon a kimenetele ugyanaz mindkét játéknak, ez utóbbit mégis sokkal nehezebb bizonyítani.

Állítás 8.8. Chooser nyeri a Picker-Chooser változatát a 4×4 tórusz játéknak.

Bizonyítás (vázlat) A teljes bizonyításhoz egy hosszú esetvizsgálatra van szükség. Noha néhány ágát a teljes játékfának le lehet vágni Beck egyik eredménye alapján [6]: Chooser nyeri a Picker-Chooser játékot a $\mathcal{H}$ halmazon, ha $T(\mathcal{H}) := \sum_{A \in E(\mathcal{H})} 2^{-|A|} < 1$.

Fontos megjegyezni, hogy fent láthattunk egy rendezést a vizsgált játékváltozatok komplexitására: könnyebb a C-P játék eredményét, mint az M-B játékét megkapni, habár (a fenti esetben legalábbis) ugyanazt adják. és sokkal nehezebb a P-C esetet meghatározni, mint a Maker-Breaker változatét.

8.2 A Chooser-Picker 7-amőba

8.2.1 A k-amőba játék

A k-amőba olyan hipergráf játék, ahol a gráf csúcsai egy végtelen négyzetrács ($\mathbb{Z}^2$) mezőinek feleltethetők meg, illetve a nyerőhalmazok k darab egymás utáni cellának (vízszintes, függőleges, vagy átlós) felelnek meg. Ha az egyik játékos megszerez egy k hosszú vonalat, akkor nyer - máskülönben a játék döntetlen. Jegyezzük meg, hogy tökéletes játékot feltételezve vagy az első játékos nyer, vagy a játék döntetlen John Nash stratégia lopásos érvelését alkalmazva [13]. További részletek a k-amőba játékról a [61, 62]-ben találhatók.

A k-amőbának mind a Maker-Maker, mind a Maker-Breaker változata $k = 6, 7$-re nyitott kérdés. Mindenki azt gondolja, hogy ezen játékok döntetlenek (Breaker nyer), de a sok erőfeszítés ellenére jelentős eredményt eddig nem ért el senki.

8.2.2 A C-P k-amőba játék

Mielőtt bebizonyítottuk a C-P 7-amőbára vonatkozó zételt, igazoltuk hogy Picker nyeri a könyebb C-P 8-amőbá játékot - ehhez a 12 mezőből álló, Zetters által alkalmazott (lásd [34]) "Z" alakú résztáblát használtunk fel.

Állítás 8.9. Picker nyeri a 8-amőba játék Chooser-Picker változatát, bármely $B \subseteq \mathbb{Z}^2$ halmazon.

Tétel 8.6. Picker nyeri a 7-amőba játék Chooser-Picker változatát, bármely A részhalmazán $\mathbb{Z}^2$-nek.

A korábban már említett 2.18 lemmát alkalmazva, Chooser elősször kiválaszt egy véges S halmazt. Tekinkjük az egész sík felbontását résztáblákra és ezeken játszunk különkülön segédjátékot. Könnyű belátni, hogy ha Picker megnyeri az összes segédjátékot, akkor Picker nyer minden olyan K táblán játszott játékot, ahol K ezen segédtáblák uniójaként áll össze. A 2.18 lemmából következik, hogy Picker nyer $S \subset K$ -en is. Egy megfelelő segédjátékokra történő felbontást kellett találnunk. A felbontás garantálja, hogy ha Picker nyer minden részjátékban, akkor Chooser nem tud hét egymás utáni cellát elfoglalni K-n.

Minden résztábla egy 4×8-as méretű téglalap, ahol a nyerőhalmazokat (a könnyebb megértés kedvéért) négy különböző táblán ábrázoltuk:

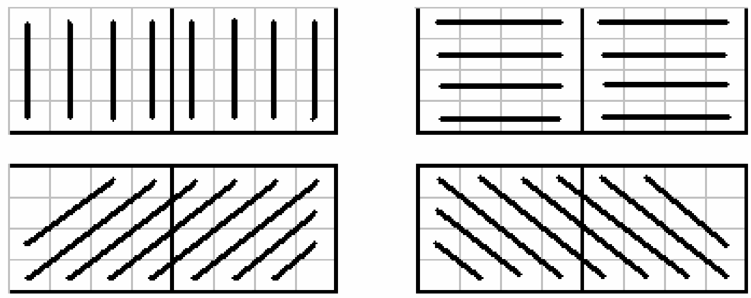

Figure 8.1: Ezek a 4×8as téglalap nyerőhalmazai. Könnyű látni, hogy pontosan egy szimmetria van benne (a dupla vonal mentén). Ezt a bizonyításban hasznosítjuk.

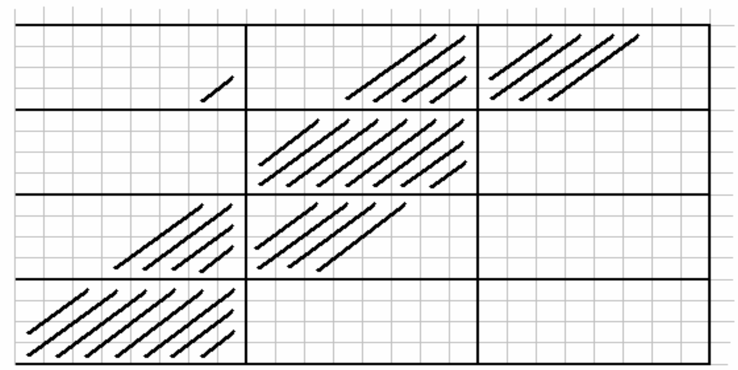

Figure 8.2: Láthatjuk, hogy hogyan következik a segédtáblákon történő játékból a döntetlen az egész táblára: sem vizszintesen, sem függőlegesen, sem átlósan (most csak egy átlós irányt részleteztünk), nincsen egymásutáni hét cella úgy, hogy ne tartalmazza egy nyerőhalmazát valamelyik segédjátáknak.

Tehát a kulcs-lemma a bizonyításunkhoz a következő.

Lemma 8.10. Picker nyeri a 4×8-as táblán definiált segédjátékot.

Megjegyzés 8.11. A M-B esetre “brute-force" számítógépes vizsgálattal megnéztük ugyanezt a segédtáblát, de az Maker nyerést adott! Tehát mi nem haszhatjuk ugyanazt a táblát, hogy belássuk, hogy a játék gyenge változatát Breaker nyeri a 7-amőbára. Természetes gondolat, hogy akkor keressünk más segédjátékokat, de ez nem igérkezik könnyű vállalkozásnak. Mindenesetre ökölszabályként érdemes először mindig a C-P esetet megvizsgálni.

8.3 A Picker-Chooser átmérőjáték

8.3.1 Gráf játékok

Számos Maker-Breaker játék van definiálva az n csúcsú teljes gráfon. A játékosok felváltva foglalnak el éleket; Maker akkor nyer, ha a részgráfjára teljesül egy előre meghatározott $\mathcal{P}$ (gyakran monoton) tulajdonság, lásd [8, 5, 12, 17]. Balogh és társai [2] bevezették az $(a : b)$ d-átmérő játékot, röviden $\mathcal{D}_d(a : b)$-t, ahol Maker pontosan akkor nyer, ha a részgráfjának az átmérője legfeljebb d. A [2] cikk legmeglepőbb eredménye az volt, hogy noha Maker elveszti a $\mathcal{D}_2(1 : 1)$ játékot, de Maker megnyeri a $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\log n)^{3/8})$ játékot.

Ez azt jelenti, hogy a játék felgyorsítása drámaian megváltoztathatja a játék kimenetelét, [61]. A végeredmény szintén sokat módosul, amikor ugyanezen játék Picker-Chooser változatát vesszük górcső alá. Fő eredményünk a következő megfigyelés, illetve az azt követő tétel:

Megfigyelés. Picker nyeri a P-C $\mathcal{D}_2(1 : 1)$ játékot K_n-en, ha $n > 22$.

Tétel 8.7. A Chooser-Picker $\mathcal{D}_2(1 : b)$ játékot Picker nyeri, ha $b < \sqrt{n/\log_2 n}/4$, míg Chooser nyer, ha $b > 3\sqrt{n}$, ha n elég nagy.

A Chooser-Picker játékok önmagukban heurisztikái a Maker-Breaker játékoknak. Ahogyan a(z) 3.6 tétel mutatja, a Maker-Breaker és a Chooser-Picker játékok nyerési feltételei gyakran egybeesnek. Ráadásul Breaker nyerése a Maker-Breaker játékban és Chooser nyerése a Picker-Chooser játékban gyakran ugyanakkor teljesül , lásd [6]. Hogy tovább vizsgálhassuk ezt a kapcsolatot, szükségünk volt a(z) 3.6 tétel elfogult változatára is. Nem kíséreltük meg a legjobb alakot leírni, a céljainkhoz elég a következő lemma.

Lemma 8.12. Picker nyeri a Chooser-Picker $(1 : b)$ elfogult játékot a $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ hipergráfon, ha

$$\frac{v}{b+1} \sum_{A \in E(\mathcal{H})} 2^{-|A|/b} < 1,$$

ahol $v = |V(\mathcal{H})|$.

Átmérő és fokszám játékok

Balogh és társai a [2] cikkben észrevették, hogy a $\mathcal{D}_2(1 : 1)$ nem esik egybe a valószínűségszámítási intuíciónkkal: Ugyanis, ha a a K_n gráf élei véletlenszerűen kerülnek Makerhez és Breakerhez, akkor majdnem biztosan 2 lesz a gráf átmérője; míg Breakernek van egy egyszerű párosítási stratégiája, amivel $n > 3$, [2]. Először vesznek egy olyan uv élt, ahol semelyik ux vagy vx élt nem foglalta el Maker; majd ha Maker elfoglal egy ux élt, akkor Breaker a vx élt foglalja el (ha vx-et már korábban elfoglalta, akkor egy tetszőleges élt választ), illetve fordítva .

A $\mathcal{D}_2(2 : 2)$ játékot játszva nincsen ilyen párosítási stratégiája Breakernek, és Maker nyeri a játékot, sőt a $\mathcal{D}_2(2 : b)$ játékot is, ahol b polimonikusan nő n-nel, ha n elég nagy:

Tétel 8.8. [2] Maker nyeri a $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\ln n)^{3/8})$ játékot. és Breaker nyeri a $\mathcal{D}_2(2 : (2+\epsilon)\sqrt{n/\ln n})$ játékot, minden $\epsilon > 0$-ra, amennyiben n elég nagy.

A 5.1 Tétel bizonyításához, szükséges a *fokszám játékok* ismerete. Székely, Beck, Balogh és társai [73, 5, 2] megmutatták, hogy ezen játékok önmagukban is érdekesek.

Ezeknél a játékoknál az egyik játékos próbál éleket minnél egyenletesebben elfoglalni, míg a másik célja hogy minél több élet foglaljon el valamelyik csúcsnál. Egy adott G gráfnál és egy előre megadott d fokszámnál, Maker és Breaker egy (a, b) elfogult játékot játszanak G élein. Maker akkor nyer, ha legalább d éle van minden csúcsnál.

Minket a $G = K_n$ eset érdekel. Balogh és társai [2] belátták a következő lemmát:

Lemma 8.13. [2] Legyen $a \leq n/(4 \ln n)$ és n elég nagy. Maker nyeri az $(a : b)$ fokszámjátékot K_n-en ha $d < \frac{a}{a+b}n - \frac{6ab}{(a+b)^{3/2}}\sqrt{n \ln n}$.

Nem akarjuk a teljes P-C (C-P) fokszám játék elméletet felépíteni, csak egy egyszerű állítást mondunk ki, mely céljainknak megfelel.

Lemma 8.14. Legyen $b < n/(8 \ln n)$ és n elég nagy. Chooser nyeri az $(1 : b)$ Chooser-Picker fokszámjátékot K_n-en, ha $d < n - 1 - 3n/b$.

A 5.1 tétel bizonyításához, legelőször a 3.9 lemmát láttuk be.

A 5.4 tétel második felét láttuk be először, vagyis, hogy Chooser nyer, ha $b > 3\sqrt{n}$, ami a 5.4 lemmából jön. Játszon Chooser a lemma szerint, akkor Pickernek legfeljebb $(3n/b) - 1$ éle lesz bármelyik csúcsot is nézzük $x \in K_n$-re, tehát a csúcsok száma, mely x-hez van kapcsolva (2-átmérőnyire) kevesebb, mint $((3n/b) - 1)^2 < n - 1$.

A tétel első felének belátásához több munka kellett. Felbontjuk a gráf csúcsait három körülbelül azonos méretű részre: X_1, X_2 és X_3. (Továbbiakban legyen $X_i = X_{i \mod 3}$, ha $i > 3$.) Az X_i csúcsai legyenek rendre $1, 2, \ldots, n/3$. [1] $E(X_i, X_j)$ legyen az élek halmaza X_i és X_j között.

Két külön játékot játszunk az egyes részeken belüli, illetve az egyes részek közötti összekötés érdekében. Az első játákban összekötjük az X_i-n belüli pontokat az $E(X_i, X_{i+1})$ éleket használva ($i = 1, 2, 3$-ra). A második játékban összekötjük X_i-et X_{i+1}-vel az X_{i+1} élein játszva.

[1]Ez lehet $\lfloor n/3 \rfloor$ és $\lceil n/3 \rceil$ is. A bizonyításban mi a $\lceil n/3 \rceil$-vel számoltunk és a $\lfloor n/3 \rfloor$ eset könnyen jön ebből.