Quasiorder lattices of universal algebras

Summary of Ph.D. Thesis

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1. Introduction

Modern algebra's most essential concept is arguably the congruence relation. The study of any algebraic structure having a nontrivial congruence can be started by studying its typically simpler factor. Moreover, for the classical algebraic structures the core of the congruence is determined by a subalgebra, again, typically a simpler structure. The natural structure formed by the congruences of an algebra is a lattice (or rather, in the case of infinitely many congruences, an *algebraic lattice*.

On the other hand, certain classes of structures–notably, lattices themselves–have congruences that are not determined by subalgebras (or any particular congruence classes). Still, the congruences of these structures are of great importance, but for lattices and semilattices in particular, there is an other kind of binary relation that is most closely associated with the structure: the so-called natural order. There is no general definition for what makes an order natural (except by untrustworthy ideologues), but a common requirement seems to be that the order should be compatible with the operations of the algebra. A notable exception is the natural order on regular semigroups (see Chapter 6 of [7]), where the relation is not generally compatible with the semigroup multiplication.

The obvious common generalization of congruences (compatible, reflexive, transitive, symmerical binary relations) and compatible orders (the same, with antisymmetry instead of symmetry) are quasiorders (the same, without symmetry). The quasiorders of an algebra $\mathbf{A}$ (unlike the compatible orders) form a lattice denoted by $\mathrm{Quo}\,\mathbf{A}$, which contains the congruence lattice (denoted by $\mathrm{Con}\,\mathbf{A}$ as a sublattice.

The goal of the thesis is the study of quasiorder lattices, and preeminently their connection to congruence lattices. Usually we will work with finite algebras, or at least algebras in locally finite varieties. The thesis is based on the papers [9, 10, 12] and the draft [11], and uses the paper [13].

Quo $\mathbf{A}$ can be naturally considered as an *involutive lattice* with the involution $\delta \mapsto \delta^{-1}$, where δ^{-1} is defined by

$$(a, b) \in \delta^{-1} \Leftrightarrow (b, a) \in \delta.$$

While this natural involution will be used numerously, it should be noted that the thesis studies quasiorder lattices as lattices, not as involutive lattices.

For any $\delta \in \operatorname{Quo}\mathbf{A}$ there correspond two equivalences: $\delta^* := \delta \wedge \delta^{-1}$ and $\delta \vee \delta^{-1}$. There is also a poset that naturally corresponds to δ: the factor of δ by δ^*. This is a poset with underlying set A/δ^*, with $(u, v) \in \delta/\delta^*$ iff $(a, b) \in \delta$ for any–or equivalently, all–a and b satisfying $a/\delta^* = u$ and $b/\delta^* = v$.

2. Lower bounded lattices

For the purpose of this summary, we will only give one of the equivalent definitions for lower boundedness of a lattice (see [4] for more). An element l of a lattice $\mathbf{L}$ is *join irreducible* if there are no elements $l_1, l_2 < l$ such that $l_1 \vee l_2 = l$. It is *completely join irreducible* if either it is the smallest element of the lattice, or there is a largest element l^* among all the elements of the lattice smaller then l. If an element is completely join irreducible, then it is also join irreducible, and for finite lattices the converse is also true. The element is *join prime* if for all l_1, l_2 satisfying $l_1 \vee l_2 \geq l$ either $l_1 \geq l$ or $l_2 \geq l$. Join primes are join irreducibles. For distributive lattices, the converse is also true.

D is a binary relation on the set of join irreducible elements of $\mathbf{L}$. It is defined by

$$aDb \Leftrightarrow a \neq b,\ (\exists c:\ a \leq b \vee c,\ \forall d < b : a \not\leq d \vee c).$$

If b is completely join irreducible, this is simplified into

$$aDb \Leftrightarrow a \neq b,\ (\exists c:\ a \leq b \vee c,\ a \not\leq b^* \vee c).$$

A lattice is *lower bounded* if it is finitely generated and the graph induced by the relation D does not contain an infinite (directed) path. It is *upper bounded* if its dual is lower bounden, and *bounded* if it is both upper and lower bounded.

An important fact is that any lower bounded lattice is *join semidistributive*, that is, it satisfies the quasi-identity

$$x \vee y = x \vee z \;\rightarrow\; x \vee y = x \vee (y \wedge z).$$

Upper bounded lattices of course satisfy the dual condition, meet semidistributivity.

3. Suborder lattices of DCC posets

As we have seen, a quasiorder can be factored into a symmetrical and an antisymmetrical part (δ^* and δ/δ^*). We will first study parts of the quasiorder lattice that only contain compatible orders. These are "lattices of posets", i.e. they are sublattices of a *suborder lattice* of a poset.

Achein in [1] proved that *any* lattice is isomorphic to a lattice of posets. However, even if the lattice is finite, the underlying set of the posets may need to be infinite. Sivak in [18] gave a characterization for a lattice to be isomorphic to a lattice of posets on a finite set (in other words, for a lattice to be embeddable into a suborder lattice of a finite poset). In [2], the authors note that this characterization precisely describe the class of finite lower bounded lattices.

Semenova in [17] proves something more general: any finite sublattice of a suborder lattice of a poset containing no infinite chain must be lower bounded. Therefore relaxing the finiteness condition (from "the poset is finite" to "it contains no infinite chain") does not make more finite lattices embeddable. (Infinite lattices are an other matter, of course: it is very easy to construct a poset without infinite chain having an infinite suborder lattice: for example, take the disjoint sum of infinitely many copies of the two-element chain).

On the other hand, if we further relax the finiteness condition, from "the poset contains no infinite chain" to "the poset contains no infinite *descending* chain" (in other words, to "the poset being a DCC poset"), then more finite lattices will be embeddable. This is a consequence of the following theorem:

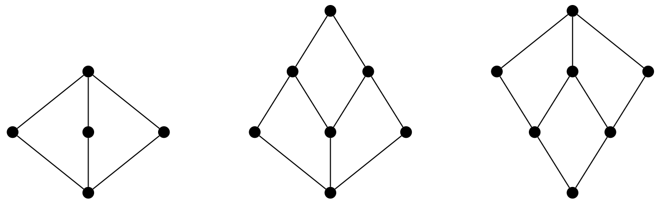

FIGURE 1. The lattices $\mathbf{M}_3$, $\mathbf{D}_1$, and $\mathbf{D}_2$

Theorem 1. *Let* $\mathbf{L}$ *be a finite lattice. Denote with* $\mathcal{C}_\mathbf{L}$ *the set of nontrivial join covers of join irreducibles, i.e. the set*

$$\{(l, l_1, \dots, l_k) \in J(L)^{k+1} : l \leq l_1 \vee l_2 \vee \cdots \vee l_k,$$
$$l \not\leq (l_1)^* \vee l_2 \vee \cdots \vee l_k,\ l \not\leq l_1 \vee (l_2)^* \vee \cdots \vee l_k, \dots, l \not\leq l_1 \vee l_2 \vee \cdots \vee (l_k)^*\}.$$

$\mathbf{L}$ *is embeddable into the suborder lattice of a DCC poset if and only if there is a mapping* $s : \mathcal{C}_\mathbf{L} \mapsto L$ *satisfying the following:*

- *for any* $(l, l_1, \dots, l_k) \in \mathcal{C}_\mathbf{L}$, $s(l, l_1, \dots, l_k) \in \{l_1, \dots, l_k\}$,
- s *is symmetrical in all but the first variable, i.e. for any permutation* $\pi \in S_k$,
$$s(l, l_1, \dots, l_k) = s(l, l_{\pi(1)}, \dots, l_{\pi(k)}),$$
- *for the binary relations*
$$T_\mathbf{L} := \{(l, l_i) : (l, l_1, \dots, l_k) \in \mathcal{C}_\mathbf{L},\ s(l, l_1, \dots, l_k) \neq l_i\}$$
and
$$U_\mathbf{L} := \mathrm{Tr}(\{(l, l) : l \in L\} \cup \{(l, l_i) : (l, l_1, \dots, l_k) \in \mathcal{C}_\mathbf{L},\ s(l, l_1, \dots, l_k) = l_i\}),$$
the relation $U_\mathbf{L} \circ T_\mathbf{L}$ *does not contain a circle.*

This theorem gives an algorithm deciding whether a finite $\mathbf{L}$ is a suborder lattice of a DCC poset. The algorithm is in $\mathcal{EXPTIME}$.

For example, it is straightforward to check that $\mathbf{M}_3$ in not embeddable, but $\mathbf{D}_1$ and $\mathbf{D}_2$ are. While $\mathbf{D}_1$ is a lower bounded lattice, so it is embeddable even into the suborder lattice of a finite poset, $\mathbf{D}_2$ is not even join semidistributive. This shows that the class of lattices embeddable into suborder lattices of DCC posets is strictly larger than the class of lattices embeddable into suborder lattices of finite posets, but still it does not contain all finite lattices.

For a lattice $\mathbf{L}$, denote with $\mathcal{CY}_{\mathbf{L}}$ the set of cycles of the D relation restricted to the *completely* join irreducible elements of $\mathbf{L}$. Introduce a binary relation on $\mathcal{CY}_{\mathbf{L}}$:

$$\begin{aligned} E_{\mathbf{L}} := \{((\beta_1, \dots, \beta_l), (\alpha_1, \dots, \alpha_k)) :& \\ \exists i : \exists j :\ \alpha_{j+1} \leq \beta_i \vee \alpha_j,\ & \alpha_{j+1} \not\leq \beta_i \vee \alpha_j^*,\ \alpha_{j+1} \not\leq \beta_i^* \vee \alpha_j\}, \end{aligned}$$

with the index j meant as modulo k and the index i as modulo l.

Theorem 2. *If $\mathbf{L}$ is embeddable into the suborder lattice of a DCC poset, then $E_{\mathbf{L}}$ does not contain a cycle.*

4. Quasiorder lattices of semilattices

The class of congruence lattices of semilattices is quite rich, satisfying no nontrivial lattice identities [6]. The class of quasiorder lattices of semilattices is even richer: while the congruence lattices are meet semidistributive, this is not the case for the quasiorder lattices.

Theorem 3. *The quasiorder lattice of* $\mathrm{FS}(3)$ *(the 3-generated free semilattice) is not a meet semidistributive lattice.*

A consequence of congruence meet semidistributivity is that the congruence lattices of semilattices do not contain lattices isomorphic to $\mathbf{M}_3$. Contrary to meet semidistributivity itself, this property translates to quasiorder lattices of semilattices–with the condition that the semilattice is finite.

Theorem 4. *If the finite algebra $\mathbf{A}$ generates a congruence meet semidistributive variety, and $\mathbf{M}$ is a simple sublattice of* $\operatorname{Quo}\mathbf{A}$*, then*

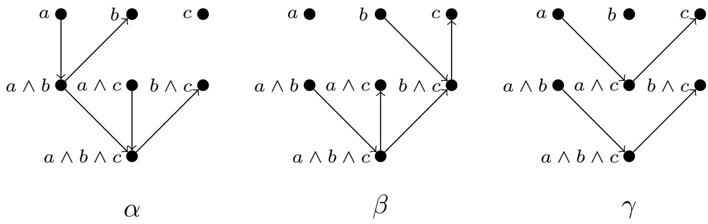

FIGURE 2. The quasiorders α, β, γ of FS(3) satisfying $\alpha \wedge \gamma = \beta \wedge \gamma < (\alpha \vee \beta) \wedge \gamma$

M *is either trivial, or it is the two-element lattice. Consequently, if* **S** *is a finite semilattice, then* Quo **S** *does not contain a sublattice isomorphic to* $\mathbf{M}_3$.

Even the second statement of this theorem needs the finiteness condition. To see this, first take a lattice of posets isomorphic to $\mathbf{M}_3$. Figure 3 shows a way to do that ($\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\gamma}_3$ are the middle elements of this lattice). If C is the underlying set of these posets, take the free semilattice FS(C). Denote the quasiorders of FS(C) generated by $\overline{\gamma}_i$ with $\gamma_i^{(0)}$.

The pairwise join of the $\gamma_i^{(0)}$ coincide, but their pairwise meets do not. So we set recursively for $k > 0$

$$\gamma_i^{(k)} = \gamma_i^{(0)} \vee (\gamma_{i-1}^{(k-1)} \wedge \gamma_{i+1}^{(k-1)}),$$

and take $\gamma_i = \bigcup_k \gamma_i^{(k)}$. Effectively if an edge is in two of the quasiorders, then we put it in the third. Thus, the pairwise meet of the γ_i will coincide. While we put in these new edges, the pairwise join does not change, so those will also coincide.

It can be shown that $\gamma_1 \neq \gamma_2$. This gives us the following theorem.

Theorem 5. Quo(FS(ω)) *contains a sublattice isomorphic to* $\mathbf{M}_3$.

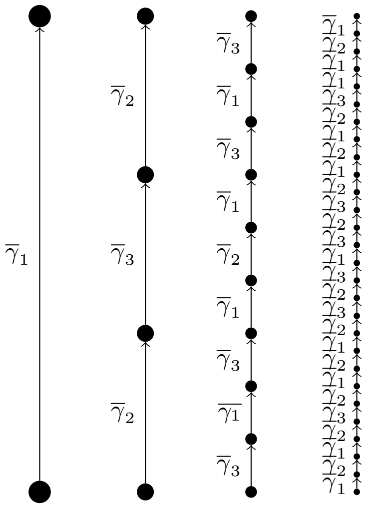

Figure 3. Three posets generating a lattice isomorphic to $\mathbf{M}_3$

5. Tame congruence theory

To establish connections between the congruence and quasiorder lattices, we will mostly use tame congruence theory (see [15]). Many, though not all, parts of it are applicable to quasiorders.

Let $\mathbf{A}$ be a finite algebra, and suppose that $\alpha \prec \beta$ holds either in $\operatorname{Con}\mathbf{A}$ or $\operatorname{Quo}\mathbf{A}$. $\mathbf{A}$ is (α, β)-*minimal* if any non-bijective unary polynomial of $\mathbf{A}$ maps any β-edge into an α-edge. Even if $\mathbf{A}$ is not (α, β)-minimal, it has an (essentially) unique corresponding (α, β)-minimal algebra.

In the congruence case, to each (α, β)-minimal algebra it corresponds a *minimal* algebra, that is, an algebra whose non-bijective polynomials are constants. By Pálfy's theorem [16], the polynomial clones of minimal algebras are isomorphic to the polynomial clone of one of the following:

(1) a unary algebra,
(2) a vector space,
(3) a two-element Boole-algebra,
(4) a two-element lattice,
(5) a two-element semilattice.

The *type* of (α, β) is a number between 1 and 5, depending on which of the above categories the corresponding minimal algebra falls into.

In the quasiorder case, there is no corresponding minimal algebra for an (α, β)-minimal algebra, but it is still possible to define the type of (α, β). We do this by reduction to congruence types.

Definition 6. Suppose that $\mathbf{A}$ is a finite (α, β)-minimal algebra, with $\alpha \prec \beta$ in $\operatorname{Quo} \mathbf{A}$.

- If $\alpha^* \prec \beta^*$ in $\operatorname{Con} \mathbf{A}$, then the quasiorder type of (α, β) will be the same as the congruence type of (α^*, β^*).
- If $\alpha^* \neq \beta^*$, but $\alpha^* \not\prec \beta^*$ in $\operatorname{Con} \mathbf{A}$, then the quasiorder type of (α, β) will be 1.
- If $\alpha^* = \beta^*$, then we take the algebra
$$\mathbf{A}_+ := \{(a, b, c) \in A^3 : (a, b), (b, c) \in \beta\},$$
and for an arbitrary $\delta \in \operatorname{Quo} \mathbf{A}$, we define $\delta_+ \in \operatorname{Con} \mathbf{A}_+$ by
$$\delta_+ := \operatorname{Tr}(\{((a, b, c), (a, b', c)) \in \mathbf{A}_+^2 : (b, b') \in \delta \cup \delta^{-1}\}).$$
Now we consider the types in the interval $[\alpha_+, \beta_+]$ of $\operatorname{Con} \mathbf{A}_+$. If there is a 4 among them, then the quasiorder type of (α, β) will be 4, otherwise, if there is a 5 among them, then the quasiorder type will be 5, and if not, then 1.

Quasiorder types have connections to the so-called pseudo-operations. We use a broader definition for these than usual for the sake of simplicity. A binary polynomial p of an algebra $\mathbf{A}$ is a *pseudo-meet operation for the element* $a \in A$ if it satisfies $p(a, x) = p(x, a) = p(x, x) = x$ for all $x \in A$. It is a *pseudo-meet operation for* (α, β) if there is an edge $(a, b) \in (\beta \backslash \alpha) \cup (\beta \backslash \alpha)^{-1}$ so that it is a pseudo-meet operation for a. Finally, the binary polynomials p and q form a *pseudo-meet–pseudo-join pair for* (α, β) if there is an edge $(a, b) \in \beta \backslash \alpha$ so that p is a pseudo-meet operation for a and q is a pseudo-meet operation for b.

The following theorem shows the connections between quasiorder types and pseudo-operations. It also suggests a way to define types with the aid of pseudo-operations.

Theorem 7. *If* $\mathbf{A}$ *is finite and minimal to* (α, β)*, where* $\alpha \prec \beta$ *in* $\operatorname{Quo} \mathbf{A}$*, then the type of* (α, β) *is*

- *3, iff* $\beta \backslash \alpha$ *is a single double edge, and there is a pseudo-meet–pseudo-join pair for it (this case is only possible if* $\alpha^* \neq \beta^*$*),*
- *4, iff* $\beta \backslash \alpha$ *is a single (directed) edge, and there is a pseudo-meet–pseudo-join pair for it,*
- *5, iff there is a pseudo-meet operation for it, but not a pseudo-meet–pseudo-join pair (and in this case, the pseudo-meet operation is for either the shared target or the shared source of all the* $\beta \backslash \alpha$*-edges),*
- *2, iff* (α^*, β^*) *is a prime congruence quotient of type 2,*
- *1 in any other case.*

There are two important conditions about the types of different covering pairs (in other words, *prime quotients*) of congruence lattices. The first is that if (α, β) and (γ, δ) are *prime perspective*, that is, $\alpha \prec \beta$, $\gamma \prec \delta$, and $\alpha, \beta, \gamma, \delta$ form a sublattice isomorphic to the direct square of the two-element lattice, then the two quotients have the same type. This is also true for quasiorders.

Theorem 8. *Suppose that* (α, β) *and* (γ, δ) *are prime perspective quotients of* $\operatorname{Quo} \mathbf{A}$*. Then the types of* (α, β) *and* (γ, δ) *coincide.*

The second condition is that the *solvability* and *strong solvability* relations on the congruence lattice of a finite algebra are congruences. For congruences μ and ν, (μ, ν) is in the solvability relation iff the interval $[\mu \wedge \nu, \mu \vee \nu]$ contains only types 1 and/or 2, it is in the strong solvability relation if the interval only contains type 1.

This is a condition that does not extend to quasiorder lattices (solvability and strong solvability are defined there analogously). Consider the semigroup $\mathbf{S}$ given by the following multiplication table:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 2 \\ 3 & 0 & 1 & 2 & 3 \end{array}$$

Proposition 9. *There are $\alpha, \beta, \gamma, \delta \in \operatorname{Quo} \mathbf{S}$ such that $\alpha \prec \beta$, $\gamma \prec \delta < \beta \vee \gamma$, but the type of (α, β) is 1 and the type of (γ, δ) is 5. Thus neither solvability nor strong solvability is a congruence on* $\operatorname{Quo} \mathbf{S}$*. Moreover, there is no congruence of* $\operatorname{Quo} \mathbf{S}$ *whose restriction to* $\operatorname{Con} \mathbf{S}$ *is the solvability relation on the congruence lattice.*

However, for a class of algebras the solvability relation is a congruence on the quasiorder lattice of the algebra, and it behaves well.

Theorem 10. *Suppose that $\mathbf{A}$ is a finite algebra in a congruence modular variety. Then the solvability and strong solvability relations on* $\operatorname{Quo} \mathbf{A}$ *coincide, and they are the congruence of* $\operatorname{Quo} \mathbf{A}$ *generated by the congruence solvability relation. Moreover, the factor of* $\operatorname{Quo} \mathbf{A}$ *by the solvability relation is a distributive lattice.*

6. Connections between congruence and quasiorder lattices

Generally speaking, a condition for the congruence lattice of an algebra does not give much information about the algebra, but if it satisfied by all the algebras in the generated variety, it is an other matter. Likewise, when studying connections between congruence and quasiorder lattices, we will usually take a variety into account. This is especially true if we use tame congruence theory, because of the following theorem.

Theorem 11. *For any $i \in \{1, 2, 3, 4, 5\}$ and any variety $\mathcal{V}$, $\mathcal{V}$ omits i for congruences iff it omits i for quasiorders.*

It is a well-known fact that if an algebra is in a congruence permutable variety, or equivalently, it has a Mal'tsev-term (a ternary term satisfying the identities $m(x, x, y) \approx m(y, x, x) \approx y$), then all its quasiorders are congruences. This condition is satisfied by all the classical algebraic structures. Instead of a Mal'tsev-term, it is actually enough for the algebra to have Hagemann-Mitchke terms (see [14]).

Czédli and Szabó showed that for any lattice $\mathbf{L}$, $\operatorname{Quo}\mathbf{L}$ is isomorphic to $(\operatorname{Con}\mathbf{L})^2$ [3]. In [12], the author and his advisor show that in a locally finite congruence distributive/modular variety, the quasiorder lattices are all distributive/modular. In the distributive case, this means that the quasiorder lattice is in the variety generated by the congruence lattice. According to Gumm [8], congruence modularity is a "composition" of congruence permutability and congruence distributivity. This suggests the following theorem:

Theorem 12. *Suppose that* $\mathbf{A}$ *is a finite algebra in a congruence modular variety. Then* $\operatorname{Con}\mathbf{A}$ *and* $\operatorname{Quo}\mathbf{A}$ *satisfy the same lattice identities.*

Corollary 13. *Suppose that* $\mathcal{P}$ *is a lattice identity so that each variety whose congruence lattices satisfy* $\mathcal{P}$ *is congruence modular. Then if all congruence lattices of a locally finite variety satisfy* $\mathcal{P}$*, then so do all the quasiorder lattices of the variety.*

A similar statement is true for join semidistributivity (which is a lattice quasi-identity rather than an identity). Congruence join semidistributivity is characterized for locally finite varieties by the variety omitting 1, 2, and 5, that is, for any $\alpha \prec \beta$ in the congruence lattice of an algebra of the variety, the type of (α, β) is either 3 or 4 (Theorem 9.11 of [15]). Congruence join semidistributivity of a variety implies that the finite algebras have lower bounded congruence lattices [5].

Theorem 14. *Suppose that* $\mathbf{A}$ *is a finite algebra in a congruence join semidistributive variety. Then* $\operatorname{Quo}\mathbf{A}$ *is a lower bounded lattice, in particular, it is join semidistributive.*

As Theorem 3 shows, the analogous statement for meet semidistributivity is not true. Theorem 4 shows that a weaker connection exists for that case. Another such weak condition exists for the omitting 1 case (this is equivalent to a congruence condition by Theorem 9.6 of [15]).

Theorem 15. *Suppose that* $\mathbf{A}$ *is a finite algebra in a variety omitting 1. Then* $\operatorname{Quo}\mathbf{A}$ *contains no nonmodular simple sublattice.*

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