Double Fourier series, generalized Lipschitz és Zygmund classes

Summary of the PhD Theses

Zoltán Sáfár

SUPERVISOR:
Ferenc Móricz DSc
PROFESSOR EMERITUS

UNIVERSITY OF SZEGED
FACULTY OF SCIENCE AND INFORMATICS
BOLYAI INSTITUTE
DOKTORAL SCHOOL IN MATHEMATICS AND SCIENCE

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Introduction

The formation theory of Fourier series can be made by middle of XVIII century, when the length l of the vibrating string taut at both ends of the shape wanted to set an arbitrary time t, under the assumption that the transverse oscillation out. Daniel Bernoulli in 1750, found that the appropriate unit of time is chosen, the solution is an

$$u(x,t) = \sum_{n=0}^{\infty} c_n(t) \sin \frac{n\pi}{l} x$$

trigonometric series. Although the classical Fourier expansions easiest, and most important case (in the section monotone and continuous functions) Dirichlet has proved a pointwise convergence, the problem is still living, as almost all convergence of classical Fourier series of the quadratically integrable functions proved Carleson only in 1966. It is understandable that even today many mathematicians may be researching this branch of the analysis.

Our goal is the generalization of the results Ferenc Móricz [10], [11], [12], [13] and [14].

The dissertation research examined three main categories of problems. The first area is the term-by-term differentiability one-variable Fourier series, and thus obtained a series of Lipschitz class covers the debt. In the next section we investigated the relationship between the two-varialbe Fourier series coefficients of the appropriate scale and function of generalized Lipschitz and Zygmund classes debt. Finally, we give a necessary and sufficient condition that a Fourier transform of bivariate functions comply with a "classical" Lipschitz and Zygmund condition.

The following four publications of the author's thesis are based on [7], [8], [15] and [17].

The markings and numbers are the same as used in the dissertation.

Differentiation of Fourier series and function classes

In this section we summarize the results of Chapter 2. This theorems generalizes the corresponding theorems proved by Boas [3] and Németh [16].

We assume that $\{c_k\} \subset \mathbb{C}$ is an absolutely convergent series, then the trigonometric series $\sum c_k e^{ikx}$ converges uniformly, and consequently it is the Fourier series of its sum $f(x)$.

In Theorem 2.1.1 we give necessary and sufficient conditions for the magnitude of Fourier coefficients of the function f in order that the function f is $r \geq 1$ times differentiable in $x \in \mathbb{T}$. Furthermore, we also show that the uniform convergence of rth formal derivative series of the Fourier series is equivalent to the continuity of the function $f^{(r)}$ on the torus.

In Theorems 2.1.2 and 2.1.3 we give sufficient conditions of the magnitude of the Fourier coefficients in order that the function $f^{(r)}$ belongs to one of the classes $\mathrm{Lip}(\alpha)$, $\mathrm{Zyg}(1)$ and $\mathrm{lip}(\alpha)$ or $\mathrm{zyg}(1)$ (where $0 < \alpha < 1$).

Finally, in Theorems 2.1.4 and 2.1.5 we show that the above condition is not only sufficient but also necessary in the case $c_k \geq 0$ or $kc_k \geq 0$. So, these conditions are best possible.

Let $\{c_k : k \in \mathbb{Z}\} \subset \mathbb{C}$ be a sequence such that

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty. \tag{1.1}$$

Then the trigonometric series

$$f(x) := \sum_{k\in\mathbb{Z}} c_k e^{ikx} \tag{1.2}$$

converges uniformly on the torus and it is the Fourier series of its sum f.

Definition 1.1.2 (Lipschitz classes). $\mathrm{Lip}(\alpha)$ consists of all functions f for which

$$|\Delta^1 f(x,h)| := |f(x+h) - f(x)| \leq Ch^\alpha =: O(h^\alpha) \quad \text{for all } x \in \mathbb{T} \text{ and } h > 0,$$

where C is a constant depending on f, but not on x and h.

The little Lipschitz class $\mathrm{lip}(\alpha)$ consists of all functions f for which

$$|\Delta^1 f(x,h)| = o(h^\alpha) \quad \text{uniformly in } x.$$

Definition 1.1.3 (Zygmund classes). $\mathrm{Zyg}(\alpha)$ consists of all continuous functions f for which

$$|\Delta^2 f(x,h)| := |f(x+h) - 2f(x) + f(x-h)| = O(h^\alpha) \quad \text{for all } x \in \mathbb{T} \text{ and } h > 0.$$

The little Zygmund class $\mathrm{zyg}(\alpha)$ consists of all continuous functions f for which

$$|\Delta^2 f(x,h)| = o(h^\alpha) \quad \text{uniformly in } x.$$

Theorem 2.1.1. *If for some $r \geq 1$,*

$$\sum_{|k|\geq n} |c_k| = o(n^{-r}),$$

then the r times formally differentiated Fourier series in (1.2) *converges at a particular point $x \in \mathbb{T}$ if and only if f is r times differentiable at x, and in this case we have*

$$f^{(r)}(x) = \sum_{k\in\mathbb{Z}} (ik)^r c_k e^{ikx}. \tag{2.1}$$

Furthermore, the r*th derivative* $f^{(r)}$ *is continuous on* $\mathbb{T}$ *if and only if the series in* (2.1) *converges uniformly on* $\mathbb{T}$.

Theorem 2.1.2. *If for some* $r \geq 1$ *and* $0 < \alpha \leq 1$,

$$\sum_{|k| \geq n} |c_k| = O(n^{-r-\alpha}), \tag{2.2}$$

then f *is* r *times differentiable on* $\mathbb{T}$, $f^{(r)} \in \mathrm{Lip}(\alpha)$ *in case* $0 < \alpha < 1$, *and* $f^{(r)} \in \mathrm{Zyg}(1)$ *in case* $\alpha = 1$.

Theorem 2.1.3. *If for some* $r \geq 1$ *and* $0 < \alpha \leq 1$,

$$\sum_{|k| \geq n} |c_k| = o(n^{-r-\alpha}), \tag{2.3}$$

then f *is* r *times differentiable on* $\mathbb{T}$, $f^{(r)} \in \mathrm{lip}(\alpha)$ *in case* $0 < \alpha < 1$, *and* $f^{(r)} \in \mathrm{zyg}(1)$ *in case* $\alpha = 1$.

Theorem 2.1.4. *Suppose that either* $kc_k \geq 0$ *for all* k *or* $c_k \geq 0$ *for all* k, *and that* f *is* r *times differentiable on* $\mathbb{T}$. *If* $f^{(r)} \in \mathrm{Lip}(\alpha)$ *for some* $0 < \alpha < 1$, *then* (2.2) *holds with this* α; *while if* $f^{(r)} \in \mathrm{Zyg}(1)$, *then* (2.2) *holds with* $\alpha = 1$.

Theorem 2.1.5. *Both statements in Theorem* 2.1.4 *remain valid if* Lip(α) *and* Zyg(1) *are replaced by* lip(α) *and* zyg(1), *respectively, and* (2.2) *is replaced by* (2.3).

Corollary 2.1.1. (i) *If for some* $r \geq 1$,

$$\sum_{|k| \leq n} |k^{r+1} c_k| = O(1), \tag{2.4}$$

then f *is* r *times differentiable on* $\mathbb{T}$ *and* $f^{(r)} \in \mathrm{Lip}(1)$.

(ii) *Suppose that* $k^{r+1} c_k \geq 0$ *for all* k *and that* f *is* r *times differentiable on* $\mathbb{T}$, *where* $r \geq 1$. *If* $f^{(r)} \in \mathrm{Lip}(1)$, *then* (2.4) *holds.*

Double Fourier series and function classes

In this section we summarize the results of Chapter 4. We generalized the single valued theorems of Ferenc Móricz [11] and [12] and we enlarged the theorems of two variables [13]. We assume that the $\{c_{kl}\} \subset \mathbb{C}$ absolute converges, so we can examine the function

$$f(x, y) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} c_{kl} e^{i(kx+ly)}.$$

In Theorem 4.2.1 we give a sufficient condition of the magnitude of Fourier coefficients of a function f to belong to the class $\text{Lip}(w_{\alpha\beta})$, where $0 < \alpha, \beta \leq 1$. This condition is also necessary in some particular cases (see Theorems 4.2.1-4.2.3 below).

The claim of Theorem 4.2.2 relates to the class $\text{Zyg}(w_{\alpha\beta})$. We give a sufficient condition for the Fourier coefficients of f function to ensure that f to belong to an extended Zygmund class, where $0 < \alpha, \beta \leq 2$.

Throughout this chapter, by $\{c_{kl} : (k,l) \in \mathbb{Z}^2\}$ we denote a sequence of complex numbers with the property

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |c_{kl}| < \infty. \tag{3.1}$$

The double trigonometric series

$$f(x,y) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} c_{kl} e^{i(kx+ly)} \tag{3.2}$$

converges uniformly. Consequently, the series in (3.2) is the Fourier series of its sum f, which is continuous on the two-dimensional torus.

We recall that a positive valued, measurable function L defined in $[a, \infty)$ ($a > 0$ arbitrary), is said to be *slowly varying* (in Karamata's sense) if for every $\lambda > 0$,

$$\frac{L(\lambda x)}{L(x)} \to 1, \quad \text{as} \quad x \to \infty.$$

Let L be a two-variable function such that

$$L(x,y) = L_1(x)L_2(y), \quad \text{where} \quad L_k(t) \to \infty \quad \text{and} \quad \frac{L_k(2t)}{L_k(t)} \to 1 \quad \text{as} \quad t \to \infty. \tag{4.1}$$

Definition 3.1.1 (Multiplicative Lipschitz classes). Let $\alpha, \beta > 0$ arbitrary and

$$\Delta^{1,1} f(x,y;h_1,h_2) = f(x+h_1, y+h_2) - f(x+h_1, y) - f(x, y+h_2) + f(x,y). \tag{3.3}$$

The class $\text{Lip}(\alpha, \beta)$ consists of all continuous functions $f(x,y)$ for which

$$|\Delta^{1,1} f(x,y;h_1,h_2)| = O(h_1^\alpha h_2^\beta)$$

for all $x, y \in \mathbb{T}; h_1, h_2 > 0$.

The class $\text{lip}(\alpha, \beta)$ consists of all continuous functions $f(x,y)$ for which

$$\lim_{h_1,h_2\to 0} h_1^{-\alpha} h_2^{-\beta} |\Delta^{1,1} f(x,y;h_1,h_2)| = 0 \quad \text{uniformly in} \quad (x,y) \in \mathbb{T}^2.$$

Definition 3.1.2 (Multiplicative Zygmund classes). Given $\alpha, \beta > 0$ and

$$\Delta^{2,2} f(x,y;h_1,h_2) =$$

$$= f(x+h_1,y+h_2) - f(x+h_1,y-h_2) - f(x-h_1,y+h_2) + f(x-h_1,y-h_2) -$$
$$-2f(x,y+h_2) - 2f(x+h_1,y) - 2f(x,y-h_2) - 2f(x-h_1,y) + 4f(x,y).$$

The class $\mathrm{Zyg}(\alpha,\beta)$ consists of all continuous functions f for which

$$|\Delta^{2,2} f(x,y;h_1,h_2)| = O(h_1^\alpha h_2^\beta)$$

for all $x,y \in \mathbb{T}; h_1,h_2 > 0$.

The class $\mathrm{zyg}(\alpha,\beta)$ consists of all continuous functions $f(x,y)$ for which

$$\lim_{h_1,h_2\to 0} h_1^{-\alpha} h_2^{-\beta} |\Delta^{2,2} f(x,y;h_1,h_2)| = 0 \quad \text{uniformly in} \quad (x,y) \in \mathbb{T}^2.$$

We will use the special series

$$c_{kl} \geq 0 \quad \text{for all} \quad k,l \geq 1, \tag{3.6}$$

and

$$c_{kl} = -c_{-k,l} = -c_{k,-l} = c_{-k,-l} \quad |k|,|l| \geq 1. \tag{3.7}$$

Definition 4.1.1 (Generalized Lipschitz classes). Given $\alpha,\beta > 0$ and a function $L(x,y)$ satisfying condition (4.1), a continuous function f is said to belong to the generalized multiplicative Lipschitz class $\mathrm{Lip}(\alpha,\beta;L)$ if

$$|\Delta^{1,1} f(x,y;h_1,h_2)| = O\Big(h_1^\alpha h_2^\beta L\Big(\frac{1}{h_1},\frac{1}{h_2}\Big)\Big)$$

for all $x,y \in \mathbb{T}; h_1,h_2 > 0$.

Given $\alpha,\beta \geq 0$ and function L satisfying condition (4.1), the function f is said to belong to $\mathrm{Lip}(\alpha,\beta;1/L)$ if

$$|\Delta^{1,1} f(x,y;h_1,h_2)| = O\Big(\frac{h_1^\alpha h_2^\beta}{L(1/h_1,1/h_2)}\Big)$$

for all $x,y \in \mathbb{T}; h_1,h_2 > 0$.

Given $\alpha,\beta \geq 0$, we denote by $W_{\alpha\beta}$ the class of all functions $w_{\alpha\beta} : [0,1]\times[0,1] \to \mathbb{R}_+$ which are nondecreasing in each variable and possess the following properties:

$$w_{\alpha\beta}(0,\delta_2) = w_{\alpha\beta}(\delta_1,0) = 0 \quad \text{for all} \quad \delta_1,\delta_2 \geq 0; \tag{4.2}$$

$$\sup_{0<\delta_1,\delta_2\leq 1} \frac{w_{\alpha\beta}(2\delta_1,\delta_2)}{w_{\alpha\beta}(\delta_1,\delta_2)} =: c_\alpha < \infty, \quad \sup_{0<\delta_1,\delta_2\leq 1} \frac{w_{\alpha\beta}(\delta_1,2\delta_2)}{w_{\alpha\beta}(\delta_1,\delta_2)} =: c_\beta < \infty, \tag{4.3}$$

$$(4.4)\qquad \liminf_{m\to\infty}\frac{w_{\alpha\beta}(2^{-m-1},\delta_2)}{w_{\alpha\beta}(2^{-m},\delta_2)} > 2^{-\alpha'},\ \limsup_{m\to\infty}\frac{w_{\alpha\beta}(2^{-m-1},\delta_2)}{w_{\alpha\beta}(2^{-m},\delta_2)} \le 2^{-\alpha}$$

for all $\alpha' > \alpha$ and $0 < \delta_2 \le 1$;

$$(4.5)\qquad \liminf_{n\to\infty}\frac{w_{\alpha\beta}(\delta_1,2^{-n-1})}{w_{\alpha\beta}(\delta_1,2^{-n})} > 2^{-\beta'},\ \limsup_{n\to\infty}\frac{w_{\alpha\beta}(\delta_1,2^{-n-1})}{w_{\alpha\beta}(\delta_1,2^{-n})} \le 2^{-\beta}$$

for all $\beta' > \beta$ and $0 < \delta_1 \le 1$.

We will define the modulus of continuity and the modulus of smoothness by

$$\omega(f;\delta_1,\delta_2) := \sup_{0<h_j\le\delta_j} ||\Delta^{1,1} f(x,y;h_1,h_2)||$$

and

$$\omega_2(f;\delta_1,\delta_2) := \sup_{0<h_j\le\delta_j} ||\Delta^{2,2} f(x,y;h_1,h_2)||,$$

where $j = 1,2$ and $||.||$ denotes the usual maximum norm.
Remark. Bary és Stečkin [1] introduced another classes of moduli of continuity which are defined by means of a function $\varphi(t) \in \Phi, 0 \le t \le \pi$ with the following four properties:

- φ is continuous on the interval $[0,\pi]$, however this was not used in the proofs of [1],
- φ is non-decreasing,
- $\varphi \ne 0$ for every $0 < t \le \pi$,
- $\varphi \to 0$ as $t \to 0$.

Definition 4.1.2 (Enlarged Lipschitz classes). Let $w_{\alpha\beta} \in W_{\alpha\beta}$ for some $\alpha,\beta \ge 0$. We define the $\mathrm{Lip}(w_{\alpha\beta})$ of continuous functions as follows:

$$\mathrm{Lip}(w_{\alpha\beta}) := \{f : \omega(f;\delta_1,\delta_2) = O(w_{\alpha\beta}(\delta_1,\delta_2))\}.$$

Definition 4.1.3 (Enlarged Zygmund classes).

$$\mathrm{Zyg}(w_{\alpha\beta}) := \{f : \omega_2(f;\delta_1,\delta_2) = O(w_{\alpha\beta}(\delta_1,\delta_2))\}.$$

Fourier series and enlarged Lipschitz and Zygmund classes of functions

Our new results are summarized in the following three theorems.

Theorem 4.2.1. *Let* $w_{\alpha\beta} \in W_{\alpha\beta}$.

(i) *If* $\{c_{kl}\} \subset \mathbb{C}$ *is such that for some* $0 < \alpha, \beta \le 1$ *we have*

$$\sum_{|k|\le m} \sum_{|l|\le n} |klc_{kl}| = O(mnw_{\alpha\beta}(m^{-1}, n^{-1})), \tag{4.8}$$

then (3.1) *is satisfied and* $f \in \mathrm{Lip}(w_{\alpha\beta})$, *where* f *is defined in* (3.2).

(ii) *Conversely, suppose that* $\{c_{kl}\} \subset \mathbb{R}$ *is such that conditions* (3.1), (3.6) *and* (3.7) *satisfied. If* $f \in \mathrm{Lip}(w_{\alpha\beta})$ *for some* $0 \le \alpha, \beta \le 1$, *then condition* (4.8) *is satisfied.*

Theorem 4.2.2. *Let* $w_{\alpha\beta} \in W_{\alpha\beta}$.

(i) *If* $\{c_{kl}\} \subset \mathbb{C}$ *is such that for some* $0 < \alpha, \beta \le 2$ *we have*

$$\sum_{|k|\le m} \sum_{|l|\le n} k^2 l^2 |c_{kl}| = O(m^2 n^2 w_{\alpha\beta}(m^{-1}, n^{-1})), \tag{4.9}$$

then (3.1) *is satisfied and* $f \in \mathrm{Zyg}(w_{\alpha\beta})$, *where* f *is defined in* (3.2).

(ii) *Conversely, suppose that* $\{c_{kl}\} \subset \mathbb{R}$ *is such that condition* (3.1) *is satisfied and*

$$c_{kl} \ge 0 \quad \text{for all} \quad |k|, |l| \ge 1. \tag{4.10}$$

If $f \in \mathrm{Zyg}(w_{\alpha\beta})$ *for some* $0 \le \alpha, \beta \le 2$, *then condition* (4.9) *is satisfied.*

Theorem 4.2.3. *Let* $w_{\alpha\beta} \in W_{\alpha\beta}$.

(i) *If* $\{c_{kl}\} \subset \mathbb{C}$ *is such that for some* $0 \le \alpha, \beta < 1$ *we have*

$$\sum_{|k|\ge m} \sum_{|l|\ge n} |c_{kl}| = O(w_{\alpha\beta}(m^{-1}, n^{-1})), \tag{4.11}$$

then $f \in \mathrm{Lip}(w_{\alpha\beta})$, *where* f *is defined in* (3.2).

(ii) *Conversely, suppose that* $\{c_{kl}\} \subset \mathbb{R}$ *is such that conditions* (3.1) *and* (4.10) *are satisfied. If* $f \in \mathrm{Zyg}(w_{\alpha\beta})$ *for some* $0 \le \alpha, \beta \le 2$, *then condition* (4.11) *is satisfied.*

Remark. In the case $0 < \alpha, \beta < 1$, Part (i) in Theorems 4.2.1 and 4.2.3 are equivalent.

In the case $0 < \alpha, \beta < 2$, Part (ii) in Theorems 4.2.2 and 4.2.3 are equivalent.

Part (ii) in Theorems 4.2.1 and 4.2.3 are not comparable.

Remark. We note that it seems to be likely that our Theorems 4.2.1-4.2.3 in the cases $0 < \alpha, \beta < 1$ can also be obtained using the theorems of Bary and Stečkin [1]. The proofs of the theorems in [15] we estimate the moduli of continuity or smothness directly in terms of the function in question. We investigate the classes $\mathrm{Lip}(w_{\alpha\beta})$ and $\mathrm{Zyg}(w_{\alpha\beta})$ for $0 \le \alpha, \beta \le 1$ and $0 \le \alpha, \beta \le 2$, respectively. Further difference is that in the following lemmas $\{a_{kl}\}$ is an arbitrary series, while the function $\varphi \in \Phi$ is non-decreasing. The exponent γ and δ may be arbitrary real number for which $\gamma > \alpha \ge 0$ and $\delta > \beta \ge 0$. Finally, the key fact in the proofs of our theorems is that Part (i) and Part (ii) are equivalent whenever $\gamma > \alpha > 0$ and $\delta > \beta > 0$.

The following three Lemmas will be of vital importance in the proofs of Theorem 4.2.1, 4.2.2 and 4.2.3:

Lemma 4.4.1. *Let* $\{a_{kl} : k, l = 1, 2, \dots\} \subset \mathbb{R}_+$ *and* $w_{\alpha\beta} \in W_{\alpha\beta}$.
(i) *If for some* $\gamma \ge \alpha > 0$ *and* $\delta \ge \beta > 0$,

$$\sum_{k=1}^{m}\sum_{l=1}^{n} k^{\gamma} l^{\delta} a_{kl} = O(m^{\gamma} n^{\delta} w_{\alpha\beta}(m^{-1}, n^{-1})), \tag{4.12}$$

then $\sum\sum a_{kl} < \infty$ *and*

$$\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} a_{kl} = O(w_{\alpha\beta}(m^{-1}, n^{-1})). \tag{4.13}$$

(ii) *Conversely, if* (4.13) *is satisfied for some* $\gamma > \alpha \ge 0$ *and* $\delta > \beta \ge 0$, *then* (4.12) *is also satisfied.*

Lemma 4.4.2. *Let* $\{a_{kl}\} \subset \mathbb{R}_+$ *and* $w_{\alpha\beta} \in W_{\alpha\beta}$.
(i) *If* (4.12) *is satisfied for some* $\delta \ge \beta > 0$, *while* γ, α *are arbitrary, then*

$$\sum_{k=m}^{\infty}\sum_{l=1}^{n} l^{\delta} a_{kl} = O(n^{\delta} w_{\alpha\beta}(m^{-1}, n^{-1})). \tag{4.29}$$

(ii) *If* (4.13) *is satisfied for some* $\delta > \beta \ge 0$, *while* γ, α *are arbitrary, then* (4.29) *is also satisfied.*

Lemma 4.4.3. *Let* $\{a_{kl}\} \subset \mathbb{R}_+$ *and* $w_{\alpha\beta} \in W_{\alpha\beta}$.
(i) *If* (4.12) *is satisfied for some* $\gamma \ge \alpha > 0$, *while* δ, β *are arbitrary, then*

$$\sum_{k=1}^{m}\sum_{l=n}^{\infty} k^{\gamma} a_{kl} = O(m^{\gamma} w_{\alpha\beta}(m^{-1}, n^{-1})). \tag{4.33}$$

(ii) *If* (4.13) *is satisfied for some* $\gamma > \alpha \ge 0$, *while* δ, β *are arbitrary, then* (4.33) *is also satisfied.*

Fourier series and generalized Lipschitz classes

The following two theorems are special cases of Theorem 4.2.1-4.2.3.

Theorem 4.3.1. *Assume* $\{c_{kl}\} \subset \mathbb{C}$ *with* (3.1), f *is defined in* (3.2) *and* L *satisfies condition* (4.1).

(i) *If for some* $0 < \alpha, \beta \le 1$,

$$\sum_{|k|\le m} \sum_{|l|\le n} |klc_{kl}| = O(m^{1-\alpha} n^{1-\beta} L(m,n)), \tag{4.6}$$

then $f \in \mathrm{Lip}(\alpha, \beta; L)$.

(ii) *Conversely, let* $\{c_{kl}\} \subset \mathbb{R}$ *be a sequence such that conditions* (3.6) *and* (3.7) *hold. If* $f \in \mathrm{Lip}(\alpha, \beta; L)$ *for some* $0 < \alpha, \beta \le 1$, *then* (4.6) *holds.*

Theorem 4.3.2. *Assume* $\{c_{kl}\} \subset \mathbb{C}$, *with* (3.1), f *is defined in* (3.2) *and* L *satisfies condition* (4.1).

(i) *If for some* $0 \le \alpha, \beta < 1$,

$$\sum_{|k|\ge m} \sum_{|l|\ge n} |c_{kl}| = O\Big(\frac{m^{-\alpha} n^{-\beta}}{L(m,n)}\Big), \tag{4.7}$$

then $f \in \mathrm{Lip}(\alpha, \beta; 1/L)$.

(ii) *Conversely, let* $\{c_{kl}\}$ *be a sequence such that conditions* (3.6) *and* (3.7) *hold. If* $f \in \mathrm{Lip}(\alpha, \beta; 1/L)$ *for some* $0 < \alpha, \beta < 1$, *then* (4.7) *holds.*

Problem. It is an open problem whether tha claim in Theorem 4.2.2 (ii) remains valid if $0 < \alpha, \beta < 1$ is replaced by $0 \le \alpha, \beta < 1$.
Remark Theorem 4.2.1 is generalization of Theorem 4.3.1, which is $w_{\alpha\beta}(\delta_1\delta_2) := \delta_1^\alpha \delta_2^\beta L(\frac{1}{\delta_1}, \frac{1}{\delta_2})$ arises easy choice.

Similarly, Theorem 4.2.3 is generalization of Theorem 4.3.2 of the Lip $(w_{\alpha\beta}) \subset \mathrm{Zyg}(w_{\alpha\beta})$ and the relation of $w_{\alpha\beta}(\delta_1\delta_2) := \frac{\delta_1^\alpha \delta_2^\beta}{L(1/\delta_1, 1/\delta_2)}$ selection.

Double Fourier transforms and function classes

In this section we summarize the results of Chapter 6. We consider complex-valued functions $f \in L^1(\mathbb{R}^2)$ and prove sufficient conditions under which the double Fourier transform $\hat{f}$ belongs to one of the multiplicative Lipschitz classes $\mathrm{Lip}(\alpha, \beta)$ for some $0 \le \alpha, \beta \le 1$, or to one of the multiplicative Zygmund classes $\mathrm{Zyg}(\alpha, \beta)$ for some $0 \le \alpha, \beta \le 2$.

In Thorem 6.2.1 we give sufficient conditions under which $\hat{f}$ belongs to the class $\mathrm{Lip}(\alpha, \beta)$, where $0 < \alpha, \beta \le 1$. This condition is also necessary in the case when $xf(x) \ge 0$ for almost every $x \in \mathbb{R}$.

In Theorem 6.2.2 we prove an analogous result in the case of Zygmund classes $\mathrm{Zyg}(\alpha, \beta)$, where $0 < \alpha, \beta \leq 2$.

Theorem 6.2.1. (i) *Suppose $f : \mathbb{R}^2 \to \mathbb{C}$ is such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and there exist some $s_0, t_0 > 0$ such that*

$$f \in L^1(\{(x, y) \in \mathbb{R}^2 : \text{either } |x| > s_0 \text{ and } |y| < t_0, \text{ or } |x| < s_0 \text{ and } |y| > t_0\}). \tag{6.1}$$

If for some $0 < \alpha, \beta \leq 1$,

$$\int_{|x|<s} \int_{|y|<t} |xyf(x, y)| \mathrm{d}x \mathrm{d}y = O(s^{1-\alpha} t^{1-\beta}), \quad s, t > 0, \tag{6.2}$$

then $f \in L^1(\mathbb{R}^2)$ and $\hat{f} \in \mathrm{Lip}(\alpha, \beta)$.

(ii) *Conversely, suppose $f \in L^1(\mathbb{R}^2)$ is such that for almost all $(x, y) \in \mathbb{R}^2_+$ we have*

$$f(x, y) = -f(-x, y) = -f(x, -y) = f(-x, -y) \geq 0; \tag{6.3}$$

in particular, if $f(x, y)$ is odd in each variable. If $\hat{f} \in \mathrm{Lip}(\alpha, \beta)$ for some $0 < \alpha, \beta \leq 1$, then (6.2) is satisfied.

Remark. The condition (6.2) is equvalent to the condition

$$f \in L^1(\{(x, y) \in \mathbb{R}^2 : |x| > s \text{ és } |y| > t\}), \quad s, t > 0. \tag{6.4}$$

Due to the assumption that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, in order to conclude $f \in L^1(\mathbb{R}^2)$ in statement (i) above, we need the fulfillment of condition (6.1).

If there exist some constants $t_0 > 0$ and $\widetilde{C}$ such that

$$\int_{|x|<s} \int_{|y|<t_0} |xf(x, y)| \mathrm{d}x \mathrm{d}y \leq \widetilde{C} s^{1-\alpha}, \quad s > 0, \tag{6.5}$$

then we also have

$$f \in L^1(\{(x, y) \in \mathbb{R}^2 : |x| > s \text{ és } |y| < t_0\}).$$

Analogously, if there exist some constants $s_0 > 0$ and $\widetilde{C}$ such that

$$\int_{|x|<s_0} \int_{|y|<t} |yf(x, y)| \mathrm{d}x \mathrm{d}y \leq \widetilde{C} t^{1-\beta}, \quad t > 0, \tag{6.6}$$

then we also have

$$f \in L^1(\{(x, y) \in \mathbb{R}^2 : |x| < s_0 \text{ és } |y| > t\}).$$

In particular, conditions (6.5) and (6.6) imply the fulfillment of condition (6.1).

Theorem 6.2.2. (i) *Suppose* $f:\mathbb{R}^2\to\mathbb{C}$ *is such that* $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ *and there exist some* $s_0,t_0>0$ *such that condition* (6.1) *is satisfied. If for some* $0<\alpha,\beta\le 2$,

$$\int_{|x|<s}\int_{|y|<t} x^2y^2|f(x,y)|\mathrm{d}x\mathrm{d}y = O(s^{2-\alpha}t^{2-\beta}),\quad s,t>0, \tag{6.7}$$

then $f\in L^1(\mathbb{R}^2)$ *and* $\hat{f}\in \mathrm{Zyg}(\alpha,\beta)$.

(ii) *Conversely, suppose* $f\in L^1(\mathbb{R}^2)$ *is such that* $f(x,y)\ge 0$ *for almost all* $(x,y)\in\mathbb{R}^2$. *If* $\hat{f}\in\mathrm{Zyg}(\alpha,\beta)$ *for some* $0<\alpha,\beta\le 2$, *then condition* (6.7) *holds.*

Remark. Again, condition (6.7) implies only the fulfillment of (6.4). Due the assumption that $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, in order to conclude $f\in L^1(\mathbb{R}^2)$ in statement (i) above, we need the fulfillment of condition (6.1).

If there exist some constants $t_0>0$ and $\widetilde{C}$ such that

$$\int_{|x|<s}\int_{|y|<t_0} x^2|f(x,y)|\mathrm{d}x\mathrm{d}y \le \widetilde{C}s^{2-\alpha},\quad s>0, \tag{6.8}$$

then we also have

$$f\in L^1(\{(x,y)\in\mathbb{R}^2 : |x|>s \text{ és } |y|<t_0\}).$$

Analogously, if there exist some constants $s_0>0$ and $\widetilde{C}$ such that

$$\int_{|x|<s_0}\int_{|y|<t} y^2|f(x,y)|\mathrm{d}x\mathrm{d}y \le \widetilde{C}t^{2-\beta},\quad t>0, \tag{6.9}$$

then we also have

$$f\in L^1(\{(x,y)\in\mathbb{R}^2 : |x|<s_0 \text{ és } |y|>t\}).$$

In particular, conditions (6.8) and (6.9) imply the fulfillment of condition (6.1).

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