Periodic Orbits and Global Dynamics for Delay Differential Equations

Abstract of Ph. D. Thesis

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Introduction

The present thesis analyzes the scalar functional differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

with parameter $\mu > 0$ and nonlinear feedback function f. Both continuously differentiable and nonsmooth, both monotone increasing and monotone decreasing nonlinearities are considered. Such equations appear in artificial neural networks [17].

The goal of this work is to describe the global attractor as thoroughly as possible for special feedback functions. If the global attractor exists, it is the compact, invariant subset of the infinite dimensional phase space $C = C([-1,0],\mathbb{R})$ that attracts all bounded solutions of the equation and therefore determines their asymptotic behavior. The investigation of the global attractor includes the study of equilibria, identification of the exact number and the stability properties of periodic orbits, and, if possible, characterization of the so-called connecting orbits. The importance of these type of results is confirmed by the Poincaré−Bendixson theorem of Mallet-Paret and Sell [11] for strictly monotone nonlinearities, which states that the omega limit set of all bounded solutions is either a single periodic orbit or a set of certain equilibrium points and connecting orbits among them.

A particular purpose of the thesis is to show that the existence of periodic orbits for equations with continuously differentiable nonlinearities can be verified by considering step feedback functions first, and then by using perturbation theorems. Equations with step feedback functions are easy to handle as certain infinite dimensional problems related to the equations (also the construction of periodic orbits) can be simplified to finite dimensional ones. However, extending results gained for step functions to smooth maps is a highly nontrivial task. A key technical property in this procedure is the hyperbolicity of the periodic orbits in question. A further purpose of the dissertation is to handle nonlinear maps "between" step functions and continuously differentiable ones, i. e. functions that are continuous but not continuously differentiable.

The present research is based on previous results [4, 6, 7, 8] of Krisztin, Walther and Wu for the positive feedback case (i.e. for continuous functions f with $xf(x) > 0$ for $x \neq 0$), on papers [14, 15, 16] of Walther and Yebdri for the negative feedback case (i.e. for continuous functions f with $xf(x) < 0$ for $x \neq 0$), and on paper [1] of Győri and Hartung. Our main analytical tools are the discrete Lyapunov function introduced by Mallet-Paret and Sell in [10], and the theory of Poincaré maps, in particular a theorem of Lani-Wayda in [9].

The thesis discusses the subsequent results in detail, which were published in [5, 12, 13].

Large-Amplitude Periodic Solutions for Monotone Positive Feedback

Chapter 3 studies Eq. (1.1) in case of positive feedback (i.e. when f is continuous and $xf(x) > 0$ for $x \neq 0$) and considers the next hypothesis.

(H1) $\mu > 0$, $f \in C^1(\mathbb{R},\mathbb{R})$ with $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, and

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$$

are five consecutive zeros of $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$ with $f'(\xi_j) < \mu < f'(\xi_k)$ for $j \in \{-2, 0, 2\}$ and $k \in \{-1, 1\}$.

Under condition (H1), $\hat{\xi}_j \in C = C([-1,0],\mathbb{R})$ defined by $\hat{\xi}_j(s) = \xi_j$ for all $-1 \leq s \leq 0$ is an equilibrium point for all $j \in \{-2,-1,0,1,2\}$. As f monotone increases, the subsets

$$C_{i,j} = \{\varphi \in C : \xi_i \leq \varphi(s) \leq \xi_j \text{ for all } s \in [-1,0]\}, \quad i \in \{-2,0\}, j \in \{0,2\},$$

of C are positively invariant under the semiflow

$$\Phi : \mathbb{R}^+ \times C \ni (t,\varphi) \mapsto x_t^{\varphi} \in C.$$

The structures of the global attractors $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ of the restrictions $\Phi|_{[0,\infty)\times C_{-2,0}}$ and $\Phi|_{[0,\infty)\times C_{0,2}}$, respectively, are (at least partially) well understood [3, 4, 6, 7, 8]. For certain nonlinearities, $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ have spindle-like structures described in [3, 6, 7, 8].

Let $\mathcal{A}$ denote the global attractor of the restriction $\Phi|_{[0,\infty)\times C_{-2,2}}$. The problem, whether under hypothesis (H1) the equality

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}$$

holds or not, arose in [7]. The first main result of the dissertation states that for special nonlinearities, the structure of $\mathcal{A}$ is more complex, and Eq. (1.1) has periodic orbits in $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$.

These periodic solutions are slowly oscillatory and of large amplitude in the following sense. Suppose $f \in C^1(\mathbb{R},\mathbb{R})$ with $f'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, and $\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$ are five consecutive zeros of $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$. Then a periodic solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) is called a large amplitude periodic solution if $x(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$. A solution $x : \mathbb{R} \to \mathbb{R}$ is slowly oscillatory if for each t, the restriction $x|_{[t-1,t]}$ has one or two sign changes. A large-amplitude slowly oscillatory periodic solution is abbreviated as an LSOP solution. An LSOP solution $x : \mathbb{R} \to \mathbb{R}$ is normalized if $x(-1) = 0$, and with some $\eta > 0$, $x(s) > 0$ for all $s \in (-1, -1+\eta)$.

Theorem 3.1.1. *There exist μ and f satisfying (H1) such that Eq.* (1.1) *has exactly two normalized LSOP solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$. For the ranges of p and q, $p(\mathbb{R}) \subsetneq q(\mathbb{R})$ holds. The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

are hyperbolic and unstable with 2 *and* 1 *Floquet multipliers outside the unit circle, respectively.*

Verifying the existence of such periodic orbits is an interesting problem as they do not arise via local bifurcation.

In Theorem 3.1.1, the nonlinear map f is "close" to the step function

$$f^{K,0}(x) = \begin{cases} -K & \text{if } x < -1, \\ 0 & \text{if } |x| \leq 1, \\ K & \text{if } x > 1. \end{cases}$$

Therefore the notion of LSOP solutions is extended to the special feedback function $f^{K,0}$. Suppose f is odd and satisfies (H1). It follows from [11] that if $x : \mathbb{R} \to \mathbb{R}$ is an LSOP solution of Eq. (1.1) with minimal period $\omega > 0$, then the following statements hold.

(i) The minimal period ω is in $(1, 2)$.
(ii) Solution x is of special symmetry, meaning that relation $x(t + \omega/2) = -x(t)$ holds for all $t \in \mathbb{R}$.
(iii) Solution x is of monotone type in the following sense: if $t_0 < t_1 < t_0 + \omega$ is set so that $x(t_0) = \min_{t\in\mathbb{R}} x(t)$ and $x(t_1) = \max_{t\in\mathbb{R}} x(t)$, then x is nondecreasing on $[t_0, t_1]$ and nonincreasing on $[t_1, t_0 + \omega]$.

These properties motivate the next definition. A periodic solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with feedback function $f^{K,0}$, $K > 0$, is said to be an LSOP solution if (i), (ii) and (iii) hold for x.

Chapter 3 contains the proof of Theorem 3.1.1, and it is organized as follows.

Set μ to be 1. The subsequent argument can be easily modified for all $\mu > 0$. As a starting point, Section 3.2 considers feedback function $f^{K,0}$ with $K > 0$, and function $f^{K,\varepsilon}$ with $K > 0$ and $\varepsilon \in (0, 1)$, where $f^{K,\varepsilon} \in C^1(\mathbb{R}, \mathbb{R})$, $K > 0$, $\varepsilon \in (0, 1)$, is an approximation of the step function $f^{K,0}$ defined such that $\left(f^{K,\varepsilon}\right)'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, and

$$f^{K,\varepsilon}(x) = \begin{cases} -K & \text{if } x < -1 - \varepsilon, \\ 0 & \text{if } |x| \leq 1, \\ K & \text{if } x > 1 + \varepsilon. \end{cases}$$

Note that $\mathbb{R} \ni \xi \mapsto -\mu\xi + f^{K,\varepsilon}(\xi) \in \mathbb{R}$ has exactly five zeros.

Fix $K > 3$. An open set U^1 in $(0,1)^3 \times [0,1)$ and a continuous map $\Sigma : U^1 \to C$ is defined so that for $\varepsilon \in (0,1)$, $U^1_\varepsilon \ni a \mapsto \Sigma(a,\varepsilon) \in C$ is smooth and injective (see Proposition 3.2.2), where U^1_ε denotes the set $\left\{a \in (0,1)^3 : (a,\varepsilon) \in U^1\right\}$. Consequently, $\Sigma(U^1_\varepsilon \times \{\varepsilon\})$ is a 3-dimensional C^1-submanifold of C for each $\varepsilon \in (0,1)$.

The aim of Subsection 3.2.1 is to construct an LSOP solution with initial segment in $\Sigma(U^1)$. There exists an open subset U^3 of U^1 such that if $f = f^{K,\varepsilon}$ with parameter $\varepsilon \in [0,1)$, then for all $(a,\varepsilon) \in U^3$, the solution $x^{\Sigma(a,\varepsilon)} : [-1,\infty) \to \mathbb{R}$ of Eq. (1.1) returns into $\Sigma(U^1)$. To be more precise, if τ is the smallest positive zero of $x^{\Sigma(a,\varepsilon)}$, then $x^{\Sigma(a,\varepsilon)}_{\tau+1} \in \Sigma(U^1)$ (see Proposition 3.2.5). This induces a smooth map $F : U^3 \to \mathbb{R}^3$ so that for all $(a,\varepsilon) \in U^3$, we have $F(a,\varepsilon) = b$ if $x^{\Sigma(a,\varepsilon)}_{\tau+1} = \Sigma(b,\varepsilon)$. If $F(a,\varepsilon) = a$ holds for some $(a,\varepsilon) \in U^3$, then the solution $x^{\Sigma(a,\varepsilon)}$ of Eq. (1.1) with $\mu = 1$ and $f = f^{K,\varepsilon}$ is an LSOP solution. Therefore the problem of finding an LSOP solution is reduced to a 3-dimensional fixed point equation depending on parameter ε. Let $K^* \approx 6.8653$ be the unique solution of

$$(K-1)(K+1)^3 = e\left(K^2 - 2K - 1\right)^2$$

on $(3,\infty)$. Then the next assertion holds.

Proposition 3.2.8. *For $K \in (3, K^*]$, equation $F(a,0) = a$ admits no solution in $U^3_0 = \{a \in \mathbb{R}^3 : (a,0) \in U^3\}$. For $K > K^*$, there is a unique $a^* \in U^3_0$ with $F(a^*,0) = a^*$.*

The fixed point a^* is hyperbolic; it is rigorously checked for $K = 7$. Thus the implicit function theorem gives the following result.

Proposition 3.2.11. *Set $K = 7$. There exits $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0,\varepsilon_0)$, $F(a,\varepsilon) = a$ has a solution $a^*(\varepsilon)$ in $U^3_\varepsilon = \{a \in \mathbb{R}^3 : (a,\varepsilon) \in U^3\}$, and $x^{\Sigma(a^*(\varepsilon),\varepsilon)} : \mathbb{R} \to \mathbb{R}$ is an LSOP solution of Eq. (1.1) with $f = f^{7,\varepsilon}$.*

Analogously to the above construction, Subsection 3.2.2 offers a second LSOP solution for $\mu = 1$ and feedback function $f = f^{7,\varepsilon}$ with $\varepsilon \in [0,\widetilde{\varepsilon}_0)$, where $\widetilde{\varepsilon}_0 > 0$ is small. The initial segment of this LSOP solution is denoted by $\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)$, and it is determined by the hyperbolic fixed point $\widetilde{a}(\varepsilon)$ of a finite dimensional map $\widetilde{F}(\cdot,\varepsilon)$.

Section 3.3 shows that the hyperbolicity of the fixed points of the 3-dimensional maps in Section 3.2 guarantees the hyperbolicity of the corresponding LSOP orbits.

Proposition 3.3.4. *The orbits defined by LSOP solutions $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ and $x^{\widetilde{\Sigma}\left(\widetilde{a}(\varepsilon),\varepsilon\right)}$ are hyperbolic with 2 and 1 Floquet multipliers outside the unit circle, respectively.*

The key fact in the proof of Proposition 3.3.4 is that a suitable Poincaré return map takes a small neighborhood of its fixed point $\Sigma(a^*(\varepsilon),\varepsilon)$ in a hyperplane of C into the 3-dimensional submanifold $\Sigma(U^3_\varepsilon \times \{\varepsilon\})$ (Proposition 3.3.1), analogously for a small neighborhood of $\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)$.

A result in paper [9] of Lani-Wayda together with Proposition 3.3.4 guarantees the existence of LSOP solutions for all nonlinearities f that are close to $f^{7,\varepsilon}$ in C^1-norm and satisfy (H1).

Proposition 3.3.5. *Set $\mu = 1$, $K = 7$.* Then for each $\varepsilon \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$, *there exists $\delta_0 = \delta_0(\varepsilon) > 0$ so that if $f \in C_b^1(\mathbb{R}, \mathbb{R})$ satisfies (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_0$, then Eq. (1.1) admits two normalized LSOP solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ with $p(\mathbb{R}) \subsetneq q(\mathbb{R})$. The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

are hyperbolic, and have 2 and 1 Floquet multipliers outside the unit circle, respectively.

After some preparatory results in Section 3.4, the exact number of LSOP solutions is investigated in Section 3.5, first for the step function $f^{K,0}$ with $K > 0$, then for $f^{7,\varepsilon}$ with $\varepsilon > 0$ small, and finally for functions f close to $f^{7,\varepsilon}$. One after another, the subsequent results are verified.

Theorem 3.5.5. *Eq. (1.1) with $\mu = 1$ and nonlinearity $f = f^{K,0}$ has no LSOP solutions for $K \in (0, K^*)$, and it admits exactly two normalized LSOP solutions for $K > K^*$.*

Proposition 3.5.6. *Let $\mu = 1$. A threshold number $\varepsilon_* \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$ can be given so that for $\varepsilon \in (0, \varepsilon_*)$, $x^{\Sigma(a^*(\varepsilon),\varepsilon)} : \mathbb{R} \to \mathbb{R}$ and $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)} : \mathbb{R} \to \mathbb{R}$ are the only normalized LSOP solutions of Eq. (1.1) with $f = f^{7,\varepsilon}$.*

Proposition 3.5.7. *Set $\mu = 1$. To each $\varepsilon \in (0, \varepsilon_*)$, there corresponds $\delta_1 = \delta_1(\varepsilon) \in (0, \delta_0(\varepsilon))$ such that if $f \in C_b^1(\mathbb{R}, \mathbb{R})$ satisfies (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1$, then Eq. (1.1) admits at most two normalized LSOP solutions.*

Summarizing the above results, Theorem 3.1.1 is obtained.

The global attractor

The global attractor is described entirely only for particular infinite dimensional systems, for example for gradient systems of parabolic equations. Chapter 4 analyzes the structure of solutions and characterizes the global attractor in the situation of Theorem 3.1.1.

A periodic solution $x : \mathbb{R} \to \mathbb{R}$ is slowly oscillatory around ξ_* with $* \in \{-1, 1\}$, if for each $t \in \mathbb{R}$, function $x - \xi_*$ has at most two sign changes on $[t-1, t]$. The nonlinearity f and the constant μ are given in Theorem 3.1.1 so that there exist periodic solutions oscillating slowly around ξ_1 and around ξ_{-1} with ranges in $(0, \xi_2)$ and in $(\xi_{-2}, 0)$, respectively [7]. There is no information about the uniqueness of these periodic solutions, but the next assertion holds.

Proposition 4.2.1. *For the map f in Theorem 3.1.1, there exist periodic solutions $x^1 : \mathbb{R} \to \mathbb{R}$ and $x^{-1} : \mathbb{R} \to \mathbb{R}$ of Eq.* (1.1) *oscillating slowly around ξ_1 and ξ_{-1} with ranges in $(0, \xi_2)$ and $(\xi_{-2}, 0)$, respectively, so that the ranges $x^1(\mathbb{R})$ and $x^{-1}(\mathbb{R})$ are maximal in the sense that $x^1(\mathbb{R}) \supset x(\mathbb{R})$ for all periodic solutions x oscillating slowly around ξ_1 with ranges in $(0, \xi_2)$; and analogously for x^{-1}.*

Set

$$\mathcal{O}_1 = \left\{ x_t^1 : t \in \mathbb{R} \right\} \text{ and } \mathcal{O}_{-1} = \left\{ x_t^{-1} : t \in \mathbb{R} \right\}.$$

Also let $\mathcal{W}^u(\mathcal{O}_p)$ and $\mathcal{W}^u(\mathcal{O}_q)$ denote the unstable sets of LSOP orbits $\mathcal{O}_p$ and $\mathcal{O}_q$, respectively. The second main result of the present thesis is the next one.

Theorem 4.1.1. *One may set μ and f satisfying (H1) such that the statement of Theorem 3.1.1 holds, and for the global attractor $\mathcal{A}$ we have the equality*

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \cup \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q).$$

Moreover, the dynamics on $\mathcal{W}^u(\mathcal{O}_p)$ and $\mathcal{W}^u(\mathcal{O}_q)$ is as follows.
For each $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$, the omega limit set $\omega(\varphi)$ is either $\left\{\hat{\xi}_{-2}\right\}$ or $\left\{\hat{\xi}_2\right\}$, and there exist heteroclinic connections from $\mathcal{O}_q$ to $\left\{\hat{\xi}_{-2}\right\}$ and to $\left\{\hat{\xi}_2\right\}$.
For each $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, $\omega(\varphi)$ is one of the sets $\left\{\hat{\xi}_{-2}\right\}$, $\left\{\hat{0}\right\}$, $\left\{\hat{\xi}_2\right\}$, $\mathcal{O}_q$, $\mathcal{O}_1$, $\mathcal{O}_{-1}$. There are heteroclinic connections from $\mathcal{O}_p$ to $\left\{\hat{\xi}_{-2}\right\}$, $\left\{\hat{0}\right\}$, $\left\{\hat{\xi}_2\right\}$, $\mathcal{O}_q$, $\mathcal{O}_1$ and $\mathcal{O}_{-1}$.

Clearly, $\mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q) \subseteq \mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$. The reverse inclusion follows from the following proposition excluding periodic solutions oscillating rapidly around 0. A solution is rapidly oscillatory if it has at least three sign changes on each interval of length 1.

Proposition 4.3.3. *To each $\varepsilon \in (0, \varepsilon_*)$, there corresponds $\delta_2 = \delta_2(\varepsilon) > 0$ such that if $\mu = 1$, $f \in C_b^1(\mathbb{R}, \mathbb{R})$ satisfies hypothesis (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_2$, then Eq.* (1.1) *with $\mu = 1$ and nonlinearity f has no periodic solutions oscillating rapidly around* 0.

$\mathcal{W}^u(\mathcal{O}_p)$ is the forward extension of $\mathcal{W}^u(p_0)$, the local unstable manifold of a suitable Poincaré return map at its fixed point p_0. Similarly for $\mathcal{W}^u(\mathcal{O}_q)$. The existence of heteroclinic orbits from $\mathcal{O}_p$ is based on the fact that $\mathcal{W}^u(p_0)$ is 2-dimensional, and it is separated into two parts by the 1-dimensional leading unstable manifold $\mathcal{W}_1^u(p_0)$. Discrete Lyapunov functionals around $\xi_{-1}, 0, \xi_1$, the Poincaré–Bendixson theorem, the theory of invariant manifolds, the monotone property of the semiflow, and elementary topological arguments are necessary to arrive at the desired result.

Slowly Oscillating Periodic Solutions for Negative Feedback

Theorem 3.1.1 brings up the question whether there exist feedback functions for which Eq. (1.1) admits an arbitrary number of slowly oscillatory periodic orbits. The problem is solved in Chapter 5 for the negative feedback case, i.e. when f is continuous and $xf(x) < 0$ for $x \in \mathbb{R} \setminus \{0\}$.

A solution $x : \mathbb{R} \to \mathbb{R}$ is slowly oscillatory in case of negative feedback if the sign changes of x are spaced at distances larger than the delay 1. Slowly oscillatory periodic solutions are abbreviated as SOP solutions. Walther [14] has given a class of Lipschitz continuous nonlinearities for which Eq. (5.1) admits an SOP solution. This result is improved by the third main theorem of the thesis.

Theorem 5.1.1. *Assume $\mu > 0$. There exists a locally Lipschitz continuous odd nonlinear map f satisfying $xf(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$, for which equation*

$$\dot{x}(t) = -\mu x(t) - f(x(t-1)) \tag{5.1}$$

admits an infinite sequence of SOP solutions $(p^n)_{n=1}^{\infty}$ with $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$ for $n \geq 0$. If f is continuously differentiable, then the corresponding periodic orbits are stable and hyperbolic.

The proof of Theorem 5.1.1 is organized as follows.

Set $\mu > 0$ and let $K > 0$ be large. First a periodic solution $u^R : \mathbb{R} \to \mathbb{R}$ is constructed in Section 5.2 for the special feedback function

$$f^R(x) = \begin{cases} -KR & \text{if } x < -R, \\ 0 & \text{if } |x| \leq R, \\ KR & \text{if } x > R \end{cases} \tag{5.4}$$

for all $R > 0$.

Then Section 5.3 introduces the function class N. Fix a constant $M > K$. For $r > 1$, $\varepsilon \in (0, r-1)$ and $\eta \in (0, M-K)$, let $N = N(r, \varepsilon, \eta)$ be the set of all continuous odd functions $f : \mathbb{R} \to \mathbb{R}$ with

$$|f(x)| < \eta \text{ for } x \in [0,1],$$

$$\left| \frac{f(x)}{r^n} \right| < M \text{ for all } x \in (r^n, r^n(1+\varepsilon)) \text{ and } n \geq 0$$

and

$$\left|\frac{f(x)}{r^n} - K\right| < \eta \text{ for all } x \in \left[r^n (1+\varepsilon), r^{n+1}\right] \text{ and } n \geq 0.$$

The elements of N restricted to $[-r^n, r^n]$, $n \geq 1$, can be viewed as perturbations of $f^{r^{n-1}}$ defined by (5.4).

For $f \in N(r, \varepsilon, \eta)$, we look for SOP solutions with initial functions in the nonempty closed convex sets $A_n = A_n(r, \varepsilon)$ defined as

$$A_n = \left\{\varphi \in C : \, r^n (1+\varepsilon) \leq \varphi(s) \leq r^{n+1} \text{ for } s \in [-1, 0), \, \varphi(0) = r^n (1+\varepsilon)\right\}$$

for each $n \geq 0$. By Proposition 5.3.1, the solutions of Eq. (5.1) with $f \in N(r, \varepsilon, \eta)$ and with initial segment in $A_n(r, \varepsilon)$ converge to u^{r^n} on $[0, 2]$ in the sense that

$$\sup_{f \in N(r,\varepsilon,\eta),\, n \geq 0,\, \varphi \in A_n(r,\varepsilon),\, t \in [0,2]} \frac{\left|x^{\varphi}(t) - u^{r^n}(t)\right|}{r^n} \to 0$$

as $r \to \infty$, $\varepsilon \to 0+$ and $\eta \to 0+$.

Based on this property, one can show that if ε and η are small enough, and r is large enough, then for all $n \geq 0$ and $\varphi \in A_n(r, \varepsilon)$, there exists $q = q(\varphi, f) \in (1, 2)$ so that $x_q^{\varphi} \in -A_n(r, \varepsilon)$ (Proposition 5.3.2). Following Walther in [14], Section 5.4 defines a return map

$$R_f^n : A_n(r, \varepsilon) \ni \varphi \mapsto -\Phi(q(\varphi, f), \varphi) \in A_n(r, \varepsilon)$$

for each $f \in N(r, \varepsilon, \eta)$ and $n \geq 0$. If R_f^n, $n \geq 0$, has a fixed point, then it is the initial segment of an SOP solution p^n of Eq. (5.1) with minimal period $2q$.

Hence to complete the proof of Theorem 5.1.1, it suffices to show that for a suitable nonlinear function $f \in N$, the maps R_f^n are strict contractions for each $n \geq 0$. Section 5.4 determines a Lipschitz constant depending on f for R_f^n, $n \geq 0$. Then Section 5.5 defines a feedback function f recursively on intervals $[-r^n, r^n]$, $n \geq 1$, so that f belongs to N, and R_f^n is a strict contraction for all $n \geq 0$. The stability and hyperbolicity of the corresponding periodic orbits follow from results in [14].

Dynamics for the Hopfield Activation Function

Chapter 6 investigates the piecewise linear Hopfield activation function

$$f : \mathbb{R} \ni x \mapsto \frac{1}{2}\left(|x+1| - |x-1|\right) = \begin{cases} 1, & x > 1, \\ x, & -1 \leq x \leq 1, \\ -1, & x < -1 \end{cases} \tag{6.1}$$

and the equation

$$\dot{x}(t) = -\mu x(t) + a f(x(t)) + b f(x(t-1)) + I \tag{6.2}$$

with

$$a, b, \mu, I \in \mathbb{R},\ \mu > 0 \text{ and } b \neq 0. \tag{6.3}$$

Considering Eq. (1.1) in this slightly more general form is motivated by applications [2].

Győri and Hartung described the dynamics for certain choices of parameters in [1] and formulated the conjecture that all solutions converge to an equilibrium as $t \to \infty$ if $b > 0$. Chapter 6 examines the truth of the conjecture for those choices of parameters that were not covered by them, namely for

$$b > 0 \text{ and } 0 < \mu = a + b - |I| \tag{6.4}$$

and for

$$b > 0 \text{ and } 0 < \mu < a + b - |I|\,. \tag{6.5}$$

The difficulty of the analysis resides in the fact that Hopfield function is neither strictly increasing nor continuously differentiable, hence the solution operator is neither injective nor differentiable everywhere. It follows that several techniques developed originally to handle strictly monotone and smooth nonlinearities cannot be used here. In particular, it is not known whether a Poincaré–Bendixson type result holds for Eq. (6.2).

The truth of the Győri-Hartung conjecture is easily verified in special case (6.4).

Theorem 6.3.1. *Consider (6.1)–(6.3). If (6.4) holds, then every solution of Eq. (6.2) tends to an equilibrium as $t \to \infty$.*

The greatest part of the chapter deals with assumption (6.5). In this case Eq. (6.2) has three equilibrium points $\hat{\xi}_+, \hat{\xi}_-, \hat{\xi}_0$ so that $\hat{\xi}_+, \hat{\xi}_-$ are stable, and $\hat{\xi}_0$ is unstable.

Subset

$$S = \left\{\varphi \in C :\ x^{\varphi} - \hat{\xi}_0 \text{ has arbitrarily large zeros}\right\}$$

of the phase space C is called separatrix. It is a 1-codimensional Lipschitz submanifold of C (Proposition 6.5.2), and it plays a key role in understanding the long-term behavior of the solutions.

The solution operators induced by the linear variational equation around $\hat{\xi}_0$ form a strongly continuous semigroup. As it is well-known, the spectrum of the generator of the semigroup consists of eigenvalues; there is one real eigenvalue λ_0, and the others form a sequence $\left(\lambda_k, \overline{\lambda_k}\right)_{k=1}^{\infty}$ of complex conjugate pairs.

If $\mu \neq a$, set $L(a,\mu) = (\mu - a)/\cos\theta$, where $\theta \in (\pi, 2\pi)$ with $\theta = (a-\mu)\tan\theta$, otherwise let $L(a,\mu) = 3\pi/2$. We focus on condition $b > L(a,\mu)$, which is equivalent to $0 < \mathrm{Re}\lambda_1 < \lambda_0$.

It is easy to see that case $b = L(a,\mu)$ serves as a counterexample to the conjecture, as in this case $\mathrm{Re}\lambda_1 = 0$, and a continuum of periodic solutions appear.

Theorem 6.3.2. *Consider (6.1)–(6.3) and (6.5).*
(i) Most of the solutions are convergent. That is, if φ is an element of $C \backslash S$, then $x_t^{\varphi} \to \hat{\xi}_+$ or $x_t^{\varphi} \to \hat{\xi}_-$ as $t \to \infty$.
(ii) Condition $b > L(a,\mu)$ implies the existence of a periodic solution $p : \mathbb{R} \to \mathbb{R}$ with minimal period $\omega \in (1,2)$.

Let $b > L(a,\mu)$. In the proof of Theorem 6.3.2 (ii), let $\mathcal{W}$ denote the forward extension of the 3-dimensional leading unstable manifold of $\hat{\xi}_0$. Then $\overline{\mathcal{W} \cap S}$, the closure of $\mathcal{W} \cap S$, is compact and invariant. One gets a more precise characterization of $\overline{\mathcal{W} \cap S}$ in Section 6.5.

Proposition 6.5.5. *If $\varphi \in \overline{\mathcal{W} \cap S} \setminus \left\{\hat{\xi}_0\right\}$ and $x = x^{\varphi} : \mathbb{R} \to \mathbb{R}$ is a solution with $x_t \in \overline{\mathcal{W} \cap S}$ for all $t \in \mathbb{R}$, then $\varphi - \hat{\xi}_0$ has at most two sign changes, and there exists a sequence $(t_n)_{-\infty}^{\infty}$ so that for all $n \in \mathbb{Z}$,*

$$t_{n+1} - t_n < 1, \quad t_{n+2} - t_n > 1,$$

$$x(t_n) = \hat{\xi}_0, \quad \dot{x}(t_{2n}) > 0, \quad \dot{x}(t_{2n+1}) < 0,$$

$$x(t) > \hat{\xi}_0 \text{ if } t \in (t_{2n}, t_{2n+1}), \quad \text{and} \quad x(t) < \hat{\xi}_0 \text{ if } t \in (t_{2n-1}, t_{2n}).$$

Section 6.6 introduces the continuous map

$$\pi_2 : C \ni \varphi \mapsto \left(\varphi(0) - \hat{\xi}_0, \varphi(-1) - \hat{\xi}_0\right) \in \mathbb{R}^2$$

and analyzes the image $\pi_2\left(\overline{\mathcal{W} \cap S}\right)$ of $\overline{\mathcal{W} \cap S}$ based on Proposition 6.5.5. Finally, a Poincaré return map is defined on a subset of $\pi_2\left(\overline{\mathcal{W} \cap S}\right)$. The fixed point of the Poincaré map guarantees the existence of a periodic orbit in $\overline{\mathcal{W} \cap S}$.

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