BENKŐ DÁVID

THE EQUILIBRIUM MEASURE AND THE SAFF CONJECTURE

(Az Egyensúlyi Mérték és Saff Sejtése)

Ph.D. disszertáció matematika analízis témában

Szegedi Tudományegyetem, 2006 Május

Témavezető: Totik Vilmos, Ph.D.

Acknowledgements

I wish to express my deepest gratitude to my advisor Vilmos Totik for teaching me the beauty of Potential Theory, for giving the topic of this dissertation, and for his invaluable help. Besides Mathematics I also benefited from knowing him in another way - his goodness and humanity became a model for me.

I am also thankful to my professors at the University of Eötvös Lóránd, the University of Szeged and the University of South Florida. Special thanks are due to my analysis professors, L. Czách, L. Hatvani, M. Ismail, L. Kérchy, M. Laczkovich, F. Móricz, L. Pintér and E. Saff. I am also grateful to my high school teachers, G. Horányi and J. S. Szászné for developing my interest in Mathematics.

Finally, I am indebted to my parents, my sister and my grandmother for their unerring support.

Remark: Research was supported by OTKA TS44782.

Table of Contents

1 Introduction

1.1 The Approximation Problem

In [28], V. Totik settled the Saff conjecture, which was a basic long standing conjecture in the theory of approximation by weighted polynomials. Let $w : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that $\lim_{x\to\pm\infty} xw(x) = 0$. Let us define q by the $w(x) = \exp(-q(x))$ equation, and call $q(x)$ to be an *external field.* The problem of Saff was the following: what $f(x)$ functions can we uniformly approximated by weighted polynomials of the form $w(x)^n P_n(x)$.

This approximation problem appears in several applications. It must be emphasized that the exponent of the weight changes with n, so this is a different (and in some sense more difficult) type of approximation than what is usually called weighted approximation. In fact, the polynomial P_n must balance exponential oscillations in w^n.

Clearly, the function which we approximate must be continuous. We shall see that exactly those continuous functions can be approximated by $w^n P_n$ which vanishes on a particular closed set Z_w. So the problem reduces to finding Z_w.

Interestingly, a measure called the *equilibrium measure associated with* w plays an important role in the problem. We will denote the support of this measure by S_w, this is a compact set.

Let us assume that $q(x)$ is a convex function. Saff conjectured that in this case a continuous $f(x)$ can be uniformly approximated by weighted polynomials if and only if it vanishes outside of the support S_w. In other worlds, Z_w is the complement of the interior of S_w.

Originally this statement was believed to be true without the convexity restriction until a counterexample was given by Totik in [29]. In [29] he also proved the conjecture under an additional smoothness condition on q.

We remark that convex external fields have the useful property that S_w is an interval.

Instead of the whole real line we may define $w(x)$ on a closed subset Σ of the real line. Suppose now that $\Sigma = [0, +\infty)$. There is a theorem which says that if $xq'(x)$ is increasing on Σ then S_w is an interval. This is again a useful statement, since there are external fields which are not convex but $xq'(x)$ is increasing.

Totik raised the following question. Assume that $xq'(x)$ is increasing on Σ and let $[a, b]$ denote the support S_w. Can we uniformly approximate continuous functions vanishing outside (a, b) by weighted polynomials $w(x)^n P_n(x)$? This was the original problem which has lead to this dissertation.

The positive answer to this question was given in [3]. There the approximation problem was solved for so called "weak convex" function, which is a large class of functions which contains both the convex functions as well as those when $xq'(x)$ is increasing. As an example, consider the external field $q(x) := x^\lambda$, where $\lambda \in (0, 1)$. Obiously q is not convex on $[0, +\infty)$ but $xq'(x)$ is increasing there. So the S_w will be an interval $[a, b]$ (which interval may even be found in terms of λ). And continuous functions vanishing outside (a, b) can be approximated by weighted polynomials $w(x)^n P_n(x)$.

In this dissertation we extend even further the validity of the above results (see Chapter 4). We define "weak convex" functions with basepoints A and B, and consider external field which belongs to this class. This class is larger than those in [3], nevertheless we will still use the "weak convex" denomination. We will also prove that "weak convex" external fields generate an equilibrium measure whose support is an interval (see Chapter 2). In particular, the support will be an interval when $\exp(q(x))$ is a convex function.

In Chapter 3 we will also consider the equilibrium problem on the unit circle. We give conditions which guarantee that the support of the equilibrium measure will be an arc of the circle.

1.2 Logarithmic Potential Theory

Let $\Sigma \subset \mathbb{R}$ be a closed set. A weight function w on Σ is said to be admissible, if it satisfies the following three conditions:

(i) w is upper semi-continuous,

(ii) $\{x \in \Sigma : \ w(x) > 0\}$ has positive capacity,

(iii) if Σ is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty, \ z \in \Sigma$.

In the approximation problem will always assume in the theorems that w is continuous, and that Σ is regular with respect to the Dirichlet problem in $\mathbf{C}\backslash\Sigma$. Condition (iii) can be relaxed, but that would make the proofs more complicated. (For related results when (iii) is not assumed see [5], [6] and [25].)

We define q by

$$w(x) =: \exp(-q(x)),$$

so $q : \Sigma \to (-\infty, \infty]$ is a lower semi-continuous function.

Let $\mathcal{M}(\Sigma)$ be the collection of all positive unit Borel measures with compact support in Σ. We define the *logarithmic potential* of $\mu \in \mathcal{M}(\Sigma)$ as

$$U^{\mu}(x) := \int \log \frac{1}{|x-t|} d\mu(t),$$

and the *weighted energy integral* as

$$I_w(\mu) := -\int\int \log(|x-y|w(x)w(y))d\mu(x)d\mu(y).$$

We will need the following basic theorem ([24], Theorem I.1.3):

Theorem 1 *Let w be an admissible (not necessarily continuous) weight on the closed set Σ and let $V_w := \inf\{I_w(\mu) : \ \mu \in \mathcal{M}(\Sigma)\}$. Then*

(a) V_w is finite,

(b) there exists a unique element $\mu_w \in \mathcal{M}(\Sigma)$ such that $I_w(\mu_w) = V_w$,

(c) setting $F_w := V_w - \int q d\mu_w$, the inequality

$$U^{\mu_w}(x) + q(x) \geq F_w$$

holds quasi-everywhere on Σ,

(d) the inequality

$$U^{\mu_w}(x) + q(x) \leq F_w \tag{1}$$

holds for all $x \in S_w := \mathrm{supp}(\mu_w)$.

Remark According to our definition, every measure in $\mathcal{M}(\Sigma)$ has compact support, so the support S_w is a compact set.

The measure μ_w is called *the equilibrium* or *extremal measure* associated with w.

Notation 2 *When we say that a property holds* inside *G - where G is a subset of $\mathbb{R}$ - we mean that the property is satisfied on every compact subset of G.*

1.3 Totik's Results

We are considering uniform approximation of continuous functions on Σ by weighted polynomials of the form $w^n P_n$, where $deg P_n \leq n$. Theorem 3 is a Stone-

Weierstrass type theorem for this kind of approximation (see [13], or see [24], Theorem VI.1.1).

Theorem 3 *There exists a closed set $Z = Z(w) \subset \Sigma$, such that a continuous function f on Σ is the uniform limit of weighted polynomials $w^n P_n$, $n = 1, 2, ...$, if and only if f vanishes on Z.*

Thus the problem of what functions can be approximated is equivalent to determining what points lie in $Z(w)$. This latter problem is intimately related to the density of μ_w. The support $S_w := \mathrm{supp}(\mu_{\mathrm{w}})$ plays a special role (see [29], or [13], or see [24], Theorem VI.1.2):

Theorem 4 *The complement of S_w belongs to Z.*

The definition of functions with smooth integrals was introduced by Totik in [28]. This was a crucial definition in solving the Saff conjecture.

Definitions 5 *We say that a function $f(x)$ has smooth integral on $R \subset \mathbb{R}$, if $f(x)$ is non-negative a.e. on R and*

$$\int_I f = (1 + o(1)) \int_J f \tag{2}$$

where $I,\ J \subset R$ are any two adjacent intervals, both of which has length $0 < \epsilon$, and $\epsilon \to 0$. The $o(1)$ term depends on ϵ and not on I and J.

Clearly, all continuous functions which have a positive lower bound have smooth integral. But $\log(1/|t|)$ also has smooth integral on $[-1/2, 1/2]$.

The next three theorems are due to Totik, [28].

Theorem 6 *Let us suppose that $[a, b]$ is a subset of the support S_w, and the extremal measure has a density v on $[a, b]$ that has a positive lower bound and smooth integral there. Then $(a, b) \cap Z(w) = \emptyset$.*

Theorem 7 *Let us suppose that* (a, b) *is a subset of the support* S_w*, and that* q *is convex on* (a, b)*. Then* μ_w *has a density* v *in* (a, b) *which has a positive lower bound and smooth integral inside* (a, b)*.*

From these two theorems follows

Theorem 8 *Let us suppose that* (a, b) *is a subset of the support* S_w*, and that* q *is convex on* (a, b)*. Then* $(a, b) \cap Z(w) = \emptyset$*. In particular, every function that vanishes outside* (a, b) *can be uniformly approximated by weighted polynomials of the form* $w^n P_n$*.*

Notice that Theorem 8 is a local result; it works for any part of the extremal support where q is convex.

1.4 Main Results

Here are some agreements we will follow.

Notation 9 *Unless otherwise noted, all intervals in the dissertation are arbitrary intervals (that is, they may be open, half open, closed, bounded or unbounded). The notation* $I = [(a, b)]$ *will mean that* I *is an interval (of any type) with endpoints* a *and* b*. The notation* $\mathrm{int}(I)$ *is used for the interior points of* I*. Absolute continuity inside* I *means that the function is absolutely continuous on any compact set which is lying in* I*.*

Notation 10 *Throughout the dissertation we agree on the following. Suppose that a function* $g(x)$ *- which is usually defined by using q'(x) - is said to be increasing (or decreasing) on a set* E*, but it is defined on a set* $F \subset E$*, where* $E \setminus F$ *has measure zero. Then by "*$f(x)$ *is increasing on E" we mean the following:*

there exists a set $G \subset F$ so that $E \backslash G$ has measure zero and $f(x)$ is increasing on G, i.e.,

$$f(x) \leq f(y), \quad \text{if } \ x < y, \ x, y \in G. \tag{3}$$

In other words, we do not require that $f(x)$ is increasing everywhere, where it is defined. (We will use the terminology "strictly increasing", if strict inequality is required in (3).)

Similarly, when we write that, say, $1/(x - A) \leq q'(x)$ on an interval (v, b), we will mean that there exists a set $G \subset (v, b)$ so that $(v, b) \backslash G$ has measure 0, $q'(x)$ exists on G and $1/(x - A) \leq q'(x)$ on G.

These agreements weaken the assumptions on the functions in the theorems, but the given proofs are correct for this modified increasing/decreasing definition as well. When reading the proofs the reader should keep in mind that we are not working on the whole interval but on a subset of the interval having full measure, even though it is not indicated.

Notation 11 *We will say that a function $g(x)$ is convex on an interval I, if*

a) $g(x)$ is absolutely continuous inside I, (so g' exists a.e. in I), and

b) $g'(x)$ is increasing on I. (By our agreement (see above) this means that there exists a set G such that $I \backslash G$ has measure zero, $g'(x)$ exists on G, and $g'(x)$ is increasing on G.)

Now we are ready to define "weak convexity". We remark that the convexity of q implies the convexity of $\exp(q)$, so the class of weak convex functions contains the convex functions.

Definition 12 *We say that a function $q : D \to \overline{\mathbb{R}}$ ($D \subset \mathbb{R}$) is weak convex on an interval $I = [(a, b)] \subset D$ ($a, b \in \overline{\mathbb{R}}$) with basepoints $A, B \in \overline{\mathbb{R}}$, $A < B$, if the following properties hold:*

(i) $I \subset [A, B]$,

(ii) q *is absolutely continuous inside* (a, b) *(so* q' *exists a.e. in* I*),*

(iii) if $a \in I$*, then*

$$\liminf_{x \to a+0} q(x) = q(a),$$

and if $b \in I$*, then*

$$\liminf_{x \to b-0} q(x) = q(b),$$

(iv) I *can be written as the disjoint union of finitely many intervals* $I_1, ..., I_n$ *such that for any interval* I_k $(1 \leq k \leq n)$*):*

$$\exp(q(x)) \qquad \textit{is convex on} \quad I_k, \quad \textit{or} \tag{4}$$

$$(x - A)(B - x)q'(x) + x \qquad \textit{is increasing on } I_k. \tag{5}$$

If $-\infty < A$ *and* $B = +\infty$ *then (5) should be replaced by:*

$$(x - A)q'(x) \quad \textit{is increasing on} \quad I_k. \tag{6}$$

If $-\infty = A$ *and* $B < +\infty$ *then (5) should be replaced by:*

$$(B - x)q'(x) \quad \textit{is increasing on} \quad I_k. \tag{7}$$

If both $-\infty = A$ *and* $B = +\infty$ *then (5) should be ignored, thus in this case* $\exp(q(x))$ *must be convex on the whole* I*.*

(v) if x_0 *is any endpoint of any* I_k *but different of* a *and* b *then:*

$$\limsup_{x \to x_0^-} q'(x) \leq \liminf_{x \to x_0^+} q'(x), \tag{8}$$

We will simply just say that q is weak convex on an interval I (without mentioning the basepoints), if q is weak convex on the interval $I = [(a,b)] \subset D$ ($a, b \in \overline{\mathbb{R}}$) with basepoints a and b.

Remark The assumptions (4), (5), (6) and (7) are meant in the broader sense (see Notation 10 and 11).

Examples: 1.) $q(x) := x^\lambda$ is weak convex on $[0, +\infty)$ for any positive λ, because $xq'(x)$ is increasing on $[0, +\infty)$.

2.) If $H \in (0,1)$ then $q(x) := \ln\sqrt{1-x^2}$ is weak convex on $[-H, H]$. (Interestingly, q is concave, moreover $\exp q$ is also concave.)

3.) For any $c > 0$, $q(x) := \cos(cx)/c^2$ is weak convex on $\Sigma := [-1, 1]$

For the proofs and for further examples, please go to Section 2.3.

The following theorem is our *main theorem*. We are considering only external fields which are weak convex on the full $[A, B]$ interval. Note also that we make the $S_w \subset (A, B)$ assumption instead of $S_w \subset [A, B]$.

Theorem 13 *Let w be a continuous admissible weight on $\mathbb{R}$. Suppose that q is weak convex on $[A, B]$ with finite basepoints A, B satisfying $S_w \subset (A, B)$. Then $Z(w) = (\text{int } S_w)^c$. Thus a continuous function $f(x)$ can be uniformly approximated by weighted polynomials $w^n P_n$ if and only if $f(x)$ vanishes outside S_w.*

We will also give several conditions on q which guarantees that the support of the equilibrium measure is an interval. Our most general theorem which guarantees that the support is an interval is the following.

Theorem 14 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and suppose that q is weak convex on the interval $I \subset \Sigma$ with basepoints $A, B \in \overline{\mathbb{R}}$ satisfying $S_w \subset [A, B]$. Then $S_w \cap I$ is an interval.*

In the next theorem we give an integral representation for the density of the equilibrium measure. For the sake of simple notations we state the next theorem for the interval $[-1,1]$, instead of $[A,B]$. (To get the general version, replace everywhere $x+1$ by $x-A$ and $1-x$ by $B-x$.)

Theorem 15 *Let $w=\exp(-q)$ be an admissible weight on $\mathbb{R}$ such that $\min S_w = -1$, $\max S_w = 1$. Suppose that q is absolutely continuous inside $(-1,1)$ and q' is bounded on $[-1,1]$. Assume further that $\liminf_{x\to-1+0} q(x) = q(-1)$, $\liminf_{x\to 1-0} q(x) = q(1)$ and with some constants $-1\le u\le v\le 1$ we have*

$$\begin{aligned} & q'(x) \le \frac{-1}{1-x} && \text{on} \quad (-1,u) \\ \frac{-1}{1-x} \le {} & q'(x) \le \frac{1}{x+1} && \text{on} \quad (u,v) \\ \frac{1}{x+1} \le {} & q'(x) && \text{on} \quad (v,1), \end{aligned}$$

$$(1-x)^2 q'(x) - x \quad \text{is increasing on} \quad (-1,u).$$

$$(1-x^2) q'(x) + x \quad \text{is increasing on} \quad (u,v).$$

$$(x+1)^2 q'(x) - x \quad \text{is increasing on} \quad (v,1).$$

Then $S_w=[-1,1]$ and the density of μ_w is $d\mu_w(t) = v(t)dt \quad$ a.e. $t\in[-1,1]$, where we define $v(t)$ for a.e. $t\in[-1,1]$ as follows:

For $t\in(-1,u)$, let

$$v(t) := \frac{1}{\pi^2\sqrt{1-t^2}}\Big[\frac{t+1}{1-t}\int_{-1}^{1}\frac{(1-s)^2q'(s)-s-[(1-t)^2q'(t)-t]}{(s-t)\sqrt{1-s^2}}ds$$

$$+\frac{2}{1-t}(\pi+\int_{-1}^{1}\frac{(1-s)q'(s)}{\sqrt{1-s^2}}ds)\Big] \tag{9}$$

For $t \in (u, v)$, *let*

$$v(t) := \frac{1}{\pi^2\sqrt{1-t^2}} \int_{-1}^{1} \frac{(1-s^2)q'(s) + s - [(1-t^2)q'(t) + t]}{(s-t)\sqrt{1-s^2}} ds. \tag{10}$$

For $t \in (v, 1)$, *let*

$$v(t) := \frac{1}{\pi^2\sqrt{1-t^2}} \Big[\frac{1-t}{t+1} \int_{-1}^{1} \frac{(s+1)^2 q'(s) - s - [(t+1)^2 q'(t) - t]}{(s-t)\sqrt{1-s^2}} ds$$

$$+ \frac{2}{t+1}(\pi - \int_{-1}^{1} \frac{(s+1)q'(s)}{\sqrt{1-s^2}} ds)\Big] \tag{11}$$

In Chapter 3 we will consider the equilibrium problem on the unit circle C. Using the unit circle we also reveal the connections between different conditions on the real line, and we give a new condition not covered by the weak convexity definition at Definition 12.

Let now $\alpha, \beta \in \mathbb{R}$ be two angles, $|\beta - \alpha| < 2\pi$. We define $\widehat{[\alpha, \beta]}$ to be the arc $[e^{i\alpha}, e^{i\beta}] \subset C$, where we go from $e^{i\alpha}$ to $e^{i\beta}$ in a counterclockwise direction. We define $\widehat{[\alpha, \alpha + 2\pi]}$ to be the full circle C.

We will prove the following two theorems:

Theorem 16 *Let* $w(z) = \exp(-q(z))$, $|z| = 1$ *be a weight on* C *and let* $I = \lfloor \gamma, \delta \rceil$ *be an interval with* $0 < \delta - \gamma \leq 2\pi$. *Assume that* q *is absolutely continuous inside* I *and*

$$\liminf_{\substack{x \to y \\ x \in I}} q(x) = q(y) \tag{12}$$

whenever y *is an endpoint of* I *with* $y \in I$. *Let* e^{ic} *be any point which is not an interior point of* $\widehat{I}$. *Let* $\widehat{[\alpha_1, \beta_1]}, ..., \widehat{[\alpha_k, \beta_k]}$ *be* $k \geq 0$ *arcs of* C. *Here, for all* $1 \leq i \leq k$, $0 < \beta_i - \alpha_i \leq 2\pi$ *and* $(S_w \cup \widehat{I}) \subset \widehat{[\alpha_i, \beta_i]}$. *Suppose further that* I

can be written as a disjoint union of $n \geq 1$ *intervals* $I_1, ..., I_n$ *and for any fixed* $1 \leq j \leq n$*, either*

$$e^{q(\theta)}\Big[2\sin\Big(\frac{\theta-c}{2}\Big)q'(\theta)-\cos\Big(\frac{\theta-c}{2}\Big)\Big]\operatorname{sgn}\Big(\sin\Big(\frac{\theta-c}{2}\Big)\Big) \tag{2.2}$$

is increasing on I_j *or for some* $1 \leq i \leq k$*:*

$$\sin\left(\frac{\theta-\alpha_i}{2}\right)\sin\left(\frac{\beta_i-\theta}{2}\right)q'(\theta)+\frac{1}{4}\sin\left(\theta-\frac{\alpha_i+\beta_i}{2}\right) \tag{2.3}$$

is increasing on I_j*. Finally we assume that*

$$\limsup_{\theta\to\theta_0^-} q'(\theta) \leq \liminf_{\theta\to\theta_0^+} q'(\theta),$$

whenever θ_0 *is an endpoint of* I_j *(*$1 \leq j \leq n$*) but not an endpoint of* I*. Then* $S_w \cap \widehat{I}$ *is an arc of* C*.*

Here sgn denotes the signum function.

Theorem 17 *For given* $k \in \mathbb{N}^+$ *let*

$$\Sigma := \cup_{i=1}^k [A_i, B_i] \subset \bar{\mathbb{R}}, \quad \textit{where} \tag{13}$$

$$-\infty < A_1 \leq B_1 < A_2 \leq B_2 < ... < A_k \leq B_k < +\infty.$$

Let $W = \exp(-Q)$ *be a weight on* Σ*,* $I \subset \Sigma$ *be an interval and assume that* Q *is absolutely continuous inside* I *and*

$$\liminf_{\substack{X \to Y \\ X \in I}} Q(X) = Q(Y) \tag{14}$$

whenever Y is an endpoint of I with $Y \in I$. Assume further that I can be written as a disjoint union of intervals $I_1, ..., I_n$ such that for any fixed $1 \leq j \leq n$ either

$$e^{Q(X)} \quad \textit{is convex on} \;\; I_j, \tag{15}$$

or for some $1 \leq i \leq k-1$

$$(X - B_i)(A_{i+1} - X)Q'(X) + X \quad \textit{is decreasing on} \;\; I_j, \tag{16}$$

or

$$(X - A_1)(B_k - X)Q'(X) + X \quad \textit{is increasing on} \;\; I_j. \tag{17}$$

Finally we assume that

$$\limsup_{X \to X_0^-} Q'(X) \leq \liminf_{X \to X_0^+} Q'(X),$$

whenever X_0 is an endpoint of I_j $(1 \leq j \leq n)$ but not an endpoint of I. Then $S_W \cap I$ is an interval.

2 Weak Convex External Fields

2.1 Introduction

Potential theory connects many areas and problems of mathematics. Examples include the electrostatic equilibrium, the weighted transfinite diameter, weighted Chebyshev polynomials, zero distribution of orthogonal polynomials, fast decreasing polynomials, incomplete polynomials, approximation by weighted polynomials, Padé approximation, Hankel determinants, and random matrices ([10], [24]).

Let $w(x)$ be an *admissible* weight function defined on a closed set $\Sigma \subset \mathbb{R}$ (precise definitions will be given below). The basic problem in weighted potential theory is to find the unit measure - called the *equilibrium measure* associated with w - which minimizes the *weighted energy integral*:

$$I_w(\mu) := -\int\int \log(|x-y|w(x)w(y))d\mu(x)d\mu(y).$$

The function $q(x) := \log(1/w(x))$ is called the *external field*.

The determination of the equilibrium measure is usually a difficult or impossible task. One should first find the support S_w of the equilibrium measure. If we can show that the support consists of N intervals, then theoretically we can use the method described in [11] to find the location of these intervals. Then, at least theoretically, the equilibrium measure may be obtained from the Riemann-Hilbert problem.

This is the reason why it is important to have conditions which guarantee that the support is an interval, or the union of several intervals. These type of conditions are also used as assumptions on the external field at many problems. For example, the convexity of $q(x)$ was assumed in the formulation of the Saff conjecture. This was a problem of approximation by weighted polynomials of the form $w^n(x)P_n(x)$, proven by Totik ([28]).

There are only a few conditions in the literature which guarantee that the support of the equilibrium measure is a single interval. The following two conditions are well known ([20]):

1.) If q is convex on the interval I, then $S_w \cap I$ is an interval.

2.) If $\Sigma := [0, +\infty)$ and $xq'(x)$ is increasing on the interval $I \subset \Sigma$ then $S_w \cap I$ is an interval.

A condition similar to 2.) is introduced in [3], which is weaker than both 1.) and 2.). In this dissertation some other conditions are given. Their combination (which we will call *weak convexity condition*) is weaker than the ones given in [3]. Other type of conditions can be found in [7], [8], [15] and [16], which guarantee that the support of the equilibrium measure is an interval, or the union of several intervals. In [10] Deift, Kriecherbauer and McLaughlin showed that for a real analytic external field the support always consists of a finite number of intervals.

The reader can find the definition of the logarithmic capacity in [24], I.1. We say that a property holds quasi-everywhere, if the set where it does not hold has capacity 0.

2.2 Results and Proofs

Let $\Sigma \subset \mathbb{R}$ be any closed set, let $w : \Sigma \to [0, +\infty)$ be *admissible*. Define $\Sigma_0 := \{x \in \Sigma : 0 < w(x)\}$

Lemma 18 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and let $I \subset \Sigma_0$ be an interval. Let μ_w be the equilibrium measure associated with w. Assume that q is* *absolutely continuous inside* I *and that*

$$\liminf_{\substack{x \to z \\ x \in I}} q(x) = q(z), \quad \text{if } z \text{ is an endpoint of } I \text{ with } z \in I. \tag{18}$$

If for some function $f : \operatorname{int}(I) \to \mathbb{R}^+$*, the function* $f(x)\frac{d}{dx}[U^{\mu_w}(x) + q(x)]$ *is strictly increasing on* $\operatorname{int}(I)$*, then* $S_w \cap I$ *is an interval.*

Proof. Suppose on the contrary that there exist $a, b \in S_w \cap I$, $a < b$: $(a, b) \cap S_w = \emptyset$.

Let $\mu := \mu_w$ denote the equilibrium measure associated with w and

$$U(x) := U^{\mu}(x) := \int_{\mathbb{R}} \ln \frac{1}{|x - t|} d\mu(t)$$

be the logarithmic potential function of μ. Since $U(x)$ is a C^{∞} function on (a, b), it is absolutely continuous on every closed subset of (a, b). Because of the Lebesgue monotone convergence theorem, $U(x)$ is continuous on $[a, b]$. (Indeed, we may assume that $\operatorname{diam}(S_w) \le 1$, so $\ln(1/|x - t|) > 0 \quad x, t \in S_w$. We split the above integral to two integrals, one with measure $\mu|_{(-\infty,a]}$ and the other with measure $\mu|_{[b,+\infty)}$. Since $U(a)$ and $U(b)$ are finite from (1), we can apply Lebesgue's theorem to the two integrals.) So $U(x)$ is absolutely continuous on $[a, b]$.

Let $R(x) := U(x) + q(x)$. By our assumption $f(x)R'(x)$ is increasing on I, i.e., there exists a set $G \subset I$ so that $I \backslash G$ has measure zero, $f(x)R'(x)$ exists for all $x \in G$, and $f(x)R'(x)$ is increasing on G. (See our agreement at Notation 10.)

It follows that we cannot find numbers $x_1, x_2 \in \operatorname{int}(I) \cap G$, $x_1 < x_2$, for which both $0 < R'(x_1)$ and $0 > R'(x_2)$ hold.

From Theorem A we have $R(x) := U(x) + q(x) \geq F_w \quad x \in (a,b)$ and $R(a), R(b) \leq F_w$. It is impossible that $R(x) = F_w$ for all $y \in (a,b) \cap G$, because then $f(x)R'(x)$ would be constant function on (a,b). So there is a $y \in (a,b) \cap G: \ F_w < R(y)$. We have:

$$0 < R(y) - F_w \leq R(y) - R(a) = R(y) - \liminf_{x \to a+0} R(x) = \limsup_{x \to a+0} \int_x^y R'(t)dt$$

which implies the existence of $x_1 \in (a,y) \cap G: \ 0 < R'(x_1)$. Similarly

$$0 > F_w - R(y) \geq R(b) - R(y) = \liminf_{x \to b-0} R(x) - R(y) = \liminf_{x \to b-0} \int_y^x R'(t)dt,$$

so there is an $x_2 \in (y,b) \cap G: \ 0 > R'(x_2)$. This is a contradiction. ∎

If q is convex, then $\exp(q)$ is also convex, so the following theorem is a generalization of the well known theorem in which the convexity of q is assumed.

Theorem 19 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and suppose that $\exp(q)$ is convex on the interval $I \subset \Sigma_0$ and satisfies condition (18) . Then $S_w \cap I$ is an interval.*

Proof. Suppose on the contrary that there exist $a, b \in S_w \cap I$, $a < b$: $(a,b) \cap S_w = \emptyset$.

Let

$$m(x,y,t) := \frac{f(y)}{t-y} + f(y)q'(y) - \frac{f(x)}{t-x} - f(x)q'(x) \tag{19}$$

Our goal is to find a non-negative function $f(x)$ such that for any fixed $x < y$, where $x, y \in (a,b)$ we have:

$$0 \leq m(x,y,t) \quad t \in \mathbb{R} \backslash (a,b) \tag{20}$$

and equality holds for at most finitely many values of t.

If we can find such a function $f(x)$, then integrating (20) with respect to $d\mu(t)$ gives

$$0 < f(y)\frac{d}{dy}[U^{\mu}(y) + q(y)] - f(x)\frac{d}{dx}[U^{\mu}(x) + q(x)] \tag{21}$$

(We have strict inequality, since μ has measure zero on any finite set.) With Lemma 18 this will prove Theorem 19.

For fixed $a < x < y < b$ let us find the infimum of $t \to m(x,y,t)$ where $t \in \mathbb{R}\backslash(a,b)$. If $0 < f(x)$ and $0 < f(y)$, but $f(x) \neq f(y)$, then the function $t \to m(x,y,t)$ has exactly one critical point in $\mathbb{R}\backslash[x,y]$, which is:

$$t_0 := \frac{y\sqrt{f(x)} - x\sqrt{f(y)}}{\sqrt{f(x)} - \sqrt{f(y)}}, \tag{22}$$

and we have

$$m(x,y,t_0) = f(y)q'(y) - f(x)q'(x) - \frac{(\sqrt{f(y)} - \sqrt{f(x)})^2}{y-x} =: I(x,y). \tag{23}$$

Simple calculus argument shows that on $\mathbb{R}\backslash(x,y)$ the one variable function $t \mapsto m(x,y,t)$ achieves its minimum at t_0 and nowhere else. (Indeed, $\lim_{t\to x^-} m(x,y,t) = \lim_{t\to y^+} m(x,y,t) = +\infty$ and $\lim_{t\to -\infty} m(x,y,t) = \lim_{t\to +\infty} m(x,y,t) = f(y)q'(y) - f(x)q'(x) > I(x,y)$, so there must be a unique global minimum at the critical point t_0.)

If either $f(x)$ or $f(y)$ equals zero, but not both of them zero, then on $\mathbb{R}\backslash(x,y)$ we still have a unique minimum at t_0, which is $I(x,y)$ (though t_0 is no longer a critical point).

Finally, if $f(x) = f(y) > 0$, then for all $t \in \mathbb{R}\backslash(a,b)$ we have

$$I(x,y) = f(y)q'(y) - f(x)q'(x) < m(x,y,t) \tag{24}$$

In conclusion,

$$I(x,y) \leq m(x,y,t) \tag{25}$$

holds for all $t \in \mathbb{R}\backslash(a,b)$, and assuming not both $f(x)$ and $f(y)$ are zero, then equality holds for at most one value of t.

To finish the proof we claim that

$$f(x) := e^{2q(x)}$$

satisfies $0 \leq I(x,y)$. Notice that if $f(x)$ is defined this way, then $f(x)q'(x) = e^{2q(x)}q'(x) = h(x)h'(x)$ where $h(x) := \sqrt{f(x)} = e^{q(x)}$. By our assumption $h(x)$ is a convex function. We have

$$I(x,y) = h(y)h'(y) - h(x)h'(x) - \frac{(h(y)-h(x))^2}{y-x}.$$

Assume $h(x) \leq h(y)$. Using $h'(x) \leq h'(y)$ we get

$$I(x,y) \geq (h(y)-h(x))\Big(h'(y) - \frac{h(y)-h(x)}{y-x}\Big).$$

Here the first factor is non-negative and the second factor is also non-negative, since $h(x)$ is a convex function.

Assume now $h(x) > h(y)$. Using $h'(x) \leq h'(y)$ we get

$$I(x,y) \geq (h(y)-h(x))\Big(h'(x) - \frac{h(y)-h(x)}{y-x}\Big).$$

Here the first factor is negative and the second factor is non-positive, since $h(x)$ is a convex function.

So in all cases $0 \leq I(x, y)$ and this together with (25) implies statement (20).

∎

The following statement follows from Theorem 19 since Proposition 24 implies that $\exp(q)$ is convex. Nevertheless, we give another proof.

Theorem 20 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and let $I \subset \Sigma_0$ be an interval. Suppose that q is absolutely continuous inside I, satisfies condition (18) and $(x-A)^2 q'(x) - x$ is increasing on the interval I, where $A \in \mathbb{R} \backslash \mathrm{int}(I)$ is a fixed real number. Then $S_w \cap I$ is an interval.*

Proof. As before, we would like to find a non-negative function $f(x)$ such that (20) holds and we have strict inequality in (20) with the exception of finitely many values of t.

As in the proof of Theorem 19 it is enough to show that $0 \leq I(x, y)$. This time let us define $f(x)$ as follows:

$$f(x) := (x - A)^2 \tag{26}$$

With this choice we have

$$\begin{aligned} I(x, y) &= f(y)q'(y) - f(x)q'(x) - \frac{(\sqrt{f(y)} - \sqrt{f(x)})^2}{y - x} \\ &\geq [(y-A)^2 q'(y) - y] - [(x-A)^2 q'(x) - x] \geq 0. \end{aligned} \tag{27}$$

since $(x - A)^2 q'(x) - x$ is an increasing function by our assumption. ∎

Theorem 21 *Let* $w = \exp(-q)$ *be an admissible weight on* $\mathbb{R}$ *and let* $I \subset \Sigma_0$ *be an interval. Suppose that* q *is absolutely continuous inside* I *and satisfies condition (18). Suppose that with some finite constants* $A < B$ *we have* $I \subset [A, B]$*,* $S_w \subset [A, B]$ *and* $(x - A)(B - x)q'(x) + x$ *is increasing on the interval* I*. Then* $S_w \cap I$ *is an interval.*

Proof. As before, we would like to find a non-negative function $f(x)$ such that (20) holds and we have strict inequality in (20) with the exception of finitely many values of t.

Let us define $f(x)$ as follows:

$$f(x) := (x - A)(B - x). \tag{28}$$

Let us assume first that $f(x) \neq f(y)$. We now show that $t_0 \notin (A, B)$ holds (t_0 is defined at (22)).

Assume that $f(x) > f(y)$. Now $t_0 \leq A$ if and only if $\sqrt{f(x)}(y - A) \leq \sqrt{f(y)}(x - A)$ which holds if and only if

$$(B - x)(y - A) \leq (B - y)(x - A). \tag{29}$$

Similarly, $t_0 \geq B$ if and only if

$$(B - x)(y - A) \geq (B - y)(x - A). \tag{30}$$

Either (29) or (30) holds, which proves that $t_0 \notin (A, B)$. The proof of this observation is similar in the case when $f(x) < f(y)$.

In the case $f(x) = f(y)$, t_0 does not exist.

For fixed $a < x < y < b$ we want to find the infimum of $m(x, y, t)$ (see (19)) where $t \in S_w$. Since $S_w \subset [A, B]$ and $(a, b) \cap S_w = \emptyset$, we have $t \in [A, B] \backslash [x, y]$. Using $t_0 \notin (A, B)$ and calculus we gain that the minimum of $m(x, y, t)$ is achieved at either $t := A$ or $t := B$ and nowhere else. It turns out that $m(x, y, A)$ and $m(x, y, B)$ has the same value which is

$$m(x, y, A) = m(x, y, B) = [(y - A)(B - y)q'(y) + y] - [(x - A)(B - x)q'(x) + x].$$

And this expression is non-negative, because of the assumption of the theorem.

Therefore $0 \leq m(x, y, t)$ holds and we have strict inequality with the exception of at most two values of t ($t := A$ and $t := B$). This proves Theorem (21). ∎

Theorem 22 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and let $I \subset \Sigma_0$ be an interval. Suppose that q is absolutely continuous inside I and satisfies condition (18). Suppose that with some constant $A \in \mathbb{R}$ we have $I \subset [A, +\infty)$, $S_w \subset [A, +\infty)$ and $(x - A)q'(x)$ is increasing on the interval I. Then $S_w \cap I$ is an interval.*

Proof. The proof of this theorem is identical to the proof of Theorem 21, with the exception that we define $f(x) := x - A$. Exactly the same way as in the proof of Theorem 21, we can see that if $f(x) \neq f(y)$, then $t_0 \notin (A, +\infty)$. We also see the same way that the minimum of $m(x, y, t)$, $t \in [A, +\infty) \backslash [x, y]$ is achieved at either A or when $t \to +\infty$. It turns out that

$$m(x, y, A) = \lim_{t \to +\infty} m(x, y, t) = (y - A)q'(y) - (x - A)q'(x),$$

which expression is non-negative. ∎

The symmetric version of Theorem 22 also holds. That version can be used when for some constant $B \in \mathbb{R}$ we have $S_w \subset (-\infty, B)$ and $(B-x)q'(x)$ is increasing on I.

In the next proposition we compare the "$\exp(q)$ is convex" and the "$(x-A)(B-x)q'(x)+x$ is increasing" conditions. When is one weaker than the other? The answer depends on whether $q'(x)$ is greater or less than $-1/(B-x)$ and $1/(x-A)$. The proof is not quite obvious due to the fact that differentiability of q' is not assumed. For simplicity let $A := -1$ and $B := 1$.

Proposition 23 *Let $J \subset [-1,1]$ be an open interval and q be a function defined on J which is absolute continuous inside J.*

a) If

$$q'(x) \in \left[\frac{-1}{1-x}, \frac{1}{x+1}\right], \quad x \in J,$$

and $\exp(q(x))$ is convex on J, then $(1-x^2)q'(x)+x$ is increasing on J.

b) If

$$q'(x) \geq \frac{1}{x+1}, \quad x \in J, \quad (\text{ or } q'(x) \leq \frac{-1}{1-x}, \quad x \in J),$$

and $(1-x^2)q'(x)+x$ is increasing on J, then $\exp(q)$ is convex on J.

Proof.

a) Let $q(x) := \exp(q(x))$, $x \in J$. Let $x, y \in J$, $x < y$ be fixed. Since q is convex, $q'(y) \geq (q(y) - q(x))/(y-x)$, from which

$$\frac{q'(y)}{q(y)} \geq \frac{1 - \frac{q(x)}{q(y)}}{y-x}. \tag{31}$$

From the convexity it also follows that

$$q(y) \geq q(x) + (y-x)q'(x). \tag{32}$$

Notice that $q(x)+(y-x)q'(x)>0$, because $q'(x)/q(x)=q'(x)>-1/(1-x)$.

Using (31) and (32) we get

$$\frac{q'(y)}{q(y)} \geq \frac{1-\frac{q(x)}{q(x)+(y-x)q'(x)}}{y-x}. \tag{33}$$

Our goal is to show that $(1-x^2)q'(x)+x \leq (1-y^2)q'(y)+y$. By (33) it is enough to show that

$$(1-x^2)\frac{q'(x)}{q(x)}+x \leq (1-y^2)\frac{1-\frac{q(x)}{q(x)+(y-x)q'(x)}}{y-x}+y.$$

After simplifications we see that this inequality is equivalent to

$$0 \leq \Big(q'(x)-\frac{-1}{1-x}\Big)\Big(\frac{1}{x+1}-q'(x)\Big),$$

which holds by the assumption of part a).

b) By symmetry we can assume that $1/(x+1) \leq q'(x), x \in J$. The function $k(x) := (1-x^2)q'(x)+x$ is increasing and $k(x) \geq 1$ for all $x \in J$. Let $x, y \in J,\ x<y$ be fixed. We have

$$q(r)=\int_x^r \frac{k(t)-t}{1-t^2}dt+C, \quad r \in J$$

and so

$$e^q(r)q'(r)=\exp\Big(\int_x^r \frac{k(t)-t}{1-t^2}dt+C\Big)\frac{k(r)-r}{1-r^2}.$$

We want to show that $e^{q(x)}q'(x) \leq e^q(y)q'(y)$, that is,

$$1 \leq \exp\Big(\int_x^y \frac{k(t)-t}{1-t^2}dt\Big)\frac{k(y)-y}{k(x)-x}\frac{1-x^2}{1-y^2}.$$

Since k is an increasing function, it is enough to show that

$$1 \leq \exp\Big(\int_x^y \frac{K-t}{1-t^2}dt\Big)\frac{K-y}{K-x}\frac{1-x^2}{1-y^2}, \tag{34}$$

where the constant K is defined by $K := k(x) \geq 1$. The right hand side of (34) can be decreased further if we replace K by 1. But if $K = 1$, we have equality in (34). Statement b) is proved.

$\blacksquare$

Proposition 24 *Let $J \subset \mathbb{R}$ be any open interval and q be a function defined on J which is absolute continuous inside J. Let $A \in \mathbb{R}\backslash J$ be arbitrary. If $(x-A)^2q'(x) - x$ is increasing on J, then* $\exp(q)$ *is convex on J.*

Proof. Assume that $k(x) := (x-A)^2q'(x) - x$ is increasing on J. Let $x, y \in J,\ x < y$ be fixed. Following the idea of the proof of Proposition 23, part b), we want to show that

$$\frac{k(x)+x}{(x-A)^2} \leq \exp\Big(\int_x^y \frac{k(t)+t}{(t-A)^2}dt\Big)\frac{k(y)+y}{(y-A)^2}. \tag{35}$$

If $k(y)+y \geq 0$, then we can decrease the right hand side of (35) by replacing the function $k(t)$ by K, and after this by replacing $k(y)$ by K, where $K := k(x)$. If, however $k(y)+y < 0$, then we can decrease the right hand side of (35) by replacing the function $k(t)$ by K^*, and we can increase the left hand side by replacing $k(x)$ by K^*, where $K^* := k(y)$. Thus, in both cases, the new inequality we want to show is of the form

$$\frac{L+x}{(x-A)^2} \leq \exp\Big(\int_x^y \frac{L+t}{(t-A)^2}dt\Big)\frac{L+y}{(y-A)^2}, \tag{36}$$

where L is a constant ($L = K$ or $L = K^*$). Using the notation

$$l(r) := \int_x^r \frac{L+t}{(t-A)^2} dt$$

we see that (36) is equivalent to $\exp(l(x))l'(x) \leq \exp(l(y))l'(y)$, so we must show that $\exp(l(r))$ is a convex function on J. But this is equivalent to $(l'(r))^2 + l''(r) \geq 0, r \in J$, as simple differentiation shows. Now direct calculation shows that

$$(l'(r))^2 + l''(r) = \frac{(L+A)^2}{(r-A)^4} \geq 0.$$

∎

Remark Differentiation shows that if q'' exists, then (4) holds if and only if

$$0 \leq (q'(x))^2 + q''(x), \quad x \in I_k,$$

and (5) holds if and only if

$$0 \leq \frac{1}{(x-A)(B-x)} + \Big(\frac{1}{x-A} - \frac{1}{B-x}\Big) q'(x) + q''(x), \quad x \in I_k, \tag{37}$$

finally, (6) holds if and only if

$$0 \leq \frac{1}{x-A} q'(x) + q''(x), \quad x \in I_k.$$

In most cases, the following statement is also true:

Let $I \subset \mathbb{R}$ be an interval with $I \subset [A_1, B_1] \subset [A_2, B_2] \subset \overline{\mathbb{R}}$. If q is weak convex on I with basepoints A_2 and B_2, then q is also weak convex on I with basepoints

A_1 and B_1. For example, this statement is true if $q''(x)$ exists and the sets

$$H := \Big\{x \in I: \; q'(x) > \frac{1}{x - A_2}\Big\} \text{ and } H^* := \Big\{x \in I: \; q'(x) > \frac{-1}{B_2 - x}\Big\}$$

consist of finitely many intervals. To see this, let us partition I to finitely many subintervals $J_1, J_2, ..., J_t$ so that for any k the open subinterval $\mathrm{int}(J_k)$ is a subset of H or H^* completely. We will assume that both A_2 and B_2 are finite. (The proof is exactly the same if at least one of them is not finite. Proposition 23 can be also stated for the case when A or B is not finite, the proof remains the same.) Let us consider J_k for a fixed k ($1 \le k \le t$). We want to show that either $\exp(q(x))$ is convex on J_k, or $(x - A_1)(B_1 - x)q'(x) + x$ is increasing on J_k. Since k is arbitrary, this will prove that q is weak convex on I with basepoints A_1 and B_1. We know that either $\exp(q(x))$ is convex on J_k, or $(x - A_2)(B_2 - x)q'(x) + x$ is increasing on J_k. We may assume that the second property holds, since if $\exp(q(x))$ is convex, then there is nothing to prove.

Case I. If $q'(x) \ge 1/(x - A_2), \; x \in J_k$ or $q'(x) \le -1/(B_2 - x), \; x \in J_k$, then $\exp(q)$ must be convex on J_k. This follows from the fact that $(x - A_2)(B_2 - x)q'(x) + x$ is increasing on J_k and from Proposition 23.

Case II. Assume now that

$$q'(x) \in \Big[\frac{-1}{B_2 - x}, \frac{1}{x - A_2}\Big], \quad x \in J_k.$$

Notice that for $x \in J_k$ we have

$$\begin{aligned} 0 \le \quad & \Big(\frac{1}{x - A_1} - \frac{1}{x - A_2}\Big)\Big(\frac{1}{B_1 - x} + q'(x)\Big) \\ + \quad & \Big(\frac{1}{B_1 - x} - \frac{1}{B_2 - x}\Big)\Big(\frac{1}{x - A_2} - q'(x)\Big), \end{aligned} \tag{38}$$

because all factors here are non-negative. From (38) we get

$$\frac{1}{(x-A_2)(B_2-x)} + \Big(\frac{1}{x-A_2} - \frac{1}{B-x_2}\Big)q'(x) + q''(x)$$

$$\leq \frac{1}{(x-A_1)(B_1-x)} + \Big(\frac{1}{x-A_1} - \frac{1}{B_1-x}\Big)q'(x) + q''(x),$$

and the left hand side of this inequality is non-negative, since $(x-A_2)(B_2-x)q'(x)+x$ is increasing on J_k. Thus the right hand side is also non-negative, i.e., $(x-A_1)(B_1-x)q'(x)+x$ is increasing on J_k.

Our most general theorem which guarantees that the support is an interval is the following.

Theorem 25 *Let* $w = \exp(-q)$ *be an admissible weight on* $\mathbb{R}$ *and suppose that* q *is weak convex on the interval* $I \subset \Sigma_0$ *with basepoints* $A, B \in \overline{\mathbb{R}}$ *satisfying* $S_w \subset [A,B]$. *Then* $S_w \cap I$ *is an interval.*

Proof. Suppose on the contrary that there exist $a, b \in S_w \cap I$, $a < b$: $(a,b) \cap S_w = \emptyset$.

We will define a positive function $f : (a,b) \to \mathbb{R}$ such that for any fixed $t \in \mathbb{R} \backslash (a,b)$:

$$g_t(x) := f(x)\Big(\frac{1}{t-x} + q'(x)\Big) \tag{39}$$

is an increasing function of x on (a,b) and it is strictly increasing with the exception of finitely many t. This will finish the proof of Theorem 25 (see the beginning of the proof of Theorem 19).

Let $I = \cup_{i=1}^{n} I_k$ be the decomposition of I to intervals, given in the definition of weak convexity.

Let $E_1 := 1$. We can find positive constants $E_2, ..., E_n$ uniquely such that the following $f(x)$ function is continuous on I. For $x \in I_k$ $(k = 1, ..., n)$ let

$$f(x) := \begin{cases} E_k \exp(2q(x)) & \text{if (4) is satisfied on } I_k \\ E_k(x-A)(B-x) & \text{if (5) is satisfied on } I_k \end{cases}. \tag{40}$$

We can assume that A and B are finite, since the proof of Theorem 25 is the the same if $A = -\infty$ or $B = +\infty$ with a slight modification. (For example, if $-\infty < A$ and $B = +\infty$, then we have to use Theorem 22 in our proof instead of Theorem 21, and we have to define $f(x)$ as

$$f(x) := \begin{cases} E_k \exp(2q(x)) & \text{if (4) is satisfied on } I_k \\ E_k(x-A) & \text{if (6) is satisfied on } I_k \end{cases},$$

instead of the definition at (40).)

Now by the proof of Theorem 19, (39) is increasing on any $I_k \cap (a, b)$ where (4) is satisfied, and strictly increasing with the exception of finitely many t. By the proof of Theorem 21, (39) is also increasing on any $I_k \cap (a, b)$ where (5) is satisfied, and strictly increasing with the exception of finitely many t.

Let I_k and I_{k+1} to be adjacent intervals, I_{k+1} being to the right of I_k. Let u denote the number which separates I_k and I_{k+1}. Let $x \in int(I_k) \cap (a, b)$ and $y \in int(I_{k+1}) \cap (a, b)$ be arbitrary inner points (assuming these are not empty sets). Using what we have just said and (8) we get:

$$g_t(x) \le \lim_{r \to u^-} g_t(r) \le \frac{f(u)}{t-u} + f(u) \limsup_{r \to u^-} q'(r)$$

$$= \frac{f(u)}{t-u} + f(u) \liminf_{r \to u^+} q'(r) \le \lim_{r \to u^+} g_t(r) \le g_t(y). \tag{41}$$

In (41) the very first and last inequality is strict inequality with the exception of finitely many t, and the set of these exceptional t values does not depend on the choice of x and y. Thus $g_t(x) \leq g_t(y)$ and with the exception of finitely many t we have $g_t(x) < g_t(y)$ for all $x \in int(I_k) \cap (a,b)$ and $y \in int(I_{k+1}) \cap (a,b)$.

Since this holds for any $k = 1, ..., n$, we conclude that $g_t(x)$ is an increasing function of x on the whole (a,b) and it is strictly increasing with the exception of finitely many t.

∎

2.3 Examples

Since our weak convexity definition is a generalization of the one given in [3], all the examples given in [3] could be repeated here. We just list some new examples to demonstrate how large the class of weak convex functions is.

1.) Let C(x) be any positive convex function on $\Sigma := \mathbb{R}$. Then $q(x) := \ln(C(x))$ is weak convex on $\Sigma := \mathbb{R}$. If we assume that $\lim_{|x|\to\infty} x/C(x) = 0$, then $w(x) := 1/C(x)$ is an admissible weight function, and S_w will be an interval by Theorem 25.

2.) $q(x) := \frac{1+c}{2}\ln(1+x) + \frac{1-c}{2}\ln(1-x)$ is weak convex on $\Sigma := [-H, H]$, where c is any constant and $H \in (0,1)$. Thus S_w is an interval. Now

$$w(x) = \frac{1}{(1-x)^{(1-c)/2}(1+x)^{(1+c)/2}}, \quad x \in [-H, H]$$

is the corresponding weight function. In particular if $c := 0$ we get that the concave function $q(x) := \ln\sqrt{1-x^2}$ is weak convex on $[-H, H]$ (now $w(x) = 1/\sqrt{1-x^2}$). (We remark that $\exp(q(x))$ is also concave.)

Notice that $(1-x^2)q'(x) + x = c$ constant, which shows that our $q(x)$ in example

2.) is weak convex with basepoints -1 and 1. Hence by the remark on Page 29 it is weak convex indeed.

3.) For any $c > 0$, $q(x) := \cos(cx)/c^2$ is weak convex on $\Sigma := [-1, 1]$, thus S_w is an interval.

Indeed, $(1 - x^2)q'(x) + x$ is increasing, because $q'(x) = -\sin(cx)/c$, and

$$\frac{d}{dx}[(1 - x^2)q'(x) + x] = \frac{2x\sin(cx)}{c} - (1 - x^2)\cos(cx) + 1 \geq \frac{2x\sin(cx)}{c} + x^2 \geq 0.$$

4.) $q(x) := x^2(x - 1/2)(x + 1/3)$ is weak convex on $\Sigma := [-1, 1]$, thus S_w is an interval.

In this example neither $\exp(q)$ is convex on the whole $[-1, 1]$, nor $(1-x^2)q'(x)+x$ is increasing on the whole $[-1, 1]$. However, $\exp(q)$ is convex on $[-1, -0.2] \cup [0.3, 1]$ and $(1 - x^2)q'(x) + x$ is increasing on $[-0.7, 0.7]$. So we can partition $[-1, 1]$ to smaller intervals (say, to $[-1, -0.2)$, $[-0.2, 0.7)$ and $[0.7, 1]$) so that one of our criteria is satisfied on any of the small intervals, as it is described at the definition of weak convexity. Hence $q(x)$ is weak convex.

5.) In [7], [8] and [16] Damelin, Dragnev and Kuijlaars studied the follwing external fields (sometimes in the context of fast decreasing polynomials):

$$q(x) := -cx^{\alpha}, \quad x \in \Sigma := [0, 1], \tag{42}$$

and

$$q(x) := -c \ \mathrm{sign}(x)|x|^{\alpha}, \ x \in \Sigma := [-1, 1], \tag{43}$$

where $c > 0$, $\alpha \geq 1$. They proved that for the external fields at (42) and at (43) the support will be the union of at most two intervals. In both cases S_w is the full interval ($S_w = [0, 1]$ and $S_w = [-1, 1]$ respectively), whenever c is small enough.

In this direction we can say the following: Let $q(x)$ be a twice differentiable function on $\Sigma := [-1, 1]$ such that $q''(x)$ is bounded and let $q(x) := cq(x)$, $x \in [-1, 1]$ where $c \in \mathbb{R}$. Then there exists $\epsilon > 0$ such that S_w is an interval whenever $|c| < \epsilon$. The proof of this statement is simple: if c is close to zero, then (37) holds, say, with $A := -2, B := 2$, so from Theorem 21 we gain that S_w is an interval (but not necessarily the full $[-1, 1]$).

Let us examine now the case when $q(x) := -cx^\alpha$, $x \in \Sigma := [0, 1]$. Let $A := 0, B := 1$, so $g(x) := (x - A)(B - x)q'(x) + x = -c\alpha(x - x^2)x^{\alpha-1} + x$. Then $g'(x) := -c\alpha^2 x^{\alpha-1} + c\alpha(\alpha + 1)x^\alpha + 1$. Notice that $g'(0) = 1$ and $g'(1) = c\alpha + 1$, which are positive values. Calculating $g''(x)$ we see that $g'(x)$ is a decreasing function on $[0, (\alpha - 1)/(\alpha + 1)]$ and increasing on $[(\alpha - 1)/(\alpha + 1), 1]$. So there exist $G_1, G_2 \in [0, 1]$, $G_1 \le G_2$, such that $g'(x)$ is non-negative on $[0, G_1] \cup [G_2, 1]$ and negative on (G_1, G_2). That is, $g(x)$ is increasing on $[0, G_1] \cup [G_2, 1]$, so both $S_w \cap [0, G_1]$ and $S_w \cap [G_2, 1]$ are intervals (empty sets are possible). But our criteria (5) cannot state anything on $[G_1, G_2]$. Using (4), too, we see that $\exp(q)$ is convex on $[G, 1]$ where $G := (\alpha - 1)/(c\alpha)$. Thus $S_w \cap [0, G_1]$ and $S_w \cap [\min(G_2, G), 1]$ are intervals (empty sets are possible), but unfortunately we cannot say anything about $S_w \cap [G_1, \min(G_2, G)]$.

2.4 Integral Representation for the Equilibrium Measure

Let $x \in \mathbb{R}$. If f is integrable on $L \setminus (x - \epsilon, x + \epsilon)$ for all $0 < \epsilon$ then the Cauchy principal value integral is defined as

$$PV \int_L f(t)dt := \lim_{\epsilon \to 0^+} \int_{L \setminus (x-\epsilon, x+\epsilon)} f(t)dt,$$

if the limit exists.

It is known that $PV\int_L g(t)/(t-x)dt$ exists for almost every $x \in \mathbb{R}$ if $g: L \to \mathbb{R}$ is integrable.

Now we are ready to prove Theorem 15.

Proof of Theorem 15 The hypotheses of Theorem 15 imply that $\exp(q)$ is convex on $(-1, u)$ and on $(v, 1)$ (see Proposition 24). Thus q is weak convex on $[-1, 1]$ and so S_w is an interval, which must be $[-1, 1]$.

According to [3] (Lemma 16), if the function

$$h(t) := \frac{1}{\pi^2\sqrt{1-t^2}} PV \int_{-1}^{1} \frac{\sqrt{1-s^2}q'(s)}{s-t} ds + \frac{1}{\pi\sqrt{1-t^2}} \tag{44}$$

is non-negative (a.e. $t \in [-1, 1]$), then $h(t)$ is the density of μ_w.

We perform the following manipulation:

$$(t+1)[(1-s)^2 q'(s) - s] + 2[(1-s)(s-t)q'(s) + s] = (1-t)[(1-s^2)q'(s) + s] \tag{45}$$

Dividing this by $(s-t)\sqrt{1-s^2}$ and integrating with respect to s (principal value integral), we get:

$$\frac{1}{\pi^2\sqrt{1-t^2}} \Big[\frac{t+1}{1-t} PV \int_{-1}^{1} \frac{(1-s)^2 q'(s) - s}{(s-t)\sqrt{1-s^2}} ds$$

$$+ \frac{2}{1-t}(\pi + \int_{-1}^{1} \frac{(1-s)q'(s)}{\sqrt{1-s^2}} ds) \Big] = h(t) \tag{46}$$

where we used the fact (see [24], formula IV.(3.20)) that

$$PV \int_{-1}^{1} \frac{1}{(s-t)\sqrt{1-s^2}} ds = 0 \tag{47}$$

and

$$PV\int_{-1}^{1}\frac{s}{(s-t)\sqrt{1-s^2}}ds$$

$$=\int_{-1}^{1}\frac{1}{\sqrt{1-s^2}}ds+tPV\int_{-1}^{1}\frac{1}{(s-t)\sqrt{1-s^2}}ds=\pi+0=\pi. \tag{48}$$

From (44) and (48) we also see that

$$\frac{1}{\pi^2\sqrt{1-t^2}}PV\int_{-1}^{1}\frac{(1-s^2)q'(s)+s}{(s-t)\sqrt{1-s^2}}ds=h(t). \tag{49}$$

Finally, using the identity

$$(1-t)[(s+1)^2q'(s)-s]-2[(s+1)(s-t)q'(s)-s]=(t+1)[(1-s^2)q'(s)+s]$$

we get the same way that

$$\frac{1}{\pi^2\sqrt{1-t^2}}\Big[\frac{1-t}{t+1}PV\int_{-1}^{1}\frac{(s+1)^2q'(s)-s}{(s-t)\sqrt{1-s^2}}ds$$

$$+\frac{2}{t+1}(\pi-\int_{-1}^{1}\frac{(s+1)q'(s)}{\sqrt{1-s^2}}ds)\Big]=h(t) \tag{50}$$

Now we claim that $h(t)$ is non-negative for a.e. $t\in[-1,1]$. Assuming $-1\neq u$, let $t\in(-1,u)$. Notice that in (46) the term

$$\pi+\int_{-1}^{1}\frac{(1-s)q'(s)}{\sqrt{1-s^2}}ds$$

is non-negative, see [3], Theorem 10.

Actually, Theorem 10 in [3] has different assumptions on q than our assumptions at Theorem 15 above. Nevertheless, the proof of Theorem 10 ([3]) works for our assumptions, if we make the following modification in the proof. Let

$A := -1, B := 1$, $a \in [A, B)$ and let

$$h(\beta) := \int_a^\beta \sqrt{\frac{x-a}{\beta-x}} q'(x) dx, \quad \beta \in (a, B].$$

We have to show that $h(\beta)$ is continuous on $(a, B]$ - which is a long argument in the proof of Theorem 10 ([3]). But now, using the boundedness of q', we can see this immediately. For this purpose let $\beta_0, \beta \in (a, B]$ and let $\beta \to \beta_0$. Without loss of generality we can assume that $\beta_0 < \beta$. Using the triangle inequality we get

$$|h(\beta) - h(\beta_0)| \leq \Big| \int_{\beta_0}^\beta \sqrt{\frac{x-a}{\beta-x}} q'(x) dx \Big| + \Big| \int_a^{\beta_0} \Big[\sqrt{\frac{x-a}{\beta-x}} - \sqrt{\frac{x-a}{\beta_0-x}} \Big] q'(x) dx \Big|$$

$$\leq C \Big[\int_{\beta_0}^\beta \sqrt{\frac{1}{\beta-x}} dx + \int_a^{\beta_0} \Big[\sqrt{\frac{1}{\beta_0-x}} - \sqrt{\frac{1}{\beta-x}} \Big] dx \Big],$$

and clearly both of these terms are going to zero as $\beta \to \beta_0$.

Let us return to the proof of Theorem 15. Let t be a number such that $q'(t)$ exists. In (46) the term

$$PV \int_{-1}^1 \frac{(1-s)^2 q'(s) - s}{(s-t)\sqrt{1-s^2}} ds$$

$$= PV \int_{-1}^1 \frac{(1-s)^2 q'(s) - s - [(1-t)^2 q'(t) - t]}{(s-t)\sqrt{1-s^2}} ds \tag{51}$$

is also non-negative which we can see as follows: if $s \in (-1, u)$, then

$$\frac{(1-s)^2 q'(s) - s - [(1-t)^2 q'(t) - t]}{s-t}$$

is clearly non-negative because $(1-x)^2 q'(x) - x$ is increasing on $(-1, u)$. Now let $s \in (u, 1)$ (if $u \neq 1$). By the assumptions of Theorem 15 we have $q'(t) \leq$

$(-1)/(1-t)$ and $q'(s) \geq (-1)/(1-s)$, from which

$$\frac{(1-s)^2 q'(s) - s - [(1-t)^2 q'(t) - t]}{s-t}$$

$$\geq \frac{(1-s)^2 \frac{-1}{1-s} - s - [(1-t)^2 \frac{-1}{1-t} - t]}{s-t} = 0$$

Since the integrand at the right hand side of (51) is non-negative it is no longer a principal value integral but a Lebesgue integral. So the "PV" sign can be dropped.

The proof of the non-negativity of $h(t)$ in the remaining cases (when $t \in (u,v)$ or $t \in (v,1)$) is similar. Thus $h(t)$ is the density, indeed. The integral formulas in Theorem 15 are coming from (46), (49) and (50) immediately (by using (47)).

∎

3 The Unit Circle and the Compactified Real Line

Chapter 3 is based on the paper [5] of Benko, Damelin and Dragnev. Damelin and Dragnev has made the following acknowledgement.

Acknowledgement D. Benko had a substantial contribution in our paper, in deciding the topic of the paper as well as in the proofs of the theorems.

S. B. Damelin (Department of Mathematics and Computer Science, Georgia Southern University, P.O. Box 8093, Statesboro, GA 30460, E-mail: damelin@gsu.cs.gasou.edu)

P. D. Dragnev (Department of Mathematical Sciences, Indiana-Purdue University, Fort Wayne, IN 46805, E-mail: dragnevp@ipfw.edu)

3.1 Some Definitions

In recent years, equilibrium measures with external fields have found an increasing number of applications in a variety of areas ranging from diverse subjects such as orthogonal polynomials, weighted Fekete points, numerical conformal mappings, weighted polynomial approximation, rational and Pade approximation, integrable systems, random matrix theory and random permutations. We refer the reader to the references [1, 2, 3, 9, 11, 17, 18, 19, 23, 24, 26, 27, 29] and those listed therein for a comprehensive account of these numerous, vast and interesting applications.

Let us recall the following important definitions from Chapter 1.2. With a compact set $\Sigma \subset \mathbb{C}$ and lower semi-continuous external field $q : \Sigma \to (-\infty, +\infty]$, we set $w := \exp(-q)$ and call w a weight associated with q, provided the set

$$\Sigma_0 := \{z \in \Sigma : \ w(z) > 0\}$$

has positive logarithmic capacity. With an external field q (or a weight w), we associate the weighted energy of a Borel probability measure μ on Σ as

$$I_w(\mu) = \int_\Sigma \int_\Sigma \log \frac{1}{|s-t| w(s) w(t)} d\mu(s) d\mu(t).$$

The equilibrium measure in the presence of an external field q, is the unique probability measure μ_w on Σ minimizing the weighted energy among all probability measures on Σ. Thus

$$I_w(\mu_w) = \min\{I_w(\mu) : \mu \in \mathcal{P}(\Sigma)\}$$

where $\mathcal{P}(\Sigma)$ denotes the class

$$\mathcal{P}(\Sigma) = \{\mu : \mu \text{ is a Borel probability measure on } \Sigma\}.$$

For more details on these topics we refer the reader to the seminal monograph of E. B. Saff and V. Totik [24].

The determination of the support S_w of the equilibrium measure μ_w is a major step in obtaining the measure. As described by Deift [11, Chapter 6], information that the support consists of $N \geq 1$ disjoint closed intervals, allows one to set up a system of equations for the endpoints, from which the endpoints may be calculated. Knowing the endpoints, the equilibrium measure may be obtained from a Riemann-Hilbert problem or, equivalently, a singular integral equation. It is for this reason that it is important to have a priori conditions on the external field q to ensure that the support is an interval or the union of a finite number of intervals. We refer the reader to the references [4, 3, 7, 8, 10, 12, 16, 22, 24, 25] for an account of advances on the equilibrium measure and support problem for one or several intervals.

In this present paper, we study supports of equilibrium measures for a general class of weights on the compactified real line and unit circle and present several conditions on the associated external field to ensure that the support of the associated equilibrium measure is one interval or one arc.

In order to present our main results, we find it convenient to introduce some needed notation and definitions.

Definition 26 *Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denote the on point compactified real line. It is a topological space which is isomorphic to the unit circle C. We will think of ∞ as $+\infty$, that is, we agree that $a < \infty$ for any $a \in \mathbb{R}$.*

We will continue to use the $-\infty$ and $+\infty$ symbols but let us keep in mind that they are the same point (which is ∞). For example, if b is a real number then $[-\infty, b]$ denotes the set $\{\infty\} \cup \{x : \ x \in \mathbb{R}, x \leq b\}$.

Let $U, V \in \bar{\mathbb{R}}, U \leq V$. In this chapter $I := \lfloor\ U, V\ \rceil \subset \bar{\mathbb{R}}$ denotes an interval which is open, closed, or half open, and has endpoints U and V. (Earlier we used the $[(U, V)]$ notation for this.) We define $[V, U] := (U, V)^c$, $(V, U) := [U, V]^c$, $(V, U] := (U, V]^c$, $[V, U) := [U, V)^c$.

Let now $\alpha, \beta \in \mathbb{R}$ be two angles, $|\beta - \alpha| < 2\pi$. We define $\widehat{\lfloor \alpha, \beta \rceil}$ to be the arc $\lfloor e^{i\alpha}, e^{i\beta} \rceil \subset C$, where we go from $e^{i\alpha}$ to $e^{i\beta}$ in a counterclockwise direction. If $\beta - \alpha = 2\pi$, let $\widehat{\lfloor \alpha, \beta \rceil}$ to be the full circle C. If $\alpha - \beta = 2\pi$, or $\alpha = \beta$, then let $\widehat{\lfloor \alpha, \beta \rceil}$ be the single point $\exp(i\alpha)$*. Finally, if $0 \leq \beta - \alpha \leq 2\pi$ and $I = \lfloor \alpha, \beta \rceil$ then define $\widehat{I}$ to be $\widehat{\lfloor \alpha, \beta \rceil}$.*

We say that $W(X)$, $X \in \bar{\mathbb{R}}$ is a weight on $\bar{\mathbb{R}}$, if

$$w(x) := \frac{W(\frac{1+x}{1-x}i)}{|1-x|}, \quad |x| = 1 \tag{52}$$

is a weight on C.

Remark A We note that this definition of weights on the real line is more general than the one given in [24] or [25], since we do not assume the existence of $\lim |X|W(X)$ as $|X| \to +\infty$. However, since $q := -\log(w)$ is bounded from below, $|X|W(X)$ must be bounded from above. In addition, studying weights on the compactified real line via weights on the unit circle C allows us to deduce several results on the supports of the equilibrium measure μ_W on the line via a general result for μ_w on the circle (see Theorems 27, 29 and 30).

In the next subsection, we describe the relation between the weighted energy problem on $\bar{\mathbb{R}}$ and on C.

3.2 Connection between the Equilibrium Problem on $\bar{\mathbb{R}}$ and on $\mathbb{C}$

We will make use of the Cayley transform between $\bar{\mathbb{R}}$ and on C as follows.

$$\bar{\mathbb{R}} \ni X \longmapsto x := \frac{X-i}{X+i} \in C \tag{53}$$

defines a bijection between $\bar{\mathbb{R}}$ and C. The inverse is

$$C \ni x \longmapsto X = \frac{1+x}{1-x} i \in \bar{\mathbb{R}} \tag{54}$$

The image of $Y, T \in \bar{\mathbb{R}}$ by the Cayley transform will be denoted by y and t.

To any measure $\mu \in \mathcal{P}(\bar{\mathbb{R}})$, we assign the Borel probability measure μ_C on C with

$$d\mu_C(x) := d\mu(X) \tag{55}$$

This mapping is a bijection between Borel probability measures on $\bar{\mathbb{R}}$ and C.

Let the weights W and w be related by (52). The weighted logarithmic potential of μ and μ_C is defined by

$$U_W^{\mu}(X) := \int \log \frac{1}{|T-X|W(T)W(X)} d\mu(T),$$

$$U_w^{\mu_C}(x) := \int \log \frac{1}{|t-x|w(t)w(x)} d\mu_C(t),$$

respectively ([25]). These are well-defined integrals (even though μ may not have compact support), as well as

$$I_W(\mu) := -\int\int \log(|X-Y|W(X)W(Y))d\mu(X)d\mu(Y).$$

From

$$|X-Y| = \Big|\frac{1+x}{1-x}i - \frac{1+y}{1-y}i\Big| = \frac{2|x-y|}{|1-x||1-y|}. \tag{56}$$

we have $|T-X|W(T)W(X) = 2|t-x|w(t)w(x)$. Thus

$$U_W^{\mu}(X) = U_w^{\mu_C}(x) - \log 2 \tag{57}$$

Integrating this we get

$$I_W(\mu) = I_w(\mu_C) - \log 2. \tag{58}$$

Since

$$W = e^{-Q}, \quad w = e^{-q} \tag{59}$$

we have the following correspondence between q and Q:

$$q(x) = Q\Big(\frac{1+x}{1-x}i\Big) + \log|1-x|,\ |x| = 1. \tag{60}$$

For convenience we will agree on the notations

$$q(\theta) := q(e^{i\theta}),\ w(\theta) := w(e^{i\theta}), \quad \theta \in \mathbb{R}.$$

Also, since

$$|1-x| = \frac{2}{|X+i|} = \frac{2}{\sqrt{1+X^2}},\ |x| = 1,\ X \in \bar{\mathbb{R}} \tag{61}$$

we have

$$Q(X) = q\Big(\frac{X-i}{X+i}\Big) + \frac{1}{2}\log(1+X^2) - \log 2,\ X \in \bar{\mathbb{R}}. \tag{62}$$

We find it more convenient to use angles instead of complex numbers on the unit circle. So let $x = e^{i\theta}$, and $y = e^{i\nu}$ for $\theta, \nu \in \mathbb{R}$.

Clearly,

$$\frac{|x-y|}{|1-x||1-y|} = \frac{|\sin\frac{\theta-\nu}{2}|}{2|\sin\theta/2||\sin\nu/2|} \ \text{ and } \ \frac{1+x}{1-x}i = -\cot\frac{\theta}{2}. \tag{63}$$

Therefore, using (63), we readily calculate that

$$I_W(\mu) =$$

$$= -\int\int \log\Big(\Big|\sin\frac{\theta-\nu}{2}\Big|\frac{W(-\cot\theta/2)}{|\sin\theta/2|}\frac{W(-\cot\nu/2)}{|\sin\nu/2|}\Big)d\mu(-\cot\frac{\theta}{2})d\mu(-\cot\frac{\nu}{2})$$

$$= -\int\int \log\Big(\Big|\sin\frac{\theta-\nu}{2}\Big|w(\theta)w(\nu)\Big)d\mu(-\cot\frac{\theta}{2})d\mu(-\cot\frac{\nu}{2}) - \log 4. \tag{64}$$

Here, we used the fact that $w(\theta) = W(-\cot\frac{\theta}{2})/(2|\sin\frac{\theta}{2}|)$ (see (52)). In addition we note that from (60) we get

$$q(\theta) = Q\left(-\cot\frac{\theta}{2}\right) + \log|\sin\frac{\theta}{2}| + \log 2. \tag{65}$$

The formulae (52)-(58) allow us to conclude the following:

$\mu \in \mathcal{P}(\bar{\mathbb{R}})$ minimizes the energy integral $I_W(\mu)$ over all probability measures on $\bar{\mathbb{R}}$ if and only if its corresponding $\mu_C \in \mathcal{P}(C)$ minimizes the energy integral $I_w(\mu_C)$ over all probability measures on C. Moreover, the support S_W is going to be an interval or a complement of an interval in $\bar{\mathbb{R}}$ if and only if the corresponding support S_w is an arc on C.

We close this section by introducing some remaining conventions which we assume henceforth.

Let $\tilde{I}$ be an arc of C. We shall say that $f : \tilde{I} \to \mathbb{R}$ is absolutely continuous inside $\tilde{I}$ if it is absolutely continuous on each compact subarc of $\tilde{I}$. (As a consequence, f' exists a.e. on $\tilde{I}$.)

Now let I be an interval or a complement of an interval in $\bar{\mathbb{R}}$. Let the arc $\tilde{I}$ be the image of I by the Cayley transform $T : \bar{\mathbb{R}} \to C$. We shall say that $f : I \to \mathbb{R}$ is absolutely continuous inside I if $f \circ T^{-1}$ is absolutely continuous inside $\tilde{I}$. (If I is a finite interval, this definition is equivalent to the usual definition of absolute continuity inside I.)

We say that a function f is increasing on an interval $I \subset \mathbb{R}$ if there exist $J \subset I$ such that the Lebesgue measure of $I \setminus J$ is zero and $f(x) \leq f(y)$ whenever $x, y \in J$, $x \leq y$. (This is a useful definition when f is defined only a.e. on I.) We define "decreasing" in a similar manner.

Moreover, we say that f is convex on an interval I if f is absolutely continuous inside I and f' is increasing on I.

We finally note that under Cayley transform (or its inverse), sets with positive capacity are transferred to sets with positive capacity.

3.3 Results on the Circle

Theorem 27 *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor \gamma, \delta \rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Assume that q is absolutely continuous inside I and*

$$\liminf_{\substack{x \to y \\ x \in I}} q(x) = q(y) \tag{66}$$

whenever y is an endpoint of I with $y \in I$. Let e^{ic} be any point which is not an interior point of $\widehat{I}$. Let $[\widehat{\alpha_1, \beta_1}], ..., [\widehat{\alpha_k, \beta_k}]$ be $k \geq 0$ arcs of C. Here, for all $1 \leq i \leq k$, $0 < \beta_i - \alpha_i \leq 2\pi$ and $(S_w \cup \widehat{I}) \subset [\widehat{\alpha_i, \beta_i}]$. Suppose further that I can be written as a disjoint union of $n \geq 1$ intervals $I_1, ..., I_n$ and for any fixed $1 \leq j \leq n$, either

$$e^{q(\theta)} \left[2\sin\left(\frac{\theta - c}{2}\right) q'(\theta) - \cos\left(\frac{\theta - c}{2}\right) \right] \operatorname{sgn}\left(\sin\left(\frac{\theta - c}{2}\right)\right) \tag{2.2}$$

is increasing on I_j or for some $1 \leq i \leq k$:

$$\sin\left(\frac{\theta - \alpha_i}{2}\right) \sin\left(\frac{\beta_i - \theta}{2}\right) q'(\theta) + \frac{1}{4} \sin\left(\theta - \frac{\alpha_i + \beta_i}{2}\right) \tag{2.3}$$

is increasing on I_j. Finally we assume that

$$\limsup_{\theta \to \theta_0^-} q'(\theta) \leq \liminf_{\theta \to \theta_0^+} q'(\theta),$$

whenever θ_0 is an endpoint of I_j $(1 \leq j \leq n)$ but not an endpoint of I. Then $S_w \cap \widehat{I}$ is an arc of C.

Here sgn denotes the signum function.

Remark B The choice of c is not important, see Remark F and the proof of Lemma 33. We also remark that if $\widehat{I}$ is the full circle, then one should check only condition (2.2) and ignore (2.3) which is a stronger assumption.

Below we give a condition which guarantees that S_w is the full circle:

Corollary 28 *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I_1 := (\gamma_1, \gamma_1 + 2\pi)$ and $I_2 := (\gamma_2, \gamma_2 + 2\pi)$ where $e^{i\gamma_1} \neq e^{i\gamma_2}$. Assume that (2.2) is increasing on I_1 where $c := \gamma_1$, and (2.2) is increasing on I_2 where $c := \gamma_2$. Then $S_w = C$.*

Proof: By Theorem 27 $S_w \cap \widehat{I_1}$ is an arc of C. Let e^{ic} be an interior point of this arc, not identical to $e^{i\gamma_2}$. Choose ρ_1, ρ_2 such that $c < \rho_2 < \rho_1 < c + 2\pi$ and both of the arcs $\widehat{(c, \rho_1)}$ and $\widehat{(\rho_2, c + 2\pi)}$ contain only one of $e^{i\gamma_1}$ and $e^{i\gamma_2}$, say, $\widehat{(c, \rho_1)}$ contains $e^{i\gamma_1}$ and $\widehat{(\rho_2, c + 2\pi)}$ contains $e^{i\gamma_2}$.

Using the first observation of Remark B, we see that (2.2) is increasing on (c, ρ_1) because (2.2) is increasing on (c, ρ_1) when at (2.2) c is replaced by γ_2. Similarly, (2.2) is increasing on $(\rho_2, c+2\pi)$ because (2.2) is increasing on $(\rho_2, c+2\pi)$ when at (2.2) c is replaced by γ_1. Thus (2.2) is increasing on $(c, c + 2\pi)$ and so $S_w = C$ by Theorem 27 and by the choice of c. $\square$

Example: The following example illustrates the theorem.

Let $q(\theta) = \cos(5\theta)\sin(3\theta)$ defined on $\Sigma = [2.9, 3.18] \cup [3.95, 4]$. (We may define w to be zero outside Σ so that w is defined on C.) We claim that both $S_w \cap \widehat{[2.9, 3.18]}$ and $S_w \cap \widehat{[3.95, 4]}$ are arcs of C. (One of them may be an empty set.)

Take $\alpha_1 = 2.9, \beta_1 = 4$ and $\alpha_2 = 3.95, \beta_2 = 3.18 + 2\pi$.

One can verify that (2.2) is satisfied on $[2.9, 3.17]$ but not on the whole $[2.9, 3.18]$. (At (2.2) c can be chosen to be any number such that e^{ic} is not an interior point of $\widehat{[2.9, 3.18]}$. Or, simply check the $(q')^2 + q'' + 1/4 \geq 0$ condition, see Remark F.) Also, using α_1 and β_1 we see that (2.3) is not satisfied on the whole $[2.9, 3.18]$. However (2.3) is satisfied on the subinterval $[3.17, 3.18]$ (see *Figure 2.1*). So the combination of the (2.2) and (2.3) conditions implies that $S_w \cap \widehat{[2.9, 3.18]}$ is an arc .

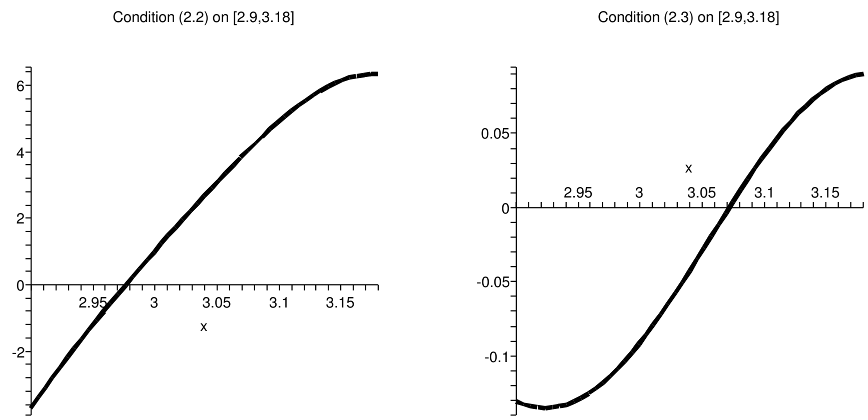

Figure 1: Conditions (2.2) and (2.3) on the interval I_1

Using α_1 and β_1 on $[3.95, 4]$ is not helpful since (2.3) is a decreasing function there. Also, (2.2) is not satisfied on the whole $[3.95, 4]$. However, (2.3) is satisfied using α_2 and β_2 on the whole $[3.95, 4]$. Theorem 27 now implies that $S_w \cap \widehat{[3.95, 4]}$ is an arc (see *Figure 2.2*). (We remark that α_2 and β_2 are not helpful on $[2.9, 3.18]$ since (2.3) is a decreasing function on $[3.17, 3.18]$.)

Remark C It is a natural question to ask what α_i and β_i numbers we should choose in order that (2.3) is as weak as possible. In most cases the following statement is true:

Let $\widehat{[\alpha, \beta]}$ and $\widehat{[\alpha', \beta']}$ $(0 < \beta - \alpha \leq 2\pi,\ 0 < \beta' - \alpha' \leq 2\pi)$ be two arcs of C

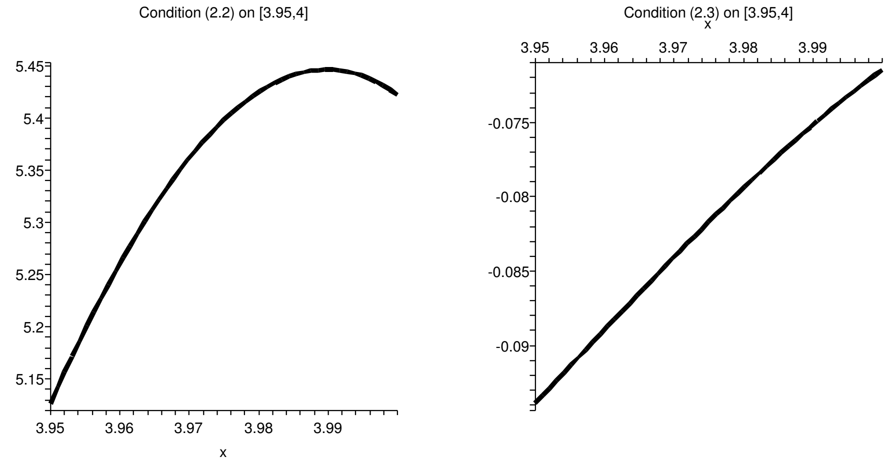

Figure 2: Conditions (2.2) and (2.3) on the interval I_2

such that $S_w \subset \widehat{[\alpha,\beta]} \subset \widehat{[\alpha',\beta']}$. Let $\widehat{I}$ be an arc contained in $\widehat{[\alpha,\beta]}$. If (2.2) or (2.3) is satisfied with α',β' then (2.2) or (2.3) is also satisfied with α,β.

For example, this statement is true if $q''(\theta)$ exists and the sets

$$H := \left\{\theta \in I : q'(\theta) > \frac{1}{2}\cot\left(\frac{\theta-\alpha'}{2}\right)\right\},\ H^* := \left\{\theta \in I : q'(\theta) > \frac{1}{2}\cot\left(\frac{\theta-\beta'}{2}\right)\right\}$$

consist of finitely many intervals. (The proof of this is similar to the proof of the statement in the remark at section 2.2.)

Theorem 27 can be effectively used when $w(z)$ is identically zero on some arcs (that is, Σ is a subset of finitely many arcs). If $w(z)$ is zero on $\widehat{[u_i,v_i]}$ $(0 < v_i - u_i < 2\pi)$, $i = 1,...,k$, then we may choose $\widehat{[\alpha_i,\beta_i]}$ to be $\widehat{[v_i,u_i]}$ in Theorem 27. This is consistent with the discussion above. For convenience we will state Theorem 29 in accordance with this remark.

3.4 Results on the Compactified Real Line

For simplicity we just state our next theorem for finite intervals (so Σ is a subset of $\mathbb{R}$). Afterwards we will explain how to modify the statement if we have an infinite interval or a complement of a finite interval.

Theorem 29 *For given $k \in \mathbb{N}^+$ let*

$$\Sigma := \cup_{i=1}^{k}[A_i, B_i] \subset \bar{\mathbb{R}}, \quad \textit{where} \tag{67}$$

$$-\infty < A_1 \leq B_1 < A_2 \leq B_2 < ... < A_k \leq B_k < +\infty.$$

Let $W = \exp(-Q)$ be a weight on Σ, $I \subset \Sigma$ be an interval and assume that Q is absolutely continuous inside I and

$$\liminf_{\substack{X \to Y \\ X \in I}} Q(X) = Q(Y) \tag{68}$$

whenever Y is an endpoint of I with $Y \in I$. Assume further that I can be written as a disjoint union of intervals $I_1, ..., I_n$ such that for any fixed $1 \leq j \leq n$ either

$$e^{Q(X)} \quad \textit{is convex on} \;\; I_j, \tag{69}$$

or for some $1 \leq i \leq k-1$

$$(X - B_i)(A_{i+1} - X)Q'(X) + X \quad \textit{is decreasing on} \;\; I_j, \tag{70}$$

or

$$(X - A_1)(B_k - X)Q'(X) + X \quad \textit{is increasing on} \;\; I_j. \tag{71}$$

Finally we assume that

$$\limsup_{X\to X_0^-} Q'(X) \le \liminf_{X\to X_0^+} Q'(X),$$

whenever X_0 *is an endpoint of* I_j $(1 \le j \le n)$ *but not an endpoint of* I*. Then* $S_W \cap I$ *is an interval.*

Remark D We remark that Theorem 29 is also valid when one interval, say, $[A_k, B_k]$ is an infinite interval or a complement of a finite interval. If $A_k > B_k$ (and, of course, $B_k < A_1$), then the conclusion of the theorem holds if (71) is replaced by the condition:

$$(X - B_k)(A_1 - X)Q'(X) + X \quad \text{is decreasing on} \;\; I_j. \tag{72}$$

If however $B_k = +\infty$ then (71) should be replaced by the condition:

$$(X - A_1)Q'(X) \quad \text{is increasing on} \;\; I_j. \tag{73}$$

Finally, if $A_1 = -\infty$ (and so $[A_1, B_1]$ is the infinite interval instead of $[A_k, B_k]$) then (71) should be replaced by the condition

$$(B_k - X)Q'(X) \quad \text{is increasing on} \;\; I_j. \tag{74}$$

At (72) and at Theorem 30 at (e) one can also consider an I which is a complement of a bounded interval. We leave the details for the reader.

Theorem 30 reveals to us the following remarkable connection between previously known conditions on Q. It also gives us a new condition (which is (e) below). As a consequence of Theorem 27 and 29 and Remark D, we now have the following general result for the case when Σ is one real interval. See also Chapter 2. Recall that for $A < B$ we define $[B, A] := (A, B)^c$.

Theorem 30 *Let W be a weight on $\mathbb{R}$ and let $I \subset \mathbb{R}$ be an interval. Assume that Q is absolutely continuous inside I and satisfies (68) . Let $A \leq B$ be finite constants and suppose that either of the following conditions below hold:*

(a) $(X - A)(B - X)Q'(X) + X$ *is increasing on* $I \subset [A, B]$, $S_W \subset [A, B]$.

(b) $(X - A)Q'(X)$ *is increasing on* $I \subset [A, +\infty)$, $S_W \subset [A, +\infty)$.

(c) $(B - X)Q'(X)$ *is increasing on* $I \subset (-\infty, B]$, $S_W \subset (-\infty, B]$.

(d) $(X - A)^2 Q'(X) - X$ *is increasing on* $I \subset \mathbb{R} \setminus \{A\}$,

(e) $(X - A)(B - X)Q'(X) + X$ *is decreasing on* $I \subset [B, A]$, $S_W \subset [B, A]$.

(f) Q *is convex on* I.

(g) $\exp(Q)$ *is convex on* I.

Then $S_W \cap I$ is an interval.

Remark E

Theoretically one should ignore (d) and (f) since (g) is a weaker assumption than both of these. Nevertheless we included them here, becasue sometimes they are easier to check.

Notice that (a) in Theorem 30 corresponds to the case of Theorem 27 when $\widehat{[\alpha, \beta]}$ is an arc of C disjoint of the point $x = 1$, (b) corresponds to the case when $\widehat{[\alpha, \beta]}$ is a proper subarc of C such that $\exp(i\beta) = 1$, (c) corresponds to the case when $\widehat{[\alpha, \beta]}$ is a proper subarc of C such that $\exp(i\alpha) = 1$, (d) corresponds to the case when $\widehat{[\alpha,\beta]}$ is the full circle C and a subcase of this is when $A=\infty$ (so $\alpha=0$ and $\beta=2\pi$) which corresponds to (f). The condition (e) corresponds to the case when $\widehat{[\alpha,\beta]}$ is a proper subarc of C which contains the point $x=1$ inside the arc. Finally, (g) is the only condition which corresponds to (2.2) and not (2.3).

Note also that if we let $A=B$ then (e) leads to condition (d), since $(X-A)(A-X)Q'(X)+X$ is decreasing if and only if $(X-A)^2Q'(X)-X$ is increasing.

One may also combine the above conditions to create a weaker condition in the spirit of Theorem 27 and 29.

3.5 Proofs

In this section, we present the proofs of our results. We find it convenient to break down our proofs into several auxiliary lemmas. Our first lemma is

Lemma 31 *Let* $w(z)=\exp(-q(z))$, $|z|=1$ *be a weight on* C *and let* $I=\lfloor\gamma,\delta\rceil$ *be an interval with* $0<\delta-\gamma\leq 2\pi$*. Let* $0<\beta-\alpha\leq 2\pi$ *and assume* $S_w\cup\widehat{I}\subset\widehat{[\alpha,\beta]}$*. Suppose* $q(\theta):=q(e^{i\theta})$ *is absolutely continuous inside* I *and satisfies (66). Moreover, assume that*

$$\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\beta-\theta}{2}\right)q'(\theta)+\frac{1}{4}\sin\left(\theta-\frac{\alpha+\beta}{2}\right) \tag{75}$$

is increasing on I*. Then* $S_w\cap\widehat{I}$ *is an arc of* C*.*

Proof: Let

$$A:=-\cot\frac{\alpha}{2},\ B:=-\cot\frac{\beta}{2},\ X:=-\cot\frac{\theta}{2}. \tag{76}$$

First let us assume that $\alpha, \beta \in (0, 2\pi)$. Thus we may assume that $0 < \alpha \leq \gamma < \theta < \delta \leq \beta < 2\pi$ and $0 < \sin(\alpha/2)$, $0 < \sin(\beta/2)$. So $A \leq X \leq B$.

From (65), we have

$$Q'\left(-\cot\frac{\theta}{2}\right) = 2\sin^2\left(\frac{\theta}{2}\right)\left(q'(\theta) - \frac{1}{2}\cot\frac{\theta}{2}\right). \tag{77}$$

Thus,

$$\begin{aligned}
&(X-A)(B-X)Q'(X) + X \qquad (78)\\
&= -\left(\cot\frac{\theta}{2} - \cot\frac{\alpha}{2}\right)\left(\cot\frac{\theta}{2} - \cot\frac{\beta}{2}\right)Q'\left(-\cot\frac{\theta}{2}\right) - \cot\frac{\theta}{2}\\
&= -\frac{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}\left(2q'(\theta) - \cot\frac{\theta}{2}\right) - \cot\frac{\theta}{2}.
\end{aligned}$$

Now we use the following identity which holds for any α, β, θ:

$$\cot\left(\frac{\theta}{2}\right)\left(\frac{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} - 1\right) = \frac{\sin(\theta - \frac{\alpha+\beta}{2})}{2\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})} - \frac{1}{2}\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right). \tag{79}$$

It follows that

$$\begin{aligned}
&(X-A)(B-X)Q'(X) + X \qquad (80)\\
&= -2\frac{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}q'(\theta) + \frac{\sin(\theta - \frac{\alpha+\beta}{2})}{2\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})} - \frac{1}{2}\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right).
\end{aligned}$$

Because $0 < \sin(\alpha/2)$, $0 < \sin(\beta/2)$, the right hand side of (80) is increasing on I if and only if (75) holds. Thus, if (75) holds then $(X-A)(B-X)Q'(X) + X$ is increasing on $\lfloor -\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2}\rceil$. Now consider the corresponding equilibrium problem on $\bar{\mathbb{R}}$ (as described earlier) and let S_W denote the corresponding equilibrium measure on $\bar{\mathbb{R}}$. Using Theorem 21 we get that $S_W \cap \lfloor -\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2}\rceil$ is an interval. It follows that $S_w \cap \widehat{I}$ is an arc of C. This proves Lemma 31 for the case when $\alpha, \beta \in (0, 2\pi)$.

Now let $\alpha \leq 2\pi \leq \beta, \beta - \alpha < 2\pi$. Note that $0 \leq \sin(\alpha/2), 0 \geq \sin(\beta/2)$. We cannot apply Theorem 21 because $B \leq A$ and X is outside $[B, A]$. However we can use the observation that condition (75) is "rotation invariant".

Let $0 < \sigma$ be a number such that

$$0 < \alpha - \sigma =: \alpha^*, \quad \beta^* := \beta - \sigma < 2\pi, \tag{81}$$

and define

$$\gamma^* := \gamma - \sigma,\ \delta^* := \delta - \sigma,$$

$$q_2(\theta) := q(\theta + \sigma). \tag{82}$$

For $w_2 = \exp(-q_2)$ and the parameters $\alpha^*, \beta^*, \gamma^*, \delta^*$, we may apply the case we studied above to get that $S_{w_2} \cap \lfloor\widehat{\gamma^*, \delta^*}\rceil$ is an arc of C. But this new equilibrium problem is isomorphic to the original one in the sense that everything (including the support) is rotated by the angle σ. It follows that $S_w \cap \widehat{I}$ is an arc of C.

Finally, we need to establish the lemma for the case when $\widehat{I}$ is the full circle. So let $\beta - \alpha := 2\pi$. Using the rotation invariance we may assume that $\alpha = 0, \beta = 2\pi$. Condition (75) is now equivalent to

$$\sin^2\left(\frac{\theta}{2}\right)q'(\theta) - \frac{1}{4}\sin\theta \quad \text{is increasing.} \tag{83}$$

Using (77) we get

$$2\sin^2(\frac{\theta}{2})q'(\theta) - \frac{1}{2}\sin\theta = Q'(-\cot\frac{\theta}{2}). \tag{84}$$

Thus $Q'(-\cot\frac{\theta}{2})$ is increasing $(0 < \theta < 2\pi)$, that is, $Q'(X)$ is increasing, and so $Q(X)$ is convex. It is well known, see [24], that in this case the support S_W is an interval. (The proof works for our more general weight.) So S_w is again an arc. We have completed the proof Lemma 31 . □

As a corollary to Lemma 31 , we have

Lemma 32 *Let W be a weight on $\bar{\mathbb{R}}$, let J be a finite interval and suppose that Q is absolutely continuous inside J and satisfies condition (68). Let $A \leq B$ be finite constants with $J \subset [B, A]$, $S_W \subset [B, A]$ and assume that $(X - A)(B - X)Q'(X) + X$ is decreasing on J. Then $S_W \cap J$ is an interval.*

Proof: Recall that $[B, A] = (A, B)^c$, see Definition 26.

We may find $\alpha < \beta$ such that $B = -\cot(\alpha/2)$, $A = -\cot(\beta/2)$ and $\beta - \alpha \leq 2\pi$. Notice

that $\sin(\alpha/2)\sin(\beta/2) < 0$ necessarily.

Let $J = \lfloor -\cot(\gamma/2), -\cot(\delta/2) \rceil$, where $\alpha \leq \gamma \leq \delta \leq \beta$ and so $\delta - \gamma \leq 2\pi$.

The left hand side of (80) is a decreasing function of X on J, and so the right hand side of (80) is a decreasing function of θ on $I := [\gamma, \delta]$. Multiply that right hand side by the negative constant $\sin(\alpha/2)\sin(\beta/2)$. In this way we get an increasing function of θ on $[\gamma, \delta]$. So condition (75) is satisfied and from Lemma 31 , we deduce that $S_w \cap \widehat{[\gamma, \delta]}$ is an arc of C. This implies immediately that $S_W \cap J$ is an interval. Lemma 32 is proved. □

Our final lemma is:

Lemma 33 *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor \gamma, \delta \rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Suppose q is absolutely continuous inside I*

and satisfies (66). Let e^{ic} be any point which is not an interior point of $\widehat{I}$. If

$$e^{q(\theta)}\Big[2\sin\Big(\frac{\theta-c}{2}\Big)q'(\theta)-\cos\Big(\frac{\theta-c}{2}\Big)\Big]\mathrm{sgn}\Big(\sin\Big(\frac{\theta-c}{2}\Big)\Big) \tag{85}$$

is increasing on I, then $S_w \cap \widehat{I}$ is an arc of C.

Remark F Whether (85) is increasing on I or not, it does not depend on the choice of c (as long as e^{ic} is not an interior point of $\widehat{I}$). The proof of this is given in the proof of Lemma 33 . We remark however that if q is twice differentiable then condition (85) is easily seen to be equivalent to

$$q'(\theta)^2 + q''(\theta) + \frac{1}{4} \geq 0, \quad \theta \in (\gamma, \delta)$$

which condition indeed does not depend on c.

We give the following example to Lemma 33 . Let Σ be one or several closed arcs on the unit circle but not the full circle. Assume the weight w is zero on the complement of Σ. Let $e^{i\rho}$ be a point in the complement of Σ, and define

$$q(\theta) := q(e^{i\theta}) := \log|\sin\frac{\theta-\rho}{2}| + d,$$

where d is an arbitrary constant. The value of c is our choice so let $c := \rho$. Then (85) is increasing on the whole of Σ (in fact it is identically zero) and therefore S_w is a set of arcs. Moreover, each arc of Σ contains at most one arc of S_w.

Proof of Lemma 33 : First we show that whether (85) is increasing on I or not, it does not depend on the choice of c. We do not assume the existance of q''.

Let $F(x)$ and $u(x)$ be two real functions on $(0,1)$ such that F is bounded and increasing, and u is non-negative and Lipschitz continous. Then there exists $E \subset (0,1)$ of full measure such that

$$\int_a^b \Big(F(x)u(x)\Big)' dx \leq (Fu)(b) - (Fu)(a) \text{ if } a, b \in E,\ a \leq b.$$

This observation easily follows from Fatou's Lemma applied to the sequence of functions $[(Fu)(x+\epsilon_n) - (Fu)(x)]/\epsilon_n,\ \ \epsilon_n \to 0^+$.

Suppose e^{ic} and e^{ic_2} are not interior points of $\widehat{I}$. Denote now (85) by $F_c(\theta)$. Let $J \subset I$ such that J has full measure and $F_c(x) \leq F_c(y)$ for all $x \leq y$, $x, y \in J$. We define the domain of F_c and q' to be J. We have

$$e^{q(\theta)}q'(\theta) = \frac{F_c(\theta) + e^{q(\theta)}\Big(\cos\frac{\theta-c}{2}\Big)\mathrm{sgn}\Big(\sin\frac{\theta-c}{2}\Big)}{2\Big|\sin\frac{\theta-c}{2}\Big|}, \quad \theta \in J, \tag{86}$$

which shows that $e^q q'$ is differentiable a.e. on J. Simple calculation gives

$$0 \leq F_c'(\theta) = 2\left|\sin\frac{\theta-c}{2}\right|\left[(e^{q(\theta)}q'(\theta))' + \frac{1}{4}e^{q(\theta)}\right] \text{ a.e. } \ \theta \in J. \tag{87}$$

Replace c by c_2 at the formula (85) and denote it by $F_{c_2}(\theta)$. Also, replace in that formula $e^q q'$ by the quotient at (86). Thus we see that with some $u(\theta)$, $v(\theta)$ functions $F_{c_2}(\theta) = F_c(\theta)u(\theta) + v(\theta)$ holds, where inside (γ, δ):

the function u is non-negative and Lipschitz continuous, F_c is increasing and bounded, and v is absolutely continuous (since e^q is absolutely continuous inside I).

So by the observation above, we have

$$\int_a^b (F_c u + v)' \leq (F_c u)(b) + v(b) - (F_c u)(a) - v(a) = F_{c_2}(b) - F_{c_2}(a)$$

for a.e. $a, b \in I$, where $a \leq b$. But this integral is non-negative, since $0 \leq F'_{c_2}$ a.e. $\theta \in I$ follows from (87) . Hence $0 \leq F_{c_2}(b) - F_{c_2}(a)$, i.e., F_{c_2} is increasing. And this is what we wanted to show.

We may assume that $c \leq \gamma < \delta \leq c + 2\pi$. Let us rotate now $\widehat{I}$ to a position such that the rotation takes e^{ic} to the point $x = 1$. Condition (85) will change accordingly to a new condition where now $c = 0$. (We denote the new rotated weight by $w = \exp(-q)$, too.) We now have to show that $S_w \cap \widehat{I}$ is an arc of C for the new S_w and new $\widehat{I}$. Once we have done that we simply rotate $\widehat{I}$ back to the original position and the proof is complete.

This argument shows that we can assume without loss of generality that $c = 0$ and $0 \leq \gamma < \delta \leq 2\pi$. Define

$$W\Big(\frac{1+x}{1-x} i\Big) := |1-x| w(x), \quad |x| = 1. \tag{88}$$

Using the arguments in Section 3.1, (88) may also be given as

$$W(X) := \frac{2w\Big(\frac{X-i}{X+i}\Big)}{\sqrt{1+X^2}}, \quad X \in \bar{\mathbb{R}}. \tag{89}$$

We define $Q(X)$ by $W(X) =: \exp(-Q(X))$. Since w is a weight on C, we know that W is a weight on $\bar{\mathbb{R}}$.

We now show that $e^{Q(X)} Q'(X)$ is increasing on

$$I^0 := \lfloor -\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2} \rceil. \tag{90}$$

Let $x = e^{i\theta}$. Note that from (65) we have

$$e^{Q(X)} = \frac{e^{q(\theta)}}{2|\sin\frac{\theta}{2}|}. \tag{91}$$

Using this and (77), for $\theta \in [0, 2\pi]$ we get

$$e^{Q(X)}Q'(X) = \frac{1}{2}e^{q(\theta)}(2\sin\frac{\theta}{2}q'(\theta) - \cos\frac{\theta}{2}). \tag{92}$$

Note that the right hand side of (92) is an increasing function of θ on I by assumption. Now we apply Theorem 19, to conclude that $S_W \cap I^0$ is an interval. (Although this theorem is formulated for weights with $\lim_{|X|\to+\infty} XW(X) = 0$, the argument in the proof may be applied word for word for the more general weights considered here. Naturally one should work with $U_W^{\mu_W}(X)$ in the proof.) Since $S_W \cap I^0$ is an interval we conclude that $S_w \cap \widehat{I}$ is an arc of C. The proof of Lemma 33 is complete. $\square$

We are now ready to present the

Proof of Theorem 27: If $\widehat{I}$ is the full circle C then it follows from the assumption that $e^{i\alpha_t} = e^{i\beta_t} = e^{ic} = e^{i\gamma}$ for all t. Now, if (2.3) is increasing on I_j then (2.2) is also increasing on I_j, as one can see. (Choose γ to be zero and use (84), (92) and the fact that the convexity of Q implies the convexity of $\exp(Q)$.) So we can get the weakest assumption if we assume that (2.2) is increasing on the whole I, and we already know from Lemma 33 that Theorem 27 holds under such an assumption. Thus, let us assume that $\widehat{I}$ is not the full circle.

As in the proof of Lemma 31 and 33 we observe that the statement of Theorem 27 is "rotation invariant". So, we may assume that $\widehat{[\gamma, \delta]}$ does not contain the $x = 1$ point and $e^{i\alpha_t} \neq 1$, $e^{i\beta_t} \neq 1$ for any t. We can also assume that $c = 0$.

Let $X = -\cot(\theta/2)$, $A_i = -\cot(\alpha_i/2)$ $B_i = -\cot(\beta_i/2)$, and $Q(X)$ be defined by (62). Let I_j be given by

$$I_j = \lfloor \xi_j, \eta_j \rceil, \quad 0 < \eta_j - \xi_j < 2\pi, \tag{93}$$

and define

$$I_j^0 := \lfloor -\cot\frac{\xi_j}{2}, -\cot\frac{\eta_j}{2} \rceil, \quad I^0 := \lfloor -\cot\frac{\gamma}{2}, -\cot\frac{\delta}{2} \rceil. \tag{94}$$

Note that I^0 is a finite subinterval of $\mathbb{R}$ and it is the disjoint union of the intervals I_j^0 $(j = 1, ..., n)$. We assume that I_j^0 is numerated from left to right. Note also that $[A_i, B_i] \supset I_j^0$ (recall Definition 26).

By assumption, for any j $(1 \leq j \leq n)$, we can find i $(1 \leq i \leq k)$, such that either

$$e^{Q(X)} \quad \text{is convex on} \quad I_j^0, \quad \text{or} \tag{95}$$

$$A_i < B_i \quad \text{and} \quad (X - A_i)(B_i - X)Q'(X) + X \quad \text{is increasing on} \quad I_j^0 \quad \text{or} \tag{96}$$

$$A_i \geq B_i \quad \text{and} \quad (X - A_i)(B_i - X)Q'(X) + X \quad \text{is decreasing on} \quad I_j^0. \tag{97}$$

((95) is coming from the argument in Lemma 33 , (96) is from Lemma 31 , and (97) is from Lemma 32 .)

Let $E_1 := 1$. We can find positive constants $E_2, ..., E_n$ (uniquely) such that the following function f is a positive continuous function inside I^0. For $x \in I_j^0$ $(j = 1, ..., n)$, let

$$f(x) := \begin{cases} E_k \exp(2Q(X)) & \text{if (95) is satisfied on } I_j^0 \\ E_k(X - A_i)(B - X_i) & \text{if (96) is satisfied on } I_j^0 \\ E_k(X - A_i)(X - B_i) & \text{if (97) is satisfied on } I_j^0. \end{cases} \tag{98}$$

Let $W := \exp(-Q)$. We can use the argument in Theorem 25 to deduce the result. For this purpose let $A = -\cot(\alpha/2)$ and $B = -\cot(\beta/2)$ be any two numbers such that $A < B$, $[A, B] \subset I^0$, $(A, B) \cap S_W = \emptyset$. Let $\mu_1 := \mu_w\Big|_{[(\alpha+\beta)/\widehat{2,(\alpha+\beta)}/2+\pi]}$, $\mu_2 := \mu - \mu_1$. Using $U_w^{\mu_w}(x) = U_w^{\mu_1}(x) + U_w^{\mu_2}(x)$ and the monotone convergence theorem it easily follows that $U_w^{\mu_w}(x)$ is absolutely continuous on $\widehat{[\alpha, \beta]}$, and so by (57) $U_W^{\mu_W}(X)$ is absolutely continuous on $[A, B]$. Also, as in Chapter 2 one can verify that

$$f(X)\frac{d}{dX}(U_W^{\mu_W}(X)) \tag{99}$$

is strictly increasing on $[A, B]$. By Lemma 18 we get that $S_W \cap [A, B]$ is an interval. It follows that $S_W \cap I^0$ is also an interval and $S_w \cap \widehat{I}$ is an arc of C. □

We conclude this section with

The Proof of Theorem 29 and Theorem 30 These follow easily using Theorem 27, Lemma 32 and the discussion in Section 3.1. □.

4 Approximation by Weighted Polynomials

4.1 Definitions and Notations

We collected below some frequently used definitions and notations in this chapter.

Definitions 34 *Let $L \subset \mathbb{R}$ and let $f: L \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.*

f is Hölder continuous with Hölder index $0 < \tau \leq 1$ if with some K constant $|f(x) - f(y)| \leq K|x-y|^\tau$, $x, y \in L$. In this case we write $f \in H^\tau(L)$.

The L^p norm of f is denoted by $||f||_p$. When $p = \infty$ we will also use the $||f||_L$ notation.

We say that an integral or limit exists if it exists as a real number.

Let $x \in \mathbb{R}$. If f is integrable on $L \setminus (x-\epsilon, x+\epsilon)$ for all $0 < \epsilon$ then the Cauchy principal value integral is defined as

$$PV \int_L f(t)dt := \lim_{\epsilon \to 0^+} \int_{L \setminus (x-\epsilon, x+\epsilon)} f(t)dt,$$

if the limit exists.

It is known that $PV \int_L g(t)/(t-x)dt$ exists for almost every $x \in \mathbb{R}$ if $g: L \to \mathbb{R}$ is integrable.

For $0 < \iota$ and $a \in \mathbb{R}$ we define

$$a_\iota^+ := \max(a, \iota) \quad \textit{and} \quad a_\iota^- := \max(-a, \iota).$$

For $a > b$ the interval $[a, b]$ is an empty set.

We say that a property is satisfied inside L if it is satisfied on all compact subsets of L.

$o(1)$ will denote a number which is approaching to zero. For example, we

may write $10^x = 100 + o(1)$ *as* $x \to 2$*. Sometimes we also specify the domain (which may change with* ϵ*) where the equation should be considered. For example,* $\sin(x) = o(1)$ *for* $x \in [\pi, \pi + \epsilon]$ *when* $\epsilon \to 0^+$*.*

Let $x \in [-1, 1]$*. Depending on the value of* $c \in [-1, 1]$ *the following integrals may or may not be principal value integrals.*

$$v_c(x) := -PV \int_{-1}^{c} \frac{\sqrt{1-t^2}}{\pi^2\sqrt{1-x^2}(t-x)} dt,$$

$$h_c(x) := PV \int_{c}^{1} \frac{\sqrt{1-t^2}}{\pi^2\sqrt{1-x^2}(t-x)} dt.$$

Define

$$B(x) := v_c(x) - h_c(x) = v_1(x) = -PV \int_{-1}^{1} \frac{\sqrt{1-t^2}}{\pi^2\sqrt{1-x^2}(t-x)} dt, \quad x \in [-1, 1].$$

$P_n(x)$ *and* $p_n(x)$ *denote polynomials of degree at most* n*.*

Recall the definition of functions with smooth integral from the introduction:

Definitions 35 *We say that* f *has smooth integral on* $R \subset L$*, if* f *is non-negative a.e. on* R *and*

$$\int_I f = (1 + o(1)) \int_J f \tag{100}$$

where $I,\ J \subset R$ *are any two adjacent intervals, both of which has length* $0 < \epsilon$*, and* $\epsilon \to 0$*. The* $o(1)$ *term depends on* ϵ *and not on* I *and* J*.*

We say that a family of functions $\mathcal{F}$ *has uniformly smooth integral on* R*, if any* $f \in \mathcal{F}$ *is non-negative a.e. on* R *and (100) holds, where the* $o(1)$ *term depends on* ϵ *only, and not on the choice of* f*,* I *or* J*.*

Cleary, if f is continuous and it has a positive lower bound on R then f has smooth integral on R. Also, non-negative linear combinations of finitely many functions with smooth integrals on R has also smooth integral on R.

From the Fubini Theorem it follows that if ν is a finite positive Borel measure on $T \subset \mathbb{R}$ and $\{v_t(x) : \ t \in T\}$ is a family of functions with uniformly smooth integral on R such that $v_t(x)$ is measurable on $T \times [a,b]$ $(t \in T, x \in [a,b])$, then

$$v(x) := \int_T v_t(x) d\nu(t)$$

has also smooth integral on R.

Finally, if $f_n \to f$ uniformly a.e. on R, f_n has smooth integral on R and f has positive lower bound a.e. on R then f has smooth integral on R.

Remark 36 $\sqrt{1-t^2} \in H^{0.5}([-1,1])$ *so* $\sqrt{1-x^2}B(x) \in H^{0.5}([-1,1])$ *by the Plemelj-Privalov Theorem ([21], §19). As a consequence,* $v_c(x)$ *and* $h_c(x)$ *exist for any* $x \in [-1,1] \setminus \{c\}$.

The following definitions and facts are well known in logarithmic potential theory (see [24] and [25]). We have disscussed them in section 1.2, but now we will also consider weights which are not "admissible".

Let $w(x) \not\equiv 0$ be a non-negative continuous function on $\bar{\mathbb{R}}$ such that

$$\lim_{x\to\infty} |x| w(x) = \alpha \in [0, +\infty) \text{ exists }. \tag{101}$$

When $\alpha = 0$, then w belongs to the class of so called "admissible" weights.

We write $w(x) = \exp(-q(x))$ and call $q(x)$ external field. If μ is a positive Borel unit measure on $\bar{\mathbb{R}}$ - in short a "probability measure", then its weighted energy is defined by

$$I_w(\mu) := \int\int \log \frac{1}{|x-y|w(x)w(y)} d\mu(x)d\mu(y).$$

The integrand is bounded from below ([25], pp. 3), so $I_w(\mu)$ is well defined and $-\infty < I_w(\mu)$. We remark that we did not assume that μ has compact support. Whenever it makes sense, we define the (unweighted) logarithmic energy of μ as $I_1(\mu)$ where 1 denotes the constant 1 function. There exists a unique probability measure μ_w - called the equilibrium measure associated with w - which minimizes $I_w(\mu)$. Also,

$$V_w := I_w(\mu_w) \quad \text{is finite,}$$

and μ_w has finite logarithmic energy when $\alpha = 0$.

If the support of μ is compact, we define its potential as

$$U^\mu(x) := \int \log \frac{1}{|t-x|} d\mu(t).$$

This definition makes sense for a signed measure ν, too, if $\int \Big| \log|t-x| \Big| d|\nu|(t)$ exists.

Let

$$S_w := \operatorname{supp}(\mu_w) \ \text{ denote the support of } \ \mu_w.$$

When $\alpha = 0$, then S_w is a compact subset of $\mathbb{R}$. In this case with some F_w constant we have

$$U^{\mu_w} + q(x) = F_w, \quad x \in S_w.$$

4.2 Lemmas

We will need Lemma 22 of [3]. We formulate it as follows:

Lemma 37 *Let $A < B < 1$, $f \in L^1[A,1]$ and $f \in H^1[A,(B+1)/2]$. Define $v^*(x) := \int_c^1 f(t)/(t-x)dt$, where $c \in [A,B]$ and $x < c$. Then*

$$v^*(x) = (f(c) + o(1)) \log \frac{1}{c-x}, \qquad as \qquad x \to c^-.$$

Here $o(1)$ depends on $c - x$ only.

Lemma 38 *Let $-1 < a < b < 1$ and $0 < \iota$ be fixed. Let $0 < \epsilon < 1/10$ and $\delta := \sqrt{\epsilon} - 2\epsilon$. Then for $x_1, x_2 \in [a,b] \cap (c-\delta, c+\delta)^c$, $|x_1 - x_2| \le \epsilon$, all the quotients*

$$\frac{v_c(x_1)_\iota^+}{v_c(x_2)_\iota^+}, \quad \frac{v_c(x_1)_\iota^-}{v_c(x_2)_\iota^-}, \quad \frac{h_c(x_1)_\iota^+}{h_c(x_2)_\iota^+}, \quad \frac{h_c(x_1)_\iota^-}{h_c(x_2)_\iota^-}$$

equal to $1 + o(1)$ as $\epsilon \to 0^+$. Here the $o(1)$ term is independent of x_1, x_2 and c.

Proof. First we consider the case when $x_1,\ x_2 \le c - \delta$. Note that for $x_1 > x_2$ we have $1/(t-x_2) < 1/(t-x_1)$, $t \in [c,1]$, whereas for $x_1 \le x_2$ we have

$$\frac{1}{t-x_2} \le \Big(1 + \frac{x_2 - x_1}{c - x_2}\Big)\frac{1}{t-x_1} = (1+o(1))\frac{1}{t-x_1}, \quad t \in [c,1].$$

Multiplying these inequalities by $\sqrt{1-t^2}/\pi^2$ and integrating on $[c,1]$ we gain

$$\frac{h_c(x_2)}{h_c(x_1)} = 1 + o(1), \tag{102}$$

where $\sqrt{1-x_2^2}/\sqrt{1-x_1^2} = 1+o(1)$ was also used. By the same argument, if $x_1,\ x_2 \geq c+\delta$, we have $v_c(x_2)/v_c(x_1) = 1+o(1)$, from which

$$\frac{v_c(x_2)_\iota^+}{v_c(x_1)_\iota^+} = 1+o(1). \tag{103}$$

Returning to the case of $x_1,\ x_2 \leq c-\delta$, from $v_c(x) = h_c(x)+B(x)$, from (102) and from $B(x_2) = B(x_1)+o(1)$ we get

$$|v_c(x_2) - v_c(x_1)| = |o(1)|(1+|v_c(x_1) - B(x_1)|)$$

$$\leq |o(1)|(|v_c(x_1)| + 1 + ||B||_{[a,b]}). \tag{104}$$

Assuming $|v_c(x_1)| \leq 1$, we have

$$|v_c(x_2)_\iota^+ - v_c(x_1)_\iota^+| \leq |v_c(x_2) - v_c(x_1)| \leq |o(1)|,$$

so (103) holds again. Finally, if $|v_c(x_1)| \geq 1$, then from (104)

$$\left|\frac{v_c(x_2)}{v_c(x_1)} - 1\right| = |o(1)|\Big(1+\frac{1+||B||_{[a,b]}}{|v_c(x_1)|}\Big) = |o(1)|,$$

from which (103) again easily follows.

The proof of the rest of our lemma is similar.

∎

Lemma 39 *Let $-1 < a < b < 1$ and $0 < \iota$ be fixed. Then the family of functions $\mathcal{F}^+ := \{v_c(x)_\iota^+ :\ c \in [-1,1]\}$ and $\mathcal{F}^- := \{v_c(x)_\iota^- :\ c \in [-1,1]\}$ have uniformly smooth integrals on $[a,b]$.*

Proof. We consider $\mathcal{F}^+$ only ($\mathcal{F}^-$ can be handled similarly). Let $c \in [-1,1]$. Let $I := [u-\epsilon, u]$, $J := [u, u+\epsilon]$ be two adjacent intervals of $[a,b]$, where $0 < \epsilon < 1/10$. We have to show that

$$\frac{\int_I v_c(t)_\iota^+ dt}{\int_J v_c(t)_\iota^+ dt} = 1 + o(1), \quad \text{as} \quad \epsilon \to 0^+,$$

where $o(1)$ is independent of I, J and c. Let $0 < \epsilon < 1/10$ and let $\delta := \sqrt{\epsilon} - 2\epsilon$ ($> \epsilon$).

Case 1: Assume $I \cup J \subset (c-\delta, c+\delta)^c$. From Lemma 38 we have $v_c(t)_\iota^+ = (1+o(1))v_c(t+\epsilon)_\iota^+$, $t \in I$. Thus $\int_I v_c(t)_\iota^+ dt = (1+o(1)) \int_J v_c(t)_\iota^+ dt$.

Case 2: Assume $(I \cup J) \cap (c-\delta, c+\delta) \neq \emptyset$. So $I \cup J \subset [c-\sqrt{\epsilon}, c+\sqrt{\epsilon}]$. Let ϵ be so small that $c \in [(a-1)/2, (b+1)/2]$. (This can be done because of our assumption of Case 2.)

Let $f(t) := \sqrt{1-t^2}/\pi^2$. Applying Lemma 37 (with $A := (a-1)/2$, $B := (b+1)/2$) we have $\sqrt{1-x^2}h_c(x) = (f(c)+o(1))(-\log|c-x|)$ for $x \in [c-\sqrt{\epsilon}, c)$ as $\epsilon \to 0^+$, which easily leads to

$$h_c(x) = (\frac{f(c)}{\sqrt{1-c^2}} + o(1))(-\log|c-x|) \text{ for } x \in [c-\sqrt{\epsilon}, c) \text{ as } \epsilon \to 0^+.$$

From here using $h_c(x) = v_c(x) - B(x)$ we get

$$v_c(x) = (\frac{f(c)}{\sqrt{1-c^2}} + o(1))(-\log|c-x|) \text{ for } x \in [c-\sqrt{\epsilon}, c) \text{ as } \epsilon \to 0^+. \quad (105)$$

Clearly, (105) also holds for $x \in (c, c+\sqrt{\epsilon}]$ (which can be seen by stating Lemma 37 for $-1 < A < B$ instead of $A < B < 1$).

$f(x)$ has a positive lower bound on $[(a-1)/2, (b+1)/2]$. So we can choose ϵ so small that the right hand side of (105) is at least ι for all possible values of c

and x. Hence $v_c(x) = v_c(x)^+_\iota$ and

$$\frac{\int_I v_c(t)^+_\iota dt}{\int_J v_c(t)^+_\iota dt} = \frac{(\frac{f(c)}{\sqrt{1-c^2}} + o(1)) \int_I \log \frac{1}{|c-t|} dt}{(\frac{f(c)}{\sqrt{1-c^2}} + o(1)) \int_J \log \frac{1}{|c-t|} dt} = (1 + o(1))^2 = 1 + o(1),$$

where we used that $\log(1/|x|)$ has smooth integral on $[-1/2, 1/2]$ ([3], Proposition 20).

∎

Lemma 40 *Let $F(x) = G(x) - H(x)$, where $F(x)$, $G(x)$, $H(x)$ are a.e. non-negative functions defined on an interval, $G(x)$ and $H(x)$ have smooth integrals and $H(x) \le (1-\eta)G(x)$ a.e. with some $\eta \in (0,1)$. Then $F(x)$ has smooth integral.*

Proof. Let I and J be two adjacent intervals of equal lengths ϵ, where ϵ is "small enough". Let $a := \int_I G$, $A := \int_J G$, $b := \int_I H$, $B := \int_J H$. By assumption

$$A = (1 + o(1))a \quad \text{and} \quad B = (1 + o(1))b, \quad \text{as } \epsilon \to 0^+ \tag{106}$$

and we have to show that $A - B = (1 + o(1))(a - b)$.

We may assume that $a - b \neq 0$, otherwise $F(x) = 0$ a.e. on I which implies $a = b = 0$ and so $A = B = 0$.

Integrating $H \le (1-\eta)G$ on I we get $b \le (1-\eta)a$, from which $(a+b)/(a-b) \le (1 + (1-\eta))/(1 - (1-\eta))$. Thus, from (106)

$$|(A - a) - (B - b)| \le |o(1)|(a + b) \le |o(1)|(a - b).$$

∎

Lemma 41 *Let $N(x)$ be a right-continuous function on $[-1,1]$ which is of bounded variation. Let $f(x) \in L^1([-1,1])$ be non-negative. Then*

$$PV \int_{-1}^{1} \frac{f(t)N(t)}{t-x} dt = -N(1)f_1(x) + \int_{(-1,1]} f_t(x) dN(t), \quad a.e.\ x \in [-1,1], \tag{107}$$

where the integral on the right hand side is a Lebesgue-Stieltjes integral and

$$f_c(x) := -PV \int_{-1}^{c} \frac{f(t)}{t-x} dt, \quad a.e.\ x \in [-1,1].$$

Proof. First let us assume that $N(x)$ is a bounded, increasing, right continuous function.

Let us denote the left hand side of (107) by $F(x)$. Since $f(x)$ and $f(x)N(x)$ are in $L^1[-1,1]$ and $N(x)$ is of bounded variation, there is a set of full measure in $(-1,1)$ where $f_1(x)$, $F(x)$ and $N'(x)$ all exist. Let x be chosen from this set. It follows that $f_c(x)$ exist for all $c \in [-1,1] \setminus \{x\}$. Also,

$$F(x) = \lim_{\epsilon \to 0^+} \Big(\int_{-1}^{x-\epsilon} \frac{f(t)N(t)}{t-x} dt + \int_{x+\epsilon}^{1} \frac{f(t)N(t)}{t-x} dt \Big). \tag{108}$$

$t \to f_t(x)$ is an absolute continuous increasing function inside $[-1,x)$ and it is an absolute continuous decreasing function inside $(x,1]$ so at (108) we can use integration by parts to get

$$\int_{-1}^{x-\epsilon} + \int_{x+\epsilon}^{1} = -f_{x-\epsilon}(x)N(x-\epsilon) + f_{-1}(x)N(-1) + \int_{(-1,x-\epsilon]} f_t(x) dN(t)$$

$$+ f_{x+\epsilon}(x)N(x+\epsilon) - f_1(x)N(1) + \int_{(x+\epsilon,1]} f_t(x) dN(t)$$

But above $f_{-1}(x) = 0$ and

$$f_{x+\epsilon}(x)N(x+\epsilon) - f_{x-\epsilon}(x)N(x-\epsilon)$$

$$= [f_{x+\epsilon}(x) - f_{x-\epsilon}(x)]N(x+\epsilon) + f_{x-\epsilon}(x)[N(x+\epsilon) - N(x-\epsilon)]. \tag{109}$$

Note that

$$f_{x+\epsilon}(x) - f_{x-\epsilon}(x) = -PV\int_{x-\epsilon}^{x+\epsilon} \frac{f(t)}{t-x}dt \to 0 \qquad \text{as } \epsilon \to 0^+,$$

since $f_1(x)$ exists.

We claim that $\epsilon f_{x-\epsilon}(x) \to 0$ (and so the second term in (109) also tends to 0 since N is differentiable at x). In other words we claim that

$$\lim_{\epsilon\to 0^+} \epsilon \int_{\epsilon}^{1} \frac{g(t)}{t}dt \to 0$$

for any integrable non-negative $g(t)$ function. Integration by parts easily yields to

$$\lim_{\epsilon\to 0^+} \epsilon \int_{\epsilon}^{1} \frac{g(t)}{t}dt = \lim_{\epsilon\to 0^+} \epsilon \int_{\epsilon}^{1} \frac{\int_0^t g(u)du}{t^2}dt$$

$$= \lim_{\epsilon\to 0^+}\left(\epsilon \int_{\epsilon}^{\rho}\right) + \lim_{\epsilon\to 0^+}\left(\epsilon \int_{\rho}^{1}\right) \leq \int_0^{\rho} g(u)du + \lim_{\epsilon\to 0^+}\left(\epsilon \int_{\rho}^{1}\right),$$

where ρ was chosen such that $\int_0^{\rho} g(u)du$ is small. And this verifies our claim.

Putting these together, we get that on one hand,

$$\lim_{\epsilon\to 0^+}\Big(\int_{(-1,x-\epsilon]} f_t(x)dN(t) + \int_{(x+\epsilon,1]} f_t(x)dN(t)\Big) \tag{110}$$

exists and equals to $F(x) + f_1(x)N(1)$, and on the other hand, (110) equals to

$$\int_{(-1,1]\setminus\{x\}} f_t(x)dN(t) = \int_{(-1,1]} f_t(x)dN(t) \tag{111}$$

by the monotone convergence theorem (which can be used since $c \to f_c(x)$ is bounded from below on $[-1,1]$ since $f_1(x)$ is finite). The the continuity of N at x allowed us to integrate on the whole $(-1,1]$ at (111). Thus (107) is proved.

Now let $N(x)$ be a right-continuous function on $[-1,1]$ which is of bounded variation. Then there exist $N_1(t), N_2(t)$ bounded increasing right continuous functions such that $N(t) = N_1(t) - N_2(t),\ t \in [-1,1]$. Almost everywhere in $[-1,1]$ the left handside of (107) is finite and (107) holds. This is also true for $N_2(x)$. Therefore (107) is true for for $N(x)$.

∎

Definition 42 *We say that a function $g(x)$ has bounded variation almost everywhere inside a set $E \subset \mathbb{R}$ if for every compact set $F \subset E$ there exists $G \subset F$ such that $F \setminus G$ has measure zero and $g(x)$ has bounded variation on G.*

Lemma 43 *Let w be an admissible weight which is absolutely continuous inside $\mathbb{R}$. Let the interval $[a_0, b_0]$ be a subset of the support S_w. If $q'(x)$ has bounded variation a.e. inside (a_0, b_0) and the extremal measure has a density V on $[a_0, b_0]$ that has positive lower bound inside (a_0, b_0) then $(a_0, b_0) \cap Z(w) = \emptyset$.*

Proof. Clearly, it is enough to prove that $(a, b)\cap Z(w) = \emptyset$ for any $[a, b] \subset (a_0, b_0)$ subinterval. So let $[a, b] \subset (a_0, b_0)$. Let us replace (a_0, b_0) by a slightly smaller open subinterval which still contains $[a, b]$. This way we achieve that $q'(x)$ has bounded variation a.e. on $[a_0, b_0]$. By Theorem 6 of Totik it is enough to show that V has smooth integral on $[a, b]$.

First we remark that the equilibrium measure μ_w is absolute continuous with respect to the Lebesgue measure and $V(t) := d\mu_w(t)/dt$ is in L^p inside (a_0, b_0) for any $1 < p < \infty$. These follows from Theorem IV.2.2 of [24], since w is absolute continuous on $[a, b]$ and $w'(x) = -w(x)q'(x)$ is in $L^p([a, b])$.

Now we need the concept of the balayage measure. Let ν be a measure on the real line and K be an interval. There is a unique measure $\overline{\nu}$ supported on K such that the total mass of $\overline{\nu}$ equals the total mass of ν and for some constant d we have $U^{\overline{\nu}}(x) = U^{\nu}(x) + d$ for every $x \in K$. $\overline{\nu}$ is called the balayage of ν onto K. Actually, the balayage process moves (sweeps) only the part of ν lying outside K, i.e.,

$$\overline{\nu} = \nu|_K + \overline{\nu|_{\mathbb{R}\setminus K}} \,. \tag{112}$$

For the second measure on the right there is a closed form (see [24], formula II.4.47), which shows that by taking balayage onto K, we add to the portion of ν lying in K a measure with a continuous density.

The relevance of the balayage to extremal fields is explained by the following: if $K \subset S_w$ is a closed interval and w_1 is the restriction of w onto K (i.e., the weight w_1 is considered on K), then the equilibrium measure μ_{w_1} associated with w_1 is the balayage of μ_w onto K. (See [24], Theorem IV.1.6(e)).

Let us restrict w to $[a_0, b_0]$. Based on what we said about the balayage and smooth integral, it is enough to prove that the equilibrium measure associated with this restricted weight function has a density V_1 which has smooth integral on $[a, b]$. Indeed, by (112), $V = V_1 - V_2$, where $0 \leq V_2$ is continuous and V has a positive lower bound on $[a, b]$, so if V_1 has smooth integral on $[a, b]$, so does v.

Therefore from now on we will assume that w is defined on $[a_0, b_0]$, i.e., $\Sigma = [a_0, b_0]$. We will continue to use v for the density of the equilibrium measure associated with this restricted w. This has positive lower bound on $[a_0, b_0]$, too. Also, from now on w denotes the restricted weight. Furthermore, because of the balayage process, the new support S_w is the interval $[a_0, b_0]$ (so V is defined on $[a_0, b_0]$).

For $x \notin (a_0, b_0)$ let $V_0(x) := 0$, and for a.e. $x \in (a_0, b_0)$ let

$$V_0(x) := \frac{PV \int_{a_0}^{b_0} \frac{\sqrt{(t-a_0)(b_0-t)}q'(t)}{t-x} dt}{\pi^2 \sqrt{(x-a_0)(b_0-x)}} + \frac{1}{\pi \sqrt{(x-a_0)(b_0-x)}}. \tag{113}$$

We now show that this is the density of μ_w, that is, $V(x) = V_0(x)$ a.e. $x \in [a_0, b_0]$.

The integral at (113) is the Hilbert transform on $\mathbb{R}$ of the function defined as $\sqrt{(t-a_0)(b_0-t)}q'(t)$ on (a_0, b_0) and 0 elsewhere. This function is in $L^p(\mathbb{R})$, so by the M. Riesz' Theorem the integral is also in $L^p(\mathbb{R})$ hence $V_0(x)$ exists for a.e. $x \in [a_0, b_0]$. Moreover, by the Hölder inequality ($1/a + 1/b = 1/c$ implies $||fg||_c \leq ||f||_a ||g||_b$) we see that $V_0 \in L_{1.9}(\mathbb{R})$, so $V_0 \in L_1(\mathbb{R})$, too.

By the proof of Lemma 16 of [3], the function V_0 satisfies $\int V_0(x)dx = 1$ and

$$\int_{a_0}^{b_0} \log|t-x| V_0(t) dt = q(x) + C, \quad x \in (a_0, b_0). \tag{114}$$

The left hand side is well defined since by the Hölder inequlaity

$$x \mapsto \int_{a_0}^{b_0} \Big| \log|t-x| \Big| |V_0(t)| dt \quad \text{is uniformly bounded on } [a_0, b_0]. \tag{115}$$

Consider the unit signed measure μ defined by $d\mu(x) := V_0(x)dx$. By (114) $U^{\mu}(x) + q(x) = -C$, $x \in (a_0, b_0)$. From this and from $U^{\mu_w}(x) + q(x) = F_w$,

$x \in [a_0, b_0]$, we get $U^{\mu}(x) = U^{\mu_w}(x)$, $x \in (a_0, b_0)$. But (115) shows that $U^{\mu^+}(x)$ and $U^{\mu^-}(x)$ are finite for all $x \in [a_0, b_0]$. So $U^{\mu^+}(x) = U^{\mu_w+\mu^-}(x)$, $x \in (a_0, b_0)$. Here μ^+ and $\mu_w + \mu^-$ are positive measures which have the same mass. μ_w, μ^- (and μ^+) all have finite logarithmic energy (see (115)), hence $\mu_w + \mu^-$ has it, too. Applying Theorem II.3.2. of [24] we get $U^{\mu^+}(z) = U^{\mu_w+\mu^-}(z)$ for all $z \in \mathbb{C}$. By the unicity theorem ([24], Theorem II.2.1.) $\mu^+ = \mu_w + \mu^-$. Hence $\mu = \mu_w$, that is, $V(x) = V_0(x)$ a.e. $x \in [a_0, b_0]$.

To keep the notations simple we will assume that $-1 < a < b < 1$, and $a_0 = -1$, $b_0 = 1$, that is, the support of μ_w is $[-1, 1]$. This can be done without loss of generality. Let $E \subset [-1, 1]$ denote the set of full measure where $q'(x)$ exists and has bounded variation on E. For $t \in [-1, 1]$ define

$$v(t) := \frac{\sqrt{1-t^2}}{\pi^2\sqrt{1-x^2}} \quad \text{and} \quad M(t) := \lim_{s\to t^+,\ s\in E} q'(s),$$

where $v(t)$ also depends on the choice of x. Clearly, $M(t) = q'(t)$ a.e., $M(t)$ has bounded variation on $[-1, 1]$ and it is right continuous. It is known that there exist $M_1(t), M_2(t)$ bounded increasing right continuous functions such that $M(t) = M_1(t) - M_2(t),\ t \in [-1, 1]$.

Applying Lemma 41 for $f(t) := v(t)$ and $N(t) := M(t)$, let us fix an $x \in [a, b]$ value for which both (107) and $d\mu_w(x) = V(x)dx$ are satisfied. (These are satisfied almost everywhere.) From (113) and Lemma 41 we have

$$V(x) = \frac{1}{\pi\sqrt{1-x^2}} + PV\int_{-1}^{1} \frac{\sqrt{1-t^2}q'(t)}{\pi^2\sqrt{1-x^2}(t-x)}dt$$

$$= \frac{1}{\pi\sqrt{1-x^2}} + PV\int_{-1}^{1} \frac{v(t)M(t)}{t-x}dt = L(x) + \int_{(-1,1]} v_t(x)dM(t),$$

where $L(x) := 1/(\pi\sqrt{1-x^2}) - M(1)B(x)$.

Let $0 < \iota$. Since $L(x)$ is a continuous function on $[a, b]$ (see Remark 36), $L(x)_\iota^+$ and $L(x)_\iota^-$ have smooth integrals on $[a, b]$. Also, by Lemma 39 $\mathcal{F}^+$ and $\mathcal{F}^-$ have uniformly smooth integrals on $[a, b]$, so all

$$\int_{(-1,1]} v_t(x)_\iota^+ dM_1(t), \quad \int_{(-1,1]} v_t(x)_\iota^- dM_1(t),$$

$$\int_{(-1,1]} v_t(x)_\iota^+ dM_2(t), \quad \int_{(-1,1]} v_t(x)_\iota^- dM_2(t)$$

have smooth integral on $[a, b]$. Therefore

$$V(x)_{(\iota)}^{(+)} := L(x)_\iota^+ + \int_{(-1,1]} v_t(x)_\iota^+ dM_1(t) + \int_{(-1,1]} v_t(x)_\iota^- dM_2(t) \quad \text{and}$$

$$V(x)_{(\iota)}^{(-)} := L(x)_\iota^- + \int_{(-1,1]} v_t(x)_\iota^- dM_1(t) + \int_{(-1,1]} v_t(x)_\iota^+ dM_2(t)$$

have smooth integrals on $[a, b]$. (These new functions are not to be mixed with $V(x)_\iota^-$ and $V(x)_\iota^-$.)

Set

$$V(x)_{(\iota)} := V(x)_{(\iota)}^{(+)} - V(x)_{(\iota)}^{(-)}.$$

Then, using $|z_\iota^+ - z_\iota^- - z| \leq \iota,\ z \in \mathbb{R}$, we get

$$|V(x)_{(\iota)} - V(x)| \leq |L(x)_\iota^+ - L(x)_\iota^- - L(x)|$$

$$+\int_{(-1,1]} |v_t(x)^+_\iota - v_t(x)^-_\iota - v_t(x)|dM_1(t) + \int_{(-1,1]} |v_t(x)^+_\iota - v_t(x)^-_\iota - v_t(x)|dM_2(t)$$

$$\leq \iota + \int_{(-1,1]} \iota dM_1(t) + \int_{(-1,1]} \iota dM_2(t)$$

$$= \iota(1 + M_1(1) - M_1(-1) + M_2(1) - M_2(-1)). \tag{116}$$

So

$$V(x)_{(\iota)} \to V(x) \ \text{ uniformly a.e. on } [a,b] \text{ as } \iota \to 0^+. \tag{117}$$

And since

$$V(x) \ \text{ has positive lower bound a.e. on } \ [a,b], \tag{118}$$

$V(x)_{(\iota)}$ has also positive lower bound a.e. on $[a,b]$, assuming ι is small enough. In addition, $v_t(x) \geq 0$ when $t \in [0,x]$, whereas $v_t(x) \geq B(x) \geq -||B||_{[a,b]}$ when $t \in (x,1]$, so $V(x)^{(-)}_{(\iota)}$ is bounded a.e. on $[a,b]$. It follows that $V(x)^{(-)}_{(\iota)} \leq (1-\eta)V(x)^{(+)}_{(\iota)}$ a.e. $x \in [a,b]$ for some $\eta \in (0,1)$.

Applying Lemma 40 we conclude that $V(x)_{(\iota)}$ has smooth integral on $[a,b]$ (if ι is small enough). Therefore $V(x)$ has smooth integral by (117) and (118).

∎

4.3 Solution of the Approximation Problem

Now we will prove our main theorem.

Theorem 44 *Let w be a continuous admissible weight on $\mathbb{R}$. Suppose that q is weak convex on $[A,B]$ with finite basepoints A, B satisfying $S_w \subset (A,B)$. Then*

$Z(w) = (\text{int } S_w)^c$. *Thus a continuous function $f(x)$ can be uniformly approximated by weighted polynomials $w^n P_n$ if and only if $f(x)$ vanishes outside S_w.*

Proof. By Theorem 25 $S_w =: [a_0, b_0]$ is an interval. Let $v(x), x \in [a_0, b_0]$ denote the density of μ_w (we have already seen that it exists). Let $[a, b] \subset (a_0, b_0)$ be arbitrary. We will show that $v(x)$ has positive lower bound and bounded variation a.e. on $[a, b]$. Once we did that we are done by Lemma 43.

By assumption $[a, b]$ can be written as the disjoint union of finitely many intervals $I_1, ..., I_n$ such that for any interval I_k $(1 \le k \le n)$:

$$\exp(q(x))q'(x) \qquad \text{is increasing on } \; I_k, \quad \text{or} \tag{119}$$

$$(x - A)(B - x)q'(x) + x \qquad \text{is increasing on } I_k. \tag{120}$$

Let $E \subset [a_0, b_0]$ be the set where we require the increasing property of $\exp(q(x))q'(x)$ or $(x - A)(B - x)q'(x) + x$. (Recall that $[a_0, b_0] \setminus E$ has measure zero.) Let $x_0 \in [a, b]$. Because of the increasing properties we required, we have

$$\liminf_{x \to x_0, \; x \in E} |q'(x)| < \infty, \qquad \limsup_{x \to x_0, \; x \in E} |q'(x)| < \infty$$

which are still valid when x_0 is an endpoint of an I_k interval, because of (8).

It follows that both $\exp(q(x))q'(x)$ and $(x-A)(B-x)q'(x)+x$ are of bounded variation a.e. on those I_k intervals where they are assumed to be increasing. Since q is absolutely continuous on $[a, b]$, $\exp(-q(x))$ has bounded variation there. And $[(x - A)(B - x)]^{-1}$ has also bounded variation on $[a, b]$. The sum and product of two functions of bounded variation is again of bounded variation. Thus $q'(x)$ has bounded variation a.e. on $[a, b]$.

Next, we show that the density $v(x), x \in [-1,1]$ of the equilibrium measure corresponding to $w = \exp(-q)$ has positive lower bound on $[a,b]$.

Let $1 < \lambda$. Define

$$q_\lambda(x) := \lambda q(x) + (\lambda - 1)\log\frac{1}{x-A}, \quad x \in [a_0, b_0],$$

and $q_\lambda(x) := +\infty$ for $x \notin [a_0, b_0]$. Then $w_\lambda := \exp(-q_\lambda)$ is an admissible weight on $\mathbb{R}$, and $S_{w_\lambda} \subset [a_0, b_0]$. In fact, w_λ may be viewed as a continuous weight on $\Sigma := [a_0, b_0]$. We claim that S_{w_λ} is an interval. For this purpose we will now show that on any I_k:

$$a) \qquad (x-A)(B-x)q_\lambda'(x) + x \text{ is increasing if}$$
$$(x-A)(B-x)q'(x) + x \text{ is increasing,}$$

$$b) \quad \exp(q_\lambda(x)) \text{ is convex if } \exp(q(x)) \text{ is convex.}$$

Note that $(x-A)(B-x)q_\lambda'(x) + x = \lambda[(x-A)(B-x)q'(x) + x] - (\lambda-1)B$, so a) is proved.

To prove b) we claim that if a non-negative function $g(x)$ is convex on a subinterval of $[0, +\infty)$ then $g(x)^\lambda x^{1-\lambda}$ is also convex there. Indeed, let $0 \le \alpha, \beta,\ \alpha + \beta = 1$, otherwise arbitrary. For the convexity we need to prove

$$\alpha g(a)^\lambda a^{1-\lambda} + \beta g(b)^\lambda b^{1-\lambda} \ge g(\alpha a + \beta b)^\lambda (\alpha a + \beta b)^{1-\lambda}.$$

It is enough to prove that

$$\alpha a A^\lambda + \beta b B^\lambda \geq (\alpha a A + \beta b B)^\lambda (\alpha a + \beta b)^{1-\lambda}, \tag{121}$$

where we introduced $A := g(a)/a, B := g(b)/b$ and used $g(\alpha a + \beta b) \leq \alpha g(a) + \beta g(b) = \alpha a A + \beta b B$. But (121) is equivalent to

$$\Big(\frac{\alpha a A^\lambda + \beta b B^\lambda}{\alpha a + \beta b}\Big)^{1/\lambda} \geq \frac{\alpha a A + \beta b B}{\alpha a + \beta b},$$

which inequality holds, since $1 < \lambda$ and both sides are a weighted average of A and B.

Now assume that $\exp(q)$ is convex on an interval $I_k \subset [a, b]$. Since $g(x) := \exp(q(x + A))$ is convex on the shifted $I_k - A \subset [0, \infty)$ interval, we have that $g(x)^\lambda x^{1-\lambda} = \exp(\lambda q(x + A) + (\lambda - 1)\log(1/x)) = \exp(q_\lambda(x + A))$ is also convex there, so b) is proved.

So by a), b) and Theorem 25, $S_{w_\lambda} \subset [a_0, b_0]$ is an interval. Let δ_A denote the unit point mass measure at A, and let $\hat{\delta}_A$ denote the balayage of δ_A onto $[a_0, b_0]$. Recall that $\hat{\delta}_A$ is a positive unit measure and it has the property that $U^{\hat{\delta}_A}(x) = U^{\delta_A}(x) + c, x \in [a_0, b_0]$.

We show that

$$\mu_w\Big|_{S_{w_\lambda}} \geq \frac{1}{\lambda}\mu_{w_\lambda} + \Big(1 - \frac{1}{\lambda}\Big)\hat{\delta}_A\Big|_{S_{w_\lambda}}. \tag{122}$$

We know that with some constant F:

$$\begin{aligned} U^{\mu_{w_\lambda}}(x) + q_\lambda(x) &\geq F \quad \text{q.e. on } [a_0, b_0], \\ U^{\mu_{w_\lambda}}(x) + q_\lambda(x) &= F \quad \text{q.e. on } S_{w_\lambda}. \end{aligned}$$

that is,

$$\begin{aligned} U^{\mu_{w_\lambda}}(x) + (\lambda-1)U^{\hat{\delta}_A}(x) + \lambda q(x) &\geq F + (\lambda-1)c \quad \text{q.e. on } [a_0, b_0], \\ U^{\mu_{w_\lambda}}(x) + (\lambda-1)U^{\hat{\delta}_A}(x) + \lambda q(x) &= F + (\lambda-1)c \quad \text{q.e. on } S_{w_\lambda}. \end{aligned}$$

Hence

$$\begin{aligned} U^{\sigma}(x) + q(x) &\geq \frac{F + (\lambda-1)c}{\lambda} \quad \text{q.e. on } [a_0, b_0], \\ U^{\sigma}(x) + q(x) &= \frac{F + (\lambda-1)c}{\lambda} \quad \text{q.e. on } S_{w_\lambda}, \end{aligned}$$

where the σ positive unit measure is given by

$$\sigma := \frac{1}{\lambda}\mu_{w_\lambda} + \Big(1 - \frac{1}{\lambda}\Big)\hat{\delta}_A.$$

We also know that

$$U^{\mu_w}(x) + q(x) = const. \quad \text{q.e. on } [a_0, b_0].$$

Since q is finite on $[a_0, b_0]$, we gain

$$U^{\sigma}(x) \geq U^{\mu_w}(x) + const. \quad \text{q.e. on } [a_0, b_0], \tag{123}$$

hence this also holds μ_w-everywhere (because μ_w has finite logarithmic energy). By the principle of domination (Theorem II.3.2 of [24]) (123) also holds for all $x \in \mathbb{C}$. Equality holds for q.e. $x \in S_{w_\lambda}$.

Let Ω be a bounded domain containing S_{w_λ}. The measure μ_{w_λ} has finite logarithmic energy. $\hat{\delta}_A$ has also finite logarithmic energy because $U^{\hat{\delta}_A}(z) \leq$

$U^{\delta_A}(z) + const.,\ z \in \mathbb{C}$ (Theorem II.4.4, [24]), and $U^{\delta_A}(z)$ is bounded on $[a_0, b_0]$. So if a set $E \subset S_{w_\lambda}$ has zero capacity then $\sigma(E) = 0$, too. So applying Theorem IV.4.5 ([24]), we see that (122) holds on the full S_{w_λ}. But the density of $\hat{\delta}_A$ has positive lower bound on $[a_0, b_0]$ (see our remark at (112)). Consequently, $v(x)$ has positive lower bound on S_{w_λ}.

Let us prove that $S_{w_\lambda} \supset [a, b]$, if λ is close to 1. Let $x_0 \in (a_0, a) =: B_0$, $x_1 \in (b, b_0) =: B_1$. Since $x_0 \in S_w$, it is known that there exists an n and a weighted polynomial $w^n(x)P_n(x),\ x \in [a_0, b_0]$ which attains its maximum value in B_0 and nowhere else on $[a_0, b_0] \setminus B_0$. Note that

$$w_\lambda(x)^n P_n(x) = w(x)^{\lambda n}(x - A)^{(\lambda-1)n} P_n(x), \quad x \in [a_0, b_0],$$

so $w_\lambda(x)^n |P_n(x)|$ uniformly approaches to $w(x)^n |P_n(x)|$ on $[a_0, b_0]$ as $\lambda \to 1^+$. So there exists $1 < \lambda_0$ such that $w_\lambda(x)^n |P_n(x)|$, $x \in [a_0, b_0]$ also attains its maximum in B_0 when $\lambda \in (1, \lambda_0)$. Therefore, by a well known property of weighted polynomials, $S_{w_\lambda} \cap B_0 \neq \emptyset$ for any $\lambda \in (1, \lambda_0)$. By the same logic, there exists $1 < \lambda_1$ such that $S_{w_\lambda} \cap B_1 \neq \emptyset$ for any $\lambda \in (1, \lambda_1)$. But S_{w_λ} is an interval. It follows that $S_{w_\lambda} \supset [a, b]$ when $\lambda \in (1, \min(\lambda_0, \lambda_1))$. Thus $v(x)$ has positive lower bound on $[a, b]$.

$\blacksquare$

References

[1] J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc., **12** (1999), pp 1119-1178.

[2] J. Baik, T. Kriecherbauer, K. McLaughlin, and P. Miller, Uniform asymptotics for polynomials orthogonal with respect to a general class of discrete weights and universality results for associated ensembles, arXiv:math.CA/0310278v1.

[3] D. Benko, Approximation by weighted polynomials, J. Approx. Theory 120 (2003), no. 1, pp 153-182.

[4] D. Benko, The support of the equilibrium measure, Acta Sci. Math. (Szeged) 70 (2004), no. 1-2, pp 35–55.

[5] D. Benko, S.B. Damelin, P.D. Dragnev On the support of the equilibrium measure for arcs of the unit circle and for real intervals, Electron. Trans. Numer. Anal. (to appear)

[6] D. Benko, A. Kroó, A Weierstrass-type theorem for homogeneous polynomials, Trans. Amer. Math. Soc. (submitted)

[7] S. B. Damelin, and A. B. J. Kuijlaars, The support of the equilibrium measure in the presence of a monomial external field on $[-1, 1]$. Trans. Amer. Math. Soc. 351 (1999), no. 11, 4561–4584.

[8] S. B. Damelin, P. D. Dragnev, and A. B. J. Kuijlaars, The support of the equilibrium measure for a class of external fields on a finite interval. Pacific J. Math. 199 (2001), no. 2, 303–320.

[9] S. B. Damelin, H. S. Jung and K. H. Kwon, Mean convergence of extended Lagrange interpolation for exponential weights, Acta Appl. Math., **76** (2003), pp 17-36.

[10] P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field. J. Approx. Theory 95 (1998), no. 3, 388–475.

[11] P. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics, 3. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. viii+273 pp. ISBN: 0-9658703-2-4; 0-8218-2695-6

[12] A. A. Gonchar and E. A. Rakhmanov, Equilibrium measure and the distribution of zeros of extremal polynomials., Mat. Sb., **125** (1984), pp 117-127 (in Russian). English translation in Math. USSR Sb., **53** (1986), pp 119-130.

[13] A. B. J. Kuijlaars, A note on weighted polynomial approximation with varying weights, J. Approx. Theory 87 (1) (1996) 112-115.

[14] A. B. J. Kuijlaars, The role of the endpoint in weighted polynomial approximation with varying weights, *Constr. Approx.*, **12**(1996), 287–301.

[15] A. B. J. Kuijlaars, and W. Van Assche, A problem of Totik on fast decreasing polynomials(with A.B.J. Kuijlaars), Constr. Approx. 14 (1998), 97-112

[16] A. B. J. Kuijlaars, and P.D. Dragnev, Equilibrium problems associated with fast decreasing polynomials. Proc. Amer. Math. Soc. 127 (1999), no. 4, 1065–1074.

[17] E. Levin and D. S. Lubinsky, Orthogonal Polynomials for Exponential Weights, Springer-Verlag, 2001.

[18] D. S. Lubinsky, An update of orthogonal polynomials and weighted approximation on the real line, Acta Appl. Math., **33** (1993), pp 121-164.

[19] H. N. Mhaskar, Introduction to the Theory of Weighted Polynomial Approximation, World Scientific, 1996.

[20] H. N. Mhaskar, and E.B. Saff, Where does the sup norm of a weighted polynomial live?, Constr. Approx., **1** (1985), 71-91

[21] N. I. Muskhelishvili, Singular Integral Equations. Dover, New York, 1992

[22] I. Pritsker, Weighted energy problem on the unit circle, Constr. Approx., to appear.

[23] I. Pritsker and R. S. Varga, Weighted polynomial approximation in the complex plane, Constr. Approx., **14** (1998), pp 475-492.

[24] E. B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, Berlin, 1997

[25] P. Simeonov, A minimal weighted energy problem for a class of admissible weights, Houston J. of Math., **31** (2005), no. 4, pp 1245-1260.

[26] B. Simon, Orthogonal Polynomials on the Unit Circle I-II, American Mathematical Society Colloquium Publications, 2004.

[27] H. Stahl and V. Totik, General Orthogonal Polynomials, Cambridge University Press, 1992.

[28] V. Totik, Weighted polynomial approximation for convex external fields, Constr. Approx. 16 (2) (2000) 261-281.

[29] V. Totik, Weighted approximation with varying weight. Lecture Notes in Mathematics, 1569. Springer-Verlag, Berlin, 1994.

[30] V. Totik, Weigthed polynomial approximation for weights with slowly varying extremal density