Stability, Convergence and Periodicity for Equations with State-Dependent Delay

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Chapter 1

Introduction

The theory of functional differential equations deals with differential equations where the right hand sides depend on delayed arguments of the unknown function. The first examples appeared about 200 years ago and were related to geometric problems. The interest in the field grew rapidly in the second half of the 20th century. The books of Myshkis [64], Bellman and Cooke [9], Krasovskii [33], Hale [23], Hale and Lunel [24], Dieckmann [14] greatly influenced the developments. Today new applications [13,29,50,51,52,54,55,58,73] also continue to arise and the involved interesting mathematical problems require modifications and further develpoments of the theory.

Over the past several years it has become apparent that there is a need for a theory of equations containing delays that are functions of the state of the system because such equations appear in applications. For example retarded equations with state-dependent delay are of interest in classical electrodynamics [15,16,17,18,20], in population models [7], in models of blood cell production [56] and of commodity price fluctuation [8].

A simple model taken from [11] is as follows: An object moves along a line, $x(t)$ denotes its position at time t. A base located at $x = -w < 0$ controls the position of the object. We assume that the base has instantaneous information on the location of the object, i.e., on $x(t)$, moreover signals controlling the object travel from the base to the object at a speed $c > 0$. In addition, we suppose that $x(t) > -w$ for all t, i.e., the object does not collide with the base. For the base it takes a unit amount of time to produce the control signal for the object. Therefore, the signal which reaches the object at time t was sent by the base at the time $t - 1 - \frac{1}{c}(x(t) + w)$. This leads to the equation

$$\dot{x}(t) = -\mu x(t) + g\left(x\left(t - 1 - \frac{w}{c} - \frac{x(t)}{c}\right)\right)$$

with positive parameters μ, w, c and a response function $g : \mathbb{R} \to \mathbb{R}$. The term $-\mu x(t)$ represents an instantaneous damping. Positive (negative) feedback with respect to the preferred position at $0 \in \mathbb{R}$ is expressed by the condition $ug(u) > 0\,(< 0)$ for all $u \neq 0$. More complicated delay functions for related control problems are obtained in [71,72].

Let $h > 0$. The function

$$\tilde{g} : C([-h,0],\mathbb{R}) \ni \phi \mapsto g\Big(\phi\Big(-1 - \frac{w}{c} - \frac{\phi(0)}{c}\Big)\Big) \in \mathbb{R}$$

is in general not smooth enough in order to define well-posed initial value problems for the above equation on all of an open subset of $C([-h,0],\mathbb{R})$. This means that the basic tools of dynamical systems theory, like linearization and local invariant manifolds, cannot be applied in a straightforward way. A reason for lack of differentiability of $\tilde{g}$ is that the evaluation map

$$C \times [-h,0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}$$

is not Lipschitz continuous. This is the main source of difficulties of the study of differential equations with state-dependent delay.

Equations with state-dependent delay in the derivative, that is the state-dependent neutral equations are also used in applications [17,19], though we still do not have a general theory for such equations. For some interesting results related to state-dependent neutral equations we refer to [25,26].

In this work we prove results for two different classes of functional differential equations with state-dependent delay contained in [4,5,6].

In Chapter 2 we consider a class of neutral differential equations with state-dependent delay. Using the parameters of the equation conditions are given for the stability, asymptotic stability and attractivity of the zero solution.

In Chapter 3 a monotone semiflow is constructed for a class of differential equations with state-dependent delay, and it is proved that the ω-limit set of all points from an open dense subset of the phase space is an equilibrium point.

In Chapter 4 for the class of differential equations considered in the previous chapter, we show the existence of a nontrivial periodic orbit and a homoclinic orbit connecting 0 to the periodic orbit.

Now let us review the problems studied and the results obtained in every chapter.

In Chapter 2 we consider the nonlinear one-dimensional neutral differential equation with state-dependent delay

$$\frac{d}{dt}\left[x(t) - px(t - r(t, x_t))\right] = -q(t)\, x(t - s(t, x_t)). \tag{1}$$

For given $\lambda_0 > 0$, let $C = C([-\lambda_0, 0], \mathbb{R})$. For given $t_0 \in \mathbb{R}_+, \omega \in \mathbb{R}_+$, $y \in C([t_0 - \lambda_0, t_0 + \omega], \mathbb{R})$ and $t \in [t_0, t_0+\omega]$, $y_t \in C$ is defined by $y_t(\tau) = y(t+\tau)$, for all $\tau \in [-\lambda_0, 0]$. Suppose that $p \in \mathbb{R}$, $q \in C(\mathbb{R}_+, \mathbb{R})$, $r \in C(\mathbb{R}_+ \times C, \mathbb{R})$, $s \in C(\mathbb{R}_+ \times C, \mathbb{R})$, and there exist r_0, $s_0 \in [0, \lambda_0]$ such that $r(\mathbb{R}_+ \times C) \subset [0, r_0]$ and $s(\mathbb{R}_+ \times C) \subset [0, s_0]$.

We mention that it is not easy to prove stability results for neutral differential equations with state-dependent delay since even the basic questions such as the existence, uniqueness and continuous dependence of solutions are still not clarified. The stability for (non-neutral) retarded differential equations is well developed. A classical exemple is the linear equation

$$\dot{x}(t) = -a(t)x(t - r(t)),$$

where α and $q > 0$ are positive constants and $a : \mathbb{R}_+ \to [0, \alpha]$, $r : \mathbb{R}_+ \to [0, q]$ are continuous functions. In case $\alpha q \leq \frac{3}{2}$ the zero solution is uniformly stable and $\frac{3}{2}$ is the best possible constant ([64,78,49]). The number $\frac{3}{2}$ also arises as an upper bound in stability conditions for nonlinear and nonautonomous equations [78,31,32]. For equations with more delay or distributed delay Krisztin [34,35] has proved an interesting result which we will use in Section 2.2 to obtain stability results for Eq. (1).

It is known that in certain cases neutral differential equations are equivalent to retarded differential equations with infinite delay [70]. This method is used in [2,21,22,43,44,45,74,75] to study stability problems for neutral equations. Our purpose in Section 2.2 is not to transform Eq. (1) to a single retarded equation with infinite delay. Such a transform may not exist here. For each fixed solution of Eq. (1) we associate a retarded equation with infinite delay, and then use the results in [34,77,78] to obtain stability conditions for Eq. (1). The main results of Section 2.2 are the following.

Theorem 2.2.2. *Assume that* $0 \leq p < 1$ *and there exists* $q_0 \in \mathbb{R}_+$ *so that* $0 \leq q(t) \leq q_0$ *for all* $t \geq 0$. *Let* $K = \{k \in \mathbb{N} : s_0 + kr_0 < \frac{1-p}{q_0}\}$.
(i) If the condition

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} + \frac{q_0^2}{2(1-p)} \sum_{k \in K} (\frac{1-p}{q_0} - s_0 - kr_0)^2 p^k \leq 3/2$$

holds and $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ *with* $t_0 \in \mathbb{R}_+$ *is a solution of Eq. (1), then*

$$\|x_t\| \leq \|x_{t_0}\| \frac{1+p}{1-p} e^{5/2} \qquad \text{for all } t \geq t_0.$$

(ii) If

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} + \frac{q_0^2}{2(1-p)} \sum_{k\in K} (\frac{1-p}{q_0} - s_0 - kr_0)^2 p^k < 3/2$$

and $\liminf_{t\to\infty} q(t) > 0$ *are satisfied, then the zero solution of Eq. (1) is asymptotically stable.*

Theorem 2.2.3. *Assume that* $0 \leq p < 1$ *and there exists* $q_0 \in \mathbb{R}_+$ *so that* $0 \leq q(t) \leq q_0$ *for all* $t \geq 0$.

(i) If the condition

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} \leq 1$$

holds and $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ *with* $t_0 \in \mathbb{R}_+$ *is a solution of Eq. (1), then*

$$\|x_t\| \leq \|x_{t_0}\| \frac{1+p}{1-p} e^{5/2} \qquad \text{for all } t \geq t_0.$$

(ii) If

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} < 1$$

and $\liminf_{t\to\infty} q(t) > 0$ *are satisfied, then the zero solution of Eq. (1) is asymptotically stable.*

In Section 2.3 we show the attractivity of the zero solution of Eq. (1) by extending results of Wu and Yu [76], given for neutral equations with constant delay to neutral equation with state-dependent delay.

The main result of Section 2.3 is the following.

Theorem 2.3.3. *Assume that* $|p| < \frac{1}{2}$, $q(t) > 0$ *for all sufficiently large* $t \in \mathbb{R}$,

$$\int_{t_0}^{\infty} q(\tau)\, d\tau = \infty,$$

$$2|p|(2-|p|) + \limsup_{t\to\infty} \int_{t-s_0}^{t} q(\tau)\, d\tau < \frac{3}{2}$$

and

(2)

for all $y \in C([-\lambda_0, \infty), \mathbb{R})$, *the function* $[0,\infty) \ni t \mapsto t - s(t, y_t) \in \mathbb{R}$ *is increasing.*

Then every solution of Eq. (1) converges to zero as $t \to \infty$.

Note that condition (2) also plays an important role in nonneutral equations with state-dependent delay [60,39]. There is a wide class of applications where

$s(t, y_t)$ is defined by a threshold, that is,

$$\int_{t - s(t, y_t)}^{t} k(y(\tau))\, d\tau = k_0,$$

where $k(t)$ is a positive continuous function and k_0 is a positive real number. In this case condition (2) holds.

We mention that Theorem 2.3.3 can be extended to the equation

$$\frac{d}{dt}\left[x(t) - px(t - r(t, x_t))\right] = f(t, x_t)$$

with appropriate conditions on f. The extension requires standard techniques.

In Chapter 3 we prove a result for monotone dynamical systems which is aplicable for the differential equation with state-dependent delay

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)), \quad r = r(x(t)), \tag{3}$$

where $\mu > 0$, f and r are smooth real functions with $f(0) = 0$ and $f' > 0$.

The theory of monotone dynamical systems was developed by Hirsch in 1980's. The infinite dimensional theory of monotone systems has been heavily influenced by the results of Matano, Smith and Thieme [66-69]. These results have a good applicability for ordinary differential equations and also for differential equations with constant delay.

First let us consider Eq. (3) in the case $r \equiv$ constant. Eq. (3) generates a semiflow $\mathcal{F}$ on the phase space $C([-r, 0], \mathbb{R})$. We introduce a closed partial order relation on $C([-r, 0], \mathbb{R})$ in the following way: $\phi \leq \psi$ whenever $\phi(s) \leq \psi(s)$ for all $s \in [-R, 0]$, $\phi < \psi$ whenever $\phi \leq \psi$ and $\phi \neq \psi$, and $\phi \ll \psi$ whenever $\phi(s) < \psi(s)$ for all $s \in [-R, 0]$. The condition $f' > 0$ guarantees that $\mathcal{F}$ is monotone, that is, for every ϕ, ψ in $C([-r, 0], \mathbb{R})$ with $\phi \leq \psi$, $\mathcal{F}(t, \phi) \leq \mathcal{F}(t, \psi)$ holds for all $t \geq 0$. It is also true that $\mathcal{F}$ is strongly order preserving (SOP), that is, $\mathcal{F}$ is monotone, and for every ϕ, ψ in $C([-r, 0], \mathbb{R})$ with $\phi < \psi$, there exist $t_0 > 0$ and open subsets $\mathcal{U}$, $\mathcal{V}$ of $C([-r, 0], \mathbb{R})$ with $\phi \in \mathcal{U}$ and $\psi \in \mathcal{V}$ such that $\mathcal{F}(t_0, \mathcal{U}) \leq \mathcal{F}(t_0, \mathcal{V})$. Then applying a result of Smith and Thieme [65,68,69], we conclude that for all elements ϕ from an open dense subset of $C([-r, 0], \mathbb{R})$ the ω-limit set $\omega(\phi)$ of ϕ is an equilibrium point.

We remark that analogous results were obtained by Smith and Thieme in [66,67] for non-quasi-monotone functional differential equations, that is, under a weaker condition than $f' > 0$.

In the case $r = r(x(t))$ the situation becomes more difficult, because it is not obvious how to choose the phase space. The questions about existence, uniqueness, and continuous dependence of solutions of Eq. (3) are also not standard, see, e.g., [62]. In Section 3.2 we show that for suitable $R > 0$ and $A > 0$, the solution of Eq. (3) defines a semiflow F on the metric space X containing Lipschitz continuous functions mapping $[-R, 0]$ into $[-A, A]$ with metric $d(\phi, \psi) = \sup_{-R\leq s\leq 0} |\phi(s) - \psi(s)|$. The constant $R > 0$ is the maximum of r on $[-A, A]$. The result of Smith and Thieme is not applicable for this semiflow generated by Eq. (3), since it does not have, in general, the SOP property. Indeed, consider two functions ϕ and ψ in the phase space such that $\phi(s) < \psi(s) < A$ for all s in $[-R, -R + \epsilon)$ and $\phi(s) = \psi(s)$ for all s in $[-R + \epsilon, 0]$, where $\epsilon > 0$. Let $\mathcal{U}$ be an open subset of the phase space with $\phi \in \mathcal{U}$. Clearly, there is a function $\alpha \in \mathcal{U}$ such that $\psi(s) < \alpha(s)$ for all $s \in [-R + \epsilon, 0]$. Let x^{ψ} and x^{α} denote the solutions of Eq. (3) with initial function ψ and α, respectively. If we also have $-r(x^{\psi}(t)) \in [-R+\epsilon, 0]$ and $-r(x^{\alpha}(t)) \in [-R + \epsilon, 0]$ for all $t \geq 0$, then it is easy to see that there exists $t_0 > 0$ such that $F(t_0, \psi) \ll F(t_0, \alpha)$. Therefore, in this case F cannot be SOP.

We observe that F satisfies the following property. F is monotone, and for every ϕ and ψ in the phase space with $\phi < \psi$ and $F(t, \phi) \neq F(t, \psi)$ for all $t \geq 0$, there exist $t_0 > 0$ and open subsets $\mathcal{U}$, $\mathcal{V}$ of the phase space with $\phi \in \mathcal{U}$ and $\psi \in \mathcal{V}$ such that $F(t_0, \mathcal{U}) \leq F(t_0, \mathcal{V})$. This is why our aim is to prove a convergence result for monotone semiflows having the above property, the so-called mildly order preserving property (MOP) instead of the SOP property. The following assumption seems to be crucial in achieving our goal. If ϕ and ψ are in a compact invariant subset of the phase space, then $\phi < \psi$ implies $F(t, \phi) \neq F(t, \psi)$ for all $t \geq 0$. This condition is satisfied for the semiflow generated by Eq. (3) as well.

In the proofs of the monotonicity the hypothesis $f' > 0$ can be weakened like in [66,67], but $f' > 0$ seems to be crucial in the verification of the property $F(t, \phi) \neq F(t, \psi)$, $t \geq 0$, for all ϕ, ψ in a compact invariant subset with $\phi < \psi$.

Section 3.1 contains a general convergence result, which is a modified version of the convergence result of Smith and Thieme [65,68]:

Theorem 3.1.1. *Consider a metric space $\mathcal{X}$ with a closed partial order relation and a semiflow Φ on $\mathcal{X}$. Assume that*

(A_1) *if x and y are in a compact invariant subset of $\mathcal{X}$, then $x < y$ implies $\Phi(t, x) \neq \Phi(t, y)$, for all $t \geq 0$,*

(A_2) *Φ is MOP,*

(A_3) *each point in $\mathcal{X}$ can be approximated either from below or from above in $\mathcal{X}$,*

(A_4) *for each x in $\mathcal{X}$, the orbit $O(x)$ of x has compact closure in $\mathcal{X}$, and*

(A_5) *for each x in $\mathcal{X}$ and for each sequence $(x_n)_1^\infty$, which approximates x either from below or from above in $\mathcal{X}$, $\bigcup_{n\geq 1}\omega(x_n)$ has compact closure in $\mathcal{X}$.*

Then the ω-limit set of all points from an open dense subset of $\mathcal{X}$ is contained in the set of equilibria.

The proof of this result can be found in Section 3.3. Section 3.2 gives explicit hypotheses on f and r ensuring the applicability of our convergence result for Eq. (3).

Theorem 3.2.16. *If f and r satisfy hypotheses*

$$(H1)\qquad \begin{cases} \mu > 0, \\ f \in C^1(\mathbb{R},\mathbb{R}),\ f(0) = 0,\ f'(u) > 0 \text{ for all } u \in \mathbb{R}, \\ \text{there exists } A > 0 \text{ such that } |f(u)| < \mu|u| \text{ for all } |u| \geq A, \\ r \in C^1(\mathbb{R},\mathbb{R}),\ r(0) = 1,\ r([-A,A]) \subset (0,\infty). \end{cases}$$

then there is an open dense subset of X such that, for each element ϕ of this subset, $\omega(\phi)$ is an equilibrium point.

Note that, it is not true in general that the ω-limit set of every point of the phase space is an equilibrium point. Krisztin, Walther and Wu [41] have shown the existence of periodic orbits in the case $r \equiv 1$ for certain μ, f, and r. A similar result is proved by Mallet-Paret and Nussbaum [60,61], Kuang and Smith [47,48], and Arino, Hadeler and Hbid [3], Krisztin and Arino [39], Walther [72] in the state-dependent delay case with a negative feedback condition. For the case $r = r(x(t))$ with a positive feedback condition Chapter 4 contains an analogous result.

In Chapter 4 we show the existence of a nontrivial periodic orbit and a homoclinic orbit connecting 0 to the periodic orbit for Eq. (3) with state-dependent delay and positive feedback. The main technical tools we use are: the result of monotone dynamical systems applicable for Eq. (3) in Chapter 3; a local unstable manifold at zero for Eq. (3) in [38]; and a discrete Lyapunov functional counting sign changes given analogously to that of [39]. We mention that it is not clear whether the applied techniques in the proofs of periodic solutions of autonomous differential equations with state-dependent delay and negative feedback (fixed point theorems, fixed point index) can be applied for the positive feedback case. Closest to the result presented in Chapter 4 is the work of Krisztin and

Arino [39], where for the negative feedback case the structure of slowly oscillating solutions is described.

In Section 4.1 we recall the hypotheses on f and r and the basic results from Chapter 3. The phase space is considered to be the space of such elements $\phi \in X$ for which the Lipschitz constant is not greater than

$$M = \max_{(u,v)\in[-A,A]\times[-A,A]} |-\mu u + f(v)|.$$

We introduce an additional condition on r to guarantee that the function $t \mapsto t - r(x(t))$ is strictly increasing. For example, the smallness of r' or concavity of r are sufficient. This monotonicity property of $t \mapsto t - r(x(t))$ plays an important role in the proofs. Then we need some results about the associated linear equation

$$\dot{x}(t) = -\mu x(t) + f'(0)x(t-1).$$

The spectrum of the infinitesimal generator of the linear semigroup defined by the above linear equation consists of a $\lambda_0 \in \mathbb{R}$ and complex conjugate pairs $\lambda_k, \overline{\lambda}_k$, for all integers $k \geq 1$, with $(2k-1)\pi < \mathrm{Im}\lambda_k < 2k\pi$ and $\mathrm{Re}\lambda_{k+1} < \mathrm{Re}\lambda_k < \lambda_0$ for all integers $k \geq 1$, and $\mathrm{Re}\lambda_k \to -\infty$ as $k \to \infty$. We assume that $\mathrm{Re}\lambda_1 > 0$.

In Section 4.2 we introduce the set S of functions ϕ in the phase space for which the solution through ϕ oscillates on $[0, \infty)$. We show that S is positively invariant, closed and there are not ϕ, ψ in S with $\phi < \psi$.

In Section 4.3 we use [38,Theorem 4.1] to show the existence of a 3-dimensional local unstable manifold which is tangent at 0 to the real generalized eigenspace of the spectral set $\{\lambda_0, \lambda_1, \overline{\lambda}_1\}$. For every element ϕ of this local unstable manifold sufficiently small there is a solution through ϕ which is defined on $(-\infty, 0]$ and stays close to 0. The forward extension of this local unstable manifold denoted by W is an invariant set. We prove that $\overline{W}$ and $\overline{W \cap S}$ are compact and invariant, and $W \cap S \setminus \{0\}$ is nonempty and is also invariant.

In Section 4.4 we define a discrete Lyapunov functional which counts the sign changes of solutions. We show that if ϕ and ψ are different elements of $\overline{W \cap S}$, then the difference $\phi - \psi$ has one or two sign changes on the interval $[-r(\phi(0)), 0]$. This fact guarantees the injectivity of a map from $\overline{W \cap S}$ into $\mathbb{R}^2$ in Section 4.5.

In Section 4.5 we prove the main result of this section:

Theorem 4.5.4.

(i) There is a periodic solution $p : \mathbb{R} \to \mathbb{R}$ of Eq. (3). The minimal period T of p satisfies $T \in (1, 2)$.

(ii) *For each* $\phi \in W \cap S \setminus \{0\}$, *there is a unique solution* $x^\phi : \mathbb{R} \to \mathbb{R}$ *of Eq. (3) such that* $x_0^\phi = \phi$, $x_t^\phi \to 0$ *as* $t \to -\infty$, $\omega(\phi) = \{p_t : t \in [0, T]\}$, *and for all* $t \in \mathbb{R}$, x_t^ϕ *has one or two sign changes on the interval* $[-r(x^\phi(t)), 0]$.

Chapter 2

3/2 Stability Theorems for Neutral Differential Equations

2.1 Preliminary results

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{N}$ be the set of real, nonnegative real, nonpositive real numbers and nonnegative integers, respectively. For given $\lambda_0 > 0$, $C = C([-\lambda_0, 0], \mathbb{R})$ denotes the Banach space of continuous functions $\phi : [-\lambda_0, 0] \to \mathbb{R}$ with norm $\|\phi\| = \sup_{-\lambda_0 \le \tau \le 0} |\phi(\tau)|$. For given $t_0 \in \mathbb{R}_+, \omega \in \mathbb{R}_+$, $y \in C([t_0 - \lambda_0, t_0 + \omega], \mathbb{R})$ and $t \in [t_0, t_0+\omega]$, $y_t \in C$ is defined by $y_t(\tau) = y(t+\tau)$, for all $\tau \in [-\lambda_0, 0]$. Suppose that $p \in \mathbb{R}$ with $|p| < 1$, $q \in C(\mathbb{R}_+, \mathbb{R})$, $r \in C(\mathbb{R}_+ \times C, \mathbb{R})$, $s \in C(\mathbb{R}_+ \times C, \mathbb{R})$, and there exist r_0, $s_0 \in [0, \lambda_0]$ such that $r(\mathbb{R}_+ \times C) \subset [0, r_0]$ and $s(\mathbb{R}_+ \times C) \subset [0, s_0]$.

Consider the nonlinear one-dimensional neutral differential equation with state-dependent delay

$$(1.1) \qquad \frac{d}{dt}\left[x(t) - px(t - r(t, x_t))\right] = -q(t)\, x(t - s(t, x_t)).$$

Note that for every bounded $y \in C(\mathbb{R}, \mathbb{R})$ the functions $\mathbb{R} \ni t \mapsto r(t, y_t) \in \mathbb{R}$ and $\mathbb{R} \ni t \mapsto s(t, y_t) \in \mathbb{R}$ are continuous and bounded.

For $(t_0, \phi) \in \mathbb{R}_+ \times C$ a function $x \in C([t_0 - \lambda_0, t_0 + \omega), \mathbb{R})$ is called a solution of Eq. (1.1) on $[t_0, t_0 + \omega)$ through (t_0, ϕ), and is denoted by $x(t_0, \phi)(\cdot)$ if $x_{t_0} = \phi$, $x_t \in C$ and the difference $x(t) - px(t - r(t, x_t))$ is differentiable and satisfies Eq. (1.1) for $t \in (t_0, t_0 + \omega)$. We assume the existence of $x(t_0, \phi)(\cdot)$ on $[t_0 - \lambda_0, \infty)$ for all $t_0 \in \mathbb{R}_+$ and $\phi \in C$, but the uniqueness of $x(t_0, \phi)(\cdot)$ is not necessarily required. Note that there are some results concerning the existence and uniqueness of solutions of Eq. (1.1) [25,30].

The zero solution of Eq. (1.1) is said to be stable if for every $\varepsilon > 0$ and $t_0 \ge 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ so that

$$|x(t_0, \phi)(t)| < \varepsilon \quad \text{for all}\, \phi \in C \text{ with } \|\phi\| < \delta \,\text{and for all}\, t \ge t_0.$$

The zero solution of Eq. (1.1) is said to be asymptotically stable (AS) if it is stable and there is $\delta_0 = \delta_0(t_0) > 0$ so that

$$x(t_0, \phi)(t) \to 0 \text{ as } t \to \infty \quad \text{for all}\, \phi \in C \text{ with } \|\phi\| < \delta_0.$$

The zero solution is said to be uniformly stable (US) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that

$$|x(t_0, \phi)(t)| < \varepsilon \quad \text{for all}\, \phi \in C \text{ with } \|\phi\| < \delta \,\text{and for all}\, t \geq t_0 \geq 0.$$

The zero solution is said to be uniformly asymptotically stable (UAS) if it is US and for every $\varepsilon > 0$ there exist $\delta_0 > 0$ and $T = T(\varepsilon) > 0$ so that

$$|x(t_0, \phi)(t)| < \varepsilon \quad \text{for all}\, \phi \in C \text{ with } \|\phi\| < \delta_0 \,\text{and for all } t \geq t_0 + T.$$

The zero solution of Eq. (1.1) is said to be attractive if every solution of Eq. (1.1) tends to zero as $t \to \infty$.

In Chapter 2 we investigate the stability and asymptotic stability of the zero solution of Eq. (1.1). We achieve this in two different ways. On one hand we associate a family of retarded equations with infinite delay with Eq. (1.1), that is, for each fixed solution of Eq. (1.1) we associate a retarded equation with infinite delay. According to a result of Krisztin [34] this retarded equation with infinite delay gives information about the stability and asymptotic behaviour of the corresponding solution of Eq. (1.1). On the other hand we extend results of Wu and Yu [76] given for neutral equations with constant delay to neutral equations with state-dependent delay.

Here let us review briefly some results of Krisztin [34], Wu and Yu [76] and Yu [79].

Let BC denote the Banach space of bounded and continuous functions $\phi : (-\infty, 0] \to \mathbb{R}$ with norm $\|\phi\|_{BC} = \sup_{\tau \leq 0} |\phi(\tau)|$. Let $a \in \mathbb{R}$ and $y : (-\infty, a] \to \mathbb{R}$ be a bounded, continuous function. For every real $t \leq a$ the function $y_t^{(-\infty,0]} \in BC$ is defined by $y_t^{(-\infty,0]}(\tau) = y(t + \tau)$, $\tau \leq 0$. For the function $f : \mathbb{R}_+ \times BC \to \mathbb{R}$ we assume that $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$, and for every $a > 0$ and for every bounded, continuous function $y : (-\infty, a] \to \mathbb{R}$ the function $[0, a] \ni t \mapsto f(t, y_t^{(-\infty,0]}) \in \mathbb{R}$ is continuous.

Consider the one-dimensional functional differential equation with infinite delay

$$x'(t) = f(t, x_t^{(-\infty,0]}). \tag{1.2}$$

Define $M : \mathbb{R}_+ \times BC \to \mathbb{R}$ by $M(u, \phi) = \max\{0, \max_{-u \leq \tau \leq 0} \phi(\tau)\}$. For a bounded, nondecreasing, left-continuous and nonconstant function μ from $\mathbb{R}_+$ to $\mathbb{R}_+$ let

$$\mu_0 = \int_0^\infty d\mu,\ \mu_1 = \int_0^\infty \tau\, d\mu(\tau),\ \mu_2 = \mu_1 + \frac{\mu_0}{2} \int_0^{1/\mu_0} \left(\frac{1}{\mu_0} - \tau \right)^2 d\mu(\tau).$$

For $(t_0,\phi)\in\mathbb{R}_+\times BC$ and $\omega>0$ a function $x\in C((-\infty,t_0+\omega),\mathbb{R})$ is called a solution of Eq. (1.2) on $[t_0,t_0+\omega)$ through (t_0,ϕ) if $x_{t_0}=\phi$ and Eq. (1.2) holds on $(t_0,t_0+\omega)$. The definitions of stability, asymptotic stability, uniform stability and uniform asymptotic stability of the zero solution of Eq. (1.2) are analogous to those for Eq. (1.1) replacing $\phi\in C$, $\|\phi\|<\delta$ and $\phi\in C$, $\|\phi\|<\delta_0$ with $\phi\in BC$, $\|\phi\|_{BC}<\delta$ and $\phi\in BC$, $\|\phi\|_{BC}<\delta_0$, respectively.

Theorem A (Krisztin [34]). *Assume that*

(1.3)
$$-\int_0^\infty M(u,\phi)\,d\mu(u)\le f(t,\phi)\le\int_0^\infty M(u,-\phi)\,d\mu(u)\quad \textit{for all } (t,\phi)\in\mathbb{R}_+\times BC.$$

(i) If $\mu_2\le 3/2$ holds and $x:\mathbb{R}\to\mathbb{R}$ is a solution of Eq. (1.2) on $[t_0,\infty)$ with $t_0\in\mathbb{R}_+$, then

$$\|x_t^{(-\infty,0]}\|_{BC}\le\|x_{t_0}^{(-\infty,0]}\|_{BC}\,e^{5/2}\qquad \textit{for all } t\ge t_0. \tag{1.4}$$

(ii) If $\mu_2<3/2$ and

$$\begin{cases}\textit{for all sequences } \{t_n\}_0^\infty \textit{ in } \mathbb{R}_+ \textit{ with } t_n\to\infty, \textit{ and } \{\phi_n\}_0^\infty \textit{ in } BC,\\ \textit{and for all } c\in\mathbb{R}\setminus\{0\}, \textit{ and } B>0 \textit{ with } \|\phi_n\|_{BC}\le B \textit{ for all } n\in\mathbb{N},\\ \textit{and } \phi_n\to c \textit{ as } n\to\infty \textit{ uniformly on compact subsets of } (-\infty,0],\\ \textit{the sequence } \{f(t_n,\phi_n)\}_0^\infty \textit{ does not converge to zero as } n\to\infty,\end{cases} \tag{1.5}$$

then the zero solution of Eq. (1.2) is UAS.

Let us mention that the theorem of Krisztin is modified here in the sense that condition (1.4) does not appear explicitly in [34] but it can be deduced from the proof of Lemma 2.3 in [34]. On the other hand, the boundedness of the sequence $\{\phi_n\}_0^\infty$ in condition (1.5) does not appear in [34] either, though it is considered to be bounded in Lemma 2.2 [34].

It is easy to see that $\mu_2\le\mu_1+\frac{1}{2}$ and we have the following corollary of Theorem A [34].

Corollary B. *If the conditions $\mu_2\le\frac{3}{2}$, $\mu_2<\frac{3}{2}$ in Theorem A are replaced by $\mu_1\le 1$, $\mu_1<1$, respectively, then the statements of Theorem A remain true.*

Applying the above results to the equation with distributed delay

$$x'(t)=\sum_{k=0}^{\infty}a_k(t)x(t-r^k(t)), \tag{1.6}$$

where $a_k \in C(\mathbb{R}_+, [0, \alpha_k])$ with $\alpha_k \in \mathbb{R}_+$ and $\sum_{k=0}^{\infty} \alpha_k < \infty$, $r^k \in C(\mathbb{R}_+, [0, q^k])$ with $q^k \in \mathbb{R}_+$ for all $k \in \mathbb{N}$, we obtain that

$$\sum_{k=0}^{\infty} \alpha_k q^k \leq 1$$

is a sufficient condition of US for Eq. (1.6) and 1 is the best possible constant (see [34]).

Wu and Yu [76,79] considered the linear neutral differential equation

$$\frac{d}{dt}\left[x(t) - px(t-r)\right] = -q(t)\,x(t-s), \;\; t \geq 0 \tag{1.7}$$

assuming $q : \mathbb{R}_+ \to \mathbb{R}$ is eventually positive, r and s are positive reals, and $p \in \mathbb{R}$.

Yu has shown that under the conditions $p \in [0, 1)$ and

$$2p(2-p) + \int_{t-s}^{t} q(\tau)\,d\tau \leq \frac{3}{2} \qquad \text{for all } t \geq 0,$$

the zero solution of Eq. (1.7) is US (see [79]).

He has also proved that under the conditions $p \in [0, 1)$,

$$\int_0^{\infty} q(\tau)\,d\tau = \infty$$

and

$$2p(2-p) + \sup_{t \geq 0} \int_{t-\,s}^{t} q(\tau)\,d\tau < \frac{3}{2} \qquad \text{for all } t \geq 0,$$

the zero solution of Eq. (1.7) is AS (see [79]).

Investigating the attractivity of the zero solution of Eq. (1.7), Wu and Yu have shown that the conditions $|p| < 1$,

$$\int_0^{\infty} q(\tau)\,d\tau = \infty$$

and

$$2|p|(2-|p|) + \limsup_{t \geq 0} \int_{t-\,s}^{t} q(\tau)\,d\tau < \frac{3}{2}$$

guarantee that every solution of Eq. (1.7) tends to zero as $t \to \infty$ (see [76]).

2.2 3/2 Stability theorems for neutral differential equations

It is known that in certain cases neutral differential equations are equivalent to retarded differential equations with infinite delay [70]. This method is used in [2,21,22,43,44,45,74,75] to study stability problems for neutral equations. Our purpose in this section is not to transform Eq. (1.1) to a single retarded equation with infinite delay. Such a transform may not exist here. For each fixed solution of Eq. (1.1) we associate a retarded equation with infinite delay and then apply Theorem A and Corollary B, which enable us to establish 3/2 stability theorems for Eq. (1.1) (see [34,77,78]).

Lemma 2.1. *Let $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ be a solution of Eq. (1.1) and define $y : [t_0, \infty) \to \mathbb{R}$ by $y(t) = x(t) - px(t - r(t, x_t))$ for $t \geq t_0$. If $y(t) \to c$ as $t \to \infty$ for some $c \in \mathbb{R}$, then $x(t) \to \dfrac{c}{1-p}$ as $t \to \infty$.*

Proof. Extend x to a function from $\mathbb{R}$ to $\mathbb{R}$ by $x(t) = x(t_0 - \lambda_0)$ for $t < t_0 - \lambda_0$. Define the map $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(t) = \begin{cases} r(t, x_t), & t \geq t_0, \\ r(t_0, x_{t_0}), & t < t_0. \end{cases}$$

Let the sequence $\{\eta^n\}_{n=0}^{\infty}$ of functions $\eta^n : \mathbb{R} \to \mathbb{R}$ be given by $\eta^0(t) = t$ for all $t \in \mathbb{R}$, and

$$\eta^n(t) = \eta^{n-1}(t - \rho(t)) \qquad \text{for all integers } n \geq 1 \text{ and } t \in \mathbb{R},$$

that is

$$\eta^n(t) = t - \rho(t) - \rho(\eta(t)) - \cdots - \rho(\eta^{n-1}(t)) \qquad \text{for all integers } n \geq 1 \text{ and } t \in \mathbb{R}.$$

We extend y to a function from $\mathbb{R}$ to $\mathbb{R}$ so that

$$y(t) = x(t) - px(t - \rho(t)) = x(t) - px(\eta(t)) \qquad \text{for all } t \in \mathbb{R}.$$

First we express x with y. For all $t \in \mathbb{R}$ we have the sequence of equalities

$$\begin{aligned} y(t) &= x(t) - px(\eta(t)), \\ py(\eta(t)) &= px(\eta(t)) - p^2 x(\eta^2(t)), \\ &\vdots \\ p^n y(\eta^n(t)) &= p^n x(\eta^n(t)) - p^{n+1} x(\eta^{n+1}(t)), \\ &\vdots \end{aligned}$$

Summing the above equalities we obtain

$$x(t) = \sum_{k=0}^{\infty} p^k y(\eta^k(t)) \qquad \text{for all } t \in \mathbb{R}, \tag{2.1}$$

which is convergent since $|p| < 1$. The convergence of $x(t)$ to $\frac{c}{1-p}$ as $t \to \infty$ follows immediately due to the fact that $x(t)$ is convergent, $y(t) \to c$ as $t \to \infty$, and $\eta^k(t) \to \infty$ as $t \to \infty$. This completes the proof. □

The main results are the following.

Theorem 2.2. *Assume that $0 \le p < 1$ and there exists $q_0 \in \mathbb{R}_+$ so that $0 \le q(t) \le q_0$ for all $t \ge 0$. Let $K = \{k \in \mathbb{N} : s_0 + kr_0 < \frac{1-p}{q_0}\}$.*
(i) If the condition

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} + \frac{q_0^2}{2(1-p)} \sum_{k \in K} (\frac{1-p}{q_0} - s_0 - kr_0)^2 p^k \le 3/2$$

holds and $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ with $t_0 \in \mathbb{R}_+$ is a solution of Eq. (1.1), then

$$\|x_t\| \le \|x_{t_0}\| \frac{1+p}{1-p} e^{5/2} \qquad \text{for all } t \ge t_0.$$

(ii) If

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} + \frac{q_0^2}{2(1-p)} \sum_{k \in K} (\frac{1-p}{q_0} - s_0 - kr_0)^2 p^k < 3/2$$

and $\liminf_{t\to\infty} q(t) > 0$ are satisfied, then the zero solution of Eq. (1.1) is AS.

Proof. (i) Let $t_0 \in \mathbb{R}_+$ and a solution $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ of Eq. (1.1) be given. Extend x to a function from $\mathbb{R}$ to $\mathbb{R}$ by $x(t) = x(t_0 - \lambda_0)$ for $t < t_0 - \lambda_0$. Define the maps $\rho : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ by

$$\rho(t) = \begin{cases} r(t, x_t), & t \ge t_0 \\ r(t_0, x_{t_0}), & t < t_0 \end{cases}, \quad \sigma(t) = \begin{cases} s(t, x_t), & t \ge t_0 \\ s(t_0, x_{t_0}), & t < t_0 \end{cases}.$$

Let the sequence $\{\eta^n\}_0^\infty$ of functions $\eta^n : \mathbb{R} \to \mathbb{R}$ be given by $\eta^0(t) = t$ for all $t \in \mathbb{R}$, and

$$\eta^n(t) = \eta^{n-1}(t - \rho(t)) \qquad \text{for all integers } n \ge 1 \text{ and } t \in \mathbb{R}.$$

Then

$$\eta^n(t) = t - \rho(t) - \rho(\eta(t)) - \cdots - \rho(\eta^{n-1}(t)) \qquad \text{for all integers } n \ge 1 \text{ and } t \in \mathbb{R}.$$

Define the function $y : \mathbb{R} \to \mathbb{R}$ by

$$y(t) = x(t) - px(t - \rho(t)) = x(t) - px(\eta(t)) \qquad \text{for all } t \in \mathbb{R}.$$

As in Lemma 2.1 expressing x with y, we get

$$x(t) = \sum_{k=0}^{\infty} p^k y(\eta^k(t)) \qquad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

Then y satisfies

$$y'(t) = -q(t) \sum_{k=0}^{\infty} p^k y(\eta^k(t - \sigma(t))) \qquad \text{for all } t > t_0. \tag{2.2}$$

Now we define the function $F : [t_0, \infty) \times BC \to \mathbb{R}$ by

$$F(t, \phi) = -q(t) \sum_{k=0}^{\infty} p^k \phi(\eta^k(t - \sigma(t)) - t).$$

Consider the retarded functional differential equation with infinite delay

$$z'(t) = F(t, z_t^{(-\infty,0]}), \qquad t > t_0. \tag{2.3}$$

Clearly, y is a solution of Eq. (2.3). Observe that Eq. (2.3) is a particular case of Eq. (1.6) with $r^k(t) = t - \eta^k(t - \sigma(t))$ and $a_k(t) = q(t)p^k$, $k \in \mathbb{N}$. Then $0 \le r^k(t) = t - \left[t - \sigma(t) - \rho(t - \sigma(t)) - \rho(\eta(t - \sigma(t))) - \cdots - \rho(\eta^{k-1}(t - \sigma(t)))\right]$ $\le s_0 + kr_0$ and $0 \le a_k(t) = q(t)p^k \le q_0 p^k$, $k \in \mathbb{N}$. Clearly, $\sum_{k=0}^{\infty} q_0 p^k < \infty$ since $0 \le p < 1$. In order to apply Theorem A to Eq. (2.3) we need a bounded, nondecreasing, left-continuous and nonconstant function μ from $\mathbb{R}_+$ to $\mathbb{R}_+$. Define the sequences $\{q^k\}_0^{\infty}$ and $\{\alpha_k\}_0^{\infty}$ by $q^k = s_0 + kr_0$ and $\alpha_k = q_0 p^k$, respectively. Let

$$\mu(t) = \begin{cases} 0, & t \in [0, q_0] \\ \alpha_0 + \alpha_1 + \cdots + \alpha_k, & t \in (q_k, q_{k+1}]. \end{cases}$$

Condition (1.3) becomes

$$-\sum_{k=0}^{\infty} M(q^k, \phi)\alpha_k \le F(t, \phi) \le \sum_{k=0}^{\infty} M(q^k, -\phi)\alpha_k \qquad \text{for all } (t, \phi) \in [t_0, \infty) \times BC,$$

that is

$$-\sum_{k=0}^{\infty} \alpha_k \max\{0, \max_{-q^k \le \tau \le 0} \phi(\tau)\} \le F(t, \phi) \le -\sum_{k=0}^{\infty} \alpha_k \min\{0, \min_{-q^k \le \tau \le 0} \phi(\tau)\}$$

which can easily be checked. We have

$$\mu_0 = \sum_{k=0}^{\infty} \alpha_k = \frac{q_0}{1-p},$$

$$\mu_1 = \sum_{k=0}^{\infty} \alpha_k\, q^k = \frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2},$$

$$\mu_2 = \frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} + \frac{q_0^2}{2(1-p)} \sum_{k\in K} p^k \left(\frac{1-p}{q_0} - s_0 - kr_0\right)^2,$$

where $K = \{k \in \mathbb{N} : s_0 + kr_0 < \frac{1-p}{q_0}\}$. Due to the assumption $\mu_2 \le 3/2$, from Theorem A (i) we conclude

$$\text{(2.4)} \qquad \|y_t^{(-\infty,0]}(t_0,\psi)\|_{BC} \le \|y_{t_0}^{(-\infty,0]}\|_{BC}\, e^{5/2} \qquad \text{for all } t \ge t_0.$$

Using the extension of x and the definition of y we infer

$$\|y_{t_0}^{(-\infty,0]}\|_{BC} \le (1+p)\|x_{t_0}\|.$$

This inequality and (2.4) combined yield

$$|y(t)| \le (1+p)\|x_{t_0}\| e^{5/2} \qquad \text{for all } t \in \mathbb{R}.$$

Applying the last inequality for $|y(t)|$ in (2.1) we obtain

$$|x(t)| \le \|x_{t_0}\| \frac{1+p}{1-p} e^{5/2} \qquad \text{for all } t \in \mathbb{R},$$

wich gives the desired extimation on $\|x_t\|$. The proof of assertion (i) is complete.

In order to show assertion (ii) it suffices to verify that every solution x of Eq. (1.1) tends to zero as $t \to \infty$ since assertion (i) implies the stability of the zero solution.

Let x be a given solution of Eq. (1.1) as in the proof for assertion (i). We define the function $y : \mathbb{R} \to \mathbb{R}$ and Eq. (2.3) with infinite delay as above. We want to apply Theorem A (ii) since the uniform asymptotic stability of the zero solution of the linear Eq. (2.3) implies, in particular, $y(t) \to 0$ as $t \to \infty$. Using Lemma 2.1 we conclude $x(t) \to 0$ as $t \to \infty$. In the proof of (i) we saw that (1.3) holds for Eq. (2.3). By our assumption in Theorem 2.2 and the formula for μ_2 in the proof of (i) yield $\mu_2 < 3/2$. So only condition (1.5) remains to be verified.

Let $\{t_n\}_0^\infty$ and $\{\phi_n\}_0^\infty$ be sequences in $\mathbb{R}_+$ and BC, respectively, and let $c \in \mathbb{R}$, $c \neq 0$ and $B > 0$ with $\|\phi_n\|_{BC} \leq B$ for all $n \in \mathbb{N}$, and $t_n \to \infty$, and $\phi_n \to c$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}_-$. Due to the fact that $\liminf_{t\to\infty} q(t) > 0$, there exist a positive real number l and a positive integer n_0 so that $q(t_n) \geq l$ for all integers $n \geq n_0$.

Select $\varepsilon > 0$ so that $\dfrac{q_0\,\varepsilon}{1-p} < \dfrac{l|c|}{4(1-p)}$ and select a positive integer N so that

$$q_0\, B\,\frac{p^{N+1}}{1-p} < \frac{l|c|}{1-p} \quad \text{and} \quad \frac{1-p^{N+1}}{1-p} > \frac{3}{4(1-p)}.$$

Choose $n_1 \in \mathbb{N}$ so that $n_1 \geq n_0$ and

$$|\phi_n(\tau) - c| < \varepsilon \qquad \text{for all } \tau \in [-s_0 - Nr_0, 0] \text{ and } n_1 \leq n \in \mathbb{N}.$$

We want to show that $F(t_n, \phi_n)$ does not converge to zero as $n \to \infty$. Having

$$-F(t_n,\phi_n) = q(t_n)\sum_{k=0}^{N} p^k c + q(t_n)\sum_{k=0}^{N} p^k[\phi_n(\eta^k(t_n - \sigma(t_n)) - t_n) - c] + q(t_n)\sum_{k=N+1}^{\infty} p^k\phi_n(\eta^k(t_n - \sigma(t_n)) - t_n),$$

we obtain for all integers $n \geq n_1$ that

$$\left|q(t_n)\sum_{k=0}^{N} p^k c\right| \geq l|c|\frac{1-p^{N+1}}{1-p} > \frac{3l|c|}{4(1-p)},$$

$$\left|q(t_n)\sum_{k=0}^{N} p^k[\phi_n(\eta^k(t_n - \sigma(t_n)) - t_n) - c]\right| \leq q_0\varepsilon\frac{1}{1-p} < \frac{l|c|}{4(1-p)},$$

and

$$\left|q(t_n)\sum_{k=N+1}^{\infty} p^k\phi_n(\eta^k(t_n - \sigma(t_n)) - t_n)\right| \leq q_0\, B\,\frac{p^{N+1}}{1-p} < \frac{l|c|}{4(1-p)}.$$

So for all integers $n \geq n_0$ we get

$$|-F(t_n,\phi_n)| > \frac{3l|c|}{4(1-p)} - \frac{l|c|}{4(1-p)} - \frac{l|c|}{4(1-p)} = \frac{l|c|}{4(1-p)} > 0,$$

which means that $F(t_n, \phi_n)$ does not converge to zero as $n \to \infty$. Therefore, condition (1.5) holds for Eq. (2.3), and the proof is complete. □

Remark. Since for each solution of Eq. (1.1) we have associated a different retarded equation with infinite delay, the uniform asymptotic stability of the zero solution of Eq. (2.3) does not imply the uniform asymptotic stability of the zero solution of Eq. (1.1).

Theorem 2.3. *Assume that* $0 \le p < 1$ *and there exists* $q_0 \in \mathbb{R}_+$ *so that* $0 \le q(t) \le q_0$ *for all* $t \ge 0$.

(i) If the condition

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} \le 1$$

holds and $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ *with* $t_0 \in \mathbb{R}_+$ *is a solution of Eq. (1.1), then*

$$\|x_t\| \le \|x_{t_0}\| \frac{1+p}{1-p} e^{5/2} \qquad \text{for all } t \ge t_0.$$

(ii) If

$$\frac{q_0\, s_0}{1-p} + \frac{q_0\, r_0\, p}{(1-p)^2} < 1$$

and $\liminf_{t\to\infty} q(t) > 0$ *are satisfied, then the zero solution of Eq. (1.1) is AS.*

The proof is analogous to that of Theorem 2.2, the only difference is that now we apply Corollary B. All we need is μ_1, which we have already calculated, and condition (1.5), which we have verified.

2.3 Attractivity for neutral differential equations

In this section we prove the attractivity of the zero solution of Eq. (1.1) by extending some results of Wu and Yu [76] to neutral equations with state-dependent delay. First we give some lemmas concerning boundedness and convergence of oscillatory solutions. For the proofs see [4].

A solution $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ of Eq. (1.1) is said to be oscillatory if x has arbitrarily large zeros.

Lemma 3.1. *Assume that* $|p| < \frac{1}{2}$, $q(t) > 0$ *for all sufficiently large* $t \in \mathbb{R}$,

$$2|p|(2-|p|) + \limsup_{t\to\infty} \int_{t-s_0}^{t} q(\tau)\, d\tau < \frac{3}{2} \tag{3.1}$$

and

(3.2)
for all $y \in C([-\lambda_0, \infty), \mathbb{R})$, *the function* $[0,\infty) \ni t \mapsto t - s(t, y_t) \in \mathbb{R}$ *is increasing.*

Then every oscillatory solution of Eq. (1.1) is bounded.

Note that condition (3.2) in Lemma 3.1 is the crutial point of the proof. Such a condition also plays an important role in nonneutral equations with state-dependent delay [60,39]. There is a wide class of applications where $s(t, y_t)$ is defined by a threshold that is

$$\int_{t - s(t, y_t)}^{t} k(y(\tau))\, d\tau = k_0,$$

where $k(t)$ is a positive continuous function and k_0 is a positive real number. In this case condition (3.2) holds.

Lemma 3.2. *Under the assumptions of Lemma 3.1, every oscillatory solution of Eq. (1.1) converges to zero as $t \to \infty$.*

The main result is the following.

Theorem 3.3. *Assume that $|p| < \frac{1}{2}$, $q(t) > 0$ for all sufficiently large $t \in \mathbb{R}$,*

$$\int_{t_0}^{\infty} q(\tau)\, d\tau = \infty, \tag{3.3}$$

$$2|p|(2 - |p|) + \limsup_{t\to\infty} \int_{t-s_0}^{t} q(\tau)\, d\tau < \frac{3}{2}$$

and

for all $y \in C([-\lambda_0, \infty), \mathbb{R})$, the function $[0, \infty) \ni t \mapsto t - s(t, y_t) \in \mathbb{R}$ is increasing.

Then every solution of Eq. (1.1) converges to zero as $t \to \infty$.

Proof. Let $t_0 \in \mathbb{R}_+$ and a solution $x : [t_0 - \lambda_0, \infty) \to \mathbb{R}$ of Eq. (1.1) be given. Using the notations $\rho(t) = r(t, x_t)$, $\sigma(t) = s(t, x_t)$ and $y(t) = x(t) - px(t - \rho(t))$ for $t \geq t_0$, we have

$$y'(t) = -q(t)x(t - \sigma(t)) \qquad \text{for all } t > t_0. \tag{3.4}$$

Two cases are to be considered:

Case I: x is oscillatory. Then the convergence to zero follows immediately from Lemma 3.2.

Case II: x is nonoscillatory. We assume that $x(t)$ is positive for all sufficiently large t. The case when $x(t)$ is negative for all sufficiently large t is similar. Our assumption implies that $y'(t) < 0$ for all sufficiently large t, say for all $t \geq T_0$, which means that $y(t)$ is decreasing on $[T_0, \infty)$. So the limit $c = \lim_{t\to\infty} y(t)$ exists and we claim that it is finite. Indeed if x is bounded then y is also bounded and thus $c \in \mathbb{R}$. In case x is not bounded assume $c = -\infty$. Then there exists $T_1 \geq t_0 + r_0$,

$T_1 > T_0$ so that $x(t) < px(t - \rho(t))$ for all $t \geq T_1$. Let $M = \sup_{t_0 \leq t \leq T_1} x(t)$ and $T_2 = \sup\{t \geq T_1 : x(\tau) \leq M \text{ for } t_0 \leq \tau \leq t\}$. Observe that $M > 0$ since $x(T_1) < px(T_1 - \rho(T_1))$. Using that x is unbounded from above we infer $T_2 < \infty$ and $M = x(T_2) < px(T_2 - \rho(T_2)) \leq pM$. From $M > 0$ we conclude $p \geq 1$. This contradiction leads to the conclusion $c \in \mathbb{R}$. Applying Lemma 2.1, we obtain the convergence of $x(t)$ to $\frac{c}{1-p}$ as $t \to \infty$. It remains to show that $c = 0$. Indeed if $c \neq 0$ then integrating (3.4) on $[t_0, \infty)$ and using (3.3), we get $c = -\infty$ which contradicts $c \in \mathbb{R}$, thereby completing the proof. □

Remark. Theorem 3.3 can be extended to the equation

$$\frac{d}{dt}\left[x(t) - px(t - r(t, x_t))\right] = f(t, x_t), \tag{3.5}$$

where $f : [0, \infty) \times C \to \mathbb{R}$. We assume that there exists a constant $H > 0$ and a continuous function $q : [t_0, \infty) \to \mathbb{R}$ such that

$$-q(t)M(\lambda, -\phi) \leq f(t, \phi) \leq q(t)M(\lambda, \phi) \text{ for all } t \geq t_0 \text{ and } \phi \in C \text{ with } \|\phi\| \leq H.$$

Then we can show that in case $|p| < \frac{1}{2}$, $q(t) > 0$ for all sufficiently large $t \in \mathbb{R}$ and

$$2|p|(2 - |p|) + \limsup_{t \to \infty} \int_{t-s_0}^{t} q(\tau)\, d\tau < \frac{3}{2}$$

there exists $h \in (0, H)$ such that for every $\phi \in C$ with $\|\phi\| \leq h$, the solution of Eq. (3.5) through (t_0, ϕ) converges to a constant as $t \to \infty$. Moreover, if for every constant mapping $c \in C$ with $\|c\| \leq H$, we have

$$\int_{t_0}^{\infty} |f(t, c)|\, dt = \infty,$$

then every solution of Eq. (3.5) through (t_0, ϕ) converges to zero as $t \to \infty$. We omit the proof because it follows the same technique as the proof of the Theorem 3.3 with appropriate modifications.

Chapter 3

Convergence of Solutions

3.1 Convergence in monotone dynamical systems

Consider the differential equation with state-dependent delay

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)), \quad r = r(x(t)), \tag{1.1}$$

where $\mu > 0$, f and r are smooth real functions with $f(0) = 0$ and $f' > 0$.

In the case $r \equiv$ constant Eq. (1.1) generates a semiflow $\mathcal{F}$ on the phase space $C([-r,0],\mathbb{R})$. The condition $f' > 0$ guarantees that $\mathcal{F}$ is monotone with respect to the pointwise ordering of the phase space, that is, for every ϕ, ψ in $C([-r,0],\mathbb{R})$ with $\phi \le \psi$, $\mathcal{F}(t,\phi) \le \mathcal{F}(t,\psi)$ holds for all $t \ge 0$. It is also true that $\mathcal{F}$ is strongly order preserving (SOP), that is, $\mathcal{F}$ is monotone, and for every ϕ, ψ in $C([-r,0],\mathbb{R})$ with $\phi < \psi$, there exist $t_0 > 0$ and open subsets $\mathcal{U}$, $\mathcal{V}$ of $C([-r,0],\mathbb{R})$ with $\phi \in \mathcal{U}$ and $\psi \in \mathcal{V}$ such that $\mathcal{F}(t_0,\mathcal{U}) \le \mathcal{F}(t_0,\mathcal{V})$. Then applying a result of Smith and Thieme [65,68,69], we conclude that the omega limit of all points from an open dense subset of the phase space is an equilibrium point.

We remark that analogous results were obtained by Smith and Thieme in [66,67] for non-quasi-monotone functional differential equations, that is, under a weaker condition than $f' > 0$.

In the case $r = r(x(t))$ the situation becomes more difficult, because it is not obvious how to choose the phase space. The questions about existence, uniqueness, and continuous dependence of solutions of Eq. (1.1) are also not standard, see, e.g., [62]. In Section 3.2 we show that for suitable $R > 0$ and $A > 0$, the solution of Eq. (3) defines a semiflow F on the metric space X containing Lipschitz continuous functions mapping $[-R,0]$ into $[-A,A]$ with metric $d(\phi,\psi) = \sup_{-R\le s\le 0} |\phi(s) - \psi(s)|$. The constant $R > 0$ is the maximum of r on $[-A,A]$. A difficulty arises: the semiflow generated by Eq. (1.1) is not in general SOP. Thus the result of Smith and Thieme is not applicable. Our aim is to prove a convergence result which is applicable for Eq. (1.1).

We give the result of Smith and Thieme first and then the modified version of this convergence result. In [65,68] a metric space $\mathcal{X}$ is considered and a closed partial order relation $\le$ on $\mathcal{X}$. For x and y in $\mathcal{X}$, $x < y$ is written whenever $x \le y$ and $x \ne y$.

A semiflow Φ is considered on $\mathcal{X}$, that is a map $\Phi : [0, \infty) \times \mathcal{X} \to \mathcal{X}$, which satisfies: (i) Φ is continuous, (ii) $\Phi(0, x) = x$ for all $x \in \mathcal{X}$, (iii) $\Phi(t, \Phi(s, x)) = \Phi(t + s, x)$ for all $t \geq 0$, $s \geq 0$ and $x \in \mathcal{X}$.

The orbit $O(x)$ of $x \in \mathcal{X}$ is defined by $O(x) = \{\Phi(t, x) : t \geq 0\}$. A point $x \in \mathcal{X}$ is called an equilibrium point if $O(x) = \{x\}$. The set of all equilibrium points of Φ is denoted by E. The omega limit set $\omega(x)$ of $x \in \mathcal{X}$ is defined by $\omega(x) = \cap_{t \geq 0} \overline{\cup_{s \geq t} \Phi(s, x)}$. Recall that $\omega(x)$ is a nonempty, compact, invariant subset of $\mathcal{X}$ and $\text{dist}(\Phi(t, x), \omega(x)) \to 0$ as $t \to \infty$ provided $\overline{O(x)}$ is a compact subset of $\mathcal{X}$. A point $x \in \mathcal{X}$ is called a quasiconvergent point if $\omega(x) \subset E$. The set of all such points is denoted by $\mathcal{Q}$. A point x is called a convergent point if $\omega(x)$ consists of a single point of E. The set of all convergent points is denoted by $\mathcal{C}$.

It is supposed that Φ is monotone, that is, for every x, y in $\mathcal{X}$ with $x \leq y$, $\Phi(t, x) \leq \Phi(t, y)$ holds for all $t \geq 0$. It is assumed that Φ is strongly order preserving (SOP), that is, Φ is monotone, and for every x, y in $\mathcal{X}$ with $x < y$, there exist $t_0 > 0$ and open subsets $\mathcal{U}$, $\mathcal{V}$ of $\mathcal{X}$ with $x \in \mathcal{U}$ and $y \in \mathcal{V}$ such that $\Phi(t_0, \mathcal{U}) \leq \Phi(t_0, \mathcal{V})$.

Assume that for each x in $\mathcal{X}$, $O(x)$ has compact closure in $\mathcal{X}$.

It is supposed that for every x in $\mathcal{X}$,

(a) there exists a sequence $(x_n)_1^\infty$ in $\mathcal{X}$ satisfying $x_n < x_{n+1} < x$ $(x < x_{n+1} < x_n)$ for all integers $n \geq 1$ and $x_n \to x$ as $n \to \infty$, and

(b) for the sequence $(x_n)_1^\infty$ with the property guaranteed by (a), $\cup_{n \geq 1} \omega(x_n)$ has compact closure in $\mathcal{X}$.

Then the result of Smith and Thieme [65,68] states that under the above assumptions $\mathcal{X} = \text{Int}\, \mathcal{Q} \cup \overline{\text{Int}\, \mathcal{C}}$. In particular Int $\mathcal{Q}$ is dense in $\mathcal{X}$.

We observe that the semiflow Φ generated by Eq. (1.1) has the following property. Φ is monotone, and for x, y in $\mathcal{X}$ with $x < y$ and $\Phi(t, x) \neq \Phi(t, y)$ for all $t \geq 0$, there exist $t_0 > 0$, and open subsets $\mathcal{U}$, $\mathcal{V}$ of $\mathcal{X}$ with $x \in \mathcal{U}$ and $y \in \mathcal{V}$ such that $\Phi(t_0, \mathcal{U}) \leq \Phi(t_0, \mathcal{V})$.

We give some notations and definitions. We shall write $x <_\Phi y$ if $x < y$ and $\Phi(t, x) \neq \Phi(t, y)$ for all $t \geq 0$. For two subsets $\{a\} = A$ and B of $\mathcal{X}$ with $A \leq B$ or $A < B$ or $A <_\Phi B$, we shall write $a \leq B$ or $a < B$ or $a <_\Phi B$.

We say that Φ is mildly order preserving (MOP) if it is monotone, and for every x, y in $\mathcal{X}$ with $x <_\Phi y$, there exist $t_0 > 0$ and open subsets $\mathcal{U}$, $\mathcal{V}$ of $\mathcal{X}$ with $x \in \mathcal{U}$ and $y \in \mathcal{V}$ such that $\Phi(t_0, \mathcal{U}) \leq \Phi(t_0, \mathcal{V})$.

Note that the difference between the MOP and SOP properties is that we have one more assumption for the MOP property, that is, $\Phi(t, x) \neq \Phi(t, y)$ for all $t \geq 0$

and for all x, y in $\mathcal{X}$ with $x < y$. Thus the MOP property is weaker than the SOP property.

We say that $x \in \mathcal{X}$ can be approximated from below (above) in $\mathcal{X}$ if there exists a sequence $(x_n)_1^\infty$ in $\mathcal{X}$ satisfying $x_n <_\Phi x_{n+1} <_\Phi x$ ($x <_\Phi x_{n+1} <_\Phi x_n$) for all integers $n \geq 1$ and $x_n \to x$ as $n \to \infty$.

Since the semiflow generated by Eq. (1.1) is MOP, we are interested in proving a convergence result for the monotone semiflow being MOP instead of SOP. Thus a natural question appears: Under what conditions can we prove such a convergence result? In Section 3.3 we prove the following theorem.

Theorem 1.1. *Consider a metric space $\mathcal{X}$ with a closed partial order relation and a semiflow Φ on $\mathcal{X}$. Assume that*

(A_1) *if x and y are in a compact invariant subset of $\mathcal{X}$, then $x < y$ implies $x <_\Phi y$,*
(A_2) *Φ is MOP,*
(A_3) *each point in $\mathcal{X}$ can be approximated either from below or from above in $\mathcal{X}$,*
(A_4) *for each x in $\mathcal{X}$, the orbit $O(x)$ of x has compact closure in $\mathcal{X}$, and*
(A_5) *for each x in $\mathcal{X}$ and for each sequence $(x_n)_1^\infty$, which approximates x either from below or from above in $\mathcal{X}$, $\cup_{n\geq 1} \omega(x_n)$ has compact closure in $\mathcal{X}$.*

Then $\mathcal{X} = \mathrm{Int}\, \mathcal{Q} \cup \overline{\mathrm{Int}\, \mathcal{C}}$. In particular $\mathrm{Int}\, \mathcal{Q}$ is dense in $\mathcal{X}$.

We mention that the differences between Theorem 1.1 and the theorem of Smith and Thieme are that assumption (A_1) in Theorem 1.1, the crutial point of the proof, does not appear in the result of Smith and Thieme, the semiflow Φ in Theorem 1.1 is MOP instead of SOP, and the definition of approximation in Theorem 1.1 differs from the one used by Smith and Thieme. An examination of the proof of the result of Smith and Thieme shows that due to assumption (A_1) the proof of Theorem 1.1 follows more or less the same line as that in [65,68].

3.2 Convergence of solutions for an Equation with State-Dependent Delay

In this section we apply Theorem 1.1 for the differential equation with state-dependent delay

$$\dot{x}(t) = -\mu x(t) + f(x(t - r(x(t)))). \tag{1.1}$$

In order to achieve our goal, we need to establish hypotheses on f and r and choose an appropriate phase space ensuring that the conditions of Theorem 1.1 are satisfied.

Consider the hypotheses

$$\text{(H1)} \qquad \begin{cases} \mu > 0, \\ f \in C^1(\mathbb{R},\mathbb{R}),\ f(0)=0,\ f'(u) > 0 \text{ for all } u \in \mathbb{R}, \\ \text{there exists } A > 0 \text{ such that } |f(u)| < \mu|u| \text{ for all } |u| \geq A, \\ r \in C^1(\mathbb{R},\mathbb{R}),\ r(0)=1,\ r([-A,A]) \subset (0,\infty). \end{cases}$$

Set $R = \max_{u\in[-A,A]} r(u)$, $M = \max_{(u,v)\in[-A,A]\times[-A,A]} |-\mu u + f(v)|$, $C = C([-R,0],\mathbb{R})$ and for $\phi \in C$ define

$$\text{lip}(\phi) = \sup\{|\phi(s)-\phi(t)|\cdot|s-t|^{-1} : s,\ t \in [-R,0],\ s \neq t\}.$$

Let G be the map defined by the right hand side of Eq. (1.1), that is, $G(\phi) = -\mu\phi(0) + f(\phi(-r(\phi(0))))$. Observe that G is not necessary defined for all $\phi \in C$. For $\phi \in C$ with $\phi(0) \notin [-A,A]$, $G(\phi)$ may not exist. Therefore consider the retraction $\rho : C \to D$ defined by $C \ni \phi \mapsto \rho(\phi) \in D$, where $D = \{\phi \in C : -A \leq \phi(t) \leq A, \text{for } -R \leq t \leq 0\}$. $\rho(\phi)$ is a function mapping $[-R,0]$ into $[-A,A]$ in the following way:

$$\text{(2.1)} \qquad \rho(\phi)(t) = \begin{cases} A & \text{for } \phi(t) \geq A, \\ \phi(t) & \text{for } -A \leq \phi(t) \leq A, \\ -A & \text{for } \phi(t) \leq -A. \end{cases}$$

Consider the function $G \circ \rho : C \to \mathbb{R}$ defined by $G(\rho(\phi)) = -\mu\rho(\phi)(0) + f(\rho(\phi)(-r(\rho(\phi)(0))))$ for all ϕ in C and the equation

$$\text{(2.2)} \qquad \dot{x}(t) = G(\rho(x_t)),$$

where $x_t \in C$, $t \geq 0$, is defined by $x_t(s) = x(t+s)$ for all $s \in [-R,0]$.

We say that a function $x : [-R,\delta) \to \mathbb{R}$, $0 < \delta \leq \infty$, is a solution of Eq. (2.2) if x is continuous on $[-R,\delta)$, $x|_{[0,\delta)}$ is differentiable on $[0,\delta)$, and x satisfies Eq. (2.2) for all $t \in [0,\delta)$. $\dot{x}(0)$ denotes the right hand derivative of x at 0.

We say that a function $x : [-R,\infty) \to [-A,A]$ is a solution of Eq. (1.1) if x is continuous on $[-R,\infty)$, $x|_{[0,\infty)}$ is differentiable on $[0,\infty)$, and x satisfies Eq. (1.1) for all $t \geq 0$.

First we show existence, uniqueness, and continuous dependence of solutions of Eq. (2.2) and then of solutions of Eq. (1.1).

We begin with proving the following claim.

Claim. *The function $G \circ \rho : C \to \mathbb{R}$ is continuous and satisfies the following property: there exist constants $a_1 > 0$ and $a_2 > 0$ such that for all $R_1 > 0$ and for all ϕ, ψ in C with $lip(\phi) \le R_1$,*

$$|G(\rho(\phi)) - G(\rho(\psi))| \le (a_1 + a_2 R_1)\|\phi - \psi\|. \tag{2.3}$$

Proof. It is easy to check that ρ is continuous; thus $G \circ \rho$ is continuous, and for all ϕ, ψ in C, we have

$$\|\rho(\phi) - \rho(\psi)\| \le \|\phi - \psi\| \tag{2.4}$$

and

$$\text{lip}(\rho(\phi)) \le \text{lip}(\phi). \tag{2.5}$$

To show (2.3), we make the estimations

$$\begin{aligned} |G(\rho(\phi)) - G(\rho(\psi))| \le \mu|\rho(\phi)(0) - \rho(\psi)(0)| + \\ |f(\rho(\phi)(-r(\rho(\phi)(0)))) - f(\rho(\psi)(-r(\rho(\psi)(0))))|. \end{aligned} \tag{2.6}$$

The function f is locally Lipschitzian because it is continuously differentiable. Thus there exists $L > 0$ such that

$$\begin{aligned} |f(\rho(\phi)(-r(\rho(\phi)(0)))) - f(\rho(\psi)(-r(\rho(\psi)(0))))| \le \\ L|\rho(\phi)(-r(\rho(\phi)(0))) - \rho(\psi)(-r(\rho(\psi)(0)))| \le \\ L|\rho(\phi)(-r(\rho(\phi)(0))) - \rho(\phi)(-r(\rho(\psi)(0)))| + \\ L|\rho(\phi)(-r(\rho(\psi)(0))) - \rho(\psi)(-r(\rho(\psi)(0)))|. \end{aligned} \tag{2.7}$$

Using (2.5) and the fact that r is locally Lipschitzian, being continuously differentiable, we obtain that there exists $N > 0$ such that

$$\begin{aligned} |\rho(\phi)(- r(\rho(\phi)(0))) - \rho(\phi)(-r(\rho(\psi)(0)))| \le \\ R_1|r(\rho(\phi)(0)) - r(\rho(\psi)(0))| \le R_1 N|\rho(\phi)(0) - \rho(\psi)(0)|. \end{aligned} \tag{2.8}$$

Combining (2.7) with (2.8) and (2.4), we find

$$|f(\rho(\phi)(-r(\rho(\phi)(0)))) - f(\rho(\psi)(-r(\rho(\psi)(0))))| \le (LR_1N + L)\|\phi - \psi\|. \tag{2.9}$$

We deduce from (2.6), (2.9), and (2.4) that

$$\begin{aligned} |G(\rho(\phi)) - G(\rho(\psi))| &\le \mu\|\phi - \psi\| + (LR_1N + L)\|\phi - \psi\| \\ &= ((\mu + L) + LNR_1)\|\phi - \psi\| = (a_1 + a_2R_1)\|\phi - \psi\|. \end{aligned}$$

Observe that the constant $a_1 + a_2 R_1$ depends on ϕ and it is independent of ψ. The proof is complete. □

For local and global existence we refer to Theorem 1.1 of [62]. Consider $\phi \in C$. Since $G \circ \rho : C \to \mathbb{R}$ is continuous, by Theorem 1.1 of [62], for some $\delta > 0$, there exists a solution $x^\phi : [-R, \delta) \to \mathbb{R}$ of Eq. (2.2) through ϕ, that is, x^ϕ is a solution of Eq. (2.2) such that $x^\phi|_{[-R,0]} = \phi$. As there exist constants c_1 and c_2 such that for all ϕ in C, $|G(\rho(\phi))| \leq c_1\|\phi\| + c_2$, by Theorem 1.1 of [62], the solution x^ϕ can be defined on $[-R, \infty)$.

For uniqueness we refer to Theorem 1.2 of [62]. Consider $\phi \in C$ with lip $(\phi) < \infty$, and for some $\delta > 0$ two solutions $y^\phi : [-R, \delta) \to \mathbb{R}$ and $z^\phi : [-R, \delta) \to \mathbb{R}$ of Eq. (2.2) through ϕ. Since $G \circ \rho$ is continuous on C and it satisfies property (2.3), by Theorem 1.2 of [62], we obtain $y^\phi(t) = z^\phi(t)$ for all t in $[-R, \delta)$. For continuous dependence of solutions of Eq. (2.2) we refer to Theorem 1.6 of [62].

We define the phase space X as the metric space of all real-valued continuous functions $\phi : [-R, 0] \to [-A, A]$ with lip$(\phi) < \infty$, where the metric is obtained from $\|\phi\| = \max_{-R \leq s \leq 0} |\phi(s)|$. We introduce a closed partial order relation on X in the following way: $\phi \leq \psi$ whenever $\phi(s) \leq \psi(s)$ for all $s \in [-R, 0]$, $\phi < \psi$ whenever $\phi \leq \psi$ and $\phi \neq \psi$, and $\phi \ll \psi$ whenever $\phi(s) < \psi(s)$ for all $s \in [-R, 0]$.

To prove existence, uniqueness, and continuous dependence of solutions of Eq. (1.1), we need the following proposition.

Proposition 2.1. *For every $\phi \in X$ the solution $x = x^\phi : [-R, \infty) \to \mathbb{R}$ of Eq. (2.2) through ϕ satisfies $x(t) \in [-A, A]$ for all $t \geq 0$.*

Proof. Set $t_1 = \sup\{s : x(t) \in [-A, A] \text{ for all } t \in [0, s]\}$. If t_1 belongs to the interval of existence, then either $x(t_1) = A$ and $\dot{x}(t_1) \geq 0$ or $x(t_1) = -A$ and $\dot{x}(t_1) \leq 0$. Suppose $x(t_1) = A$ and $\dot{x}(t_1) \geq 0$. Since r is positive on $[-A, A]$, we have $x(t_1 - r(x(t_1))) \leq A$. The monotonicity of f yields $f(x(t_1 - r(x(t_1)))) \leq f(A)$. The solution x satisfies Eq. (2.2), that is $\dot{x}(t_1) = -\mu\rho(x_{t_1})(0) + f(\rho(x_{t_1})(-r(\rho(x_{t_1})(0))))$. As $x_{t_1}(0) \in [-A, A]$ and $x_{t_1}(-r(x_{t_1}(0))) \in [-A, A]$, according to the definiton of ρ, we obtain $\rho(x_{t_1})(0) = x_{t_1}(0)$ and $\rho(x_{t_1})(-r(\rho(x_{t_1})(0))) = x_{t_1}(-r(x_{t_1}(0)))$. Thus, $\dot{x}(t_1) = -\mu x(t_1) + f(x(t_1 - r(x(t_1))))$. The assumption $-\mu A + f(A) < 0$ implies $\dot{x}(t_1) < 0$. This is a contradiction. A similar argument leads to a contradiction in the case $x(t_1) = -A$ and $\dot{x}(t_1) \leq 0$. Therefore $x(t) \in [-A, A]$ for all $t \geq 0$. □

Proposition 2.2. *For every $\phi \in X$ there is a unique solution $x = x^{\phi} : [-R, \infty) \to [-A, A]$ of Eq. (1.1) such that $x|_{[-R,0]} = \phi$.*

Proof. Let $\phi \in X$. Then there exists a solution $x = x^{\phi} : [-R, \infty) \to \mathbb{R}$ of Eq. (2.2) through ϕ. By Proposition 2.1, $x(t) \in [-A, A]$ for all $t \geq 0$. Hence $\rho(x_t) = x_t$ and $G(\rho(x_t)) = G(x_t)$ for all $t \geq 0$. Thus, $x : [-R, \infty) \to [-A, A]$ is a solution of Eq. (1.1) with $x|_{[-R,0]} = \phi$. Consider a solution $y = y^{\phi} : [-R, \infty) \to [-A, A]$ of Eq. (1.1) such that $y|_{[-R,0]} = \phi$. Since $y(t) \in [-A, A]$ for all $t \geq 0$, by the definition of ρ, $y_t = \rho(y_t)$, and $G(y_t) = G(\rho(y_t))$ for all $t \geq 0$. Thus y is also a solution of Eq. (2.2) through ϕ; therefore $x(t) = y(t)$ for all t in the interval of existence. □

Proposition 2.3. *Let $\epsilon \geq 0$, ϕ, $\psi \in X$, $x = x^{\phi} : [-R, \infty) \to [-A, A]$ be the unique solution of equation $\dot{x}(t) = G(x_t)$ such that $x|_{[-R,0]} = \phi$, and $y = y^{\psi} : [-R, \infty) \to [-A, A]$ be the unique solution of equation $\dot{y}(t) = G(y_t) + \epsilon$ such that $y|_{[-R,0]} = \psi$. Then there exists a constant $c > 0$ independent of ϵ, ϕ, ψ such that*

$$|x(t) - y(t)| \leq e^{ct}\|\phi - \psi\| + (e^{ct} - 1)\frac{\epsilon}{c} \quad \textit{for all } t \geq 0.$$

Proof. Since $x(t) \in [-A, A]$ and $y(t) \in [-A, A]$ for all $t \geq 0$, $G(x_t) = G(\rho(x_t))$ and $G(y_t) = G(\rho(y_t))$ for all $t \geq 0$. Thus x is the solution of equation $\dot{x}(t) = G(\rho(x_t))$ with $x|_{[-R,0]} = \phi$, and y is the solution of equation $\dot{y}(t) = G(\rho(y_t)) + \epsilon$ with $y|_{[-R,0]} = \psi$. By Theorem 1.6 of [62], there exists $c > 0$ such that we have the desired estimation for $|x(t) - y(t)|$. □

We define the map F by $[0, \infty) \times X \ni (t, \phi) \mapsto x_t^{\phi} \in X$, where x^{ϕ} denotes the solution of Eq. (1.1) through ϕ.

Proposition 2.4. *The map F is a semiflow on X, that is:*
(i) F is continuous,
(ii) $F(0, \phi) = \phi$ for all $\phi \in X$,
(iii) $F(t, F(s, \phi)) = F(t + s, \phi)$ for all $t \geq 0$, for all $s \geq 0$ and for all $\phi \in X$.

Proof. Proof of (i). The continuity of F in the first variable is obvious. To prove the continuity of F in the second variable, consider the solutions $x = x^{\phi}$ and $y = x^{\phi_0}$ of Eq. (1.1) through ϕ and $\phi_0 \in X$, respectively. Applying Proposition 2.3 with $\epsilon = 0$, we obtain that there exists a constant $c > 0$ such that $|x(t) - y(t)| \leq e^{ct}\|\phi - \phi_0\|$ for all $t \geq 0$. Hence $\|x_t - y_t\| \leq e^{ct}\|\phi - \phi_0\|$ for all $t \geq 0$, which implies the continuity of F in the second variable. It is easy to see that (ii) and (iii) are satisfied. □

Proposition 2.5. *The semiflow F is monotone, that is, $F(t,\phi) \le F(t,\psi)$ whenever $\phi \le \psi$ and $t \ge 0$.*

Proof. Let ϕ, ψ be in X with $\phi \le \psi$, and $\epsilon_0 > 0$ such that $-\mu A + f(A) + \epsilon_0 < 0$. For all ϵ in $(0, \epsilon_0)$, consider the equation

$$\dot{z}(t) = -\mu z(t) + f(z(t - r(z(t)))) + \epsilon. \tag{2.10}$$

First we show that the solution of Eq. (2.10) through ψ exists and it is unique. Consider the retraction ρ defined by (2.1), the map $G(\rho(\phi)) = -\mu\rho(\phi)(0) + f(\rho(\phi)(-r(\rho(\phi)(0))))$ for all $\phi \in C$, and the equation

$$\dot{z}(t) = G(\rho(z_t)) + \epsilon. \tag{2.11}$$

We mention that the definitions of solutions of Eq. (2.10) and Eq. (2.11) are the same as those for Eq. (1.1) and Eq. (2.2), respectively. Since the function defined by the right hand side of Eq. (2.11) is continuous, it satisfies property (2.3), and there exist constants c_1 and c_2 such that for all $\phi \in C$, $|G(\rho(\phi)) + \epsilon| \le c_1 \|\phi\| + c_2$, by Theorem 1.1 of [62], we obtain that there exists a unique solution $z^\epsilon : [-R, \infty) \to \mathbb{R}$ of Eq. (2.11) with $z_0^\epsilon = \psi$. We prove that $z^\epsilon(t) \in [-A, A]$ for all $t \ge 0$. Set $t_1 = \sup\{s : z^\epsilon(t) \in [-A, A] \text{ for all } t \in [0, s]\}$. If t_1 belongs to the interval of existence, then either $z^\epsilon(t_1) = A$ and $\dot{z}^\epsilon(t_1) \ge 0$ or $z^\epsilon(t_1) = -A$ and $\dot{z}^\epsilon(t_1) \le 0$. Suppose $z^\epsilon(t_1) = A$ and $\dot{z}^\epsilon(t_1) \ge 0$. Since r is positive, we have $z^\epsilon(t_1 - r(z^\epsilon(t_1))) \le A$. The monotonicity of f yields $f(z^\epsilon(t_1 - r(z^\epsilon(t_1)))) \le f(A)$. As $z_{t_1}^\epsilon(0) \in [-A, A]$ and $z_{t_1}^\epsilon(-r(z_{t_1}^\epsilon(0))) \in [-A, A]$, we infer

$$\begin{aligned} \dot{z}^\epsilon(t_1) &= -\mu\rho(z_{t_1}^\epsilon)(0) + f(\rho(z_{t_1}^\epsilon)(-r(\rho(z_{t_1}^\epsilon)(0)))) + \epsilon \\ &= -\mu z^\epsilon(t_1) + f(z^\epsilon(t_1 - r(z^\epsilon(t_1)))) + \epsilon \le -\mu A + f(A) + \epsilon_0. \end{aligned}$$

The assumption $-\mu A + f(A) + \epsilon_0 < 0$ implies $\dot{z}^\epsilon(t_1) < 0$. This is a contradiction. A similar argument leads to a contradiction in the case $z^\epsilon(t_1) = -A$ and $\dot{z}^\epsilon(t_1) \le 0$. Therefore, $z^\epsilon(t) \in [-A, A]$ for all $t \ge 0$. Hence $\rho(z_t^\epsilon) = z_t^\epsilon$ and $G(\rho(z_t^\epsilon)) = G(z_t^\epsilon)$ for all $t \ge 0$. Thus, $z^\epsilon : [-R, \infty) \to [-A, A]$ is a solution of Eq. (2.10) through ψ. To prove uniqueness, consider a solution $y^\epsilon : [-R, \infty) \to [-A, A]$ of Eq. (2.10) with $y_0^\epsilon = \psi$. Since $y^\epsilon(t) \in [-A, A]$ for all $t \ge 0$, y^ϵ is a solution of Eq. (2.11) through ψ. Therefore $z^\epsilon(t) = y^\epsilon(t)$ for all $t \in [-R,\infty)$. Thus $z^\epsilon : [-R,\infty) \to [-A,A]$ is the unique solution of Eq. (2.10) through ψ. Denote by x and y the solutions of Eq. (1.1) with $x_0 = \phi$ and $y_0 = \psi$. Proposition 2.3 implies $z^\epsilon \to y$ uniformly on compact subsets of $[-R,\infty)$ as $\epsilon \to 0$. Therefore, in order to conclude the monotonicity of F, it suffices to show that $x(t) < z^\epsilon(t)$ for all $t > 0$ and for all ϵ in $(0,\epsilon_0)$. Fix an ϵ. If $\phi(0) = \psi(0)$ then $x(0) = z^\epsilon(0)$. So $\dot{z}^\epsilon(0) - \dot{x}(0) = f(\psi(-r(x(0)))) - f(\phi(-r(x(0)))) + \epsilon$. As $\phi(-r(x(0))) \leq \psi(-r(x(0)))$, the assumption $f' > 0$ implies $f(\phi(-r(x(0)))) \leq f(\psi(-r(x(0))))$. Consequently, $\dot{z}^\epsilon(0) - \dot{x}(0) \geq \epsilon > 0$. Hence there exists $\delta > 0$ so that $x(t) < z^\epsilon(t)$ for all t in $(0,\delta)$. The existence of such a δ in the case $\phi(0) < \psi(0)$ follows immediately from the continuity of z^ϵ and x. Therefore, by way of contradiction we can choose $s > 0$ such that $x(t) < z^\epsilon(t)$ for $0 < t < s$ and $x(s) = z^\epsilon(s)$. Clearly, $\dot{x}(s) \geq \dot{z}^\epsilon(s)$. On the other hand we have

$$\dot{z}^\epsilon(s) = -\mu z^\epsilon(s) + f(z^\epsilon(s - r(z^\epsilon(s)))) + \epsilon. \tag{2.12}$$

As $\phi \leq \psi$ and r is positive on $[-A,A]$, $x(s - r(x(s))) \leq z^\epsilon(s - r(x(s))) = z^\epsilon(s - r(z^\epsilon(s)))$. The assumption $f' > 0$ implies $f(x(s - r(x(s)))) \leq f(z^\epsilon(s - r(z^\epsilon(s))))$. Consequently,

$$-\mu x(s) + f(x(s - r(x(s)))) < -\mu z^\epsilon(s) + f(z^\epsilon(s - r(z^\epsilon(s)))) + \epsilon. \tag{2.13}$$

Combining (2.12) with (2.13), we conclude $\dot{z}^\epsilon(s) > \dot{x}(s)$. This contradiction completes the proof. □

Our next goal is to prove that if ϕ and ψ belong to some compact invariant subset of X and $\phi < \psi$, then $\phi <_F \psi$. In order to achive this purpose we need the following two lemmas.

Lemma 2.6. *If $x : \mathbb{R} \to [-A,A]$ and $y : \mathbb{R} \to [-A,A]$ are two solutions of Eq. (1.1), then the difference $z = x - y$ satisfies the linear equation*

$$\dot{z}(t) = a(t)z(t) + b(t)z(t - r(x(t))) \tag{2.14}$$

for all $t \in \mathbb{R}$, where $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are continuous and bounded functions defined by

$$\begin{aligned} a(t) = -\mu - &\int_0^1 f'[sy(t - r(x(t))) + (1-s)y(t - r(y(t)))]ds \times \\ &\int_0^1 y'[s(t - r(x(t))) + (1-s)(t - r(y(t)))]ds \times \\ &\int_0^1 r'[sx(t) + (1-s)y(t)]ds \end{aligned} \tag{2.15}$$

and

$$b(t) = \int_0^1 f'[sx(t - r(x(t))) + (1 - s)y(t - r(x(t)))]ds. \tag{2.16}$$

Proof. Consider the solutions $x : \mathbb{R} \to [-A, A]$ and $y : \mathbb{R} \to [-A, A]$ of Eq. (1.1). For the difference $z = x - y$, we have

$$\begin{aligned} \dot{z}(t) &= -\mu z(t) + f(x(t - r(x(t)))) - f(y(t - r(y(t)))) \\ &= -\mu z(t) + f(x(t - r(x(t)))) - f(y(t - r(x(t)))) + \\ &\quad f(y(t - r(x(t)))) - f(y(t - r(y(t)))) \qquad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Using the equality

$$\begin{aligned} f(u) - f(v) &= \int_0^1 \frac{d}{ds} f(su + (1 - s)v)ds \\ &= \int_0^1 f'(su + (1 - s)v)ds \times (u - v) \qquad \text{for all } u, v \in \mathbb{R}, \end{aligned} \tag{2.17}$$

we obtain, for $t \in \mathbb{R}$

(2.18)

$$\begin{aligned} \dot{z}(t) = -\mu z(t) + &\int_0^1 f'[sx(t - r(x(t))) + (1 - s)y(t - r(x(t)))]ds \times \\ &z(t - r(x(t))) + \int_0^1 f'[sy(t - r(x(t))) + (1 - s)y(t - r(y(t)))]ds \times \\ &[y(t - r(x(t))) - y(t - r(y(t)))], \end{aligned}$$

$$\begin{aligned} &y(t - r(x(t))) - y(t - r(y(t))) = \\ &- \int_0^1 y'[s(t - r(x(t))) + (1 - s)(t - r(y(t)))]ds \times [r(x(t)) - r(y(t))], \end{aligned} \tag{2.19}$$

$$r(x(t)) - r(y(t)) = \int_0^1 r'[sx(t) + (1 - s)y(t)]ds \times z(t). \tag{2.20}$$

Combining (2.18) with (2.19) and (2.20), it follows that z satisfies Eq. (2.14), where $a(t)$ and $b(t)$ are continuous, bounded functions defined by (2.15) and (2.16). The proof is complete. □

Remark 2.7. *If $x : [-R, \infty) \to [-A, A]$ and $y : [-R, \infty) \to [-A, A]$ are two solutions of Eq. (1.1), then the difference $z = x - y$ satisfies Eq. (2.14) for $t \geq R$,* *where $a : [R,\infty) \to \mathbb{R}$ and $b : [R,\infty) \to \mathbb{R}$ are continuous and bounded functions defined by (2.15) and (2.16).*

Lemma 2.8. *If $x : \mathbb{R} \to [-A, A]$ and $y : \mathbb{R} \to [-A, A]$ are two solutions of Eq. (1.1) on $\mathbb{R}$ and $x_0 = y_0$, then $x(t) = y(t)$ for all $t \in \mathbb{R}$.*

Proof. Set $z = x - y$. By Lemma 2.6, z satisfies Eq. (2.14), with $a(t)$ and $b(t)$ given by (2.15) and (2.16). Moreover, $b(t) > 0$ for all $t \in \mathbb{R}$ since $f' > 0$. Define $v(t) = z(t)e^{-\int_0^t a(s)ds}$ for all $t \in \mathbb{R}$. Multiplying Eq. (2.14) by $e^{-\int_0^t a(s)ds}$, we infer

$$\dot{v}(t) = b(t)z(t - r(x(t)))\, e^{-\int_0^t a(s)ds} \qquad \text{for all } t \in \mathbb{R}. \tag{2.21}$$

The definition of $v(t)$ yields

$$z(t - r(x(t))) = v(t - r(x(t)))\, e^{\int_0^{t-r(x(t))} a(s)ds} \qquad \text{for all } t \in \mathbb{R}.$$

Thus $v(t)$ satisfies the linear equation

$$\dot{v}(t) = c(t)v(t - r(x(t))) \qquad \text{for all } t \in \mathbb{R}, \tag{2.22}$$

where $c(t)$ is defined by $c(t) = b(t)\, e^{-\int_{t-r(x(t))}^{t} a(s)ds}$ for all $t \in \mathbb{R}$, and $c(t)$ is continuous and bounded, and $c(t) > 0$ for all $t \in \mathbb{R}$. In order to show that $z(t) = 0$ for all $t \in \mathbb{R}$, it suffices to prove that $v(t) = 0$ for all $t \in \mathbb{R}$. Note that $v(t) = 0$ for all $t \geq -R$ due to the uniqueness of solutions and $x_0 = y_0$. Set $\tau = \inf\{t : v(s) = 0 \text{ for all } s \geq t\}$. We claim that $\tau = -\infty$. Otherwise $-\infty < \tau \leq -R$. We have $v(t) = 0$ for all $t \geq \tau$. Therefore $\dot{v}(t) = 0$ for all $t \geq \tau$. Then, by Eq. (2.22), $v(t - r(x(t))) = 0$ for all $t \geq \tau$, which using the definition of τ, implies $\tau = \inf\{t - r(x(t)) : t \geq \tau\}$. On the other hand using the assumption $r(u) > 0$ for all $u \in [-A, A]$, we obtain $\inf\{t - r(x(t)) : t \geq \tau\} \leq \tau - r(x(\tau)) < \tau$. This contadiction shows that $\tau = -\infty$. Consequently, $v(t) = 0$ for all $t \in \mathbb{R}$ and the lemma is proved. □

In the proofs of the above lemmas it is important that the delay r depends only on $x(t)$ and not on x_t. The hypothesis $f' > 0$ seems to be also crucial.

Corollary 2.9. *Let B be a compact invariant subset of X, where invariance means that for any $\phi \in B$, there exists a solution x^ϕ of Eq. (1.1) on $\mathbb{R}$ with $x_0^\phi = \phi$ and $x_t^\phi \in B$ for all $t \in \mathbb{R}$. If $\phi, \psi \in B$ with $\phi \neq \psi$, then $x_t^\phi \neq x_t^\psi$ for all $t \in \mathbb{R}$.*

If $x : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (1.1) with $x_0 = \phi$, then we also denote this solution by x^ϕ. This should not cause confusion.

The semiflow F has the following property.

Proposition 2.10. *If ϕ, $\psi \in X$ with $\phi <_F \psi$, then there exists $t_0 > 0$ such that $F(t,\phi) \ll F(t,\psi)$ for all $t \geq t_0$.*

Proof. Consider ϕ, $\psi \in X$ with $\phi <_F \psi$, and the solutions $x = x^\phi : [-R,\infty) \to [-A,A]$ and $y = x^\psi : [-R,\infty) \to [-A,A]$ of Eq. (1.1) through ϕ and ψ, respectively. The monotonicity of F implies $x(t) \leq y(t)$ for all $t \geq -R$. Our aim is to prove that $x(t) < y(t)$ for all $t \geq 2R$. Then it follows that $x_t(s) < y_t(s)$ for all $s \in [-R,0]$ and for all $t \geq 3R$, that is, $F(t,\phi) \ll F(t,\psi)$ for all $t \geq 3R$. Set $z = x - y$. By Remark 2.7, z satisfies Eq. (2.14) for $t \geq R$ with $a(t)$ and $b(t)$ defined by (2.15) and (2.16) for $t \geq R$. We define $v(t) = z(t)\,\mathrm{e}^{-\int_R^t a(s)ds}$ for all $t \geq R$. Instead of (2.21) here we obtain

$$\dot{v}(t) = b(t)z(t - r(x(t)))\,\mathrm{e}^{-\int_R^t a(s)ds} \qquad \text{for all } t \geq R. \tag{2.23}$$

$v(t)$ satisfies the equation

$$\dot{v}(t) = c(t)v(t - r(x(t))) \qquad \text{for all } t \geq R, \tag{2.24}$$

where $c(t)$ is defined by $c(t) = b(t)\,\mathrm{e}^{-\int_{t-r(x(t))}^t a(s)ds}$ for all $t \geq R$. The assumption $\phi <_F \psi$ implies $x_{2R} \neq y_{2R}$, that is, $z_{2R} \neq 0$. By the definition of v, it follows that $v_{2R} \neq 0$. Since $v(t) \leq 0$ for all $t \geq -R$, there exists $u \in [R,2R]$ such that $v(u) < 0$. Having $v(t - r(x(t))) \leq 0$ for all $t \geq R$ and $c(t) > 0$, by (2.24), we deduce $\dot{v}(t) \leq 0$ for all $t \geq R$. Therefore $v(t) \leq v(u) < 0$ for all $t \geq u$. Hence $x(t) < y(t)$ for all $t \geq 2R$. The proof is complete. □

Proposition 2.5 and Proposition 2.10 imply that F is MOP.

Proposition 2.11. *The semiflow F is MOP, that is, it is monotone, and for every ϕ, ψ in X with $\phi <_F \psi$, there exist $t_0 > 0$ and neighbourhoods $\mathcal{U}$ of ϕ and $\mathcal{V}$ of ψ such that $F(t_0,\mathcal{U}) \leq F(t_0,\mathcal{V})$.*

Proof. We have already shown the monotonicity of F in Proposition 2.5. Consider ϕ, ψ in X with $\phi <_F \psi$. Proposition 2.10 implies that there exists $t_0 > 0$ such that $F(t_0,\phi) \ll F(t_0,\psi)$. There are neighbourhoods $\widetilde{\mathcal{U}}$ of $F(t_0,\phi)$ and $\widetilde{\mathcal{V}}$ of $F(t_0,\psi)$ such that $\widetilde{\mathcal{U}} \ll \widetilde{\mathcal{V}}$. By the continuity of $F(t_0,\cdot)$, there exist neighbourhoods $\mathcal{U}$ of ϕ and $\mathcal{V}$ of ψ such that $F(t_0,\mathcal{U}) \subset \widetilde{\mathcal{U}}$ and $F(t_0,\mathcal{V}) \subset \widetilde{\mathcal{V}}$. Consequently, $F(t_0,\mathcal{U}) \ll F(t_0,\mathcal{V})$. Hence $F(t_0,\mathcal{U}) \leq F(t_0,\mathcal{V})$ as required. □

As the proof of Proposition 2.11 shows, we have also proved that for all ϕ, ψ in X with $\phi <_F \psi$, there exist $t_0 > 0$ and neighbourhoods $\mathcal{U}$ of ϕ and $\mathcal{V}$ of ψ such that $F(t_0, \mathcal{U}) \ll F(t_0, \mathcal{V})$. Since t_0 can be choosen $3R$, by Proposition 2.10, we conclude the following property of F.

Remark 2.12. *If ϕ, $\psi \in X$ with $\phi <_F \psi$, then there exist neighbourhoods $\mathcal{U}$ of ϕ and $\mathcal{V}$ of ψ such that $F(3R, \mathcal{U}) \ll F(3R, \mathcal{V})$.*

Our next goal is to prove that each point in X can be approximated either from below or from above in X. In order to do this, we need one more property of the semiflow F.

Proposition 2.13. *If ϕ, $\psi \in X$ with $\phi \ll \psi$ and $t \geq 0$, then $F(t, \phi) \ll F(t, \psi)$.*

Proof. Consider ϕ, $\psi \in X$ with $\phi \ll \psi$, and the solutions $x = x^{\phi} : [-R, \infty) \to [-A, A]$ and $y = x^{\psi} : [-R, \infty) \to [-A, A]$ of Eq. (1.1) through ϕ and ψ, respectively. Suppose by way of contradiction that there exists $s > 0$ such that $x(t) < y(t)$ for all $0 \leq t < s$ and $x(s) = y(s)$. Clearly, $\dot{x}(s) \geq \dot{y}(s)$. We have $x(s - r(x(s))) < y(s - r(x(s))) = y(s - r(y(s)))$ since $\phi \ll \psi$ and r is positive on $[-A, A]$. The assumption $f' > 0$ implies $f(x(s - r(x(s)))) < f(y(s - r(y(s))))$. Therefore $\dot{x}(s) - \dot{y}(s) = f(x(s - r(x(s)))) - f(y(s - r(y(s)))) < 0$. This contradiction completes the proof. □

Proposition 2.14. *Each point in X can be approximated either from below or from above in X.*

Proof. Let ϕ be in X. Define $\phi_n(s) = \min\{A, \phi(s) + \frac{1}{n}\}$ and $\psi_n(s) = \max\{-A, \phi(s) - \frac{1}{n}\}$ for all $s \in [-R, 0]$ and $n \in \mathbb{N} \setminus \{0\}$. We prove that ϕ can be approximated either from below by a subsequence of $(\psi_n)_1^{\infty}$ or from above by a subsequence of $(\phi_n)_1^{\infty}$ in X. Clearly, ϕ_n and ψ_n are in X for all $n \in \mathbb{N} \setminus \{0\}$, and $\phi_n \to \phi$ and $\psi_n \to \phi$ as $n \to \infty$. First we show that for every $n \in \mathbb{N} \setminus \{0\}$, either $\psi_n <_F \phi$ or $\phi <_F \phi_n$. As for every $n \in \mathbb{N} \setminus \{0\}$, $\psi_n \ll \phi_n$, Proposition 2.13 implies that for every $n \in \mathbb{N} \setminus \{0\}$ and $t \geq 0$, $F(t, \psi_n) \ll F(t, \phi_n)$. Hence it follows that for every $n \in \mathbb{N} \setminus \{0\}$, either $\psi_n <_F \phi$ or $\phi <_F \phi_n$. Therefore there is a subsequence (n_k) such that either $\psi_{n_k} <_F \phi$ for all $k \in \mathbb{N} \setminus \{0\}$ or $\phi <_F \phi_{n_k}$ for all $k \in \mathbb{N} \setminus \{0\}$. Without loss of generality assume that $\phi <_F \phi_{n_k}$ for all $k \in \mathbb{N} \setminus \{0\}$. By Remark 2.12, for all $k \in \mathbb{N} \setminus \{0\}$, there is a neighbourhood $\mathcal{U}_k$ of ϕ such that $F(3R, \mathcal{U}_k) \ll F(3R, \phi_{n_k})$. Since $\phi_{n_k} \to \phi$ as $k \to \infty$, we obtain that for all $k \in \mathbb{N} \setminus \{0\}$, there exists $l \in \mathbb{N} \setminus \{0\}$ such that $\phi_{n_{k+l}} \in \mathcal{U}_k$. Consequently, for all

$k \in \mathbb{N} \setminus \{0\}$, there exists $l \in \mathbb{N} \setminus \{0\}$ such that $F(3R, \phi_{n_{k+l}}) \ll F(3R, \phi_{n_k})$. This implies that for all $k \in \mathbb{N} \setminus \{0\}$, there exists $l \in \mathbb{N} \setminus \{0\}$ such that $\phi_{n_{k+l}} <_F \phi_{n_k}$. Now it is clear how to choose a subsequence $(\phi_{n_{k_j}})$ such that it approximates ϕ from above. □

It remains to show the following.

Proposition 2.15. *For each ϕ in X, the orbit $O(\phi)$ of ϕ has compact closure in X. Furthermore, for each ϕ in X and for each sequence $(\phi_n)_1^\infty$, which approximates ϕ either from below or from above in X, $\cup_{n\geq 1}\omega(\phi_n)$ has compact closure in X.*

Proof. The first assertion follows by the Arzèla–Ascoli theorem using the fact that $\text{lip}(x_t^\phi) \leq M$ for all $t \geq R$. For the second assertion notice that, for all ϕ in X, $\omega(\phi)$ is a nonempty, compact, and invariant set, and, for all ϕ in X, $\omega(\phi)$ is contained in the set of all functions ψ in X with $\text{lip}(\psi) \leq M$. Thus, for a sequence $(\phi_n)_1^\infty$, which approximates ϕ either from below or from above in X, $\cup_{n\geq 1}\omega(\phi_n)$ is also included in the set of all functions ψ in X with $\text{lip}(\psi) \leq M$, which by the Arzèla–Ascoli theorem is compact. Hence $\overline{\cup_{n\geq 1}\omega(\phi_n)}$ is compact in X as well. □

The set of equilibrium points of F is $E = \{\phi \in X : \phi(s) = \phi(0)$ for all $s \in [-R, 0]$, and $f(\phi(0)) = \mu\phi(0)\}$. Note that for all ϕ_1, $\phi_2 \in E$ with $\phi_1 \neq \phi_2$, we have either $\phi_1 < \phi_2$ or $\phi_2 < \phi_1$. According to Claim 2 (Section 3) an omega limit set cannot contain two points ϕ_1, ϕ_2 such that $\phi_1 < \phi_2$ or $\phi_2 < \phi_1$. Then it follows that for all ϕ in X, the set $\omega(\phi) \cap E$ has at most a single point. Therefore the set of quasiconvergent points Q coincides with the set of convergent points C. Consequently, Theorem 1.1 states in this special case:

Theorem 2.16. *Under hypotheses (H1) on f and r, $X = \overline{Int C}$, that is, $Int C$ is dense in X.*

Note that in general $X \neq C$. Krisztin, Walther and Wu [41] have shown the existence of periodic orbits in the case $r \equiv 1$ for certain μ, f, and r. A similar result is proved by Mallet-Paret and Nussbaum [60,61], Kuang and Smith [47,48], and Arino, Hadeler and Hbid [3] in the stat-dependent delay case with a negative feedback condition. For the case $r = r(x(t))$ with a positive feedback condition Chapter 4 contains an analogous result. Krisztin and Arino [39] have shown that there exists a smooth disk of nonquasiconvergent points for the case $r = r(x(t))$ with negative feedback condition. A similar result is expected for Eq. (1.1) in the positive feedback case.

3.3 Proof of the main result

Hereafter we suppose that assumptions (A_1)–(A_5) of Theorem 1.1 are satisfied. The proof of Theorem 1.1 consists of the following steps.

Claim 1.

(i) If $\Phi(T,x) \geq x$ for some $T > 0$, then $\omega(x)$ is a T-periodic orbit.

(ii) If $\Phi(t,x) \geq x$ for t belonging to some nonempty open subset of $(0,\infty)$, then there exists $p \in E$ such that $\Phi(t,x) \to p$ as $t \to \infty$.

(iii) If $\Phi(T,x) >_\Phi x$ for some $T > 0$, then there exists $p \in E$ such that $\Phi(t,x) \to p$ as $t \to \infty$.

Proof. The proofs of assertion (i) and (ii) can be found in Smith [65, Theorem 2.1]. To prove (iii) consider $x <_\Phi \Phi(T,x)$. Since Φ is MOP, there exist neighbourhoods $\mathcal{U}$ of x, $\mathcal{V}$ of $\Phi(T,x)$, and $t_0 > 0$ such that $\Phi(t_0,\mathcal{U}) \leq \Phi(t_0,\mathcal{V})$. As there exists $\epsilon > 0$ such that $\Phi(t,x) \in \mathcal{V}$ for all $t \in (T-\epsilon, T+\epsilon)$, it follows that $\Phi(t_0,x) \leq \Phi(t_0,\Phi(t,x))$ for all $t \in (T-\epsilon,T+\epsilon)$. Case (ii) implies $\Phi(t,x) \to p \in E$ as $t \to \infty$. □

Claim 2. *An omega limit set cannot contain two points x and y such that $x < y$.*

Proof. Suppose by way of contradiction that there are x, y in $\omega(z)$ such that $x < y$. Then $x <_\Phi y$ because $\omega(z)$ is a compact, invariant subset of $\mathcal{X}$. As Φ is MOP, there exist neighbourhoods $\mathcal{U}$ of x, $\mathcal{V}$ of y, and $t_0 > 0$ such that $\Phi(t_0,\mathcal{U}) \leq \Phi(t_0,\mathcal{V})$. Choose $t_1 > 0$ such that $\Phi(t_1,z) \in \mathcal{U}$ and $t_2 > t_1$ such that $\Phi(t_2,z) \in \mathcal{V}$. Since $\Phi(t,z) \in \mathcal{V}$ for all $t \in (t_2-\epsilon, t_2+\epsilon)$ and for some $\epsilon \in (0, t_2-t_1)$, it follows that $\Phi(t_0+t_1,z) \leq \Phi(t_0,\Phi(t,z)) = \Phi(t-t_1,\Phi(t_0+t_1,z))$ for all $t \in (t_2-\epsilon,t_2+\epsilon)$. By Claim 1(ii), $\Phi(t,z) \to p \in E$ as $t \to \infty$. Thus $\omega(z) = p$ and $x = y$, which is a contradiction. □

An immediate consequence of Claim 2 is that an omega limit set cannot contain a maximal (minimal) element.

Claim 3. *If $a \in \omega(x)$ and $\omega(x) \leq a$ $(a \leq \omega(x))$, then $\omega(x) = a$.*

Proof. Consider $a \in \omega(x)$ and $\omega(x) \leq a$. Suppose that there exits $b \in \omega(x)$ such that $b \neq a$. Then $b < a$. Since b and a are in $\omega(x)$, we have obtained a contradiction to Claim 2. □

Claim 4. *If $x <_\Phi y$, $t_k \to \infty$, $\Phi(t_k,x) \to p$, and $\Phi(t_k,y) \to p$ as $k \to \infty$, then $p \in E$.*

Proof. Let $x <_\Phi y$. Since Φ is MOP, there exist neighbourhoods $\mathcal{U}$ of x, $\mathcal{V}$ of y and $t_0 > 0$ such that $\Phi(t_0, \mathcal{U}) \le \Phi(t_0, \mathcal{V})$. Let $\delta > 0$ be so small that $\{\Phi(s,x) : 0 \le s \le \delta\} \subset \mathcal{U}$ and $\{\Phi(\tilde{s},y) : 0 \le \tilde{s} \le \delta\} \subset \mathcal{V}$. Then

$$\Phi(t_0 + s, x) \le \Phi(t_0 + \tilde{s}, y) \qquad \text{for all } s \text{ and } \tilde{s} \text{ in } [0,\delta]. \tag{3.1}$$

Consider $\tilde{s} = 0$ in (3.1). Thus, $\Phi(t_0 + s, x) \le \Phi(t_0, y)$ for all s in $[0,\delta]$. The monotonicity of Φ implies $\Phi(t_k - t_0 + t_0 + s, x) \le \Phi(t_k - t_0 + t_0, y)$ for all s in $[0,\delta]$ and for all large k. Thus $\Phi(s, \Phi(t_k, x)) \le \Phi(t_k, y)$ for all s in $[0,\delta]$ and for all large k. Passing to the limit as $k \to \infty$, we infer $\Phi(s,p) \le p$ for all s in $[0,\delta]$. Considering $s = 0$ in (3.1) and arguing as above, we obtain $p \le \Phi(\tilde{s}, p)$ for all $\tilde{s}$ in $[0,\delta]$. Thus, $\Phi(\tilde{s},p) = p$ for all $0 \le \tilde{s} \le \delta$ and therefore, for all $\tilde{s} \ge 0$, so $p \in E$. □

Claim 5. *If $x <_\Phi y$ then $\omega(x) \cap \omega(y) \subset E$.*

Proof. Consider $p \in \omega(x) \cap \omega(y)$. Then there exists a sequence $(t_k)_1^\infty$ such that $t_k \to \infty$ and $\Phi(t_k, x) \to p$ as $k \to \infty$. $(\Phi(t_k, y))_1^\infty$ is a sequence in the compact set $\overline{O(y)}$. By passing to a subsequence if necessary, we can assume that $\Phi(t_k, y) \to q$ as $k \to \infty$. The monotonicity of Φ implies $\Phi(t_k, x) \le \Phi(t_k, y)$ for all integers $k \ge 1$. Letting $k \to \infty$, we find that $p \le q$. The case $p < q$ contradicts Claim 2, since p, $q \in \omega(y)$. Hence $p = q$ and by Claim 4, $p \in E$. □

Claim 6. *Let K_1 and K_2 be compact subsets of $\mathcal{X}$ satisfying $K_1 <_\Phi K_2$. Then there are open sets $\mathcal{U}$ and $\mathcal{V}$, with $K_1 \subset \mathcal{U}$ and $K_2 \subset \mathcal{V}$, and $t_1 > 0$, $\epsilon > 0$ such that $\Phi(t + s, \mathcal{U}) \le \Phi(t, \mathcal{V})$ for all $t \ge t_1$ and for all $0 \le s < \epsilon$.*

Proof. Fix an x in K_1. Since Φ is MOP, for each $y \in K_2$, there exist neighbourhoods $\mathcal{U}_y$ of x, $\mathcal{V}_y$ of y, and $t_y > 0$ such that $\Phi(t, \mathcal{U}_y) \le \Phi(t, \mathcal{V}_y)$ for all $t \ge t_y$. As K_2 is compact and$\{\mathcal{V}_y\}_{y \in K_2}$ is an open cover of K_2, we may choose a finite subcover, $K_2 \subset \cup_{i=1}^n \mathcal{V}_{y_i}$, where $y_i \in K_2$ for all $1 \le i \le n$. Set $\widetilde{\mathcal{V}} = \cup_{i=1}^n \mathcal{V}_{y_i}$, $\widetilde{\mathcal{U}} = \cap_{i=1}^n \mathcal{U}_{y_i}$ and $\widetilde{t} = \max_{1 \le i \le n} t_{y_i}$. Then $\Phi(t, \widetilde{\mathcal{U}}) \subset \Phi(t, \mathcal{U}_{y_i}) \le \Phi(t, \mathcal{V}_{y_i})$ for all $t \ge \widetilde{t}$ and for all $1 \le i \le n$. It follows that $\Phi(t, \widetilde{\mathcal{U}}) \le \Phi(t, \widetilde{\mathcal{V}})$ for all $t \ge \widetilde{t}$. Denote $\widetilde{\mathcal{V}}_x = \widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{U}}_x = \widetilde{\mathcal{U}}$ to emphasize the dependence of these open sets on the point $x \in K_1$. Similarly, $\widetilde{t}_x = \widetilde{t}$. We have obtained that for each $x \in K_1$, there exist neighbourhoods $\widetilde{\mathcal{U}}_x$ of x, $\widetilde{\mathcal{V}}_x$ of K_2, and $\widetilde{t}_x > 0$ such that $\Phi(t, \widetilde{\mathcal{U}}_x) \le \Phi(t, \widetilde{\mathcal{V}}_x)$ for all $t \ge \widetilde{t}_x$. Again, as $\{\widetilde{\mathcal{U}}_x\}_{x \in K_1}$ is an open cover of K_1, we may extract a finite subcover, $K_1 \subset \cup_{i=1}^m \widetilde{\mathcal{U}}_{x_i}$, where $x_i \in K_1$ for all $1 \le i \le m$. Set $\mathcal{U} = \cup_{i=1}^m \widetilde{\mathcal{U}}_{x_i}$, $\mathcal{V} = \cap_{i=1}^m \widetilde{\mathcal{V}}_{x_i}$, and $t_1 = \max_{1 \le i \le m} \widetilde{t}_{x_i}$. Since $\Phi(t, \mathcal{V}) \subset \Phi(t, \widetilde{\mathcal{V}}_{x_i})$ and $\Phi(t, \widetilde{\mathcal{U}}_{x_i}) \le \Phi(t, \widetilde{\mathcal{V}}_{x_i})$ for all $t \geq t_1$ and for all $1 \leq i \leq m$, we conclude that $\Phi(t, \widetilde{\mathcal{U}}_{x_i}) \leq \Phi(t, \mathcal{V})$ for all $t \geq t_1$ and for all $1 \leq i \leq m$. Thus, $\Phi(t, \mathcal{U}) \leq \Phi(t, \mathcal{V})$ for all $t \geq t_1$. In order to obtain the stronger conclusion of the claim, we observe that, by the continuity of Φ, for each $x \in K_1$, there exist $\epsilon_x > 0$ and a neighbourhood $\mathcal{W}_x$ of x such that $\Phi([0, \epsilon_x) \times \mathcal{W}_x) \subset \mathcal{U}$. As $\{\mathcal{W}_x\}_{x \in K_1}$ is an open cover of K_1, we may choose a finite subcover, $K_1 \subset \cup_{i=1}^m \mathcal{W}_{x_i}$. Denote $\mathcal{U}' = \cup_{i=1}^m \mathcal{W}_{x_i}$ and $\epsilon = \min_{1 \leq i \leq m} \epsilon_{x_i}$. If $x \in \mathcal{U}'$ and $0 \leq s < \epsilon$, then $x \in \mathcal{W}_{x_i}$ for some i. Thus $\Phi(s, x) \in \mathcal{U}$. Therefore, $\Phi([0, \epsilon) \times \mathcal{U}') \subset \mathcal{U}$ and then $\Phi(s, \mathcal{U}') \subset \mathcal{U}$ for all $0 \leq s < \epsilon$. It follows that $\Phi(t + s, \mathcal{U}') \subset \Phi(t, \mathcal{U}) \leq \Phi(t, \mathcal{V})$ for all $t \geq t_1$ and for all $0 \leq s < \epsilon$. □

Claim 7. *If $x < y$, $t_k \to \infty$, $\Phi(t_k, x) \to a$, $\Phi(t_k, y) \to b$ as $k \to \infty$ and $a < b$, then $O(a) <_\Phi b$.*

Proof. For $u \in \overline{O(x)}$, $v \in \overline{O(y)}$ with $u \leq v$, define

$$J(u, v) = \sup\{\tilde{s} \geq 0 : \Phi(t, u) \leq v,\ 0 \leq t \leq \tilde{s}\}.$$

Our aim is to prove that $J(a, b) = +\infty$. First we verify two properties of $J(u, v)$.

(P_1) $J(\Phi(t, u), \Phi(t, v))$ is monotone nondecreasing in t.

To show (P_1), it suffices to establish $J(\Phi(t, u), \Phi(t, v)) \geq J(u, v)$ for all $t \geq 0$. We have $\Phi(s, u) \leq v$ for all $0 \leq s \leq J(u, v)$. The monotonicity of Φ implies $\Phi(s, \Phi(t, u)) \leq \Phi(t, v)$ for all $0 \leq s \leq J(u, v)$ and $t \geq 0$. Thus, $J(\Phi(t, u), \Phi(t, v)) \geq J(u, v)$ for all $t \geq 0$.

(P_2) If $u_k \leq v_k$, $u_k \in \overline{O(x)}$, $v_k \in \overline{O(y)}$, and $u_k \to u$, $v_k \to v$, then $\limsup_{k \to \infty} J(u_k, v_k) \leq J(u, v)$.

If $J(u, v) = \infty$, then the assertion is obvious. Assume $J(u, v) < \infty$. Suppose by way of contradiction there exists $\epsilon > 0$ such that $\limsup_{k \to \infty} J(u_k, v_k) - \epsilon > J(u, v)$. Let (k_i) be a sequence in $\mathbb{N}$ with $k_i \to \infty$ as $i \to \infty$ such that $\limsup_{k \to \infty} J(u_k, v_k) = \lim_{i \to \infty} J(u_{k_i}, v_{k_i})$. Consequently, $J(u, v) + \epsilon < J(u_{k_i}, v_{k_i})$ for all large i. From the definition of $J(u_{k_i}, v_{k_i})$ it follows that $\Phi(s, u_{k_i}) \leq v_{k_i}$ for $0 \leq s \leq J(u, v) + \epsilon$ and for all large i. Letting $i \to \infty$, we obtain $\Phi(s, u) \leq v$ for $0 \leq s \leq J(u, v) + \epsilon$, which contradicts the definition of $J(u, v)$.

Denote $\alpha = \lim_{t \to \infty} J(\Phi(t, x), \Phi(t, y))$, which exists in $[0, \infty]$ according to (P_1). By (P_2), we obtain $\alpha \leq J(a, b)$. Suppose $J(a, b) < \infty$. For $0 \leq \tilde{s} \leq J(a, b)$, $\Phi(\tilde{s}, a) \leq b$. Moreover, $\Phi(\tilde{s}, a) < b$ for $0 \leq \tilde{s} \leq J(a, b)$. Otherwise $b \in \omega(x)$, by the invariance of $\omega(x)$, $a \in \omega(x)$, and $a < b$, in contradiction to Claim 2. We assert that $\Phi(\tilde{s}, a) <_\Phi b$ for $0 \leq \tilde{s} \leq J(a, b)$. Indeed, by the invariance of $\omega(x)$, $O(a) \subset \omega(x)$.

So $\Phi(\tilde{s}, a)$ is in $\omega(x) \cup \omega(y)$ for $0 \le \tilde{s} \le J(a,b)$ and b is also in $\omega(x) \cup \omega(y)$. As $\omega(x) \cup \omega(y)$ is compact and invariant, we deduce $\Phi(\tilde{s}, a) <_\Phi b$ for $0 \le \tilde{s} \le J(a,b)$. Set $\mathcal{K} = \{\Phi(\tilde{s}, a) : 0 \le \tilde{s} \le J(a,b)\}$. $\mathcal{K}$ is compact and $\mathcal{K} <_\phi b$. Thus, Claim 6 implies that there exist $t_1 > 0$, $\epsilon > 0$ and open sets $\mathcal{U}$, $\mathcal{V}$ with $\mathcal{K} \subset \mathcal{U}$ and $b \in \mathcal{V}$ such that $\Phi(t+s, \mathcal{U}) \le \Phi(t, \mathcal{V})$ for all $t \ge t_1$ and $0 \le s < \epsilon$. Since $\Phi(t_k, y) \to b$ as $k \to \infty$, there exists an integer k_0 such that $\Phi(t_k, y) \in \mathcal{V}$ for all $k \ge k_0$. As $\Phi(t_k, x) \to a$ as $k \to \infty$, $\Phi(\tilde{s}, \Phi(t_k, x)) \to \Phi(\tilde{s}, a)$ uniformly in $\tilde{s} \in [0, J(a,b)]$ as $k \to \infty$. Consequently, there exists $k_1 > 0$ such that $\Phi(\tilde{s}, \Phi(t_k, x)) \in \mathcal{U}$ for all $k \ge k_1$ and for all $\tilde{s} \in [0, J(a,b)]$. We infer $\Phi(t+s, \Phi(\tilde{s}, \Phi(t_k, x))) \le \Phi(t, \Phi(t_k, y))$ for all $t \ge t_1$, for all $0 \le s < \epsilon$, for all $k \ge k_2 = \max\{k_0, k_1\}$, and for all $\tilde{s} \in [0, J(a,b)]$. On rearranging the arguments, we conclude $\Phi(\tilde{s} + s, \Phi(t + t_k, x)) \le \Phi(t + t_k, y)$ for all $t \ge t_1$, for all $k \ge k_2$ and for all $0 \le s + \tilde{s} < \epsilon + J(a,b)$. It follows that $J(\Phi(t + t_k, x), \Phi(t + t_k, y)) \ge J(a,b) + \epsilon$ for all $t \ge t_1$ and for all $k \ge k_2$. Letting $k \to \infty$, we obtain $\alpha \ge J(a,b) + \epsilon$. But $J(a,b) \ge \alpha$, which provides a contradiction. Hence $J(a,b) = \infty$. Then $O(a) < b$, that is, $\Phi(\tilde{s}, a) < b$ for all $\tilde{s} \ge 0$. Otherwise, as we have shown above, we get a contradiction to Claim 2. Moreover, $O(a) <_\Phi b$. Indeed, by the invariance of $\omega(x)$, $\Phi(\tilde{s}, a) \in \omega(x)$ for all $\tilde{s} \ge 0$, thus, $\Phi(\tilde{s}, a) \in \omega(x) \cup \omega(y)$ for all $\tilde{s} \ge 0$, and b is also in $\omega(x) \cup \omega(y)$. The compactness and invariance of $\omega(x) \cup \omega(y)$ implies the desired assertion. □

Claim 8. *If u, $v \in \mathcal{X}$ and there exists $x \in \omega(u)$ such that $x < \omega(v)$, then $\omega(u) <_\Phi \omega(v)$. Similarly, if there exists $x \in \omega(u)$ such that $\omega(v) < x$, then $\omega(v) <_\Phi \omega(u)$.*

Proof. First note that $x \in \omega(u)$, $x < \omega(v)$ implies $x <_\Phi \omega(v)$. Indeed, we have $x \in \omega(u) \cup \omega(v)$ and $y \in \omega(u) \cup \omega(v)$ for all y in $\omega(v)$. By the compactness and invariance of $\omega(u) \cup \omega(v)$, $x <_\Phi y$ for all $y \in \omega(v)$, that is, $x <_\Phi \omega(v)$. Applying Claim 6, we obtain that there exist $t_0 > 0$ and neighbourhoods $\mathcal{U}$ of x and $\mathcal{V}$ of $\omega(v)$ such that $\Phi(t_0, \mathcal{U}) \le \Phi(t_0, \mathcal{V})$. Since $\omega(v) \subset \mathcal{V}$ and $\omega(v)$ is invariant, $\Phi(t_0, \mathcal{U}) \le \omega(v)$. As $x \in \omega(u)$, there exists $t_1 > 0$ such that $\Phi(t_1, u) \in \mathcal{U}$. Thus, $\Phi(t_0 + t_1, u) \le \omega(v)$. The monotonicity of Φ and invariance of $\omega(v)$ imply $\Phi(t_0 + t_1 + s, u) \le \omega(v)$ for all $s \ge 0$. Hence $\omega(u) \le \omega(v)$. We assert that $\omega(u) < \omega(v)$. Suppose that there exists z in $\omega(u) \cap \omega(v)$. Due to the fact that $z \le \omega(v)$ and $z \in \omega(v)$, by Claim 3, we find that $z = \omega(v)$. Similarly, since $\omega(u) \le z$ and $z \in \omega(u)$, we get $z = \omega(u)$. On the other hand $x < \omega(v)$ implies $x < z$, and $x \in \omega(u)$ implies $x = z$, which is impossible. Finally, $\omega(u) <_\Phi \omega(v)$ because of the compactness and invariance of $\omega(u) \cup \omega(v)$. □

Claim 9. *If $x < y$, $t_k \to \infty$, $\Phi(t_k, x) \to a$, $\Phi(t_k, y) \to b$ as $k \to \infty$ and $a < b$, then $\omega(x) <_\Phi \omega(y)$.*

Proof. According to Claim 7, $O(a) <_\Phi b$. Hence $\omega(a) \leq b$. We assert that $\omega(a) < b$. Suppose $b \in \omega(a)$. Then $\omega(a) = b$, by Claim 3. Since $a < \omega(a)$ and $a \in \omega(x)$, Claim 8 implies $\omega(x) <_\Phi \omega(a)$. This is impossible as $\omega(a) \subset \omega(x)$. Consequently, $\omega(a) < b$. Due to the fact that b is in $\omega(y)$, by Claim 8, we obtain $\omega(a) <_\Phi \omega(y)$. Since every $z \in \omega(a)$ belongs to $\omega(x)$ as well, Claim 8 gives $\omega(x) <_\Phi \omega(y)$. □

Claim 10. *If $x <_\Phi y$ then either*
(a) $\omega(x) <_\Phi \omega(y)$ or
(b) $\omega(x) = \omega(y) \subset E$.

Proof. If $\omega(x) = \omega(y)$, according to Claim 5, we obtain $\omega(x) = \omega(y) \subset E$. If $\omega(x) \neq \omega(y)$, then we may suppose that there exists $q \in \omega(y) \setminus \omega(x)$. The other case is treated similarly. There exists a sequence (t_k) such that $t_k \to \infty$ and $\Phi(t_k, y) \to q$ as $k \to \infty$. Since $(\Phi(t_k, x))_1^\infty$ is a sequence in the compact set $\overline{O(x)}$, we may assume, by passing to a subsequence if necessary, that $\Phi(t_k, x) \to p$ as $k \to \infty$. The monotonicity of Φ implies $\Phi(t_k, x) \leq \Phi(t_k, y)$ for all k. Letting $k \to \infty$, we get $p \leq q$. We assert that $p < q$. Indeed, if $p = q$, then $q \in \omega(x)$, which is a contradiction. Thus, by Claim 9, it follows that $\omega(x) <_\Phi \omega(y)$. □

Claim 11. *If $x_0 \in \mathcal{X}$ can be approximated from below in $\mathcal{X}$ by a sequence $(\tilde{x}_n)_1^\infty$, then there exists a subsequence $(x_n)_1^\infty$ of $(\tilde{x}_n)_1^\infty$ such that $x_n <_\Phi x_{n+1} <_\Phi x_0$ for all integers $n \geq 1$, with $x_n \to x_0$ as $n \to \infty$, satisfying one of the following properties.*
(a) There exists $u_0 \in E$ such that

$$\omega(x_n) <_\Phi \omega(x_{n+1}) <_\Phi u_0 = \omega(x_0) \qquad \text{for all integers } n \geq 1$$

and

$$\lim_{n\to\infty} dist(u_0, \omega(x_n)) = 0.$$

(b) There exists $u_0 \in E$ such that

$$\omega(x_n) = u_0 <_\Phi \omega(x_0) \qquad \text{for all integers } n \geq 1.$$

If $u \in E$ and $u <_\Phi \omega(x_0)$, then $u \leq u_0$.

(c) $\omega(x_n) = \omega(x_0) \subset E$ *for all integers* $n \geq 1$.

An analogous result holds if x_0 can be approximated from above in $\mathcal{X}$. Claim 11 describes three alternatives for the point x_0: x_0 can be a convergent point by (a), or x_0 can be a quasiconvergent point by (c), or x_0 can belong to the closure of the set of convergent points according to (b).

Proof. By Claim 10, there exists a subsequence $(x_n)_1^\infty$ of $(\tilde{x}_n)_1^\infty$ such that either $\omega(x_n) = \omega(x_{n+1}) \subset E$ for all integers $n \geq 1$ or $\omega(x_n) <_\Phi \omega(x_{n+1})$ for all integers $n \geq 1$.

Consider the case $\omega(x_n) <_\Phi \omega(x_{n+1})$ for all integers $n \geq 1$, which is equivalent to $\omega(x_n) < \omega(x_{n+1})$ for all integers $n \geq 1$, by the invariance and compactness of $\omega(x_n) \cup \omega(x_{n+1})$. It follows that $\omega(x_n) <_\Phi \omega(x_0)$ for all integers $n \geq 1$. Indeed if there exists $n_0 \geq 1$ such that $\omega(x_{n_0}) = \omega(x_0)$, then $\omega(x_n) = \omega(x_0)$ for all $n \geq n_0$, which is a contradiction. Set $\Omega = \{y : y = \lim_{n\to\infty} y_n,\ y_n \in \omega(x_n)\} \subset \overline{\cup_{n\geq 1}\omega(x_n)}$. Ω is nonempty due to the fact that $(y_n)_1^\infty$ is a monotone sequence in the compact set $\overline{\cup_{n\geq 1}\omega(x_n)}$. We claim that Ω consists of a single element, that is, $\Omega = \{u_0\}$. Indeed, if there are y and u in Ω so that $y_n \to y$ and $u_n \to u$ as $n \to \infty$, where $y_n,\ u_n \in \omega(x_n)$, then $\omega(x_n) <_\Phi \omega(x_{n+1})$ implies $y_n < u_{n+1}$ and $u_n < y_{n+1}$ for all integers $n \geq 1$. Letting $n \to \infty$, we infer $y \leq u$ and $u \leq y$, that is, $y = u$. We claim that $u_0 \in E$. Consider $y_n \in \omega(x_n)$. Then $y_n \to u_0$ as $n \to \infty$. By the continuity of Φ, $\Phi(t, y_n) \to \Phi(t, u_0)$ as $n \to \infty$. Since $\Phi(t, y_n) \in \omega(x_n)$ by the invariance of $\omega(x_n)$, we obtain $\Phi(t, y_n) \to u_0$ as $n \to \infty$. Thus, $\Phi(t, u_0) = u_0$ for all $t \geq 0$. It follows from the definition of Ω and the compactness of $\overline{\cup_{n\geq 1}\omega(x_n)}$ that $\lim_{n\to\infty} \operatorname{dist}(u_0, \omega(x_n)) = 0$. Finally, $\omega(x_n) <_\Phi \omega(x_0)$ for all integers $n \geq 1$ implies $u_0 \leq \omega(x_0)$. If $u_0 \in \omega(x_0)$, then by Claim 3, we get $\omega(x_0) = u_0$, which is case (a). Suppose $u_0 \notin \omega(x_0)$. Then $u_0 < \omega(x_0)$, that is, $u_0 <_\Phi \omega(x_0)$ by the invariance and compactness of $\{u_0\} \cup \omega(x_0)$. Claim 6 implies that there is a neighbourhood $\mathcal{W}$ of $\omega(x_0)$ and $t_0 > 0$ such that $u_0 = \Phi(t, u_0) \leq \Phi(t, \mathcal{W})$ for all $t \geq t_0$. There exists $t_1 > 0$ such that $\Phi(t_1, x_0) \in \mathcal{W}$. By the continuity of $\Phi(t_1, \cdot)$, there is an integer $n_0 \geq 1$ such that $\Phi(t_1, x_n) \in \mathcal{W}$ for all $n \geq n_0$. Consequently, $u_0 \leq \Phi(t - t_1, \Phi(t_1, x_n))$ for all $t \geq t_0 + t_1$ and for all $n \geq n_0$. Letting $t \to \infty$, we obtain $u_0 \leq \omega(x_n)$ for all $n \geq n_0$. Since $\omega(x_{n_0+1}) <_\Phi \omega(x_k)$ for all large k, it follows that $\omega(x_{n_0+1}) \leq u_0$. Thus, $u_0 \leq \omega(x_{n_0}) <_\Phi \omega(x_{n_0+1}) \leq u_0$, which is a contradiction.

Consider the case $\omega(x_n) = \omega(x_{n+1}) \subset E$ for all integers $n \geq 1$. As $x_n <_\Phi x_0$, Claim 10 implies that either $\omega(x_n) = \omega(x_0) \subset E$ for all integers $n \geq 1$, which is case (c), or $\omega(x_n) <_\Phi \omega(x_0)$ for all integers $n \geq 1$. Suppose $\omega(x_n) <_\Phi \omega(x_0)$ for all integers $n \geq 1$. Let $u_0 \in \omega(x_1) = \omega(x_n) \subset E$. Consequently, $u_0 <_\Phi \omega(x_0)$. Arguing exactly as above, we obtain that there exist an integer $n_0 \geq 1$ and $t_2 > 0$ such that $u_0 \leq \Phi(t, x_n)$ for all $t \geq t_2$ and for all $n \geq n_0$. Then $u_0 \leq \omega(x_n)$ for all $n \geq n_0$. Since $u_0 \in \omega(x_n)$, by Claim 3, $\omega(x_n) = u_0$ for all $n \geq 1$. Finally, if $u \in E$ and $u <_\Phi \omega(x_0)$, then arguing as above, we find that $u \leq \omega(x_n)$ for all $n \geq n_0$. As $\omega(x_n) = u_0$, it follows that $u \leq u_0$. □

The next result gives some information which strengthens the assertion concerning case (b) of Claim 11.

Claim 12. *In case (b) of Claim 11 we have in addition the following properties:*

(i) There exist a neighbourhood O of u_0, t_0, $t_1 > 0$, and an integer $n \geq 1$ such that

$$\Phi(t, O) \leq \Phi(t + t_1, x_n) \qquad \text{for all } t \geq t_0.$$

(ii) There is a neighbourhood $\mathcal{U}$ of x_0 with the following property: for each $x \in \mathcal{U}$ with $x <_\Phi x_0$, there exist a neighbourhood $\mathcal{V}$ of x in $\mathcal{U}$, an integer N, and $T > 0$ such that

$$u_0 \leq \Phi(t, \mathcal{V}) \leq \Phi(t, x_N) \qquad \text{for all } t \geq T.$$

(iii) $x_0 \in \overline{\mathrm{Int}\, C}$.

Proof. Proof of (i). In case (b), we have $u_0 <_\Phi \omega(x_0)$. Thus Claim 6 implies that there exist a neighbourhood $\mathcal{W}$ of $\omega(x_0)$, O of u_0, and $t_0 > 0$ such that $\Phi(t, O) \leq \Phi(t, \mathcal{W})$ for all $t \geq t_0$. There exists $t_1 > 0$ such that $\Phi(t_1, x_0) \in \mathcal{W}$. By the continuity of $\Phi(t_1, \cdot)$, it follows that there exists an integer $n \geq 1$ such that $\Phi(t_1, x_n) \in \mathcal{W}$. Then $\Phi(t, O) \leq \Phi(t + t_1, x_n)$ for all $t \geq t_0$.

Proof of (ii). We choose a neighbourhood $\mathcal{U}$ of x_0 such that $\Phi(t_1, \mathcal{U}) \subset \mathcal{W}$. Let $x \in \mathcal{U}$ with $x <_\Phi x_0$. Since Φ is MOP, there exist a neighbourhood $\mathcal{V}$ of x with $\mathcal{V} \subset \mathcal{U}$, $\mathcal{N}$ of x_0, and $t_2 > 0$ such that $\Phi(t, \mathcal{V}) \leq \Phi(t, \mathcal{N})$ for all $t \geq t_2$. As there is an integer N such that $x_N \in \mathcal{N}$, we get $\Phi(t, \mathcal{V}) \leq \Phi(t, x_N)$ for all $t \geq t_2$. By (i), $u_0 = \Phi(t, u_0) \in \Phi(t, O) \leq \Phi(t, \mathcal{W})$ for all $t \geq t_0$. Due to the fact that $\Phi(t_1, \mathcal{V}) \subset \Phi(t_1, \mathcal{U}) \subset \mathcal{W}$, we obtain $u_0 \in \Phi(t, O) \leq \Phi(t + t_1, \mathcal{V})$ for all $t \geq t_0$. Hence $u_0 \leq \Phi(t, \mathcal{V}) \leq \Phi(t, x_N)$ for all $t \geq T$, where $T = t_0 + t_1 + t_2$.

Proof of (iii). Since $\omega(x_N) = u_0$, $\Phi(t, x_N) \to u_0$ as $t \to \infty$. Thus, by (ii), we obtain $\omega(v) = u_0$ for all v in $\mathcal{V}$. Therefore, for all $x \in \mathcal{U}$ with $x <_\Phi x_0$, we get

$\omega(x) = u_0$ and $x \in \operatorname{Int} C$. If we consider the sequence $(x_n)_1^\infty$, which approximates x_0 from below, then $x_n \in \operatorname{Int} C$ for all large n. Hence $x_0 \in \overline{\operatorname{Int} C}$. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose $x_0 \in \mathcal{X} \backslash \operatorname{Int} \mathcal{Q}$. Then there exists a sequence $(y_n)_1^\infty$ such that $y_n \in \mathcal{X} \setminus \mathcal{Q}$ and $y_n \to x_0$ as $n \to \infty$. By assumption (A_3) of Theorem 1.1, for each n, y_n can be approximated either from below or from above in $\mathcal{X}$. Consider the former case as the latter case is similar. Using Claim 11, we obtain, by passing to a subsequence if necessary, that for each n, there exists a sequence $(x_m^n)_{m=1}^\infty$ such that $x_m^n <_\Phi x_{m+1}^n <_\Phi y_n$ for all integers $m \geq 1$ and $x_m^n \to y_n$ as $m \to \infty$. For each n, $y_n \notin \mathcal{Q}$; therefore case (b) of Claim 11 must hold. Claim 12(iii) implies that for each n, $y_n \in \overline{\operatorname{Int} C}$. Hence $x_0 \in \overline{\operatorname{Int} C}$, which completes the proof. □

Chapter 4

Periodic Solutions and Connecting Orbits

4.1 Basic facts

Consider the differential equation with state-dependent delay

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)), \quad r = r(x(t)), \tag{1.1}$$

where $\mu > 0$, f and r are smooth real functions with $r(0) = 1$ and $f' > 0$.

In this chapter we prove that there are a nontrivial periodic orbit and a homoclinic orbit connecting 0 to the periodic orbit. Eq. (1.1) with constant delay, i.e., $r \equiv 1$, was widely studied in the monograph [41] and in the papers [36,37,40,42,46]. In these works the fine structure of the global attractor is described by using recent results of the geometric theory of infinite dimensional dynamical systems. The situation considered in the present paper is more complicated. Although most of the ideas from the above mentioned results can be applied, nontrivial modifications are necessary in the standard techniques. The main technical tools are as follows: the result of monotone dynamical systems applicable for Eq. (1.1) in Chapter 3; a local unstable manifold at zero for Eq. (1.1) in [38]; and a discrete Lyapunov functional counting sign changes given analogously to that of [39].

The following definitions and notations will be used in this chapter. The symbols $\mathbb{N}$ and $\mathbb{R}_+$ denote the nonnegative integers and reals, respectively. $\mathbb{R}$ and $\mathbb{Z}$ stand for the set of all reals and all integers, respectively.

The distance of two sets M and N is defined as

$$\operatorname{dist}(M, N) = \sup_{m \in M} \operatorname{dist}(m, N) = \sup_{m \in M} \inf_{n \in N} d(m, n).$$

A trajectory of a map $g : M \to N$, $M \subset N$, is a finite or infinite sequence $(x_j)_{j \in I \cap \mathbb{Z}}$, $I \subset \mathbb{R}$ an interval in M, with $x_{j+1} = g(x_j)$ for all $j \in I \cap \mathbb{Z}$ with $j + 1 \in I \cap \mathbb{Z}$.

If $\mathcal{E}$ is a Banach space and $\delta > 0$, then $B_\delta(\mathcal{E})$ denotes the open ball in $\mathcal{E}$ with radius δ and center at 0.

A simple closed curve is a continuous map c from a compact interval $[a, b] \subset \mathbb{R}$, $a < b$, into $\mathbb{R}^n$ so that $c|_{[a,b)}$ is injective and $c(a) = c(b)$. The set of values of a simple closed curve c, or trace, is denoted by $|c|$. The Jordan curve theorem guarantees that the complement of the trace of a simple closed curve c in $\mathbb{R}^2$ consists of two nonempty connected open sets, one bounded and the other unbounded, and $|c|$ is the boundary of each of these components. We denote the bounded component by $\text{int}(c)$ and the unbounded one by $\text{ext}(c)$.

Spectra of continuous linear maps $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ are defined as spectra of their complexifications. If a decomposition $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ into closed linear subspaces is given, then $Pr_{\mathcal{F}} : \mathcal{E} \to \mathcal{E}$ and $Pr_{\mathcal{G}} : \mathcal{E} \to \mathcal{E}$ denote the associated projection operators along $\mathcal{G}$ onto $\mathcal{F}$ and along $\mathcal{F}$ onto $\mathcal{G}$, respectively.

For given reals a, b with $a < b$, $C([a,b],\mathbb{R})$ denotes the Banach space of continuous functions $\phi : [a,b] \to \mathbb{R}$ with the norm given by $\|\phi\|_{C([a,b],\mathbb{R})} = \max_{a \le t \le b} |\phi(t)|$.

$C^1([a,b],\mathbb{R})$ is the Banach space of all C^1-maps $\phi : [a,b] \to \mathbb{R}$ with the norm given by

$$\|\phi\|_{C^1([a,b],\mathbb{R})} = \|\phi\|_{C([a,b],\mathbb{R})} + \|\dot{\phi}\|_{C([a,b],\mathbb{R})}.$$

Let $\Phi : \mathbb{R}_+ \times \mathcal{E} \to \mathcal{E}$ be a semiflow. A set $\mathcal{A} \subset \mathcal{E}$ is called positively invariant if $\Phi(\mathbb{R}_+ \times \mathcal{A}) \subset \mathcal{A}$. It is called invariant (resp. negatively invariant) if for every $x \in \mathcal{A}$ there is a complete phase curve, i.e., a map $\gamma : \mathbb{R} \to \mathcal{E}$ with $\gamma(t+s) = \Phi(t, \gamma(s))$ for all $s \in \mathbb{R}$ and $t \ge 0$, which satisfies $\gamma(0) = x$ and $\gamma(\mathbb{R}) \subset \mathcal{A}$ (resp. $\gamma((-\infty, 0]) \subset \mathcal{A}$).

We recall the hypotheses from Chapter 3:

$$\text{(H1)} \qquad \begin{cases} \mu > 0, \\ f \in C^1(\mathbb{R},\mathbb{R}),\ f(0) = 0,\ f'(u) > 0 \text{ for all } u \in \mathbb{R}, \\ \text{there exists } A > 0 \text{ such that } |f(u)| < \mu|u| \text{ for all } |u| \ge A, \\ r \in C^1(\mathbb{R},\mathbb{R}),\ r(0) = 1,\ r([-A,A]) \subset (0,\infty). \end{cases}$$

As in the previous chapter set

$$R = \max_{u \in [-A,A]} r(u),\ M = \max_{(u,v) \in [-A,A] \times [-A,A]} |-\mu u + f(v)|,\ C = C([-R,0],\mathbb{R}),$$

and for $\phi \in C$ define

$$\text{lip}(\phi) = \sup\{|\phi(s) - \phi(t)| \cdot |s-t|^{-1} : s,\ t \in [-R,0],\ s \ne t\}.$$

The set $K = \{\phi \in C : 0 \le \phi\}$ is a convex cone in C. We have $(0,\infty)K \subset K$, $K \cap (-K) = \{0\}$, and $\overset{\circ}{K} = \{\phi \in C : \phi(s) > 0 \text{ for all } s \in [-R,0]\}$ is the interior of K. We introduce a closed partial order relation on C in the same way as in Chapter 3, that is:

$\phi \leq \psi$ whenever $\psi - \phi \in K$,

$\phi < \psi$ whenever $\phi \leq \psi$ and $\phi \neq \psi$, and

$\phi \ll \psi$ whenever $\psi - \phi \in \overset{o}{K}$.

The relations $\phi \geq \psi$, $\phi > \psi$ and $\phi \gg \psi$ are defined analogously.

Let the subspace of elements ϕ of C with $\text{lip}(\phi) \leq M$ and $\phi(s) \in [-A, A]$, $s \in [-R, 0]$, be denoted by Y. Then Y is a complete metric space. By the Arzèla–Ascoli theorem, Y is compact.

We recall some basic properties from Chapter 3. Observe that $Y \subset X$, where X is defined in the previous chapter. Clearly the results obtained for X in Chapter 3 remain valid for Y.

Proposition 1.1.

(i) For every $\phi \in Y$, there is a unique solution $x^\phi : [-R, \infty) \to [-A, A]$ of Eq. (1.1) through ϕ, that is x^ϕ is a solution of Eq. (1.1) and $x^\phi|_{[-R,0]} = \phi$.

(ii) The map $F : \mathbb{R}_+ \times Y \ni (t, \phi) \mapsto x_t^\phi \in Y$ defines a continuous semiflow on Y.

(iii) F is monotone, that is, $F(t, \phi) \leq F(t, \psi)$ whenever $\phi \leq \psi$ and $t \geq 0$.

(iv) If ϕ, $\psi \in Y$ with $\phi < \psi$ and $F(t, \phi) \neq F(t, \psi)$ for all $t \geq 0$, then $F(t, \phi) \ll F(t, \psi)$ for all $t \geq 3R$.

(v) If ϕ, $\psi \in Y$ with $\phi \ll \psi$ and $t \geq 0$, then $F(t, \phi) \ll F(t, \psi)$.

(vi) If ϕ, $\psi \in Y$ with $\phi \ll \psi$, then there exists $\phi^* \in Y$ such that $\phi \ll \phi^* \ll \psi$ and $x_t{}^{\phi^*} \to e$ as $t \to \infty$, where $e \in Y$ is an equilibrium point.

(vii) If $c \in \mathbb{R}$ and $x : (c, \infty) \to [-A, A]$, $y : (c, \infty) \to [-A, A]$ are two solutions of Eq. (1.1), then the difference $z = x - y$ satisfies the linear equation

$$\dot{z}(t) = a(t)z(t) + b(t)z(t - r(x(t))), \tag{1.2}$$

for all $t \in (c + 2R, \infty)$, where $a : (c + 2R, \infty) \to \mathbb{R}$ and $b : (c + 2R, \infty) \to \mathbb{R}$ are continuous and bounded functions defined by

$$\begin{aligned} a(t) = -\mu - &\int_0^1 f'[sy(t - r(x(t))) + (1 - s)y(t - r(y(t)))]ds \times \\ &\int_0^1 y'[s(t - r(x(t))) + (1 - s)(t - r(y(t)))]ds \times \\ &\int_0^1 r'[sx(t) + (1 - s)y(t)]ds \end{aligned} \tag{1.3}$$

and

$$b(t) = \int_0^1 f'[sx(t - r(x(t))) + (1 - s)y(t - r(x(t)))]ds. \tag{1.4}$$

The statement also holds in case $c = -\infty$, when we have $(-\infty, \infty)$ instead of (c, ∞) and $(c + 2R, \infty)$.

(viii) If $x : \mathbb{R} \to [-A, A]$ and $y : \mathbb{R} \to [-A, A]$ are two solutions of Eq. (1.1) and $x_0 = y_0$, then $x(t) = y(t)$ for all $t \in \mathbb{R}$.

If $x : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (1.1) with $x_0 = \phi$, then we also denote this solution by x^ϕ. This should not cause confusion.

Hereafter we need the increasing property of the function $t \mapsto t - r(x(t))$, where x is a solution of Eq. (1.1) with values in $[-A, A]$. Either one of the following two hypotheses guarantees the desired property of the above function on some interval.

(H2)
$$|r'(u)| < \frac{1}{M} \quad \text{for all } u \in [-A, A].$$

(H2′)
$$\begin{cases} r \in C^2([-A, A], \mathbb{R}) \text{ and there exists } c \in (0, 1) \\ \text{with } r''(u) \leq c\mu(r'(u))^2 \text{ for all } u \in [-A, A]. \end{cases}$$

Condition (H2′) was introduced by Mallet-Paret and Nussbaum [60]. The advantage of (H2′) comparing to (H2) is that it is independent of f. In the remaining part of the paper we always assume that, in addition to (H1), either (H2) or (H2′) holds.

Lemma 1.2. *Let $x : \mathbb{R} \to [-A, A]$ be a solution of Eq. (1.1). Suppose $\dot{x}(\rho) = 0$ for some $\rho \in \mathbb{R}$. Then $\frac{d}{dt}(t - r(x(t))) > 0$ for all $t \geq \rho$.*

The proof is the same as that of [39, Lemma 2.5].

Now consider the space C and the linear equation

(1.5)
$$\dot{x}(t) = -\mu x(t) + f'(0)x(t-1).$$

Although the map $C \ni \phi \mapsto -\mu\phi(0) + f'(0)\phi(-r(\phi(0))) \in \mathbb{R}$ is not, in general, differentiable, Eq. (1.5) can be considered as the linearization of Eq. (1.1) at 0 (see Cooke and Huang [12] and also [10,27,71]).

For each $\phi \in C$, Eq. (1.5) has a unique solution $x^\phi : [-R, \infty) \to \mathbb{R}$ through ϕ. Solutions of Eq. (1.5) define the C_0-semigroup $T : \mathbb{R}_+ \times C \to C$ given by $T(t, \phi) = x_t^\phi$. The spectrum of the generator of the semigroup $(T(t))_{t \geq 0}$ coincides with the zeros of the characteristic function $\mathbb{C} \ni \lambda \mapsto \lambda + \mu - f'(0)e^{-\lambda} \in \mathbb{C}$. According to [14] all zeros are simple. There is one real zero λ_0; the others form a sequence of complex conjugate pairs $(\lambda_k, \overline{\lambda}_k)$, $k \geq 1$, with $(2k-1)\pi < \text{Im}\lambda_k < 2k\pi$ and $\text{Re}\lambda_{k+1} < \text{Re}\lambda_k < \lambda_0$ for all integers $k \geq 1$, and $\text{Re}\lambda_k \to -\infty$ as $k \to \infty$.

We assume as in [41] that

$$\text{(H3)} \qquad f'(0) > \frac{\mu}{\cos\theta_\mu} \quad \text{for } \theta_\mu \in (\frac{3\pi}{2}, 2\pi) \quad \text{with } \theta_\mu = \mu \tan\theta_\mu.$$

According to [41], (H3) is equivalent to $\mathrm{Re}\lambda_1 > 0$.

Choose β with $\max\{0, \mathrm{Re}\lambda_2\} < \beta < \mathrm{Re}\lambda_1$. The real generalized eigenspace $C_{\beta<}$ of the generator associated with λ_0, λ_1 and $\overline{\lambda}_1$ is 3-dimensional and it is given by the segments of solutions $\mathbb{R} \ni t \mapsto e^{\lambda_0 t} \in \mathbb{R}$ and $\mathbb{R} \ni t \mapsto e^{\mathrm{Re}\lambda_1 t}(a\cos(\mathrm{Im}\lambda_1)t + b\sin(\mathrm{Im}\lambda_1)t)$, a, $b \in \mathbb{R}$, of Eq. (1.5). Let $C_{<\beta}$ denote the real generalized eigenspace of the generator associated with the pairs $(\lambda_k, \overline{\lambda}_k)$, $k \geq 2$.

We have a decomposition $C = C_{\beta<} \oplus C_{<\beta}$ into generalized eigenspaces of the generator of the semigroup given by Eq. (1.5). $C_{\beta<}$ and $C_{<\beta}$ are $T(t)$ invariant for $t \geq 0$. $T(t)$ can be extended to a group on $C_{\beta<}$.

4.2 The separatrix

Let S denote the set of $\phi \in Y$ such that x^ϕ oscillates on $[0, \infty)$, that is

$$S = \{\phi \in Y : (x^\phi)^{-1}(0) \text{ is not bounded from above}\}.$$

Then S is positively invariant. Since

$$Y \setminus S = \cup_{t \geq 0} F(t, \cdot)^{-1}(\mathring{K} \cup (-\mathring{K}))$$

is open, it follows that S is closed. The set S is a separatrix for the semiflow F in the sense that its complement $Y \setminus S$ splits into the set of initial data for solutions which are positive on some unbounded interval, and into the set of initial data for solutions which are negative on some unbounded interval.

Proposition 2.1. *(Nonordering of S). For all ϕ, ψ in Y with $\phi < \psi$ and $F(t, \phi) \neq F(t, \psi)$ for all $t \geq 0$, either $\phi \notin S$ or $\psi \notin S$.*

Proof. Assume that S contains elements $\tilde{\phi}$, $\tilde{\psi}$ with $\tilde{\phi} < \tilde{\psi}$ and $F(t, \tilde{\phi}) \neq F(t, \tilde{\psi})$ for all $t \geq 0$. Then the positive invariance of S and Proposition 1.1(iv) yield $\phi = x^{\tilde{\phi}}_{3R} \ll \psi = x^{\tilde{\psi}}_{3R}$ with ϕ, ψ in S. Using Proposition 1.1(vi), we find ϕ^*, $\psi^* \in Y$ such that $\phi \ll \phi^* \ll \psi^* \ll \psi$, $x^{\phi^*}_t \to e_1$ and $x^{\psi^*}_t \to e_2$ as $t \to \infty$, where e_1, $e_2 \in Y$ are equilibrium points. Denote $x = x^{\psi^*}$, $y = x^{\phi^*}$ and $z = x - y$. Proposition 1.1(v) implies $0 \ll z_t$ for all $t \geq 0$. By Proposition 1.1(vii) with $c = -R$, we find that z satisfies Eq. (1.2) with $a(t)$ and $b(t)$ defined by (1.3) and (1.4).

We show that at least one of the ω-limit sets of ϕ^* and ψ^* consists of a non-zero equilibrium point. Suppose that both x and y converge to 0 as $t \to \infty$. Then $a(t) \to -\mu$ and $b(t) \to f'(0)$ as $t \to \infty$. From (H3), we obtain $\mu < f'(0)$. Hence there is $\epsilon > 0$ such that $f'(0) - \epsilon > \mu + \epsilon$. It follows that there exists $T \geq R$ so that

$$a(t) > -\mu - \epsilon \text{ for all } t \geq T, \text{ and } b(t) > f'(0) - \epsilon \text{ for all } t \geq T. \tag{2.1}$$

The choice of ϵ ensures the existence of a real number $\lambda^* > 0$ such that

$$\lambda^* = -\mu - \epsilon + (f'(0) - \epsilon)e^{-R\lambda^*}.$$

For all $\delta \in \mathbb{R}$, $\delta e^{\lambda^* t}$ is a solution of the equation

$$\dot{v}(t) = -(\mu + \epsilon)v(t) + (f'(0) - \epsilon)v(t - R).$$

Fix $\delta > 0$ so that $z(t) > \delta e^{\lambda^* t}$ for all $t \in [T, T + R]$. If the assertion $z(t) > \delta e^{\lambda^* t}$ for all $t \geq T$ is not true, then there exists $t^* > T + R$ such that $z(t^*) = \delta e^{\lambda^* t^*}$ and $z(t) > \delta e^{\lambda^* t}$ for all $t \in [T, t^*)$. Clearly, $\dot{z}(t^*) - \delta\lambda^* e^{\lambda^* t^*} \leq 0$. On the other hand using (2.1), $z(t^*) = \delta e^{\lambda^* t^*}$ and $z(t^* - r(x(t^*))) > \delta e^{\lambda^*(t^* - r(x(t^*)))} > \delta e^{\lambda^*(t^* - R)}$, we find

$$\begin{aligned}
\dot{z}(t^*) - \delta\lambda^* e^{\lambda^* t^*} =& [a(t^*) + \mu + \epsilon]z(t^*) + b(t^*)z(t^* - r(x(t^*))) \\
& - (f'(0) - \epsilon)\delta e^{\lambda^*(t^* - R)} \\
=& [a(t^*) + \mu + \epsilon]z(t^*) + [b(t^*) - (f'(0) - \epsilon)]z(t^* - r(x(t^*))) \\
& + [f'(0) - \epsilon][z(t^* - r(x(t^*))) - \delta e^{\lambda^*(t^* - R)}] > 0,
\end{aligned}$$

a contradiction. Therefore $z(t) > \delta e^{\lambda^* t}$ for all $t \geq T$, which contradicts the boundedness of z.

Assume that $p \in \omega(\psi^*)$ is a non-zero equilibrium point. As the equilibrium points of F are constant functions, either $0 \ll p$ or $p \ll 0$. If $0 \ll p$ there is $t_0 > 0$ so that $0 \ll x_{t_0}^{\psi^*}$. Proposition 1.1(v) yields $x_{t_0}^{\psi^*} \ll x_{t_0}^{\psi}$, and thus $0 \ll x_t^{\psi}$ for all $t \geq t_0$. Hence $\psi \notin S$ in contradiction to $\psi \in S$. If $p \ll 0$ there is $t_0 > 0$ so that $x_{t_0}^{\psi^*} \ll 0$. Proposition 1.1(v) implies $x_{t_0}^{\phi} \ll x_{t_0}^{\psi^*}$, and thus $x_t^{\phi} \ll 0$ for all $t \geq t_0$. Therefore $\phi \notin S$ in contradiction to $\phi \in S$. Similarly, if $p \in \omega(\phi^*)$ is a non-zero equilibrium point, then we obtain a contradiction. □

4.3 An unstable set of zero

Recall that we have a decomposition

$$C = C_{\beta<} \oplus C_{<\beta},$$

where $\max\{0, \mathrm{Re}\lambda_2\} < \beta < \mathrm{Re}\lambda_1$, $C_{\beta<}$ is the real generalized eigenspace of the generator given by the spectral set $\{\lambda_0, \lambda_1, \overline{\lambda}_1\}$, while $C_{<\beta}$ is the real generalized eigenspace of the generator for the complementary spectral set $\{\lambda_k, \overline{\lambda}_k : k \geq 2\}$.

According to [38,Theorem 4.1], there exist positive numbers δ_1, δ_2, a continuously differentiable map $w : B_{\delta_1}(C_{\beta<}) \to C_{<\beta}$ such that the graph of the map w (the local unstable β-manifold)

$$W_\beta = \{\psi + w(\psi) : \psi \in B_{\delta_1}(C_{\beta<})\} \subset C$$

has the following properties:

(i) $w(0) = 0$, $w(B_{\delta_1}(C_{\beta<})) \subset B_{\delta_2}(C_{<\beta})$, $Dw(0) = 0$, $B_{\delta_1}(C_{\beta<}) \subset Y$, $w(B_{\delta_1}(C_{\beta<})) \subset Y$, $W_\beta \subset Y$.

(ii) W_β is invariant in the sense that there exists $\delta_3 > 0$ so that for all $\phi \in W_\beta$ with $\|\phi\| < \delta_3$, there is a solution x^ϕ of Eq. (1.1) on $(-\infty, 0]$ such that $x_0^\phi = \phi$, $x_t^\phi \in W_\beta$ for all $t \leq 0$, and $e^{-\beta t}x_t^\phi \to 0$ as $t \to -\infty$.

(iii) There are $\delta_4 > 0$, $\delta_5 > 0$ such that if $\phi \in B_{\delta_4}(C)$ and there exists a solution x^ϕ of Eq. (1.1) on $(-\infty, 0]$ so that $e^{-\beta t}x_t^\phi \in B_{\delta_5}(C)$ for all $t \leq 0$, and $e^{-\beta t}x_t^\phi \to 0$ as $t \to -\infty$, then $\phi \in W_\beta$.

Denote $\widehat{W_\beta} = W_\beta \cap \{\phi \in C : \|\phi\| < \delta_3\}$ and define the set $W = F(\mathbb{R}_+ \times \widehat{W_\beta})$. Clearly, $W \subset Y$. Now we establish some properties of $W \cap S$, the closure $\overline{W}$ of W and the closure $\overline{W \cap S}$ of $W \cap S$.

Proposition 3.1.
(i) $\overline{W}$ *and* $\overline{W \cap S}$ *are compact and invariant subsets of* Y.
(ii) $W \cap S \setminus \{0\}$ *is nonempty and invariant.*

Proof. The proof of (i). The compactness of $\overline{W}$ and $\overline{W \cap S}$ follows from the compactness of Y. Consider $\phi \in \overline{W}$ and a sequence $(\phi_n)_1^\infty$ in W such that $\phi_n \to \phi$ as $n \to \infty$. By the definition of W and by Proposition 1.1(viii), for each $n \in \mathbb{N}$ there exists a unique solution $x^n : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0^n = \phi_n$ and $x_t^n \in W$ for all $t \in \mathbb{R}$. Since $W \subset Y$, and Y is compact, by the diagonalization procedure there is a subsequence $(x^{n_k})_{k=1}^\infty$ of $(x^n)_1^\infty$ such that $(x^{n_k})_{k=1}^\infty$ uniformly converges to a continuous function $y : \mathbb{R} \to [-A, A]$ as $k \to \infty$ on each compact subset of $\mathbb{R}$. Clearly, $y_0 = \phi$. Using Eq. (1.1) it easily follows that y is also differentiable and satisfies Eq. (1.1) on $\mathbb{R}$. By Proposition 1.1(viii), y is unique. From $x_t^{n_k} \in W$ we find $y_t \in \overline{W}$ for all $t \in \mathbb{R}$. Therefore, $\overline{W}$ is invariant.

Now let $\phi \in \overline{W \cap S}$ and a sequence $(\phi_n)_1^\infty$ in $W \cap S$ with $\phi_n \to \phi$ as $n \to \infty$. The definition of W guarantees the existence of solutions $x^n : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0^n = \phi_n$ and $x_t^n \in W$ for all $t \in \mathbb{R}$. It also follows that $x_t^n \in S$ for all $t \in \mathbb{R}$ because of the definition of S and the monotonicity of F. A subsequence of $(x^n)_1^\infty$ converging uniformly on compact subsets of $\mathbb{R}$ can be constructed as above. We obtain a solution y as above. Clearly, $y_t \in \overline{W \cap S}$ for all $t \in \mathbb{R}$ and $y_0 = \phi$. Thus, $\overline{W \cap S}$ is also invariant.

The proof of (ii). Let $e^{\lambda_0 \cdot}$ denote the element $[-R, 0] \ni s \mapsto e^{\lambda_0 s} \in \mathbb{R}$ of $C_{\beta<}$. Obviously, we find $\epsilon_0 > 0$ so that

$$\epsilon e^{\lambda_0 \cdot} + w(\epsilon e^{\lambda_0 \cdot}) \in \widehat{W}_\beta \quad \text{for } |\epsilon| < \epsilon_0.$$

From $Dw(0) = 0$ it follows that

$$\frac{\| w(\epsilon e^{\lambda_0 \cdot})\|}{\| \epsilon e^{\lambda_0 \cdot}\|} \to 0 \quad \text{as } \epsilon \to 0.$$

Hence, for small $\epsilon > 0$

$$\begin{aligned} \epsilon e^{\lambda_0 \cdot} + w(\epsilon e^{\lambda_0 \cdot}) &\gg 0 \quad \text{and} \\ -\epsilon e^{\lambda_0 \cdot} + w(-\epsilon e^{\lambda_0 \cdot}) &\ll 0. \end{aligned}$$

Fix such an $\epsilon > 0$, and define

$$\begin{aligned} \psi &= \epsilon e^{\lambda_0 \cdot} + w(\epsilon e^{\lambda_0 \cdot}) \quad \text{and} \\ \eta &= -\epsilon e^{\lambda_0 \cdot} + w(-\epsilon e^{\lambda_0 \cdot}). \end{aligned}$$

There is a continuous curve $\gamma : [0, 1] \to C$ with $\gamma(0) = \eta$, $\gamma(1) = \psi$, $\gamma(s) \in \widehat{W}_\beta$ for all $s \in [0, 1]$, and $\gamma(s) \neq 0$ for all $s \in [0, 1]$. The sets

$$\begin{aligned} \mathcal{J}^+ &= \{s \in [0, 1] : x_t^{\gamma(s)} \gg 0 \quad \text{for all sufficiently large } t\} \quad \text{and} \\ \mathcal{J}^- &= \{s \in [0, 1] : x_t^{\gamma(s)} \ll 0 \quad \text{for all sufficiently large } t\} \end{aligned}$$

are open, nonempty and disjoint subsets of $[0, 1]$. From the connectedness of $[0, 1]$, it follows that there exists $s^* \in [0, 1] \setminus (\mathcal{J}^+ \cup \mathcal{J}^-)$. Clearly, $\gamma(s^*) \in \widehat{W}_\beta \cap S \setminus \{0\} \subset W \cap S \setminus \{0\}$.

To prove the invariance of $W \cap S \setminus \{0\}$ consider $\phi \in W \cap S \setminus \{0\}$. By the definition of W, there is a solution $x^\phi : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0^\phi = \phi$, and $x_t^\phi \in W$ for all $t \in \mathbb{R}$. By the definition of S and the monotonicity of F it is clear that $x_t^\phi \in S$ for all $t \in \mathbb{R}$. Proposition 1.1(viii) gives $x_t^\phi \neq 0$ for all $t \in \mathbb{R}$. □

Proposition 3.2. *Let ϕ and ψ be in $\overline{W \cap S}$. If $\phi(s) = \psi(s)$ for all $s \in [-r(\phi(0)), 0]$, then the solution $x^\phi : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) through ϕ coincides with the solution $x^\psi : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) through ψ.*

Proof. Since ϕ and ψ are in $\overline{W \cap S}$, there exist the solutions $x^\phi : \mathbb{R} \to \mathbb{R}$ and $x^\psi : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) through ϕ and ψ, respectively. By the invariance of $\overline{W \cap S}$, $x_t^\phi, x_t^\psi \in \overline{W \cap S} \subset \overline{W} \cap S \subset S$ for all $t \in \mathbb{R}$. If $\dot{x}^\phi$ has no zero on an interval $(-\infty, T]$ then either $x_t^\phi \gg 0$ or $x_t^\phi \ll 0$ for some $t < 0$. In either case Proposition 1.1(v) and $F(\cdot, 0) \equiv 0$ imply $\phi \notin S$, a contradiction. Analogous statement holds for x^ψ.

Therefore there are arbitrarily large negative numbers ρ and σ so that $\dot{x}^\phi(\rho) = 0$ and $\dot{x}^\psi(\sigma) = 0$. By Lemma 1.2, the functions $t \mapsto t - r(x^\phi(t)))$ and $t \mapsto t - r(x^\psi(t)))$ are strictly increasing on $\mathbb{R}$. In particular, $t - r(x^\phi(t))) \geq -r(\phi(0))$ and $t - r(x^\psi(t))) \geq -r(\psi(0)) = -r(\phi(0))$ follows for all $t \geq 0$. Therefore $\phi|_{[-r(\phi(0)),0]}$ and $\psi|_{[-r(\psi(0)),0]}$ determine the values of $x^\phi(t)$ and $x^\psi(t)$ for all $t > 0$ uniquely. Uniqueness is shown in the same way as in the proof of Proposition 1.1(i). Hence $x^\phi(t) = x^\psi(t)$ for all $t \in [-r(\phi(0)), \infty)$. Using Proposition 1.1(viii), we obtain $x^\phi(t) = x^\psi(t)$ for all $t \in \mathbb{R}$. The proof is complete. □

4.4 A discrete Lyapunov functional

In this section first we define a discrete integer-valued Lyapunov functional. For equations with constant delay Mallet-Paret [57] introduced a discrete Lyapunov functional. A more general version is contained in [63]. The state-dependent delay requires a modified version of the functional. We have to count sign changes of solutions x of Eq. (1.1) on intervals of the form $[t - r(x(t)), t]$ instead of on intervals with fixed length. For Eq. (1.1) with condition $f' < 0$, a discrete Lyapunov functional is introduced in [39]. In our case, when $f' > 0$, the definition of a Lyapunov functional is similar to that of [39], and its properties are analogous to the case $f' < 0$.

Let $[a, b]$ be an interval and ϕ be a real valued continuous function defined on an interval containing $[a, b]$ such that $\phi|_{[a,b]} \not\equiv 0$. Then the numbers of sign changes $\mathrm{sc}(\phi, [a,b])$ of ϕ on $[a,b]$ is 0 if either $\phi(s) \geq 0$ for all $s \in [a,b]$ or $\phi(s) \leq 0$ for all $s \in [a,b]$; otherwise $\mathrm{sc}(\phi, [a,b])$ is given by

$$\mathrm{sc}(\phi, [a,b]) = \sup\{k : \text{ there is } s_0 < s_1 < \ldots < s_k \text{ such that } s_i \in [a,b] \text{ for } i = 0,1,\ldots,k, \text{ and } \phi(s_i)\phi(s_{i+1}) < 0 \text{ for } i = 0,1,\ldots,k-1\}.$$

Define the functional $V : C([a,b], \mathbb{R}) \setminus \{0\} \to 2\mathbb{N} \cup \{\infty\}$ by

$$V(\phi, [a,b]) = \begin{cases} \mathrm{sc}(\phi, [a,b]) & \text{if } \mathrm{sc}(\phi, [a,b]) \text{ is even or } \infty, \\ \mathrm{sc}(\phi, [a,b]) + 1 & \text{if } \mathrm{sc}(\phi, [a,b]) \text{ is odd.} \end{cases}$$

Define the set

$$H_{[a,b]} = \{\phi \in C^1([a,b], \mathbb{R}) : \phi(b) \neq 0 \text{ or } \phi(a)\dot{\phi}(b) > 0, \quad \phi(a) \neq 0 \text{ or } \dot{\phi}(a)\phi(b) < 0, \quad \text{all zeros of } \phi \text{ in } (a,b) \text{ are simple}\}.$$

Some useful properties of V are contained in the next two lemmas. With a suitable modification, the proofs follow closely those of [39, Lemmas 4.1 and 4.2]. Therefore we omit them.

Lemma 4.1.

(i) V is lower semi-continuous in the following sense. If ϕ, ϕ_n are nonzero continuous functions on the intervals $[a,b]$, $[a_n, b_n]$, respectively, and

$$\max_{s \in [a,b] \cap [a_n, b_n]} |\phi_n(s) - \phi(s)| \to 0, \; a_n \to a, \; b_n \to b \quad \text{as } n \to \infty,$$

then

$$V(\phi, [a,b]) \leq \liminf_{n \to \infty} V(\phi_n, [a_n, b_n]).$$

(ii) If $\phi \in H_{[a,b]}$ then $V(\phi, [a,b]) < \infty$.

(iii) If $\phi \in C^1([a-\delta, b+\delta], \mathbb{R})$ for some $\delta > 0$ and $\phi|_{[a,b]} \in H_{[a,b]}$, then there is $\gamma \in (0, \delta)$ such that

$$|a - c| < \gamma, \; |b - d| < \gamma, \; \psi \in C^1([c,d], \mathbb{R}), \; \| \psi - \phi \|_{C^1([c,d],\mathbb{R})} < \gamma$$

imply

$$V(\psi, [c,d]) = V(\phi, [a,b]).$$

Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$, $\tau : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $\beta(t) > 0$, $\tau(\mathbb{R}) \subset [0, R]$, and the function $\mathbb{R} \ni t \mapsto t - \tau(t) \in \mathbb{R}$ is strictly increasing on $\mathbb{R}$. Let $v : \mathbb{R} \to \mathbb{R}$ be a continuous function which is continuously differentiable on $\mathbb{R}$ and satisfies

$$\dot{v}(t) = \alpha(t)v(t) + \beta(t)v(t - \tau(t)) \tag{4.1}$$

for all $t \in \mathbb{R}$.

Lemma 4.2. *Assume that α, β, τ, v are given as above, moreover for all $t \in \mathbb{R}$ $v|_{[t-\tau(t),t]}$ is not identically zero. Then*
(i) $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ imply $V(v, [t_1 - \tau(t_1), t_1]) \geq V(v, [t_2 - r(t_2), t_2])$;
(ii) $t \in \mathbb{R}$ and $V(v, [s-\tau(s), s]) = V(v, [t-\tau(t), t]) < \infty$ for all $s \in [t-3R, t]$ imply $v|_{[t-\tau(t),t]} \in H_{[t-\tau(t),t]}$.

The aim of this section is to establish the values of the Lyapunov functional for differences of functions in $\overline{W \cap S}$.

Proposition 4.3. *If ϕ, $\psi \in \overline{W \cap S}$ with $\phi \neq \psi$ then $V(\phi - \psi, [-r(\phi(0)), 0]) = 2$.*

Proof. Let $\phi, \psi \in \overline{W \cap S}$ with $\phi \neq \psi$. It follows that $\phi(s) \neq \psi(s)$ for some $s \in [-r(\phi(0)), 0]$, since otherwise, by Proposition 3.2, $x^\phi = x^\psi$ on $\mathbb{R}$, a contradiction.

We have $V(\phi - \psi, [-r(\phi(0)), 0]) \geq 2$. Indeed, let $V(\phi - \psi, [-r(\phi(0)), 0]) = 0$. The difference $x^\phi - x^\psi$ satisfies Eq. (4.1) with $\tau(t) = r(x^\phi(t))$, $\alpha(t) = a(t)$ and $\beta(t) = b(t)$, where $a(t)$ and $b(t)$ are defined by (1.3) and (1.4). Thus, using Lemma 4.2(i), we get $V(x_t^\phi - x_t^\psi, [-r(x^\phi(t)), 0]) = 0$ for all $t \geq 0$. Therefore, $x_t^\phi < x_t^\psi$ or $x_t^\phi > x_t^\psi$ for all $t \geq R$. Hence $x_R^\phi < x_R^\psi$ or $x_R^\phi > x_R^\psi$. Then, by Proposition 1.1(iv), $x_{4R}^\phi \ll x_{4R}^\psi$ or $x_{4R}^\phi \gg x_{4R}^\psi$. By $\overline{W \cap S} \subset \overline{W} \cap \overline{S} = \overline{W} \cap S$ and by the invariance of S, x_{4R}^ϕ and x_{4R}^ψ are in S, which contradicts Proposition 2.1.

Let ϕ, $\psi \in W \cap S$ with $\phi \neq \psi$. To prove $V(\phi - \psi, [-r(\phi(0)), 0]) \leq 2$, by the monotone property of V it suffices to show that there exists a sequence $(t_n)_0^\infty$ with $t_n \to -\infty$ as $n \to \infty$, and $V(x_{t_n}^\phi - x_{t_n}^\psi, [-r(x^\phi(t_n)), 0]) \leq 2$ for all $n \in \mathbb{N}$. Set $y = x^\phi - x^\psi$. Using Proposition 1.1(vii) with $c = -\infty$, we obtain that y satisfies Eq. (1.2) with $a(t)$ defined by (1.3) and $b(t)$ by (1.4).

The definition of W implies $y(t) \to 0$ as $t \to -\infty$. Hence, there exists a sequence $(t_n)_0^\infty$ with $t_n \to -\infty$ as $n \to \infty$, and

$$|y(t_n)| = \sup_{s \leq 0} |y(t_n + s)|.$$

The functions $z^n : (-\infty, 0] \to \mathbb{R}$, $n \in \mathbb{N}$, given by

$$z^n(t) = \frac{y(t_n + t)}{|y(t_n)|}$$

satisfy

$$|z^n(t)| \leq 1 = |z^n(0)| \quad \text{for all } t \leq 0, \tag{4.2}$$

and

$$\dot{z}^n(t) = a(t_n+t)z^n(t) + b(t_n+t)z^n(t - r(x^\phi(t_n+t))), \quad \text{for all } t \leq 0.$$

Clearly, $b(t_n+t) \to f'(0)$ and $a(t_n+t) \to -\mu$ as $n \to \infty$ uniformly on $(-\infty, 0]$. Using the Arzèla-Ascoli theorem and the equations for z^n, we obtain a subsequence $(z^{n_k})_{k=0}^{\infty}$ of $(z^n)_0^\infty$ and a C^1 function $Z : (-\infty, 0] \to \mathbb{R}$ such that

$$z^{n_k}(t) \to Z(t),\ \dot{z}^{n_k}(t) \to \dot{Z}(t) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$, and Z satisfies

$$\dot{Z}(t) = -\mu Z(t) + f'(0)Z(t-1) \text{ for all } t \leq 0.$$

It follows that $|Z(t)| \leq 1 = |Z(0)|$ for all $t \leq 0$.

We claim that $V(Z_t, [-1,0]) \leq 2$ for all $t \leq 0$. Using the definition of W and the invariance of W_β, we obtain that $x^\phi_{t_n}$ and $x^\psi_{t_n}$ are in W_β for all sufficiently large $n \in \mathbb{N}$. Therefore, for all sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} x^\phi_{t_n} &= \chi_n + w(\chi_n), \\ x^\psi_{t_n} &= \eta_n + w(\eta_n), \end{aligned} \tag{4.3}$$

where χ_n, η_n are in $B_{\delta_1}(C_{\beta<})$ and $w(\chi_n)$, $w(\eta_n)$ are in $B_{\delta_2}(C_{<\beta})$. We have $z_0^{n_k} = Pr_{C_{\beta<}}\, z_0^{n_k} + Pr_{C_{<\beta}}\, z_0^{n_k}$. The definition of $z_0^{n_k}$ and (4.3) imply that

$$Pr_{C_{\beta<}}\, z_0^{n_k} = \frac{\chi_{n_k} - \eta_{n_k}}{|y(t_{n_k})|} \quad \text{and} \quad Pr_{C_{<\beta}}\, z_0^{n_k} = \frac{w(\chi_{n_k}) - w(\eta_{n_k})}{|y(t_{n_k})|}.$$

Note that $\|\,\mathrm{Pr}_{C_{\beta<}}\, z_0^{n_k}\| \neq 0$, and $\chi_{n_k}, \eta_{n_k} \to 0$ as $k \to \infty$. Since w is continuously differentiable and $Dw(0) = 0$, we infer

$$\frac{\|\, Pr_{C_{<\beta}}\, z_0^{n_k}\|}{\|\, Pr_{C_{\beta<}}\, z_0^{n_k}\|} = \frac{\|\, w(\chi_{n_k}) - w(\eta_{n_k})\|}{\|\, \chi_{n_k} - \eta_{n_k}\|} \to 0 \quad \text{as } k \to \infty.$$

From (4.2), we obtain that $Pr_{C_{\beta<}}\, z_0^{n_k}$ is bounded. Therefore, we get $Pr_{C_{<\beta}}\, z_0^{n_k} \to 0$ as $k \to \infty$. We conclude that $\lim_{k\to\infty} z_0^{n_k} = \lim_{k\to\infty} Pr_{C_{\beta<}}\, z_0^{n_k} \in C_{\beta<}$. Consequently, $Z_0 \in C_{\beta<}$. Since $C_{\beta<}$ is invariant under $T(t)$, $t \geq 0$, and backward uniqueness holds for the solutions of $\dot{x}(t) = -\mu x(t) + f'(0)x(t-1)$, it follows that $Z_t \in C_{\beta<}$ for all $t \leq 0$. The definition of $C_{\beta<}$ implies that $Z(t) = c_0 e^{\lambda_0 t} + c_1 e^{\mathrm{Re}\lambda_1 t} \sin((\mathrm{Im}\lambda_1)t + c_2)$ for some $(c_0, c_1, c_2) \in \mathbb{R}^3$ with $c_0^2 + c_1^2 \neq 0$.

If $c_1 = 0$, then $c_0 \neq 0$ and thus $V(Z_t) = 0$ for all $t \leq 0$. If $c_1 \neq 0$, then $Z(t) = c_1 e^{\mathrm{Re}\lambda_1 t}[\sin((\mathrm{Im}\lambda_1)t + c_2) + \frac{c_0}{c_1} e^{(\lambda_0 - \mathrm{Re}\lambda_1)t}]$, and for all sufficiently large negative t the sign changes of Z are determined by the dominant term $\sin((\mathrm{Im}\lambda_1)t + c_2)$. Using $\mathrm{Im}\lambda_1 \in (\pi, 2\pi)$, we find $V(Z_t, [-1,0]) = 2$ for all sufficiently large negative t. From Lemma 4.2(ii), it follows that there exists $T \leq 0$ such that $Z_T|_{[-1,0]} \in H_{[-1,0]}$. By Lemma 4.1(iii) and $r(0) = 1$,

$$V(Z_T, [-1,0]) = V(z_T^{n_k}, [-r(x^\phi(t_{n_k} + T)), 0]) \quad \text{for all sufficiently large } k.$$

Since

$$V(z_T^{n_k}, [-r(x^\phi(t_{n_k} + T)), 0]) = V(y_{t_{n_k}+T}, [-r(x^\phi(t_{n_k} + T)), 0]),$$

we find that $V(y_{t_{n_k}+T}, [-r(x^\phi(t_{n_k} + T)), 0]) \leq 2$ for all sufficiently large k, which completes the proof of $V(\phi - \psi, [-r(\phi(0)), 0] \leq 2$. Consequently, $V(\phi - \psi, [-r(\phi(0)), 0]) = 2$ for all ϕ, ψ in $W \cap S$ with $\phi \neq \psi$.

Let ϕ, $\psi \in \overline{W \cap S}$ with $\phi \neq \psi$, and choose two sequences of points ϕ_n, $\psi_n \in W \cap S$, $n \in \mathbb{N}$, with $\phi_n \neq \psi_n$, $\phi_n \to \phi$ and $\psi_n \to \psi$ as $n \to \infty$. We know that $V(\phi_n - \psi_n, [-r(\phi_n(0)), 0]) = 2$. From Lemma 4.1(i), we conclude

$$V(\phi - \psi, [-r(\phi(0)), 0]) \leq \liminf_{n\to\infty} V(\phi_n - \psi_n, [-r(\phi_n(0)), 0]) = 2.$$

Therefore for all ϕ, $\psi \in \overline{W \cap S}$ with $\phi \neq \psi$ we have $V(\phi - \psi, [-r(\phi(0)), 0]) = 2$. The proof is complete. □

Proposition 4.4. *If $\phi \in \overline{W \cap S} \setminus \{0\}$ and $x = x^\phi$, then $x_t|_{[-r(x(t)),0]} \in H_{[-r(x(t)),0]}$ for all $t \in \mathbb{R}$, and there exists a sequence $(t_n)_{-\infty}^{\infty}$ such that for all $n \in \mathbb{Z}$ we have*

$$\begin{aligned} &t_{n+1} - t_n < 1, \quad t_{n+2} - t_n > 1, \\ &x(t_n) = 0, \ \dot{x}(t_{2n}) > 0, \ \dot{x}(t_{2n+1}) < 0, \\ &x(t) > 0 \quad \text{for all } t \in (t_{2n}, t_{2n+1}), \\ &x(t) < 0 \quad \text{for all } t \in (t_{2n-1}, t_{2n}). \end{aligned}$$

Proof. Let $\phi \in \overline{W \cap S} \setminus \{0\}$. By the invariance of $\overline{W \cap S}$, the solution $x = x^\phi : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0 = \phi$ satisfies $x_t \in \overline{W \cap S}$ for all $t \in \mathbb{R}$. By Proposition 1.1(viii) $x_t \neq 0$ for all $t \in \mathbb{R}$. As $0 \in \overline{W \cap S}$, Proposition 4.3 yields $V(x_t, [-r(x(t)), 0]) = 2$ for all $t \in \mathbb{R}$. Using Lemma 4.2(ii), it follows that $x_t|_{[-r(x(t)),0]} \in H_{[-r(x(t)),0]}$ for all $t \in \mathbb{R}$. By the definition of $H_{[-r(x(t)),0]}$, all zeros of x are simple. Consequently, there is an increasing sequence $(t_n)_{-\infty}^{\infty}$ so that for all $n \in \mathbb{Z}$, $x(t_n) = 0$, $\dot{x}(t_{2n}) > 0$, $\dot{x}(t_{2n+1}) < 0$, $x(t) > 0$ for all $t \in (t_{2n}, t_{2n+1})$ and $x(t) < 0$ for all $t \in (t_{2n-1}, t_{2n})$. The definition of $H_{[-r(x(t)),0]}$ and $V(x_t, [-r(x(t)), 0]) = 2$ for all $t \in \mathbb{R}$ imply $t_{n+1} - r(x(t_{n+1})) < t_n$, that is $t_{n+1} - t_n < r(x(t_{n+1})) = 1$ for all $n \in \mathbb{Z}$. Similarly, we have $t_{n+2} - r(x(t_{n+2})) > t_n$ for all $n \in \mathbb{Z}$, that is $t_{n+2} - t_n > r(x(t_{n+2})) = 1$ for all $n \in \mathbb{Z}$. The proof is complete. □

4.5 Dynamics on $\overline{W \cap S}$

This section contains the main result. Namely, we prove that the ω-limit set of all points in $W \cap S \setminus \{0\}$ is a nontrivial periodic orbit. First, we need some preparatory results.

We begin with the continuous map

$$\Pi : C \ni \phi \mapsto (\phi(0), \phi(-r(\phi(0)))) \in \mathbb{R}^2.$$

Proposition 5.1. *The restriction of* Π *to* $\overline{W \cap S}$ *is injective.*

Proof. Consider $\phi, \psi \in \overline{W \cap S}$ with $\phi \neq \psi$. By Proposition 4.3, $V(x_t^\phi - x_t^\psi, [-r(x^\phi(0)), 0]) = 2$ for all $t \in \mathbb{R}$. Lemma 4.2(ii) implies $(\phi - \psi)|_{[-r(\phi(0)),0]} \in H_{[-r(\phi(0)),0]}$. Therefore, $\Pi\phi \neq \Pi\psi$, and the proof is complete. □

As Π is continuous and $\overline{W \cap S}$ is compact, Π maps $\overline{W \cap S}$ onto its range $\Pi(\overline{W \cap S}) \subset \mathbb{R}^2$ homeomorphically. Let $\Pi^{-1} : \Pi(\overline{W \cap S}) \to C$ be the map given by the inverse of $\overline{W \cap S} \ni \phi \mapsto \Pi\phi \in \Pi(\overline{W \cap S})$.

Let $\chi_0 \in \Pi(\overline{W \cap S})$ and $\psi = \Pi^{-1}(\chi_0)$. By the invariance of $\overline{W \cap S}$, $x_t^\psi \in \overline{W \cap S}$ for all $t \in \mathbb{R}$. The curve

$$\chi : \mathbb{R} \ni t \mapsto \Pi x_t^\psi = (x^\psi(t), x^\psi(t - r(x^\psi(t)))) \in \mathbb{R}^2$$

is C^1-smooth and has its range in $\Pi(\overline{W \cap S})$. We call this curve the canonical curve through χ_0.

Proposition 5.2. *The canonical curves through* $\chi_0 \in \Pi(W \cap S \setminus \{0\})$ *are injective.*

Proof. Consider $\chi_0 \in \Pi(W \cap S \setminus \{0\})$. Then $\psi \in W \cap S \setminus \{0\}$. Thus $x_t^\psi \neq 0$ for all $t \in \mathbb{R}$, and by the definition of W, $x_t^\psi \to 0$ as $t \to -\infty$. We infer that $x_{t_1}^\psi \neq x_{t_2}^\psi$ for all t_1, t_2 with $t_1 < t_2$. Otherwise x^ψ is a $t_2 - t_1$ periodic solution, which contradicts $x_t^\psi \to 0$ as $t \to -\infty$. The proof is complete. □

The images of the closed hyperplane

$$\mathcal{H} = \{\phi \in C : \phi(0) = 0\}$$

and the subsets

$$\mathcal{H}_+ = \{\phi \in \mathcal{H} : \phi(-r(\phi(0))) = \phi(-1) > 0\},$$
$$\mathcal{H}_- = \{\phi \in \mathcal{H} : \phi(-r(\phi(0))) = \phi(-1) < 0\}$$

of $\mathcal{H}$ under Π are

$$\{(u,v) \in \mathbb{R}^2 : u = 0\},$$

$$v_+ = \{(u,v) \in \mathbb{R}^2 : u = 0,\ v > 0\},\ \text{and}\ v_- = \{(u,v) \in \mathbb{R}^2 : u = 0,\ v < 0\},$$

respectively.

The canonical curves through $\chi_0 \neq (0,0)$ intersects $v_- \cup v_+$ transversally. Indeed, for every $\chi_0 \in \Pi(\overline{W \cap S}) \setminus \{(0,0)\}$ and $\psi = \Pi^{-1}(\chi_0)$, and for every $t \in \mathbb{R}$ with $\Pi x_t^\psi \in v_+ (\in v_-)$, the first component $D_1 \Pi x_t^\psi 1$, that is $\dot{x}^\psi(t)$, of the tangent vector of Πx_t^ψ satisfies

$$\dot{x}^\psi(t) = f(x^\psi(t-1)) > 0\,(< 0).$$

We introduce the intersection map

$$c : (v_- \cup v_+) \cap \Pi(\overline{W \cap S}) \to (v_- \cup v_+) \cap \Pi(\overline{W \cap S})$$

as follows. For $\chi_0 \in (v_- \cup v_+) \cap \Pi(\overline{W \cap S})$ and $\psi = \Pi^{-1}(\chi_0) \in (\mathcal{H}_- \cup \mathcal{H}_+) \cap \overline{W \cap S}$, Proposition 4.4 shows that there is a smallest zero $z_1 = z_1(\psi)$ of $x^\psi : \mathbb{R} \to \mathbb{R}$ in $(0, \infty)$; we set

$$c(\chi_0) = \Pi x_{z_1}^\psi = (0, x^\psi(z_1 - 1)).$$

Analogously we can use the largest zero $z_{-1} = z_{-1}(\psi)$ of $x^\psi : \mathbb{R} \to \mathbb{R}$ in $(-\infty, 0)$ with $\psi = \Pi^{-1}(\chi_0)$, $\chi_0 \in (v_- \cup v_+) \cap \Pi(\overline{W \cap S})$ to define the map

$$c_- : (v_- \cup v_+) \cap \Pi(\overline{W \cap S}) \to (v_- \cup v_+) \cap \Pi(\overline{W \cap S})$$

by $c_-(\chi_0) = \Pi x_{z_{-1}}^\psi$. It follows that c is continuous, and bijective with $c^{-1} = c_-$. Moreover, $c(v_+ \cap \Pi(\overline{W \cap S})) = v_- \cap \Pi(\overline{W \cap S})$ and $c(v_- \cap \Pi(\overline{W \cap S})) = v_+ \cap \Pi(\overline{W \cap S})$ and the map

$$\rho : v_+ \cap \Pi(\overline{W \cap S}) \ni \chi_0 \mapsto c(c(\chi_0)) \in v_+ \cap \Pi(\overline{W \cap S})$$

is continuous, and bijective with $\rho^{-1}(\chi_0) = c^{-1}(c^{-1}(\chi_0))$. It is convenient to write $\chi_0 \prec \hat{\chi}_0$ for elements in $\{(0,u) \in \mathbb{R}^2 : u \in \mathbb{R}\}$ if and only if the second components satify $[\chi_0]_2 \prec [\hat{\chi}_0]_2$.

The following results follow closely those in [41] for Eq. (1.1) with constant delay.

Proposition 5.3.

(i) For all χ_0, $\hat{\chi}_0$ in $v_+ \cap \Pi(\overline{W \cap S})$ with $\chi_0 \prec \hat{\chi}_0$, $\rho(\chi_0) \prec \rho(\hat{\chi}_0)$.

(ii) For every trajectory $(\chi_n)_{-\infty}^{\infty}$ of ρ in $v_+ \cap \Pi(\overline{W \cap S})$, there exist χ_- and χ_+ in $\overline{v}_+ \cap \Pi(\overline{W \cap S})$ so that

$$\chi_n \to \chi_- \text{ as } n \to -\infty, \quad \chi_n \to \chi_+ \text{ as } n \to \infty.$$

In case $\chi_- \neq 0$ ($\chi_+ \neq 0$) the solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0 = \Pi^{-1}(\chi_-)$ ($x_0 = \Pi^{-1}(\chi_+)$) is periodic. For

$$\mathcal{O}_{\Pi^{-1}(\chi_-)} = \left\{x_t^{\Pi^{-1}(\chi_-)} : t \in \mathbb{R}\right\} \quad \text{and} \quad \mathcal{O}_{\Pi^{-1}(\chi_+)} = \left\{x_t^{\Pi^{-1}(\chi_+)} : t \in \mathbb{R}\right\},$$

$$dist\left(x_t^{\Pi^{-1}(\chi_0)}, \mathcal{O}_{\Pi^{-1}(\chi_-)}\right) \to 0 \text{ as } t \to -\infty$$

and

$$dist\left(x_t^{\Pi^{-1}(\chi_0)}, \mathcal{O}_{\Pi^{-1}(\chi_+)}\right) \to 0 \text{ as } t \to \infty.$$

Proof. The proof of (i). Assertion (i) follows from $c(\hat{\chi}_0) \prec c(\chi_0)$ for all χ_0, $\hat{\chi}_0$ in $(v_- \cup v_+) \cap \Pi(\overline{W \cap S})$ with $\chi_0 \prec \hat{\chi}_0$. In order to derive this statement, consider χ_0 and $\hat{\chi}_0$ in $(v_- \cup v_+) \cap \Pi(\overline{W \cap S})$ with $\chi_0 \prec \hat{\chi}_0$. In case $\chi_0 \in v_-$ and $\hat{\chi}_0 \in v_+$ we have $c(\hat{\chi}_0) \in v_-$ and $c(\chi_0) \in v_+$, thus $c(\hat{\chi}_0) \prec c(\chi_0)$.

Consider the case $\chi_0 \in v_+$ and $\hat{\chi}_0 \in v_+$. Let $x : \mathbb{R} \to \mathbb{R}$ and $\hat{x} : \mathbb{R} \to \mathbb{R}$ denote the solutions of Eq. (1.1) with $x_0 = \Pi^{-1}(\chi_0)$ and $\hat{x}_0 = \Pi^{-1}(\hat{\chi}_0)$. The canonical curves χ and $\hat{\chi}$ defined by $\chi(t) = (x(t), x(t - r(x(t))))$ and $\hat{\chi}(t) = (\hat{x}(t), \hat{x}(t - r(\hat{x}(t))))$ for all $t \in \mathbb{R}$, satisfy $\chi(0) = \chi_0$, $\hat{\chi}(0) = \hat{\chi}_0$, and $0 = x(0) = \hat{x}(0)$, $0 < x(-1) < \hat{x}(-1)$. By Proposition 4.4, $x_0|_{[-1,0]} \in H_{[-1,0]}$ and $\hat{x}_0|_{[-1,0]} \in H_{[-1,0]}$. Thus $\dot{x}(0) > 0$, $\dot{\hat{x}}(0) > 0$, and for the smallest positive zero z_1 and $\hat{z}_1$ of x and $\hat{x}$, respectively, we have $0 < x$ in $(0, z_1)$, $0 < \hat{x}$ in $(0, \hat{z}_1)$, $\dot{x}(z_1) < 0$, $x(z_1 - 1) < 0$, $\dot{\hat{x}}(z_1) < 0$, $\hat{x}(\hat{z}_1 - 1) < 0$.

The restriction $\chi|_{[0,z_1]}$ and the line segment $\lambda : [0,1] \ni s \mapsto \chi_0 + s(c(\chi_0) - \chi_0) \in \mathbb{R}^2$ from $\chi(0) = \chi_0 = \lambda(0)$ to $\chi(z_1) = c(\chi_0) = \lambda(1)$ form a simple closed curve ζ. The set $\{(u,v) \in \mathbb{R}^2 : u < 0, \text{ or } u = 0 \text{ and } v < x(z_1 - 1), \text{ or } u = 0 \text{ and } x(-1) < v\}$ belongs to $\text{ext}(\zeta)$. In particular $\hat{\chi}(0) \in \text{ext}(\zeta)$.

We show $\hat{\chi}([0, \hat{z}_1]) \cap \chi([0, z_1]) = \emptyset$. Otherwise there exit $\hat{t} \in (0, \hat{z}_1]$ and $t \in (0, z_1]$ with $\Pi \hat{x}_{\hat{t}} = \hat{\chi}(\hat{t}) = \chi(t) = \Pi x_t$. By the injectivity of Π, $\hat{x}_{\hat{t}} = x_t$, which implies $\hat{x}(s) = x(s + t - \hat{t})$ for all $s \in \mathbb{R}$. In case $\hat{t} < t$, $0 = \hat{x}(0) = x(t - \hat{t})$, a contradiction to $0 < x$ in $(0, z_1)$. In case $t < \hat{t}$, $0 = x(\hat{t} - t + t - \hat{t}) = \hat{x}(\hat{t} - t)$, a contradiction to $0 < \hat{x}$ in $(0, \hat{z}_1)$. Therefore $\hat{x} = x$; in particular $\hat{x}_0 = x_0$, and $\hat{\chi}_0 = \chi_0$, in contradiction to

$\chi_0 \prec \hat{\chi}_0$. Using $0 < \hat{x}$ in $(0, \hat{z}_1)$ and $\hat{x}(\hat{z}-1) < 0$, we infer $\hat{\chi}([0,\hat{z}_1]) \cap \lambda([0,1]) = \emptyset$. Then $\hat{\chi}([0,\hat{z}_1]) \cap |\zeta| = \emptyset$. Since $\hat{\chi}_0 \in \text{ext}(\zeta)$, we obtain $\hat{\chi}([0,\hat{z}_1]) \subset \text{ext}(\zeta)$. The injectivity of c yields $c(\hat{\chi}_0) \neq c(\chi_0)$. Therefore, we conclude that $c(\hat{\chi}_0) \prec c(\chi_0)$.

In case $\chi_0 \in v_-$ and $\hat{\chi}_0 \in v_-$ one can proceed analogously.

The proof of (ii). Let a trajectory $(\chi_n)_{-\infty}^{\infty}$ of ρ in $v_+ \cap \Pi(\overline{W \cap S})$ be given. The statement in (i) implies that the second components form a monotone sequence in $\mathbb{R}$. As $\overline{W \cap S}$ is compact and Π is continuous, the set $\Pi(\overline{W \cap S})$ is compact in $\mathbb{R}^2$. It follows that there exist χ_- and χ_+ in $\overline{v}_+ \cap \Pi(\overline{W \cap S})$ such that

$$\chi_n \to \chi_- \text{ as } n \to -\infty \text{ and } \chi_n \to \chi_+ \text{ as } n \to \infty.$$

Observe $\overline{v}_+ = v_+ \cup \{(0,0)\}$.

If $\chi_- \neq 0$ then $\rho(\chi_-) = \chi_-$ by the continuity of ρ. From the definition of ρ and the injectivity of Π, we conclude that the solution through $\Pi(\chi_-)$ is a nontrivial periodic solution of Eq. (1.1). The argument in case $\chi_+ \neq 0$ is the same.

Now consider the canonical curve $\chi : \mathbb{R} \ni t \mapsto \Pi x_t^{\Pi^{-1}(\chi_0)} \in \mathbb{R}^2$. The definition of ρ implies that there is a sequence $(s_n)_{-\infty}^{\infty}$ such that $\chi(s_n) = \chi_n$ and $1 < s_{n+1} - s_n < 2$ for all $n \in \mathbb{Z}$. By the continuity of Π, we have

$$x_{s_n}^{\Pi^{-1}(\chi_0)} = \Pi^{-1}(\chi_n) \to \Pi^{-1}(\chi_+) \text{ as } n \to \infty.$$

Let $\epsilon > 0$. By the continuous dependence of solutions of Eq. (1.1) on the initial data, there is $\delta > 0$ such that $\phi \in C$ and $\| \phi - \Pi^{-1}(\chi_+)\| < \delta$ imply $\sup_{t\in[0,T]} \inf_{s\in[0,2]} \| x_t^{\Pi^{-1}(\chi_+)} - x_s^{\phi}\| < \epsilon$, where T denotes the minimal period of the solution $x^{\Pi^{-1}(\chi_+)}$. That is

$$\text{dist}(\mathcal{O}_{\Pi^{-1}(\chi_+)}, \{x_s^{\phi} : 0 \le s \le 2\}) < \epsilon.$$

Choose $n_0 \in \mathbb{N}$ so that for $n_0 \le n \in \mathbb{N}$,

$$\| \Pi^{-1}(\chi_n) - \Pi^{-1}(\chi_+)\| < \delta.$$

Thus, for $n_0 \le n \in \mathbb{N}$

$$\text{dist}\left(\mathcal{O}_{\Pi^{-1}(\chi_+)}, \left\{x_s^{\Pi^{-1}(\chi_n)} : 0 \le s \le 2\right\}\right) < \epsilon, \text{ that is}$$
$$\text{dist}\left(\mathcal{O}_{\Pi^{-1}(\chi_+)}, \left\{x_t^{\Pi^{-1}(\chi_0)} : s_n \le t \le 2 + s_n\right\}\right) < \epsilon.$$

Since $1 < s_{n+1} - s_n < 2$ for all $n \in \mathbb{Z}$, it follows that

$$\text{dist}\left(\mathcal{O}_{\Pi^{-1}(\chi_+)}, x_t^{\Pi^{-1}(\chi_0)}\right) < \epsilon \quad \text{for all } t \ge s_{n_0}.$$

The proof is complete. □

Theorem 5.4.

(i) There is a periodic solution $p : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) such that $p_t \in \overline{W \cap S}$ for all $t \in \mathbb{R}$. The minimal period ω of p satisfies $1 < \omega < 2$.

(ii) For each $\phi \in W \cap S \setminus \{0\}$ the phase curve $\mathbb{R}_+ \ni t \mapsto F(t, \phi) \in Y$ tends to the periodic orbit $\mathcal{O} = \{p_t : t \in [0, \omega]\}$ as $t \to \infty$.

Proof. The proof of (i). Let $\psi \in W \cap S \setminus \{0\}$. By the invariance of $W \cap S \setminus \{0\}$ and Proposition 4.4, there is $s \geq 0$ with $x_s^\psi \in \mathcal{H}_+ \cap W \cap S$.

Let $\chi_0 = \Pi x_s^\psi = (0, x^\psi(s-1))$, and consider the trajectory $(\chi_n)_{-\infty}^{\infty}$ of ρ in $v_+ \cap \Pi(W \cap S)$. The definition of W yields $x_t^\psi \to 0$ as $t \to -\infty$. This fact implies $\chi_n \to 0$ as $n \to -\infty$. Note that $\chi_0 \neq \rho(\chi_0)$. Otherwise the solution x^ψ of Eq. (1.1) is periodic, which contradicts $x_t^\psi \to 0$ as $t \to -\infty$. Therefore, using Proposition 5.3(i), the sequence $(\chi_n)_{-\infty}^{\infty}$ is monotone. Moreover it is strictly increasing since $\chi_n \to 0$ as $n \to -\infty$. Then $\chi_+ = \lim_{n \to \infty} \chi_n$ satisfies $\chi_+ \in v_+ \cap \Pi(\overline{W \cap S})$. From Proposition 5.3(ii) it follows that the solution $p : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $p_0 = \Pi^{-1}(\chi_+)$ is periodic. Let $\omega > 0$ be its minimal period. As $\chi_+ \in v_+ \cap \Pi(\overline{W \cap S})$ and $\overline{W \cap S}$ is invariant, we obtain $p_0 \in \mathcal{H}_+$ and $p_t \in \overline{W \cap S}$ for all $t \in \mathbb{R}$. The statement about the minimal period ω is a consequence of Proposition 4.4 and $\rho(\chi_+) = \chi_+$.

The proof of (ii). Let $\phi \in W \cap S \setminus \{0\}$. There is $\hat{s} \geq 0$ with $x_{\hat{s}}^\phi \in \mathcal{H}_+ \cap W \cap S$. Let $\hat{\chi}_0 = \Pi x_{\hat{s}}^\phi$ and consider the trajectory $(\hat{\chi}_n)_{-\infty}^{\infty}$ of ρ in $v_+ \cap \Pi(W \cap S)$. In the same way as for $(\chi_n)_{-\infty}^{\infty}$, it follows that $\hat{\chi}_n \to 0$ as $n \to -\infty$, and $(\hat{\chi}_n)_{-\infty}^{\infty}$ is a strictly increasing sequence. The monotonicity of ρ and $\lim_{n \to -\infty} \chi_n = \lim_{n \to -\infty} \hat{\chi}_n = 0$ imply that there is an integer k such that either

$$\chi_n = \hat{\chi}_{n+k} \quad \text{for all } n \in \mathbb{Z}, \text{ or}$$

$$\chi_n \prec \hat{\chi}_{n+k} \prec \chi_{n+1} \prec \hat{\chi}_{n+k+1} \quad \text{for all } n \in \mathbb{Z}.$$

In both cases $\lim_{n \to \infty} \hat{\chi}_n = \lim_{n \to \infty} \chi_n = \chi_+$. Then Proposition 5.3(ii) yields $F(t, \phi) \to \mathcal{O}$ as $t \to \infty$. The proof is complete. □

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Summary

The theory of functional differential equations deals with differential equations where the right hand sides depend on delayed arguments of the unknown function. The first examples appeared about 200 years ago and were related to geometric problems. The interest in the field grew rapidly in the second half of the 20th century. Over the past several years it has become apparent that there is a need for a theory of differential equations with state-dependent delay because such equations appear in several applications (in classical electrodynamics, in population models, in models of blood cell production).

Equations with state-dependent delay in the derivative, that is the state-dependent neutral equations are also used in applications though we still do not have a general theory for such equations.

Consider the differential equation with state-dependent delay

$$\dot{x}(t) = -\mu x(t) + f(x(t - r(x(t)))),$$

where f and r are smooth real functions. Let $h > 0$. The function

$$C([-h,0],\mathbb{R}) \ni \phi \mapsto -\mu\phi(0) + f(\phi(-r(\phi(0)))) \in \mathbb{R}$$

is in general not differentiable. Therefore the basic tools of dynamical systems theory, like linearization and local invariant manifolds, cannot be applied in a straightforward way. This shows the main source of difficulties of the study of differential equations with state-dependent delay.

In this work we prove results for two different classes of functional differential equations with state-dependent delay. We use the notations: $C = C([-h,0],\mathbb{R})$; if $x : I \to \mathbb{R}$ is a continuous function on the interval I and t, $t - h \in I$, then $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $s \in [-h,0]$.

In Chapter 2 we consider the nonlinear one-dimensional neutral differential equation with state-dependent delay

$$(1) \qquad \frac{d}{dt}\left[x(t) - px(t - r(t,x_t))\right] = -q(t)\,x(t - s(t,x_t)),$$

where $p \in \mathbb{R}$, $q : [0,\infty) \to [q_0,\infty)$, $r : [0,\infty) \times C \to [0,r_0]$, and $s : [0,\infty) \times C \to [0,s_0]$ with $q_0 \geq 0$ and r_0, $s_0 \leq h$. Using the parameters of the equation, conditions are given for the stability and attractivity of the zero solution of Eq. (1).

When $p = 0$ and the function s is independent of x_t then according to a result of Myshkis, Yorke and Lillo, the assumption $q_0 s_0 \leq \frac{3}{2}$ implies the uniform stability

of the zero solution, and $\frac{3}{2}$ is the best possible constant. The number $\frac{3}{2}$ also arises as an upper bound in our stability condition for Eq. (1).

A consequence of the main result of this chapter is the following:

$$\text{if } 0 \le p < 1, \quad \frac{1-p}{q_0} < s_0 \text{ and } \frac{q_0 s_0}{1-p} + \frac{q_0 r_0 p}{(1-p)^2} \le \frac{3}{2}$$

then the zero solution of Eq. (1) is stable.

In Chapter 3 we prove a result of monotone dynamical systems applicable for the differential equation with state-dependent delay

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)), \quad r = r(x(t)), \tag{2}$$

where $\mu > 0$, f and r are smooth real functions with $f(0) = 0$ and $f' > 0$.

In the case $r \equiv$ constant the semiflow generated by Eq. (2) satisfies a certain monotonicity condition, that is, it is strongly order preserving. Therefore applying a result of Smith and Thieme, we conclude that the ω-limit set of all points from an open dense subset of the phase space is an equilibrium point.

In the case $r = r(x(t))$ the semiflow generated by Eq. (2) in the phase space $X \subset C$ of Lipschitz continuous functions is monotone, but it is not strongly order preserving. Thus the result of Smith and Thieme is not applicable. We prove a convergence result under a weaker monotonicity condition than the strong order preserving property, and we show that under certain hypotheses on r the ω-limit set of all points from an open dense subset of the phase space is an equilibrium point.

Note that, it is not true in general that the ω-limit set of every point of the phase space is an equilibrium point. Krisztin, Walther and Wu have shown the existence of periodic orbits in the case $r \equiv 1$ for certain μ, f, and r. A similar result is proved by Mallet-Paret and Nussbaum, Kuang and Smith, Arino, Hadeler and Hbid, Krisztin and Arino, Walther in the state-dependent delay case with a negative feedback condition. For the case $r = r(x(t))$ with a positive feedback condition Chapter 4 contains an analogous result.

In Chapter 4 we show that there is a nontrivial periodic orbit of Eq. (2). First an unstable set W of zero is constructed by forward extension of a local unstable manifold at zero. Then it is proved that for each nonzero $\phi \in W$, for which the solution x^ϕ through ϕ oscillates on $[0, \infty)$, $x^\phi(t) \to 0$ as $t \to -\infty$, and x_t^ϕ tends to the periodic orbit as $t \to \infty$. Moreover, x^ϕ has one or two sign changes on the interval $[t - r(x^\phi(t)), t]$ for all $t \in \mathbb{R}$.

Összefoglaló

A retardált funkcionál-differenciálegyenletek olyan folyamatokat modelleznek, amelyek változására múltbeli állapotaik is hatással vannak. Az első példák ilyen egyenletekre geometriai problémákban jelentek meg mintegy 200 éve. Az utóbbi ötven évben rohamosan nőtt az érdeklődés a funkcionál-differenciálegyenletek elmélete iránt. Az egyre szélesebb körű alkalmazások hatására szükségessé vált az állapotfüggő retardálású funkcionál-differenciálegyenletek elméletének kidolgozása. Az állapotfüggő neutrális differenciálegyenletek, azaz az olyan differenciálegyenletek amelyekben a derivált is tartalmaz a rendszer állapotától függő retardálást, szintén gyakorlati alkalmazással bírnak, annak ellenére, hogy ezekre az egyenletekre még nincs egy általánosan kidolgozott elmélet.

Tekintsük az

$$\dot{x}(t) = -\mu x(t) + f(x(t - r(x(t))))$$

állapotfüggő retardálású differenciálegyenletet, ahol $\mu > 0$ és f, r sima valós függvények. Legyen $h > 0$. A

$$C([-h, 0], \mathbb{R}) \ni \phi \mapsto -\mu\phi(0) + f(\phi(-r(\phi(0)))) \in \mathbb{R}$$

leképezés általában nem differenciálható. Ezért a dinamikus rendszerek elméletének általános eszközei mint a linearizálás, invariáns sokaságok nem alkalmazhatók a szokásos módon. Mindez az állapotfüggő retardálású differenciálegyenletek tanulmányozásában felmerülő nehézségekre utal.

Az értekezésben az állapotfüggő retardálású funkcionál-differenciálegyenletek két különböző osztályára bizonyítunk eredményeket. Bevezetjük az alábbi jelöléseket: $C = C([-h, 0], \mathbb{R})$; ha $x : I \to \mathbb{R}$ az I intervallumon folytonos függvény és t, $t-h \in I$, akkor az $x_t \in C$ függvény az $x_t(s) = x(t+s)$ képlettel definiált minden $s \in [-h, 0]$ esetén.

A 2. fejezetben az alábbi nemlineáris állapotfüggő neutrális differenciálegyenletet tekintjük:

$$\frac{d}{dt}\left[x(t) - px(t - r(t, x_t))\right] = -q(t)\, x(t - s(t, x_t)), \tag{1}$$

ahol $p \in \mathbb{R}$, $q : [0, \infty) \to [q_0, \infty)$, $r : [0, \infty) \times C \to [0, r_0]$, $s : [0, \infty) \times C \to [0, s_0]$, $q_0 \geq 0$ és r_0, $s_0 \leq h$. Az egyenletben adott paraméterek azon tartományát becsüljük, ahol az (1) egyenlet $x = 0$ megoldása stabil.

Ha $p = 0$ és az s retardálás nem függ az x_t-től, akkor Myshkis, Yorke és Lillo egy jól ismert eredménye alapján a $q_0 s_0 \leq \frac{3}{2}$ feltétel teljesülése esetén az

$x = 0$ megoldás egyenletesen stabil, és $\frac{3}{2}$ nem helyettesíthető nagyobb számmal. Kimutatjuk, hogy az (1) egyenletre is érvényes egy ún. $\frac{3}{2}$-es stabilitási tétel.

A fejezet fő eredményének egy következménye alapján,

$$\text{ha } 0 \le p < 1, \; \frac{1-p}{q_0} < s_0 \text{ és } \frac{q_0 s_0}{1-p} + \frac{q_0 r_0 p}{(1-p)^2} \le \frac{3}{2},$$

akkor az (1) egyenlet $x = 0$ megoldása stabil.

A 3. fejezetben a monoton dinamikus rendszerekre vonatkozó olyan eredményt bizonyítunk, amely alkalmazható az alábbi állapotfüggő retardálású differenciálegyenletre:

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)), \quad r = r(x(t)), \tag{2}$$

ahol $\mu > 0$, f és r sima valós függvények, $f(0) = 0$ és $f' > 0$.

Az $r \equiv$ konstans esetben a (2) egyenlet által generált szemidinamikus rendszer rendelkezik egy bizonyos monotonitási tulajdonsággal: erősen rendezéstartó. Így Smith és Thieme egy eredménye alapján a fázistér egy nyitott és sűrű halmazához tartozó pontok ω-limesz halmaza egy egyensúlyi helyzetből áll.

Az $r = r(x(t))$ esetben a Lipschitz folytonos függvények $X \subset C$ fázisterén a (2) egyenlet által generált szemidinamikus rendszer monoton ugyan, de nem erősen rendezéstartó. Így Smith és Thieme eredménye nem alkalmazható. Az erős rendezéstartásnál gyengébb monotonitási feltétel mellett bizonyítunk konvergencia eredményt, és megmutatjuk, hogy az r függvényre tett bizonyos feltételek teljesülése esetén, a fázistér egy nyitott és sűrű halmazához tartozó pontok ω-limesz halmaza egy egyensúlyi helyzetből áll. Megemlítjük, hogy a fázistér minden pontja nem rendelkezik azzal a tulajdonsággal, hogy az ω-limesz halmaza egy egyensúlyi helyzet. Krisztin, Walther és Wu kimutatta periodikus pályák létezését $r = 1$ esetben. Hasonló eredményt bizonyított Mallet-Paret és Nussbaum, Kuang és Smith, Arino, Hadeler és Hbid, Krisztin és Arino, Walther állapotfüggő késleltetés esetére egy negatív visszacsatolási feltétel mellett. Az $r = r(x(t))$ esetben egy pozitív visszacsatolási feltétellel a 4. fejezet tartalmaz hasonló eredményt.

A 4. fejezetben kimutatjuk a (2) egyenlet egy nemtriviális periodikus pályájának létezését. Előbb 0-nak egy W-vel jelölt instabil halmazát konstruáljuk meg a 0 egy lokális instabil sokaságának pozitív irányban való kiterjesztésével. Majd bebizonyítjuk, hogy minden olyan 0-tól különböző $\phi \in W$ esetén, amelyekből induló x^{ϕ} megoldások oszcillálnak a $[0, \infty)$ intervallumon, $x^{\phi}(t) \to 0$ $t \to -\infty$ esetén, és x_t^{ϕ} tart a periodikus pályához $t \to \infty$ esetén. Sőt az is igaz, hogy az x^{ϕ} megoldás előjelváltásainak a száma egy vagy kettő a $[t - r(x^\phi(t)), t]$ intervallumon minden $t \in \mathbb{R}$ esetén.