The dyadic Cesàro and Copson operators

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PhD thesis

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Introduction

The systematic investigations of the convergence problems of trigonometric Fourier series was started in the middle of the 19th century. Among others, the classical convergence tests of *Dirichlet* and *Dini* as well as the first results relating to the divergence of Fourier series were obtained. In 1876 *du Bois Raymond* observed that there exists a continuous function which cannot be reconstructed from its Fourier series by means of the convergence of the partial sums. More exactly, he presented a continuous function, whose Fourier series diverges at some point. This example gave rise to a number of attempts to find other reasonable methods for evaluating Fourier series. In 1900 *Lipót Fejér* proved that the first arithmetic means of the partial sums of the Fourier series of anz continuous function converge to the function even uniformly. The same convergence result was known before for the Abel-Poisson means.

In connection with Fejér's theorem the following question arises: What happens to the Fourier series if we form the arithmetic means of the Fourier coefficients instead of forming the arithmetic means of the partial sums of its Fourier series. It is known that in the case of a complete orthogonal system there is a one-to-one correspondence between the expanded functions and the sequence of their Fourier coefficients. Given a periodic integrable function f, in symbol: $f \in L^1_{2\pi}$, we form the sequence cosisting of the arithmetic means of its Fourier coefficients. The question is whether there exists a function $g \in L^1_{2\pi}$ whose Fourier coefficients coincide with these arithmetic means.

Hardy raised first this question in 1928 and proved that the $L^p_{2\pi}$ spaces for $1 \leq p < \infty$ are invariant with respect to this averaging process (see [24]). In other words, if $f \in L^p_{2\pi}$ for some $1 \leq p < \infty$, then the function g, whose Fourier coefficients coincide with the corresponding arithmetic means of the Fourier coefficients of f, also belongs to the same space $L^p_{2\pi}$. The operator which maps the function $f \in L^p_{2\pi}$ to the function $g \in L^p_{2\pi}$ is called the *Cessàro operator* acting on $L^p_{2\pi}$, while its adjoint is called the *Copson operator*. We note that the terminology is not consistent in the literature. The term "Hardy operator" is offen used in place of "Cesàro operator".

In the last few years, the study of these operators has come into fashion again. *R. Bellman*, *B. I. Golubov* ,*D. V. Giang*,*F. Móricz*, *V. Rodin*, *A. G. Sisakis*, *K. Sternpak* and others have proved a number of interesting theorems of these operators.

In our PhD thesis, we study the behavior of these operators in the case of Walsh-Fourier series. The methodsused here are essentially different from those used in the case of trigonometric series. In the classical case, integration by parts and ordinary derivative play base roles. In our investigations, the classical derivative is replaced by dyadic derivative, while the classical integral is replaced by dyadic integral (antiderivative). For example, an analouge of the integration by parts holds no longer in the dyadic setting. The dyadic integral is defined in terms of a convolution with a function which can be handled with some difficulity. As a result, the computations are more compliated than in the case of the classical integral. This fact explains that the proofs are longer and more technical for the Walsh series than those for the trigonometric ones.

We define the Cesàro and Copson operators on the largest possible subclass of Walsh series, which can be identified with the space of dyadic martingales. Then we consider the restrictions of these operators onto several important subspaces. A new representation of the Walsh-Dirichlet kernel plays a crucial role in our investigations, in which the notion of dzadic derivative is also used.

We deal with four versions of the dzadic Cesàro and Copson operators. Namely, the ones defined on the space of martingales, the ones defined on the one-dimensional and two-dimensional spaces of integrable functions, respectively, and their continuous extensions to the positive real half-axis. In the first three cases, the Cesàro operator is defined by means of the first arithmetic means of the Walsh-Fourier coefficients of the martingales or functions, respectively. In the fourth case, the Walsh-Fourier transform is substituted for the Walsh-Fourier coefficient. The functions in question are defined on the unit interval $\mathbb{I} := \mathbb{I}^1 := [0,1)$, on the unit square $\mathbb{I}^2 := \mathbb{I} \times \mathbb{I}$, and on the positive real half-axis $\mathbb{R}_+ := [0,+\infty)$, respectively. These operators can be represented in the form of integral operators on the corresponding L^1 spaces, which are a special cases of the so-called local convolution operators introduced in Chapter 3. Beside the Cesàro operator, we also study these local convolution operators on the spaces $L^1(\mathbb{I}^j)$ $(j = 1,2)$, $L^1(\mathbb{R}_+)$ as well as on the spaces L^p, Hardy and BMO. We formulate those properties of the local convolution operators, from which the boundedness of these operators follows on the spaces mentioned above.

Dyadic Cesàro and Copson operators on L^p spaces

First of all we introduce the *dyadic Cesàro* and *Copson operators*. Where it is possible, we treat the one-dimensional case along with the two-dimensional case. For the proofs of the theorems of this chapter, see [11], [12], [13]. The operators mentioned above are defined on the broadest possible space, on the space of the dyadic martingales $\mathcal{M}_0(\mathbb{I}^j)$. In the case $j = 1$, set

$$\mathcal{M}_0(\mathbb{I}) := \{F \in \mathcal{M}(\mathbb{I}) : \hat{F}(0) = 0\}, \tag{3.1}$$

and in the case $j = 2$ set

$$\mathcal{M}_0(\mathbb{I}^2) := \{F \in \mathcal{M}(\mathbb{I}^2) : \hat{F}(k,\ell) = 0 \quad k,\ell \in \mathbb{N}, \min(k,\ell) = 0\}. \tag{3.2}$$

The Walsh-Fourier coefficients of the dyadic martingale $F = (f_n, n \in \mathbb{N}^j)$ $(j = 1,2)$ are defined by the limit

$$\hat{F}(k) = \lim_{\min(n_1,n_2)\to\infty} \int_0^1 f_n w_k dx \quad (k \in \mathbb{N}^j).$$

It is easy to see that if $k < 2^{n_0}$ and $n \geq n_0$, then

$$\int_0^1 f_n(x) w_k(x) dx = \int_0^1 f_{n_0}(x) w_k(x) dx,$$

consequently, this limit exists and

$$\hat{F}(k) = \int_0^1 f_{n_0}(x) w_k(x) dx \quad (k < 2^{n_0}).$$

In the two-dimensional case, the existence of the limit can be justified similarly. The Walsh series of the dyadic martingale F is denoted by the symbol

(3.3)
$$f \sim S(F) := \sum_{k \in \mathbb{N}^j} \hat{F}(k) w_k \quad (j = 1, 2).$$

Since there is a one-to-one correspondence between the Walsh-series, the coefficient sequences, and the dyadic martingales, for all dyadic martingales $F \in \mathcal{M}_0(\mathbb{I}^j)$ there is exactly one dyadic martingale $G \in \mathcal{M}_0(\mathbb{I}^j)$ such that the Walsh-Fourier coefficients of which are the arithmetic mean of the Walsh-Fourier coefficients of the dyadic martingale F, namely for which

(3.4)
$$\hat{G}(n) := 0 \ (n \in \mathbb{N}^j, \wedge n = 0), \ \hat{G}(n) := \frac{1}{n^*} \sum_{k=0}^{n-1} \hat{F}(k) \quad (n \in \mathbb{P}^j),$$

where $n^* = n_1 \cdot n_2$, $\wedge n = \min(n_1, n_2)$, if $n \in \mathbb{P}^2$, and $\wedge n = n$, if $n \in \mathbb{P}$ $\mathbb{P} = 1, 2, \ldots$. The operator $\widetilde{C} : \mathcal{M}_0(\mathbb{I}^j) \to \mathcal{M}_0(\mathbb{I}^j)$, which is defined by the instruction

(3.5)
$$\widetilde{C}F := G \quad (F \in \mathcal{M}_0(\mathbb{I}^j))$$

is called *dyadic Cesàro operator*. It is obvious that in this manner we defined a linear operator on $\mathcal{M}_0(\mathbb{I}^j)$.

We interpret the adjoint operator $\widetilde{C}^*$ of the operator $\widetilde{C}$ in the first step on the space of the stationary martingales $\mathcal{M}_S(\mathbb{I}^j)$, that is, we consider such martingales, whose terms are constant from a certain index (there is $n_0 \in \mathbb{N}$ such that $f_n = f_{n_0}$, for $n > n_0$). We define the martingale $G := \widetilde{C}^* F$ $(F \in \mathcal{M}_S(\mathbb{I}^j))$ in such a way that its Walsh coefficients are

(3.6)
$$\hat{G}(n) := \sum_{k : n < k} \frac{\hat{F}(k)}{k^*} \quad (n \in \mathbb{N}^j).$$

Since $\hat{F}(k) = 0$ except of finitely many index $k \in \mathbb{N}^j$ hold, the above mentioned sum makes sense. The mapping $\widetilde{C}^*$ is called the *Copson operator*. We introduce the bilinear functional (2.1.7)

$$\langle F, G \rangle = \sum_{k=0}^{\infty} \hat{F}(k) \hat{G}(k) = \lim_{n \to \infty} \int_0^1 f_n g_n \quad (F = (f_n, n \in \mathbb{N}) \in \mathcal{M}_0, G = (g_n, n \in \mathbb{N}) \in \mathcal{M})$$

on the space $\mathcal{M}(\mathbb{I}^j) \times \mathcal{M}_0(\mathbb{I}^j)$. The name adjoint operator is explained by the following

Lemma 3.1. *The adjoint operator of the Cesàro operator* $\widetilde{C}$ *is the Copson operator* $\widetilde{C}^*$ *defined under* (3.6) *with respect to the bilinear functional* (2.1.7), *namely*

$$\langle \widetilde{C}F, G\rangle = \langle F, \widetilde{C}^*G\rangle \quad (F \in \mathcal{M}_0(\mathbb{I}^j), G \in \mathcal{M}_S(\mathbb{I}^j)). \tag{3.7}$$

We study the operators $\widetilde{C}$ and $\widetilde{C}^*$ on several subspaces of the dyadic martingales. We give first an integral form of the Cesàro operator by showing that the martingale $\widetilde{C}F = G = (g_n, n \in \mathbb{N}^j)$ is of the form

$$g_n(x) = \int_{\mathbb{I}^j} f_n(t)K_n^{\langle j\rangle}(x,t)\,dt \quad (x \in \mathbb{I}^j, F = (f_n, n \in \mathbb{N}^j) \in \mathcal{M}_0(\mathbb{I}^j))m, \tag{3.8}$$

where the kernel function $K_n^{\langle j\rangle}$ can be expressed in terms of the modified dyadic difference operator. Namely, denote by χ_s the characteristic function of the interval $J_s := [2^{-s}, 2^{-s+1})$ $(s \in \mathbb{P}^j)$ and let

$$W_n^{\langle j\rangle} := S_{2^n}(W^{\langle j\rangle}) \quad (n \in \mathbb{N}^j) \tag{3.9}$$

be the 2^nth partial sum of the series

$$W^{\langle j\rangle} := \sum_{k\in\mathbb{P}^j} \frac{w_k}{k^*} \quad (j = 1, 2).$$

We show that the kernel function $K_n^{\langle j\rangle} \in L^1(\mathbb{I}^j \times \mathbb{I}^j)$ is of the form

$$K_n^{\langle j\rangle}(x,t) := \sum_{s\in\mathbb{P}^j} \chi_s(t)(\Delta_{s-1}^- W_n^{\langle j\rangle})(x \dotplus t) \quad (x,t \in \mathbb{I}^j), \tag{3.10}$$

where in the case $j = 1$ the modified dyadic one-parameter difference operator is defined by

$$(\Delta_n^- f)(x) := \sum_{k=0}^{n-1} 2^{k-1}(f(x) - f(x \dotplus 2^{-k-1})) - 2^{n-1}(f(x) - f(x \dotplus 2^{-n-1})).$$

(For the definition of the two-parameter operator, see [3], [12].) We denote the dyadic addition by the symbol $\dotplus$. Since the supports of the functions χ_s are disjoint, the sum in (3.10) converges absolutely at all points $(x,t) \in \mathbb{I}^j \times \mathbb{I}^j$. It is not difficult to verify that the sum in (3.10) is convergent in the $L^1(\mathbb{I}^j \times \mathbb{I}^j)$-norm, and that $K_n^{\langle j\rangle} \in L^1(\mathbb{I}^j \times \mathbb{I}^j)$.

Beside the kernel functions just introduced we shall use the function

$$K^{\langle j\rangle}(x,t) := \sum_{s\in\mathbb{P}^j} \chi_s(t)(\Delta_{s-1}^- W^{\langle j\rangle})(x \dotplus t) \quad (x,t \in \mathbb{I}^j). \tag{3.10'}$$

Similarly to that what we have said above, we have $K^{\langle j\rangle} \in L^1(\mathbb{I}^j)$. Motivated by these we define the integral operators

$$(\mathcal{K}_n^{\langle j\rangle} h)(x) := \int_{\mathbb{I}^j} h(t)(K_n^{\langle j\rangle}(x,t)\,dt, \quad (\mathcal{K}^{\langle j\rangle} h)(x) := \int_{\mathbb{I}^j} h(t)K^{\langle j\rangle}(x,t)\,dt \tag{3.11}$$

$(x \in \mathbb{I}^j, h \in L_0^1(\mathbb{I}^j), j = 1, 2)$, where $L_0^1(\mathbb{I}^j)$ consists of the functions in $L^1(\mathbb{I}^j)$, whose integral is zero over the interval $\mathbb{I}^j$.

Lemma 3.2.. *The Cesàro transform* $\widetilde{C}F := G = (g_n, n \in \mathbb{N})$ *of the dyadic martingale* $F = (f_n, n \in \mathbb{N}^j) \in \mathcal{M}_0(\mathbb{I}^j)$ *can be represnted in the form*

$$g_n = \mathcal{K}_n^{\langle j \rangle} f_n \quad (n \in \mathbb{N}^j). \tag{3.12}$$

The operators $\mathcal{K}_n^{\langle j \rangle} : L^1(\mathbb{I}^j) \to L^1(\mathbb{I}^j)$ $(n \in \mathbb{N}^j)$ *are uniformly bounded, and the operator* $\mathcal{K}^{\langle j \rangle} : L^1(\mathbb{I}^j) \to L^1(\mathbb{I}^j)$ *is bounded, that is, there exists a constant* $C > 0$ *such that*

$$\|\mathcal{K}_n^{\langle j \rangle} h\|_1 \leq C\|h\|_1 \quad \text{and} \quad \|\mathcal{K}^{\langle j \rangle} h\|_1 \leq C\|h\|_1 \quad (h \in L^1(\mathbb{I}^j), n \in \mathbb{N}^j). \tag{3.13}$$

In the following we consider the Cesàro operator on the subspace of the space of the L^1-bounded martingales:

$$\mathcal{M}^1(\mathbb{I}^j) := \{F = (f_n, n \in \mathbb{N}^j) : \sup_{n \in \mathbb{N}^j} \|f_n\|_1 < \infty\}.$$

This subspace can be identified with the space of the functions of bounded variation on the interval $\mathbb{I}^j$, and the corresponding Walsh series can be identified with the Walsh-Fourier-Stielties series of the functions of bounded variation.

Theorem 3.1. *The restriction of the Cesàro operator* $\widetilde{C}$ *onto the space of the* L^1*-bounded martingales is a bounded linear operator on* $\mathcal{M}^1(\mathbb{I}^j)$.

Taking into account the connection between the spaces $\mathcal{M}^1(\mathbb{I}^j)$ and $\mathcal{BV}(\mathbb{I}^j)$, we obtain the following

Corollary 3.1. *Let* $\Phi \in \mathcal{BV}(\mathbb{I}^j)$ *be a function of bounded variation, and*

$$a_k := \int_{\mathbb{I}^j} w_k \, d\Phi \ (k \in \mathbb{N}^j)$$

the Walsh-Fourier-Stieltjes coefficients of Φ. *Then there exists a function* $\Psi \in \mathcal{BV}(\mathbb{I}^j)$ *of bounded variation such that for the Walsh-Fourier-Stieltjes coefficients we have*

$$\int_{\mathbb{I}^j} w_k \, d\Psi = \frac{1}{k^*} \sum_{0 \leq \ell < k} a_\ell \quad (k \in \mathbb{P}^j).$$

In the following we shall restrict the domain of the Cesàro operator. We proved in Lemma 3.1 that the integral operator $\mathcal{K}^{\langle j \rangle} : L^1(\mathbb{I}^j) \to L^1(\mathbb{I}^j)$, which is defined by means of the kernel function $K^{\langle j \rangle}$ (see (3.10')), is bounded. We show that this operator is precisely the Cesàro operator on the space $L^1_0(\mathbb{I}^j)$.

Theorem 3.2. *The operator* $\mathcal{K}^{\langle j \rangle} : L^1_0(\mathbb{I}^j) \to L^1(\mathbb{I}^j)$ *is identical with the restriction of the operator* $\widetilde{C}$ *onto the space* $L^1_0(\mathbb{I}^j)$, *that is,*

$$(\widehat{\mathcal{K}^{\langle j \rangle} f})(k) = \frac{1}{k^*} \sum_{\ell < k} \hat{f}(\ell) \quad (k \in \mathbb{P}^j, f \in L^1_0(\mathbb{I}^j), j = 1, 2). \tag{3.2.1}$$

There is a strong connection between the one-parameter Cesàro operator $\widetilde{C}$, which is defined on the space of martingales, and the Cesàro operator C, which is defined on the space of integrable functions. since for all martingales $F = (f_n, n \in \mathbb{N})$ we have

$$(\widetilde{C}F)_n = E_n(Cf_n) \quad (n \in \mathbb{N}), \tag{3.2.3}$$

where

$$(E_n f)(x) := \frac{1}{|I_n(x)|} \int\limits_{I_n(x)} f(t)dt \quad (x \in \mathbb{I}, n \in \mathbb{N})$$

is the conditional expectation of f. Here $I_n(x)$ denotes the dyadic interval of length 2^{-n}, which contains x. The following notation is suggested by (3.2.3). Let $\Phi : L^1(\mathbb{I}) \to L^1(\mathbb{I})$ be bounded linear operator. We call the operator $\widetilde{\Phi}$ defined on the spaces of martingales for which

$$(\widetilde{\Phi}F)_m := E_m(\Phi f_m) \quad (m \in \mathbb{N}, F = (f_n, n \in \mathbb{N}) \in \mathcal{M}) \tag{3.2.4}$$

holds, the *diagonal extension* of Φ to $\mathcal{M}$.

We studied the following problem: under what conditions imposed on the diagonal extension $\widetilde{\Phi}$, we obtain dyadic martingales from diagonal martingales. In connection with this we introduce the following notion. We say that the operator $\Phi : L^1(\mathbb{I}) \to L^1(\mathbb{I})$ is *spectrum-preserving* if $(\Phi f)\widehat{\ }(k) = 0$ $(k = 0, ..., 2^m - 1)$ follows from the conditions that $f \in L^1$, $m \in \mathbb{N}$ and $\hat{f}(k) = 0$ $(k = 0, ..., 2^m - 1)$.

It can be proved that if the operator Φ is spectrum-preserving, then the operator $\widetilde{\Phi}$ maps martingales to martingales, and the operator $\widetilde{\Phi}$ is the extension of the operator Φ from $L_0^1(\mathbb{I})$ to $\mathcal{M}_0(\mathbb{I})$. Indeed, if the function $f \in L^1$ is integrable, then $\widetilde{\Phi}F = (E_m(\Phi f), m \in \mathbb{N})$ (see [15]).

Starting with the properties of the Cesàro operator, we introduce a new class of operators, that of the *local convolution operators* which will be denoted by the symbol $\mathcal{N}^{\langle j \rangle}$ $(j = 1, 2)$. We define the elements of $\mathcal{N}^{\langle j \rangle}$ by a sequence of dyadic convolution operators $\Phi_n^{\langle j \rangle} f := f * \phi_n^{\langle j \rangle}$ $(n \in \mathbb{N}^j)$. We assume that the functions $\phi_k^{\langle j \rangle}$ $(k \in \mathbb{N})$ are integrable, and in the two-parameter case the functions in question are the Kronecker product of the one-parameter functions, namely,

$$\phi_n^{\langle 2 \rangle} = \phi_{n_1}^{\langle 1 \rangle} \times \phi_{n_2}^{\langle 1 \rangle} \quad (n = (n_1, n_2) \in \mathbb{N}^2). \tag{3.3.1}$$

The operators $\Phi^{\langle j \rangle} \in \mathcal{N}^{\langle j \rangle}$ is of the form

$$\Phi^{\langle j \rangle} f := \sum_{n \in \mathbb{P}^j} \Phi_n^{\langle j \rangle}(\chi_n f) \quad (f \in L_0^1(\mathbb{I}^j), j = 1, 2), \tag{3.3.2}$$

where χ_n $(n \in \mathbb{P}^j)$ is the characteristic function of the interval J_n.

The convolution operators $\Phi_n^{\langle j\rangle}$ map the class of the Walsh polynomials onto itself, and

$$\langle \Phi_n^{\langle j\rangle} f, g\rangle = \langle f, \Phi_n^{\langle j\rangle} g\rangle \quad (f \in L^1(\mathbb{I}^j), g \in \mathcal{P}^{\langle j\rangle}, n \in \mathbb{N}^j), \tag{3.3.3}$$

where

$$\langle f, g\rangle = \int_{\mathbb{I}^j} f(t)g(t)dt$$

denotes the usual inner product of f and g. The class $\mathcal{N}^{\langle j\rangle}$ of operators contains the convolution operators. Namely, if $\phi_1^{\langle j\rangle} = \ldots = \phi_n^{\langle j\rangle} = \ldots = \phi^{\langle j\rangle}$, then $\Phi^{\langle j\rangle} f = f * \phi^{\langle j\rangle}$.

The maximal operator of the sequence $(\Phi_n^{\langle j\rangle}, n \in \mathbb{N}^j)$ of operators is denoted by $\Phi_*^{\langle j\rangle}$, and defined by

$$\Phi_*^{\langle j\rangle} f := \sup_{n\in\mathbb{N}^j} |f| * |\phi_n^{\langle j\rangle}|. \tag{3.3.4}$$

It can be verified that a local convolution operator is spectrum-preserving, if the Walsh-Fourier coefficients $\hat{\phi}_j(k)$ $(k < 2^{j-1})$ are independent from j.

The following theorem applies to the operator $\Phi^{\langle j\rangle}$.

Theorem 3.3. *i) If for the sequence $\phi_n^{\langle j\rangle}$ $(n \in \mathbb{N}^j)$ of functions generating the operator $\Phi^{\langle j\rangle} \in \mathcal{N}^{\langle j\rangle}$ we have*

$$M := \sup_{n\in\mathbb{P}^j} \|\phi_n^{\langle j\rangle}\|_1 < \infty, \tag{3.3.5}$$

then $\Phi^{\langle j\rangle}$ is a bounded linear operator from $L^1(\mathbb{I}^j)$ to $L^1(\mathbb{I}^j)$, and

$$\|\Phi^{\langle j\rangle} f\|_1 \le M\|f\|_1 \quad (f \in L_0^1(\mathbb{I}^j)). \tag{3.3.6}$$

ii) Let $1 < p < \infty$, and $1/p + 1/p' = 1$. If $\Phi_*^{\langle j\rangle}$ is a bounded operator from $L^{p'}(\mathbb{I}^j)$ to $L^{p'}(\mathbb{I}^j)$, that is, if for some constant $M_p^* > 0$ we have

$$\|\Phi_*^{\langle j\rangle} g\|_{p'} \le M_p^* \|g\|_{p'} \quad (g \in L^{p'}(\mathbb{I}^j)), \tag{3.3.7}$$

then $\Phi^{\langle j\rangle}$ is a bounded linear operator from $L^p(\mathbb{I}^j)$ to $L^p(\mathbb{I}^j)$, and

$$\|\Phi^{\langle j\rangle} f\|_p \le M_p^* \|f\|_p \quad (f \in L^p(\mathbb{I}^j)). \tag{3.3.8}$$

If we apply this theorem for the Cesàro operator, then we get the following

Theorem 3.4.

i) *The Cesàro operator is a bounded linear operator from $L^p(\mathbb{I}^j)$ to $L^p(\mathbb{I}^j)$ if $1 \le p < \infty$.*

ii) *The Cesàro operator is not bounded from $L^\infty(\mathbb{I}^j)$ to $L^\infty(\mathbb{I}^j)$.*

We define the adjoint operator C^* of the Cesàro operator on the set of the Walsh-polynomials $\mathcal{P}(\mathbb{I}^j)$. Since

$$Cf \in L^1_0(\mathbb{I}^j) \quad \text{if} \quad f \in L^1_0(\mathbb{I}^j), \quad \text{and} \quad C^*g \in \mathcal{P}(\mathbb{I}^j), \quad \text{if} \quad g \in \mathcal{P}(\mathbb{I}^j),$$

by Lemma 3.1, (3.7) and (2.1.7) we get that

$$\langle f, C^*g\rangle = \langle Cf, g\rangle \quad (f \in L^1_0(\mathbb{I}^j), g \in \mathcal{P}(\mathbb{I}^j)). \tag{3.4.1}$$

Relying onthis and using the well-known duality principle, we show that the operator C^* can be extended from the subspace $\mathcal{P}(\mathbb{I}^j)$ to a bounded operator $C^* : L^p(\mathbb{I}^j) \to L^p(\mathbb{I}^j)$ if $1 < p < \infty$. In the case $p = \infty$ we consider the closure of the subspace $\mathcal{P}(\mathbb{I}^j)$ with respect to the L^∞-norm, instead of the space $L^\infty(\mathbb{I}^j)$, and we denote this space by $X^\infty(\mathbb{I}^j)$, and in case $0 < p < \infty$ let be $X^p(\mathbb{I}^j) := L^p(\mathbb{I}^j)$.

Theorem 3.5. *There exists a constant $C_p > 0$ depending only on p for all $1 < p \leq \infty$ such that*

$$\|C^*g\|_p \leq C_p\|g\|_p \quad (g \in \mathcal{P}(\mathbb{I}^j)). \tag{3.4.2}$$

The Copson operator can be extended from $\mathcal{P}(\mathbb{I}^j)$ to a bounded linear operator $C^* : X^p(\mathbb{I}^j) \to X^p(\mathbb{I}^j)$, such that

$$(\widehat{C^*g})(k) = \sum_{k<\ell} \frac{\hat{g}(\ell)}{\ell^*} \quad (g \in X^p(\mathbb{I}^j), k \in \mathbb{I}^j). \tag{3.4.3}$$

For the function $w_1 \in L^1(\mathbb{I})$, we have

$$(C^*w_1)\hat{}(k) = \sum_{k<\ell} \frac{1}{\ell} = \infty \quad (k \in \mathbb{N}).$$

This shows that the condition $p > 1$ in Theorem 3.5 can not be replaced by the condition $p \geq 1$.

The dyadic Cesàro and Copson operators on the dyadic Hardy- and BMO spaces

In this paragraph we consider the dyadic Cesàro operator on the dyadic Hardy space $H(\mathbb{I})$, and its adjoint operator, the dyadic Copson operator is considered on the closure of the space of the dyadic step functions with respect to the BMO-norm, on the space VMO($\mathbb{I}$). We show that these operators are bounded. These statements follow from those statements that concern the local convolution operators, which are the consequences of the properties of the kernel functions. For the proofs of the theorems of this chapter, see [11], [15].

We consider first the one-parameter operators introduced in the previous section, which are of the form (3.3.2) and belong to $\mathcal{N}^{\langle 1\rangle}$. In the following we also assume that the generating sequence $\phi_n := \phi_n^{\langle 1\rangle}$ $(n \in \mathbb{N})$ of the operator $\Phi := \Phi^{\langle 1\rangle}$ obeys the condition (3.3.5). Then

$$\Phi f = \sum_{n\in\mathbb{P}} (\chi_n f) * \phi_n, \tag{4.1.1}$$

and this series is convergent in L^1-norm. By Theorem 3.3 $\Phi : L^1(\mathbb{I}) \to L^1(\mathbb{I})$ is a bounded operator, and the inequality (3.3.6) holds. The following theorem is connected with the restriction of this operator onto the Hardy space $H(\mathbb{I}) \subset L^1(\mathbb{I})$.

Theorem 4.1. *Assume that for the generating sequence ϕ_n $(n \in \mathbb{N})$ of the operator $\Phi \in \mathcal{N}^{\langle 1\rangle}$ condition (3.3.5) holds. If one of the following three conditions*

i) $\hat{\phi}_n(k) = 0$ $(0 \le k < 2^n, n \in \mathbb{P})$,
ii) $\phi_n = D_{2^n}$ $(n \in \mathbb{P})$,
iii) $\phi_n = 2^n(S_{2^n}W - S_{2^{n-1}}W)$

holds, then Φ is a bounded linear operator from $H^1(\mathbb{I})$ to $H^1(\mathbb{I})$, and

$$\|\Phi f\|_{H^1} \le M_1 \|f\|_{H^1} \quad (f \in H^1(\mathbb{I})), \tag{4.1.2}$$

where M_1 is a constant, which depends on the constant M occurring in condition (3.3.6).

Applying this theorem yields the following

Theorem 4.2. *The dyadic Cesàro operator is bounded from $H^1(\mathbb{I})$ to $H^1(\mathbb{I})$, that is, there exists a $C > 0$ constant such that*

$$\|\mathcal{C}f\|_{H^1} \le C\|f\|_{H^1} \quad (f \in H^1(\mathbb{I})).$$

The BMO space is an intermediate one between the spaces L^p $(p < \infty)$ and L^∞:

$$L^\infty(\mathbb{I}) \subset \mathrm{BMO}(\mathbb{I}) \subset L^p(\mathbb{I}) \ (p < \infty).$$

We have seen that $\mathcal{C} : L^p(\mathbb{I}) \to L^p(\mathbb{I})$ is bounded if $p < \infty$, and is not bounded if $p = \infty$. It comes naturally the question: What can we say about the restriction of $\mathcal{C}$ onto the space BMO? The following theorem answers this problem.

Theorem 4.3. *The Cesàro operator is not bounded from the space VMO($\mathbb{I}$) to the space BMO($\mathbb{I}$).*

We start with the Walsh-series

$$\begin{aligned} f(x) &= a_1 \frac{w_1(x)}{2} + a_2 \frac{w_2(x) + w_3(x)}{2^2} + \cdots + a_n \frac{w_{2^{n-1}}(x) + \cdots + w_{2^n-1}(x)}{2^n} + \cdots \\ &= \sum_{n=1}^{\infty} a_n 2^{-n} \sum_{k=2^{n-1}}^{2^n-1} w_k(x), \end{aligned}$$

where $a_n = 1/\sqrt{n}$ $(n \in \mathbb{P})$. It can be verified that $f \in \mathrm{VMO}$, and $Cf \notin \mathrm{BMO}$.

It follows from duality considerations, similarly as in Theorem 3.5, that the dyadic Copson operator is bounded from $\mathrm{BMO}(\mathbb{I})$ to $\mathrm{BMO}(\mathbb{I})$, and is not bounded from $H(\mathbb{I})$ to $H(\mathbb{I})$.

In the following, we shall give sufficient conditions for the boundedness of the diagonal extensions of the local convolution operators on the dyadic Hardy spaces H^p $(1/2 < p \leq 1)$. At the same time, we present a new proof in the case $p = 1$ for Theorem 4.1.

Let $\phi = (\phi_n, n \in \mathbb{N})$ be a sequence of integrable functions. We introduce the following quasi-norm for $0 < p \leq 1$:

$$\|\phi\|_{\langle p \rangle} := \sup_{n \in \mathbb{N}} \left(\sum_{I \in \mathcal{I}_n} \left(\int_I |\phi_n(t)|\, dt \right)^p \right)^{1/p}. \tag{4.2.1}$$

In particular, if $p = 1$ and ϕ is a martingale, then $\|\phi\|_{\langle 1 \rangle}$ is equivalent to the usual $L^1(\mathbb{I})$-norm of ϕ.

Theorem 4.4. *Let $1/2 < p \leq 1$. Assume that $\widetilde{\Phi}$ is a diagonal extension of the local convolution operator Φ (see (3.2.4)), and for the generating sequence $\phi = (\phi_n, n \in \mathbb{N})$ one of the following conditions holds:*

(i) $\hat{\phi}_n(k) = 0 \quad (0 \leq k < 2^n, n \in \mathbb{N}), \quad \|\phi\|_{\langle p \rangle} < \infty;$
(ii) $\phi_n = D_{2^n}$ $(n \in \mathbb{P})$;
(iii) $\phi_n = 2^n(S_{2^n}W - S_{2^{n-1}}W)$.

Then $\widetilde{\Phi}$ is bounded on the space $H^p(\mathbb{I})$, that is, there exists a constant C_p depending only on p such that

$$\|\widetilde{\Phi} F\|_{H^p} \leq C_p \|F\|_{H^p}$$

holds for all $F \in H^p(\mathbb{I})$.

By what has been said before Theorem 3.4, the local convolution operators playing role in Theorem 4.4 are spectrum-preserving, which means that their diagonal extension is an $\mathcal{M} \to \mathcal{M}$ operator, indeed.

It can be proved that the Cesàro operator can be represented as a sum of three operators, which statisfy one of the conditions of Theorem 4.4 if $1/2 < p \leq 1$.

Theorem 4.5. *The dyadic Cesàro operator is bounded on the dyadic Hardy space $H^p(\mathbb{I})$ if $1/2 < p \leq 1$.*

The dyadic Cesàro operator on Lipschitz spaces

In this paragraph we consider the dyadic Cesàro operator on Lipschitz spaces. We prove that the operator in question is bounded on the space $\mathrm{Lip}(\alpha, p)$ if $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\alpha < 1/p$. Furthermore, we show that this condition can not be weakened if $p > 1$. For the proofs of these theorems, see [14].

The L^p-*modulus of continuity* $(1 \le p \le \infty)$ of an $f \in L^p[0,1)$ is defined by

$$\omega^p(f,\delta) := \sup_{y \le \delta} \|f - \tau_y f\|_p \quad (\delta > 0),$$

where τ means the dyadic translation operator. For arbitrary $\alpha > 0$ we call the space

$$\mathrm{Lip}(\alpha, p) := \{f \in L^p : \omega^p(f,\delta) = O(\delta^\alpha) \text{ as } \delta \to 0\} \quad (1 \le p \le \infty)$$

Lipschitz space (or Hölder space). It is known (see Sch-W-S-P [30] p. 189 Th.3) that a function f belongs to $\mathrm{Lip}(\alpha,p)$ $(1 \le p \le \infty)$ if and only if

$$\|f - E_n f\|_p = O(2^{-n\alpha}), \quad n \to \infty \quad (\alpha > 0). \tag{5.1.1}$$

We introduce the norm

$$\|f\|_{\mathrm{Lip}(\alpha,p)} := \sup_{n \in \mathbb{N}} 2^{n\alpha} \|f - E_n f\|_p. \tag{5.1.2}$$

Consequently, the function f belongs to $\mathrm{Lip}(\alpha,p)$ if and only if $\|f\|_{\mathrm{Lip}(\alpha,p)} < \infty$.

In the following, we will estimate the 2^nth partial sum at the point zero of the Walsh-Fourier series of the functions belonging to the spaces L^p and $\mathrm{Lip}(\alpha,p)$, respectively.

Lemma 5.1. *If $f \in L^p[0,1)$, then*

$$|(E_n f)(0)| \le \sum_{k=0}^{n-1} 2^{(k+1)/p} \|f - E_k f\|_p + |\hat{f}(0)| \quad \left(1 \le p \le \infty, \frac{1}{\infty} := 0\right).$$

Corollary 5.1. *If $\alpha < 1/p$, $1 \le p < \infty$, $f \in Lip(\alpha,p)$ and $\hat{f}(0) = 0$, then*

$$|(E_n f)(0)| \le M_{p,\alpha} \cdot 2^{n(1/p-\alpha)} \|f\|_{\mathrm{Lip}(\alpha,p)},$$

where the constant $M_{p,\alpha}$ depends only on p and α.

For the L^p-norm of the partial sums

$$W_n = \sum_{k=1}^{2^n-1} \frac{w_k}{k} \quad (n \in \mathbb{N})$$

of the Walsh-Fourier series of the kernel function of the dyadic integral W the following estimates are true:

Lemma 5.2. *If $n \in \mathbb{N}$, $1 \le p \le \infty$, then*

$$\|W_{n+1} - W_n\|_p \ge \frac{1}{8} 2^{-n/p}.$$

Lemma 5.3. *If $1 < p < \infty$, then there exist contstants $0 < c_p \le C_p < \infty$, for which*

$$c_p 2^{-n/p} \le \|W - W_n\|_p \le C_p 2^{-n/p},$$

and if $p = 1$, there exits a constant C_1, for which

$$\|W - W_n\|_1 \le C_1 2^{-n}.$$

The following theorem estabilishes the connection between the dyadic Cesàro operator and the operator of the conditional expectation.

Lemma 5.4. *If $f \in L_0^1(\mathbb{I})$, then*

$$Cf - E_n(Cf) = C(f - E_n f) + (E_n f)(0) \cdot (W - W_n) \quad (n \in \mathbb{P}).$$

By means of the above mentioned lemmas we prove the following two theorems.

Theorem 5.1. *If $0 < \alpha \le 1$, $1 \le p < \infty$ and $\alpha < 1/p$, then the Cesàro operator is bounded on the space $Lip(\alpha, p)$.*

In the following theorem we prove that the condition $\alpha < 1/p$ can not be weakened if $p > 1$.

Theorem 5.2. *For all $1 < p < \infty$ there exists a function $f \in Lip(1/p, p)$ such that Cf does not belong to the space $Lip(1/p, p)$.*

We show that for the function

$$f := \sum_{n=0}^{\infty} \frac{r_n D_{2^n}}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}-1} w_k$$

we have $f \in \mathrm{Lip}(1/p, p)$ if $p > 1$, and $Cf \notin \mathrm{Lip}(1/p, p)$.

The dyadic Cesàro operator on $\mathbb{R}_+$

We consider the continuous equivalent of the Cesàro operator in this section by replacing the Walsh-Fourier coefficient by the Walsh-Fourier transform. We give an integral form of the Cesàro operator in the form of a sum of local convolution operators. Similarly to the previous sections, we consider the interval $\mathbb{R}_+ := [0, +\infty)$, in place of the dyadic field (which is represented by this interval) in our investigations. For the proof of the theorems of this chapter see [16].

We interpret the Walsh-Fourier-transform with the help of the *generalized Walsh functions* ψ_y $(y \in \mathbb{R}_+)$ according to the characters of the dyadic field. Namely, let

$$\psi_y(x) = (-1)^{\sum_{j=-\infty}^{\infty} x_j y_{-j-1}},$$

where the $x_j, y_j \in \{0,1\}\,(j \in \mathbb{Z})$ are *the binary coefficients* of the numbers $x, y \in \mathbb{R}_+$ in the dyadic representations

$$x = \sum_{k\in\mathbb{Z}} \frac{x_k}{2^{k+1}}, \quad y = \sum_{k\in\mathbb{Z}} \frac{y_k}{2^{k+1}}.$$

In particular, if

$$x = \frac{x_0}{2} + \frac{x_1}{2^2} + \cdots + \frac{x_j}{2^{j+1}} + \cdots \in \mathbb{I}, \quad \text{and} \quad y = y_1 + y_2 2 + \cdots + y_{-j-1}2^j + \cdots \in \mathbb{N},$$

then it follows from the interpretation (2.2.5) of the Walsh functions that

$$w_y(x) = \psi_y(x) \quad (x \in \mathbb{I}, y \in \mathbb{N}),$$

and thus the functions ψ_y can be regarded as the extensions of the functions w_n. It follows easily from the definition that if $k \in \mathbb{N}$, then the function ψ_k is periodic of period 1.

The Walsh-Fourier transform of the function $f \in L^1(\mathbb{R}_+)$ is defined by

$$\hat{f}(x) := \int_0^\infty f(t)\psi_x(t)dt \quad (x \in \mathbb{R}_+). \tag{6.1.2}$$

If the support of f is a subset of $\mathbb{I}$, and $y \in \mathbb{N}$, then we get back the Walsh-Fourier coefficient of f. It is known that the Walsh-Fourier transform can be extended from $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ to a unitary map $\mathcal{F} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, and

$$(\mathcal{F}f)(x) = \lim_{n\to\infty} \int_0^{2^n} f(t)\psi_x(t)\,dt \quad \text{for a.e. } x \in \mathbb{R}_+.$$

In the following we will use that decomposition of the interval $[0,t]$ given in the following

Lemma 6.1. *If $t_k, k \in \mathbb{Z}$ are the binary coefficients of $t \in \mathbb{R}_+$, and*

$$A_k := [2^{-k-1}, 2^{-k}) \quad (k \in \mathbb{Z}),$$

then

$$\bigcup_{k\in\mathbb{Z}, t_k=1} (t \dotplus A_k) = [0,t)$$

holds for almost every $t \in \mathbb{R}_+$, where $t \dotplus A_k = \{t \dotplus x : x \in A_k\}$.

The *generalized Walsh-Dirichlet kernel* is defined by

$$D_t^\circ(y) = \int_0^t \psi_x(y)dx \quad (t, y \in \mathbb{R}_+).$$

If $t = n \in \mathbb{N}$, then D_t vanishes outside the interval $\mathbb{I}$, and it is equal to the ordinary Walsh-Dirichlet kernel on $\mathbb{I}$. Namely,

$$D_t^\circ(y) = \begin{cases} \sum_{k=0}^{n-1} w_k(y) = D_n(y) & (y \in \mathbb{I}), \\ 0 & (y \in [1,\infty)) \end{cases}$$

for all $t = n \in \mathbb{N}$, and

$$D_{2^k}^\circ = 2^k \chi_{[0,2^{-k})} \tag{6.1.5}$$

for all $k \in \mathbb{Z}$, where D_n denotes the ordinary Walsh-Dirichlet kernel (see Sch-W-S-P [30] p. 428 Ch.9.4).

We prove that for every function $f \in L^1(\mathbb{R}_+)$ there exists exactly one function $g \in L^1(\mathbb{R}_+)$, for which

$$\hat{g}(x) = \frac{1}{x}\int_0^x \hat{f}(u)\,du \quad (x > 0). \tag{6.2.1}$$

The map C from $L^1(\mathbb{R}_+)$ to $L^1(\mathbb{R}_+)$ defined by $Cf := g$ is called the *dyadic Cesàro operator* on $L^1(\mathbb{R}_+)$. We shall show that C can be written as the sum of certain local dyadic wavelet operators. To describe these kernels of the wavelet operators, we start with the Walsh-Fourier transform of the function $a(t) := (t - [t])/t$ $(t \in \mathbb{R}_+)$:

$$A(x) := (\mathcal{F}a)(x) = \lim_{n\to\infty} \int_0^{2^n} \frac{t-[t]}{t} w_x(t)\,dt \quad (x \in \mathbb{R}_+). \tag{6.2.2}$$

We shall prove in Lemma 6.2 that $A \in L^1(\mathbb{R}_+)$. To this end we introduce three new functions:

$$f_n(x) := \sum_{j=-\infty}^{n} 2^{j-n} \sum_{i=j}^{n} D_{2^i}^\circ(x \dotplus 2^{-j}),$$

$$g_n(x) := \sum_{j=-\infty}^{n} 2^{j} \sum_{i=n}^{\infty} 2^{-i} D_{2^i}^\circ(x \dotplus 2^{-j}),$$

$$h_n(x) := \sum_{i=n}^{\infty} 2^{n-i+2} D_{2^i}^\circ(x),$$

where $n \in \mathbb{Z}$, and $x \in \mathbb{R}_+$.

Lemma 6.2. *For all $x \in \mathbb{R}_+$ and $n \in \mathbb{Z}$ we have*

$$|2^n A(2^n x)| \le f_n(x) + g_n(x) + h_n(x), \tag{6.2.3}$$

consequently $A \in L^1(\mathbb{R}_+)$.

In the proof of the most important theorems of this paragraph we shall use the following

Lemma 6.3. *The maximal functions*

$$F^*h := \sup_{n\in\mathbb{Z}} |h| * |f_n| \quad \text{and} \quad G^*h := \sup_{n\in\mathbb{Z}} |h| * |g_n|$$

are of weak type $(1,1)$ *and of strong type* (∞,∞).

Let $a(x) := (x - [x])/x$,

(6.2.10) $$v(x) := 2a(x) - a(2^{-1}x), \ V(x) := (\mathcal{F}v)(x) \quad (x \in \mathbb{R}_+),$$

and denote by χ_n $(n \in \mathbb{Z})$ the characteristic function of the interval $[2^{-n-1}, 2^{-n})$. Then by (6.2.2) and by the well-known properties of the Walsh-Fourier transform we get that

$$V(x) = 2(A(x) - A(2x)) \quad (x > 0).$$

We introduce the wavelet operator

(6.2.11) $$(\mathcal{W}_n f)(x) := 2^n \int_0^\infty f(t)V(2^n(x \dot{+} t))\,dt \quad (f \in L^1, n \in \mathbb{Z}),$$

and then we form the following local convolution operator as follows:

(6.2.12) $$\mathcal{W}f := \sum_{n\in\mathbb{Z}} \mathcal{W}_n(f\chi_n) \quad (f \in L^1).$$

It is easy to see that the series in (6.2.12) converges a.e. as well as in L^1-norm. We prove in Theorem 6.2 that the operators $\mathcal{W}$ and $\mathcal{C}$ are equivalent on the space $L^1(\mathbb{R}_+)$. To prove the boundedness of the operator $\mathcal{W}$ in $L^p := L^p(\mathbb{R}_+)$, we use the maximal operator

(6.2.13) $$(\mathcal{V}f)(x) := \sup_{n\in\mathbb{Z}} 2^n \int_0^\infty |f(x \dot{+} t)V(2^n t)|\,dt \quad (x \in \mathbb{R}_+).$$

Denote by $\|f\|_p$ the L^p-norm of $f \in L^p(\mathbb{R}_+)$.

Theorem 6.1. *The maximal operator* $\mathcal{V}$ *is of weak type* $(1,1)$ *and of strong type* (q,q) *for every* $1 < q \leq \infty$, *that is,*

(6.2.14) $$\|\mathcal{V}f\|_q \leq C'_q \|f\|_q \quad (f \in L^q, 1 < q \leq \infty),$$

where the constant C'_q *depends only on* q.

Theorem 6.2. *The operator* $\mathcal{W}$ *is bounded from* L^1 *to* L^1 *and coincides with the dyadic Cesàro operator* $\mathcal{C}$, *that is,*

(6.2.16) $$\left(\widehat{\mathcal{W}f}\right)(x) = \frac{1}{x}\int_0^x \hat{f}(t)\,dt \quad (x > 0, f \in L^1).$$

Moreover, $\mathcal{W}$ is bounded from L^p to L^p for every $1 \le p < \infty$, that is,

$$\|\mathcal{W}f\|_p \le C_p \|f\|_p \quad (f \in L^p), \tag{6.2.17}$$

where $C_p = C'_q$, $1/p + 1/q = 1$ and C'_q is from (6.2.14). The operator $\mathcal{W}$ is not bounded on L^∞.

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