Characterisation of geometries in projective-metric spaces

Ph.D. abstract

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1 Antecedents

At the Second International Congress of Mathematicians (1900) David Hilbert raised a range of decisive problems for the mathematics of the 20th century. Fourth in the row was the identification of those geometries, emerging as common generalisation of classical non-Euclidean geometries, the straight line in which possesses the *"shortest path"* property.

Starting with the works of his students, P. Funk [6] and G. Hamel [8], thanks to the work of many other mathematicians, such as W. Blaschke [2], A. V. Pogolerov [18] and Zoltán Szabó [19], we are able to produce all the "*projective-metric*" geometries. These are far too many to investigate one by one, however, Busemann pointed out [4] that two most important type of them can be separated, the Minkowski and the Hilbert type, as isometries of straight lines in these two cases are projectivities (namely affinities, in case of Minkowski geometry).

Minkowski geometries can be known more thoroughly, since a significant deal of mathematicians knows them normed vector spaces, and a rather rich literature musters their properties and characteristics [1, 20], for example, that a Minkowski plane is Euclidean if and only if *for any vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *of the unit circle,* $(\mathbf{x}+\mathbf{y}) \perp_b (\mathbf{x}-\mathbf{y})$, where index $\mathcal{B}$ refers to Birkhoff perpendicularity [10].

Hilbert geometries are less familiar[1], however, their researches fall into line with new and diverse results time and again [17].

By way of example, hyperbolic geometry is characterised among Hilbert geometries by any of the following statements:

- perpendicularity is symmetric; in higher dimensions, as well [11], [3];
- the metric is locally Euclidean [10];
- there exists axial reflection in every straight line [3];
- curvature is non-positive in all points of the geometry [15];
- area of asymptotic triangles is constant in the geometry [5].

The aim of our research was to extend these results with characterisations, where geometric configurations do the lead.

2 Methods and knowledges applied

We apply in the theses, therefore present, *Minkowski geometries* (which are generalisations of the Euclidean geometry), the *metric ratio* and *Brkhoff perpendicularity*. We apply and present, as well, Hilbert geometries and the *hyperbolic ratio*, furthermore, properties of some fundamental configurations of hyperbolic geometry.

[1] Elsősorban ezért mutatjuk be a Hilbert-geometriákat kicsit részletesebben.

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Beyond these, apply and introduce the notion of *bundles*; the notion of *bisectoral centre* and *orthocentre* in triangles; furthermore the notion of *Menelaus, respectively Ceva triplets* of points and real numbers.

As they are not generally known in hyperbolic case, however we utilise them, proofs will be given for the following theorems, in the well known Cayley–Klein, respectively Poincaré modell.

Theorem 2.1. (Hyperbolic Menelaus' theorem[16, p. 467–468]) *A triplet* (C', A', B') *of the trigon* $ABC\triangle$ *in the hyperbolic space is of Menelaus type if and only if the triple* $(\langle A, B; C'\rangle, \langle B, C; A'\rangle, \langle C, A; B'\rangle)$ *is of Menelaus type.*

Theorem 2.2. (Hyperbolic Ceva's theorem [16, p. 467–468]) *A triplet* (C', A', B') *of the trigon* $ABC\triangle$ *in the hyperbolic space is of Ceva type if and only if the triple* $(\langle A, B; C'\rangle, \langle B, C; A'\rangle, \langle C, A; B'\rangle)$ *is of Ceva type.*

Theorem 2.3. (Theorem on hyperbolic bisectoral centre [16, p. 350]) *In hyperbolic space, perpendicular bisectors of every trigon belong to a bundle.*

Theorem 2.4. (Theorem on hyperbolic orthocentre [9, Theorem 3]) *In hyperbolic space, altitudes of every trigon belong to a bundle.*

In the course of presenting the most important characteristics of *Hilbert geometries*, we show about the logarithm of the cross ratio that fulfils the triangle inequality, and in fact determines a metric in the defining domain. We point out that a local Minkowski metric arises from the metric at each point, and the Hilbert geometry is a Finsler manifold with this metric. Perpendicularity is defined via this metric, which is in accordance with the Birkhoff perpendicularity. We show that a line is perpendicular to an another one exactly when the latter one is concurrent to the tangents taken at the endpoints. Observe that every collineation, keeping the base domain invariant, is an isometry of the Hilbert geometry, furthermore, *each isometry of the Hilbert geometry is a restriction of some collineation of the projective plane on the base domain, keeping the base domain invariant* (see [3, (22.10)]).

3 Exposition of the main results

Main results presented in the theses have been appeared in three publications [KKc, KKh, Km], in addition, we refer to a joint publication [KKp], furthermore some other results [KKq]. These all, without exception, give characterisations of classical Euclidean, respectively hyperbolic geometry among Minkowski, respectivey Hilbert geometries.

The first technical result in the course of the preliminaries gives the strict monotony of the hyperbolic ratio.

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Lemma 3.1. [KKc, Lemma 2.3] *Let A, B and C be collinear points in the Hilbert geometry given by $\mathcal{H}$, and let $AB \cap \partial\mathcal{H} = \{P, Q\}$ such that A is between P and B. Consider a Euclidean coordinate system of the line AB such that the coordinate of P is 0, and the coordinate of A is 1. Let the coordinates of the points Q, B and C be q, b and c with the assumptions $q > b > 1$ and $0 < c < q$. Then, for the hyperbolic ratio we have*

$$\langle A, B; C\rangle = \frac{c-b}{(c-1)\sqrt{b}}\sqrt{1+\frac{b-1}{q-b}}.$$

Use of cross ratio makes possible the unified projective discussion of the classical Ceva's and Menelaus' theorems (see [KKp, 2.2. tétel]). For this end, original Ceva configuration is supplemented with straight lines and points in the projective plane, obtained by extension of the affine plane (see Figure 3.1).

Theorem 3.2. ([KKp, 2.2. Tétel]) *Let f_A, f_B and f_C be straight lines through the corresponding vertices of the trigon $ABC\triangle$, and let X, Y and Z be points one by one on the sides of the trigon $ABC\triangle$. Let the points A', B' and C' be one by one the intersections of lines f_A, f_B and f_C with the straight lines of the opposite sides.*

(1) *If f_A, f_B and f_C are concurrent, and points X, Y and Z are collinear, then*

$$(A, B; C', Z)(B, C; A', X)(C, A; B', Y) = -1. \tag{3.1}$$

(2) *If (3.1) fulfils, then points X, Y, Z are collinear if and only if straight lines f_A, f_B and f_C are concurrent.*

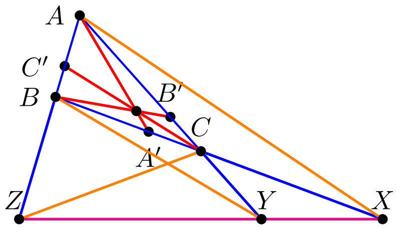

Figure 3.1. Projektív Ceva-konfiguráció

The theorems which characterise the Euclidean, respectively hyperbolic geometries among Minkowski, respectively Hilbert geometries, were obtained via comparing smooth curves with ellipses. In the course of this, *John–Löwner ellipses* of strictly convex domains have a prominent role, hence we review the basic statements regarding tangent points of circumscribed John–Löwner ellipses (of minimum area), and prove two lemmas about position of these tangent points, respectively triangles they determine. Particular statements of the following lemmas are applied in the later proofs.

Lemma 3.3. ([KKc] Lemma 2.2) *For any non-empty, open, convex $\mathcal{H}$ set in the plane, which is not an ellipse, there is an ellipse $\mathcal{E}$ containing $\mathcal{H}$ such that the set $\partial\mathcal{E} \setminus \mathcal{H}$ has at least six different points, and the set $\mathcal{E} \setminus \mathcal{H}$ is not empty.*

Lemma 3.4. ([KKh, Lemma 3.3]) *Let $\mathcal{H}$ be a convex domain in the plane. Then*

(1) *there exists a circumscribed ellipse n $\mathcal{E}$ of $\mathcal{H}$ with at least three different tangent points E_1, E_2, E_3 in the set $\partial\mathcal{H} \cap \partial\mathcal{E}$ such that the closed triangle $E_1E_2E_3\triangle$ contains the centre C of the ellipse $\mathcal{E}$, and*

(2) *if $\mathcal{H} \not\equiv \mathcal{E}$, then these tangent points can be chosen such that in all neighbourhood of one of them condition $\partial\mathcal{H} \setminus \partial\mathcal{E} \neq \emptyset$ fulfils.*

Let t_1, t_2 and t_3 be the common tangents of the two shapes at points E_1, E_2, and E_3, respectively. Then

(3) *if triangle $E_1E_2E_3\triangle$ contains point C internally, then t_1, t_2, t_3 form a trigon, with vertices $M_1 = t_2 \cap t_3$, $M_2 = t_3 \cap t_1$ and $M_3 = t_1 \cap t_2$ such that this trigon contains ellipse $\mathcal{E}$ internally, excepts points E_1, E_2, E_3;*

(4) *if some side of triangle $E_1E_2E_3\triangle$ contains C, say $C \in \overline{E_2E_3}$, then the three tangents determine such a half strip in the plane between parallel lines, vertices (possibly ideal ones) of which are $M_2 = t_1 \cap t_3$, $M_3 = t_2 \cap t_1$, ant this contains internally ellipse $\mathcal{E}$, except tangent points E_1, E_2, E_3.*

If B_1, B_2, B_3 one by one are midpoints of segments $\overline{E_2E_3}$, $\overline{E_3E_1}$ and $\overline{E_1E_2}$, then

(5) *lines M_iB_i $(i = 1, 2, 3)$ meet in the point C.*

In the course of the proof of our theorems we have to compare smooth, convex curves in the plane that have common tangent and sign of curvature at a point, and one is in the interior of the other. These results were obtained applying the usual apparatus of differential geometry [13].

Lemma 3.5. ([KKh, Lemma 3.4]) *Let $\mathbf{r}, \mathbf{p}\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ be twofold continuously differentiable curves for small values $\varepsilon > 0$ such that $\mathbf{p}(\tau) = p(\tau)\mathbf{u}_\tau$, and $\mathbf{r}(\tau) = r(\tau)\mathbf{u}_\tau$, where $p, r\colon (-\varepsilon, \varepsilon) \to \mathbb{R}_+$, furthermore, $r(\tau)/p(\tau)$ takes the minimum value 1 exclusively at $\tau = 0$.*

Then tangent lines of the curves $\mathbf{r}$, respectively $\mathbf{p}$ at the points $\mathbf{r}(\tau)$, respectively $\mathbf{p}(\tau)$ meet each other in a point $\mathbf{m}(\tau)$ which tends to the point $\mathbf{p}(0)$ as $\tau \to 0$, namely such that this point is on the same side of the straight line $\overline{\mathbf{Op}(\tau)}$ as $\mathbf{p}(0)$.

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Lemma 3.6. ([Km, Lemma 2.4]) *Let* $\mathbf{r}, \mathbf{p}\colon [0;1] \to \mathbb{R}^2$ *continuously differentiable curves with non-vanishing derivative.*

(1) *If* (i) $\mathbf{r} \parallel \mathbf{p}$, (ii) $\dot{\mathbf{r}} \parallel \dot{\mathbf{p}}$, *and these curves meet each other, then* $\mathbf{r} = \mathbf{p}$.

(2) *If* $\dot{\mathbf{r}}(0) \parallel \dot{\mathbf{p}}(0)$, *and* $\dot{\mathbf{r}}(1) \parallel \dot{\mathbf{p}}(1)$, *there there exists a value* $t_0 \in (0,1)$ *such that* $\dot{\mathbf{r}}(t_0) \parallel \dot{\mathbf{p}}(t_0)$.

3.1 Characterisation of hyperbolic geometry among Hilbert geometries

The question was the following: what consequences in Hilbert geometry the fulfilment of the statement with Ceva triplets and triples of Ceva type imply.

Theorem 3.7. (Characterisation of Ceva type) [KKc, Theorem 3.1] *In a Hilbert geometry, for every trigon* $ABC\triangle$ *there exists a Ceva triplet* (C', A', B') *such that triple* $(\langle A, B; C'\rangle, \langle B, C; A'\rangle, \langle C, A; B'\rangle)$ *is of Ceva type if and only if the geometry is hyperbolic.*

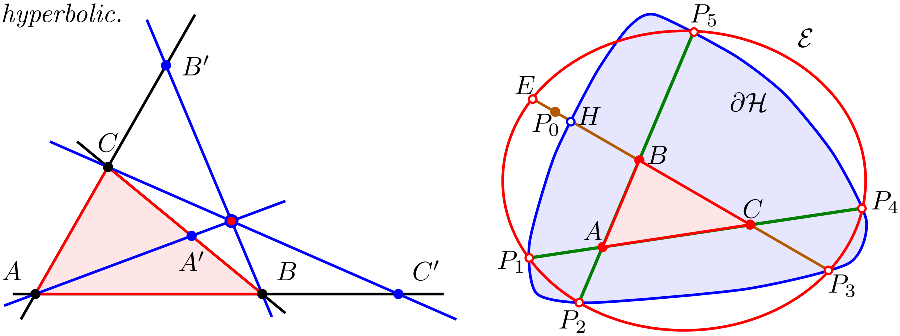

Figure 3.2. Ceva configuration and a triangle for the counterexample

According to the opportunity provided by [7, Lemma 12.1, p. 226], in the course of the proof, we can restrict ourselves to the plane. The key of the proof is to taking six points on the curve defining the Hilbert geometry by use of Lemma 3.3 regarding John–Löwner ellipse such that — under the indirect assumption of $\partial\mathcal{H}$ being not an ellipse — five out of them are on an ellipse while the sixth is an internal point of it. With a proper pairing we get a non-degenerate triangle such that two sides belong to straight lines which are hyperbolic and of Hilbert type at the same time, while the hyperbolic straight line of the third side includes that of Hilbert type (see Figure 3.2). According to Lemma 3.1, hyperbolic ratios on the first two lines are equal while different on the third one. Therefore, to a Ceva triplet, according to the hyperbolic Ceva's theorem, belongs a triple of Ceva type with $+1$ as a product of ratios, while the product of the ratios according to the Hilbert geoemtry can not be $+1$, hence the triple of numbers cannot be of Ceva type in the Hilbert geometry.

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The characterisation property of the Menelaus' theorem follows analogously.

Theorem 3.8. (Characterisation of Menelaus type) [KKc, Theorem 3.2] *In a Hilbert geometry, for every trigon $ABC\triangle$ there exists a Menelaus triplet (C', A', B') such that triple $(\langle A, B; C'\rangle, \langle B, C; A'\rangle, \langle C, A; B'\rangle)$ is of Menelaus type if and only if the geometry is hyperbolic.*

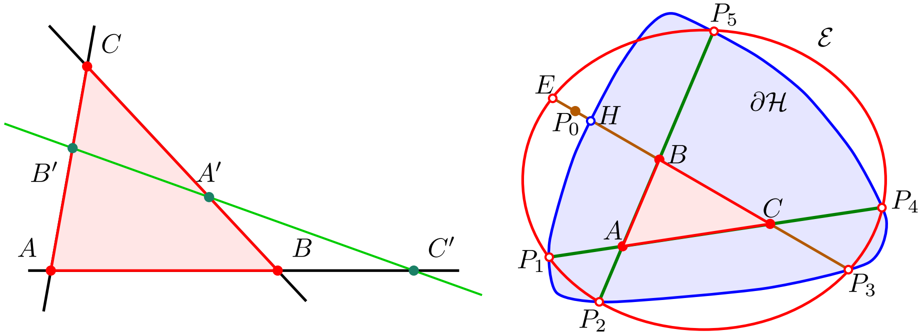

Figure 3.3. Meelaus configuration and a triangle for the counterexample

Recall that similar problem for Minkowski geometries is not brought up as a matter of fact, since the metric ratio and the affine ration are identical in such a geometry, therefore Ceva's and Menelaus' theorems fulfil exactly the same way as in the Euclidean geometry.

The next step is the investigation of the natural question: Is it possible a similar characterisation with perpendicular bisectors, respectively altitudes?

Before formulate the answer we have to mention that in the case of Hilbert geometries the inverse of the Hilbert perpendicularity, the $\mathcal{H}$-perpendicularity, is used for the perpendicular bisectors and later for the atitudes, as well, because we could not get any result in the case of Birkhoff perpendicularity, and we could not find a similar result in the literature, neither.

While perpendicular bisectors of a trigon incident to the centre of the circumcircle in the Euclidean geometry, a trigon in the hyperbolic plane does not have a circumcircle generally — however, the perpendicular bisectors of the sides of a trigon form a bundle the same way, hence one can speak of existence of *bisectoral centre*. The situation is the same in the case of the altitudes — they form a pencil both in Eucliden and hyperbolic geometry, hence one can speak of *orthocentre* this case (Theorem 2.4).

The natural question arises: *Under what conditions is the bundle property fulfilled? And if so, what can be stated about the geometry?*

We arrived an answer through two theorems which are interesting on their own, as well.

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Let us mention first one of our results on ellipse characterisation. This is a geometric characterisation[2] of ellipses built on harmonic divide, and applies to the following configuration.

Configuration 3.9. For different points E_i $(i = 1, 2, 3)$ on the oval $\partial\mathcal{H}$ denote ℓ_i the lines E_jE_k $(i, j, k$ are different), and $t_i^{\mathcal{H}}$ the tangents to $\mathcal{H}$ through points E_i. Finally, denote $f_i^{\mathcal{H}}$ the line, that is divided harmonically from the line $t_i^{\mathcal{H}}$ by the lines $\ell_j = E_kE_i$ and $\ell_k = E_iE_j$ $(j, k = 1, 2, 3)$. ▲

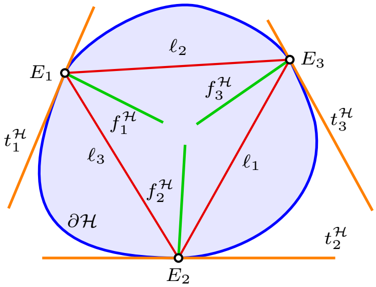

Figure 3.4. Configuration for the ellipse characterisation

Theorem 3.10. [KKh, Theorem 4.2] *Let us consider a Configuration 3.9.*

(i) *If $\mathcal{H}$ is an ellipse then the straight lines $f_1^{\mathcal{H}}, f_2^{\mathcal{H}}, f_3^{\mathcal{H}}$ are concurrent.*

(ii) *If $f_1^{\mathcal{H}}, f_2^{\mathcal{H}}, f_3^{\mathcal{H}}$ are straight lines in a bundle for an arbitrary chose of points $E_1, E_2, E_3 \in \partial\mathcal{H}$, then $\mathcal{H}$ is an ellipse.*

The proof of part (i) can be read off from the Figure 3.5.

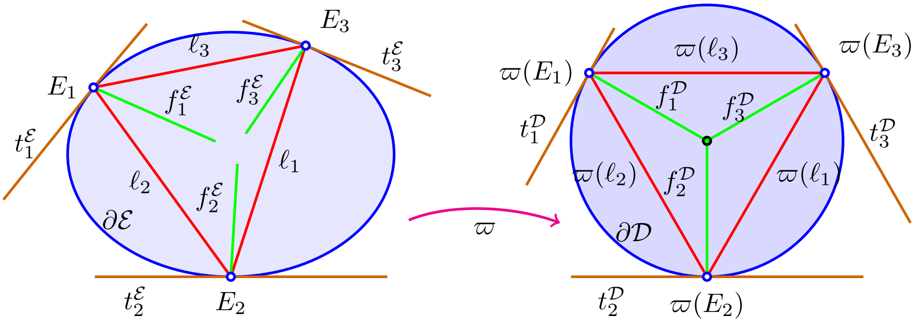

Figure 3.5. ϖ transforms the ellipse $\mathcal{E}$ into disc $\mathcal{D}$, the triangle $E_1E_2E_3\triangle$ into a regular one.

[2]It turned out later that its dual — via Ceva's and Menelaus' theorems — is equivalent to a former result of Segre, concerning finite geometries.

In order to prove (ii), transform the configuration so that two out of the three chosen points be diametrical, with tangents lines perpendicular to their position vectors. Then take an ellipse that has common tangents with the curve at these points, and the third chosen point is common, as well (see Figure 3.6).

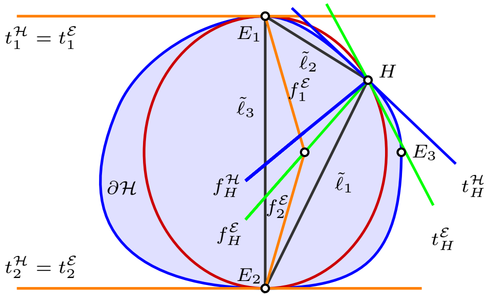

Figure 3.6. Introducing harmonic bundle through a point

Now we prove that the tangent is common at the third point, as well, finally show that our curve is an ellipse with centre in the origin.

Our second important result is a theorem of Ceva type on triangles inscribed an oval, which is proven for the extended Configuration 3.9 of the Configuration (3.11).

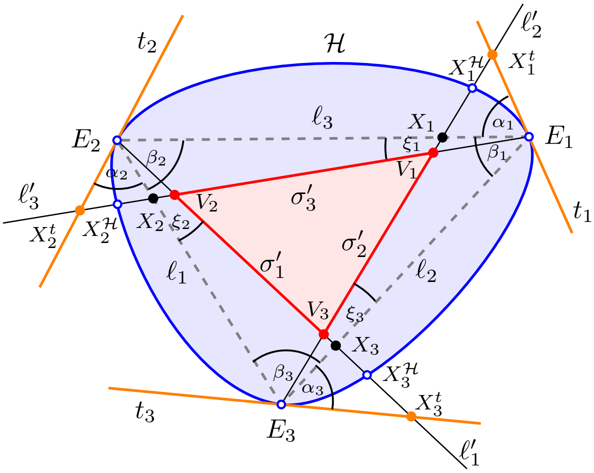

Figure 3.7. Extended Configuration 3.9

Configuration 3.11. Configuration 3.9 is extended as follows.

Let point X_i be close to the point E_i on the open segment $\sigma_i = \overline{E_j E_k}$ for all $i = 1, 2, 3$, where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, and denote straight lines E_2X_3, E_3X_1 and E_1X_2 one by one ℓ_1', ℓ_2' and ℓ_3'.

Take the following points $V_1 = \ell'_2 \cap \ell'_3$, $V_2 = \ell'_3 \cap \ell'_1$, $V_3 = \ell'_1 \cap \ell'_2$ of intersections. Let us introduce the notations $\sigma'_1 = \overline{V_2V_3}$, $\sigma'_2 = \overline{V_3V_1}$ and $\sigma'_3 = \overline{V_1V_2}$ for the open segments determined by these points of intersections. Furthermore, let us take the points of intersection $X_1^t = t_1 \cap \ell'_3$, $X_2^t = t_2 \cap \ell'_1$, $X_3^t = t_3 \cap \ell'_2$, respectively $X_1^{\mathcal{H}} = \partial\mathcal{H} \cap (\ell'_2 \setminus \{E_3\})$, $X_2^{\mathcal{H}} = \partial\mathcal{H} \cap (\ell'_3 \setminus \{E_1\})$, $X_3^{\mathcal{H}} = \partial\mathcal{H} \cap (\ell'_1 \setminus \{E_2\})$ on the tangent lines, respectively on the curve $\partial\mathcal{H}$. These points of intersections come into existence when points X_i are chosen sufficiently close to points E_i $(i = 1, 2, 3)$.

Finally, let the magnitude of the angles $X_2E_1E_2\angle$, $X_3E_2E_3\angle$, $X_1E_3E_1\angle$ be one by one ξ_1, ξ_2, ξ_3, while that of angles $X_1^tE_1E_2\angle$, $X_2^tE_2E_3\angle$, $X_3^tE_3E_1\angle$ be one by one α_1, α_2, α_3, and that of the angles $E_3E_1E_2\angle$, $E_1E_2E_3\angle$, $E_2E_3E_1\angle$ be one by one β_1, β_2, β_3.

▲

Theorem 3.12. [KKh, Theorem 4.4] *Consider a Configuration 3.11. For each $i = 1, 2, 3$ denote B_i the Euclidean midpoint of the segment σ_i, and $B_i^{\mathcal{H}}$ the $\mathcal{H}$-midpoint of the open segment σ'_i. The straight lines f_1, f_2, f_3 belong to one bundle if and only if the points X_1, X_2 és X_3 can be chosen for any $\varepsilon, \delta > 0$ such that*

$$|B_1^{\mathcal{H}} - B_1| + |B_2^{\mathcal{H}} - B_2| + |B_3^{\mathcal{H}} - B_3| < \varepsilon,$$
$$|X_1 - E_1| + |X_2 - E_2| + |X_3 - E_3| < \delta.$$

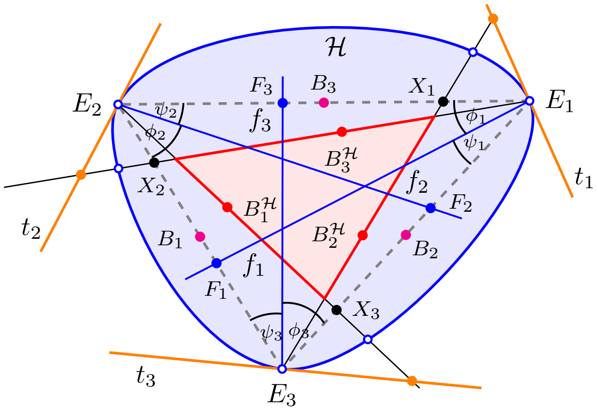

Figure 3.8. Construction with the midpoints

Calculations necessary for the proof of the theorem can be carried out applying cross ratios figured out by means of angles and lengths indicated on Figure 3.8.

The answer for the question, regarding the bisectoral centre, in the case of Hilbert geometry is given by the following theorem.

Theorem 3.13. ([KKh, Theorem 5.1]) *Every triangle has a bisectoral centre in a Hilbert geometry if and only if the geometry is hyperbolic.*

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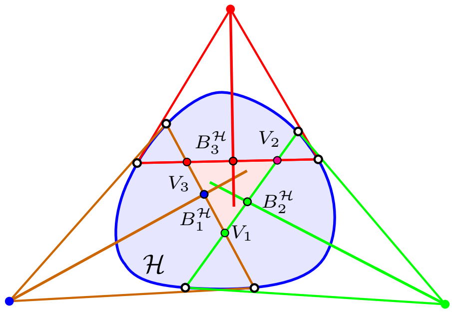

Figure 3.9. Perpendicular bisectors of a trigon in Hilbert geometry

The indirect assumption in the proof of the theorem is that domain $\mathcal{H}$ determining the geometry is not an ellipsoid. With respect to the statement [3, Lemma 12.1, p. 226] it is enough to work in the plane. According to part (1) of Lemma 3.4, there exists an ellipse $\mathcal{E}$ of minimum area, circumscribed $\mathcal{H}$, with at lest three different tangent points E_1, E_2, E_3 in $\partial\mathcal{H} \cap \partial\mathcal{E}$, and with the property that the closed trigon $E_1E_2E_3\triangle$ contains the origin . Hence one can consider the configuration described in part (4) of Lemma 3.4, which can be obtained by a projection given in part (3) of the same lemma. Given the necessary points and lines we can apply Lemma 3.1, then for the points of intersection with verified existence, proceeding by Lemma 3.5 obtain a configuration such that Theorem 3.10 and Theorem 3.12 guarantee appropriate convergence for midpoints of Euclidean, respectively Hilbert type. Finally we arrive at the existence of a trigon for which the intersection of certain halfplanes of its perpendicilar bisectors is empty, while one can construct a point which is in all the three halfplanes. This is a contradiction with the starting assumption.

As we had seen the characterising property of the existence of bisectoral centre via the investigation of perpendicular bisectors, we have expected, and actually obtained similar result for the altitudes.

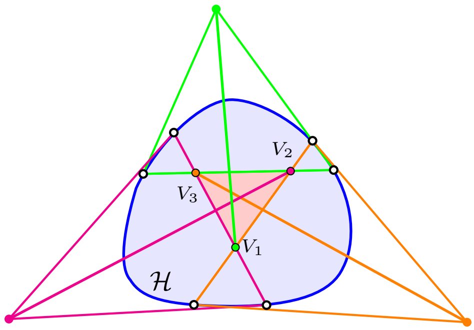

Figure 3.10. Altitudes of trigons in Hilbert geometry

Theorem 3.14. ([KKh, Theorem 5.2]) *If every triangle has a bisectoral centre in a Hilbert geometry, then it is hyperbolic.*

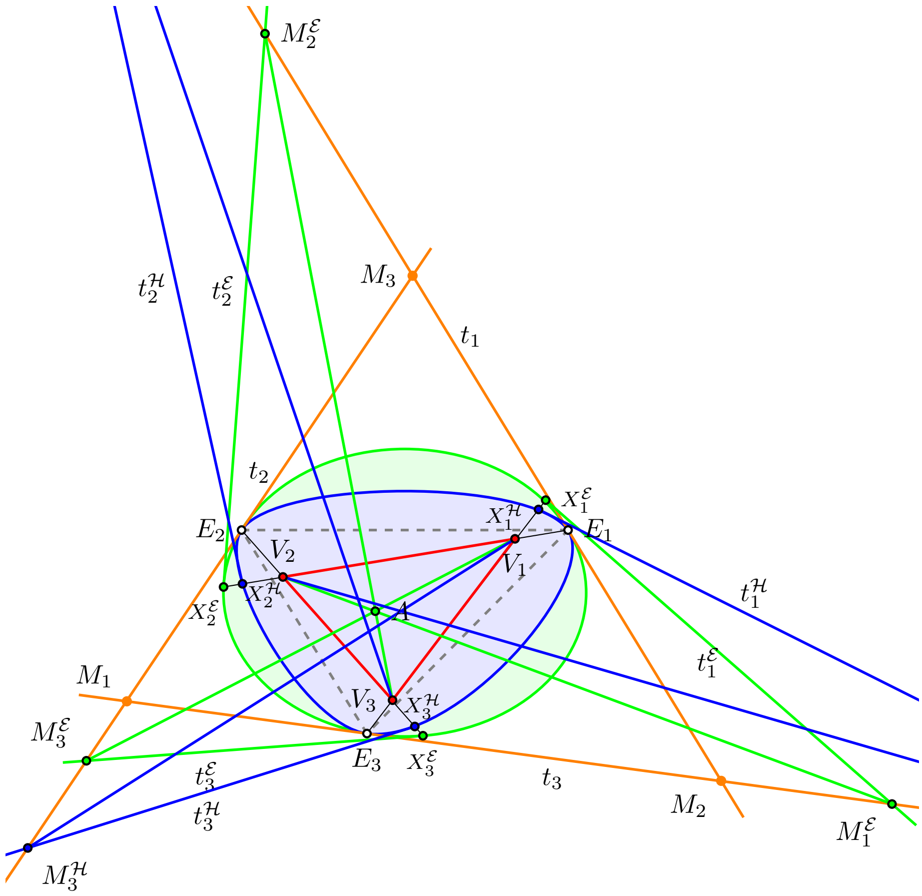

Figure 3.11. Triangle with altitudes intersecting internally

We arrive a contradiction assuming that $\mathcal{H}$ is not an ellipse, considering a construction similar to the one employed in the previous theorem (see Figure 3.11). With regard to the previously referred lemmas it is enough to prove in two dimensions, moreover, we can restrict the proof to a suitable configuration. The Euclidean points of intersections, obtained in the respective steps, lead to a point that should be in the intersection of three given halfplanes of perpendicular bisectors of Hilbert type; however this intersection is demonstrably empty.

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3.2 Characterisation of Euclidean geometry among Minkowski geometries

Thereafter, we looked for theorems in Minkowski geometries analogous to theorems regarding bisectoral centre and orthocentre in Hilbert geoemtries. This time we came at results in the case of Birkhoff perpendicularity and its inverse, i.e. for left-, respectively right-perpendicularity, as well.

Let us start with the problem of bisectoral centre. We prove first that right-perpendicular perpendicular bisectors pass trough one point (the right bisectoral centre) for every triangle if and only if the geometry is Euclidean. Next we also show that right-perpendicular altitudes pass trough one point (right orthocentre) in a Minkowski geometry exactly when it is Euclidean.

Theorem 3.15. ([Km, Theorem 3.1]) *The right-perpendicular bisectors are concurrent fo every triangle in a Minkowski geometry if and only if it is Euclidean.*

According to [3, Lemma 12.1, p. 226.], it is enough to prove in the plane.

Our indirect assumption is that the indicatrix has a two dimensional section which is not an ellipse. By Lemma 3.4 there exist three different points in the intersection of the ellipse and the indicatrix. With respect to the central symmetry, the intersections of the two curves appear in pairs symmetric to the centre, hence we have at least four different points. We arrive at a contradictions in specific cases according to the number and the position of the pairs of points (perpendicularity of their straight lines), via investigating a properly constructed trigon (for a specific case see Figure 3.12).

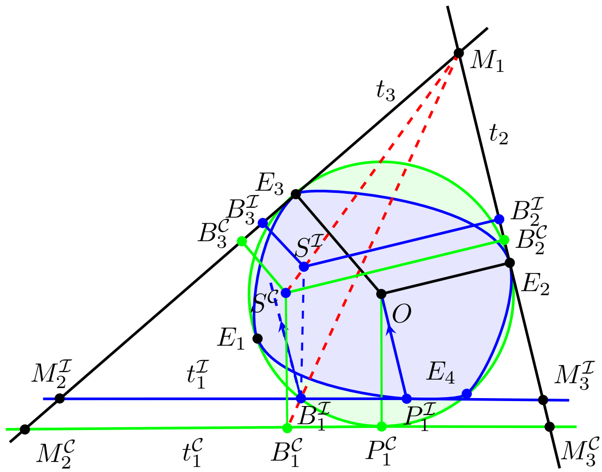

Figure 3.12. Right-perpendicular bisectors when $S^{\mathcal{C}} \neq P_1^{\mathcal{C}}$

In course of the proof we apply the previously proven Lemma 3.6, which shows that two perpendicular bisectors of a trigon meet the third one in points on opposite rays with common starting point.

The concurency of right-altitudes is a characterising property, as well.

Theorem 3.16. ([Km, Theorem 3.2]) *The right-altitudes are concurrent for every triangle in a Minkowski geometry if and only if it is Euclidean.*

The proof restrict the procedure to a suitable two dimensional configuration based on the first two out of the 11 main steps of the proof of the previous theorem. Then proceed with dividing the demonstration to cases corresponding the number of common points of the indicatrix and the John–Löwner ellipse, respectively the position of the pairs of common points.

We show also this time that two altitudes out of three of a trigon meet the third one in different points. For this end investigate the angle of the indicatrix and the ellipse. As a consequence of Lemma 3.5, the comparison of the altitude of the respective angles lead to this contradiction.

In the case of left-bisectoral centre and left-orthocentre for the indirect verification of the characterising statements, the proper trigon can not be produced directly by a suitable choice of the tangents to the indicatrix and the John–Löwner ellipse, however, it is possible to select them so that we could construct a triangle, leading to contradiction, from three lines parallel with them (see the right hand side of Figure 3.13).

Theorem 3.17. ([Km, Theorem 4.1]) *The left-perpendicular bisectors are concurrent for every triangle in a Minkowski geometry if and only if it is Euclidean.*

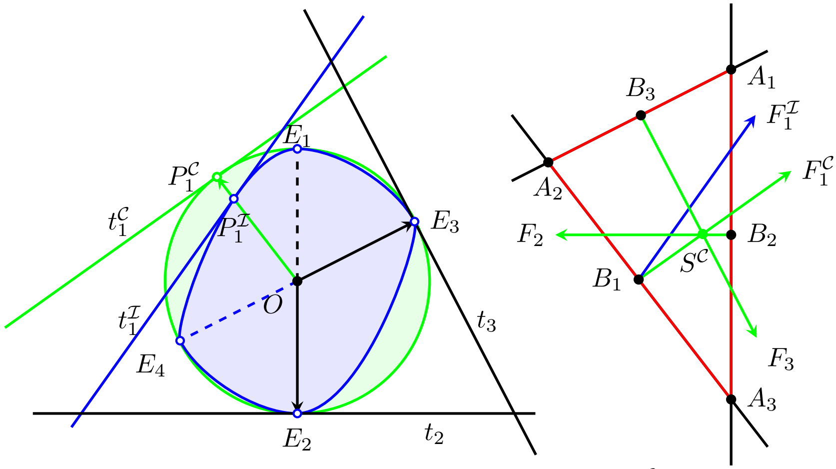

Figure 3.13. Left-perpendicular bisectors when $S^{\mathcal{C}} \neq B_1$.

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According to the previous remark, the proof can start with steps of the theorem on right-perpendicular bisectors. In the cases differentiated according to the number and position of the common points of the indicatrix and the John–Löwner ellipse reasoning can be analogous (identical in some cases) to the steps of the previous theorem. In the indirect proof the contradiction is obtained by showing the difference of certain points of intersection.

Finally, we close with the case of the left-perpendicular altitudes, which can be proven this time also with differentiating cases, and construction a configuration where the altitudes of a trigon can not pass trough one point (see Figure 3.14).

Theorem 3.18. ([Km, Theorem 4.2]) *The left-altitudes are concurrent for every triangle in a Minkowski geometry if and only if it is Euclidean.*

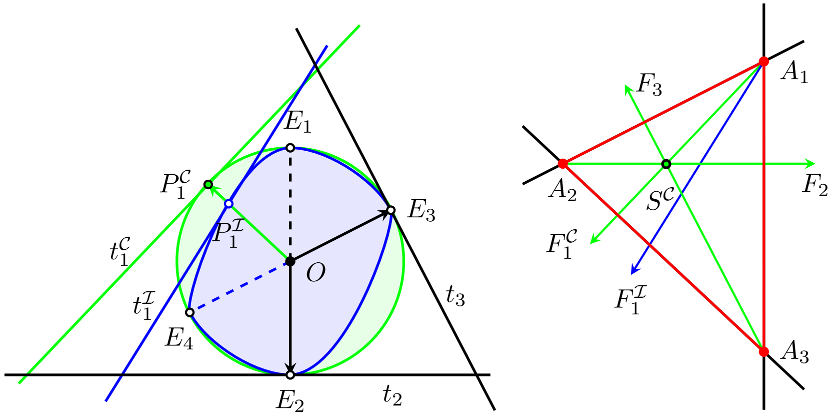

Figure 3.14. Left-perpendicularity when $\mathbf{s}^{\mathcal{C}} \neq A_2$

4 Summary and perspective

We presented the outcomes of our research on the two most important projective-metric geometries, namely the Hilbert and Minkowski geometries, which are the direct generalisations of the classical Euclidean, respectively hyperbolic geometries. These results characterise the respective classical geometries among their projective-metric generalisations via comparison of metric and affine properties of triangles.

Continuing the researches we investigated weather a projective metric is a classical one if a hyperbola is a quadratic curve in it. Presentation of the answers, however, is beyond the framework of present theses, therefore we only mention that we proved [KKq] for both the Minkowski and Hilbert geometries that a hyperbola is quadratic if and only if the geometry is Euclidean, respectively hyperbolic.

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