Smoothness of Green's Functions and Density of Sets

by

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To my parents

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ABSTRACT

We investigate local properties of the Green function of the complement of a compact set E.

First we consider the case $E \subset [0,1]$ in the extended complex plane (c.f. [16]). We extend results of V. Andrievskii, L. Carleson and V. Totik which claim that the Green function satisfies the $1/2$-Hölder condition locally at the origin if and only if the density of E at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0,1]$. We give an integral estimate on the density in terms of the Green function and extend the results to the case $E \subset [-1,1]$. In this case the maximal smoothness of the Green function is Hölder-1 and a similar integral estimate and necessary and sufficient condition hold as well.

The second part of the paper is joint work with Vilmos Totik (c.f. [15]). A characterization is given for compact sets $E \subset \mathbf{C}$ whose Green function satisfies the Lipschitz (or Hölder-1) condition. It is shown that this Lipschitz condition is equivalent to a Lipschitz type condition on the equilibrium measure and to the Markov inequality $||P_n'||_E \leq Cn||P_n||_E$ for any polynomial P_n of degree $\leq n$. We also give an example for such a set with infinitely many connected components.

In the third part of the paper we consider the case when E is a compact set in $\mathbf{R}^d$, $d > 2$ (c.f. [17]). We give a Wiener type characterization for the Hölder continuity of the Green function, thus extending a result of L. Carleson and V. Totik. The obtained density condition is necessary, and it is sufficient as well, provided E satisfies the cone condition. It is also shown that the Hölder condition for the Green function at a boundary point can be equivalently stated in terms of the equilibrium measure and the solution to the corresponding Dirichlet problem. The results solve a long standing open problem - raised by Maz'ja in the 1960's - under the simple cone condition.

Chapter 1

Introduction

The continuity of Green's functions at boundary points has been extensively studied for a long time. The aim of this research is to give conditions for the stronger Hölder continuity in terms of the geometry of the set. We consider both the planar and the higher dimensional case. For the concepts and notions in this Chapter see Section 2.1.

Suppose that $E \subset \mathbf{C}$ is a compact set with positive logarithmic capacity $\operatorname{cap}(E) > 0$. Let $\Omega := \overline{\mathbf{C}} \setminus E$, where $\overline{\mathbf{C}} := \{\infty\} \cup \mathbf{C}$ is the extended complex plane. Denote by $g_\Omega(z) = g_\Omega(z, \infty)$, $z \in \Omega$, the Green function of Ω with pole at ∞. We extend g_Ω to $\partial\Omega$ in the usual way by

$$g_\Omega(z, \infty) = \limsup_{w \to z,\ w \in \Omega} g_\Omega(w, \infty),$$

and to $\overline{\mathbf{C}} \setminus \overline{\Omega}$ by setting $g_\Omega(z, \infty) = 0$ there. This way g_Ω becomes a subharmonic function on $\mathbf{C}$. We are interested in the behavior of g_Ω at a regular boundary point.

Suppose that 0 is a regular point of E, i.e., $g_\Omega(z)$ is continuous at 0 and $g_\Omega(0) = 0$. First consider the case $E \subset [0, 1]$. The monotonicity of the Green function yields

$$g_\Omega(z) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(z), \quad z \in \mathbf{C} \setminus [0, 1],$$

that is, if E has the "highest density" at 0, then g_Ω has the "highest smoothness" at the origin. In particular

$$g_\Omega(-r) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(-r) > \sqrt{r}, \quad 0 < r < 1.$$

In this regard, we would like to explore properties of E whose Green function has the “highest smoothness" at 0, that is, E conforming to the following condition

$$g_\Omega(z) \leq C|z|^{1/2}, \qquad z \in \mathbf{C},$$

which is known to be the same as

$$g_\Omega(-r) \leq Cr^{1/2}, \qquad 0 < r < 1 \tag{1.0.1}$$

(c.f. [1, Theorem 3.6]). Various sufficient conditions for (1.0.1) in terms of metric properties of E are stated in [5], where the reader can also find further references.

There are compact sets $E \subset [0,1]$ of linear Lebesgue measure 0 with property (1.0.1) (see e.g. [5, Corollary 5.2]), hence (1.0.1) may hold, though the set E is not dense at 0 in terms of linear measure. On the contrary, V. Andrievskii [2] proved that if E satisfies (1.0.1) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrary close to the density of $[0,1]$ in that neighborhood, i.e. (1.0.1) implies

$$\lim_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = \frac{1}{4}. \tag{1.0.2}$$

In Chapter 2 we will prove a general integral estimate for the density via the Green function.

For $0<\varepsilon<1/2$ we set (see [5])

$$E_\varepsilon(t) = (E\cap[0,t]) \cup [0,\varepsilon t] \cup [(1-\varepsilon)t, t]. \tag{1.0.3}$$

L. Carleson and V. Totik [5] have characterized the optimal smoothness in terms of a Wiener type condition. They proved

Theorem 1.0.1. *Let $\varepsilon < 1/3$. E satisfies* (1.0.1) *if and only if*

$$\sum_k \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(2^{-k}))}{2^{-k}} \right) < \infty.$$

This theorem plays the same role for Lip 1/2 smoothness as Wiener's theorem for continuity. The proof of Theorem 1.0.1 in [5], due to L. Carleson, was based on Poisson's formula. There is an alternative approach: using the technique of balayage; and with it we prove the integral variant of Carleson's Theorem, from which Andrievskii's theorem (1.0.2) easily follows (see Lemma 2.7.1).

Andrievskii also constructed a regular compact set $E \subset [0,1]$ such that

$$\lim_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2-\varepsilon}} = 0, \quad 0<\varepsilon<\frac{1}{2}$$

holds but

$$\liminf_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = 0. \tag{1.0.4}$$

Furthermore he proved that conversely, (1.0.2) does not imply (1.0.1).

Now let's turn to the case $E\subset[-1,1]$. In this case

$$g_\Omega(ir) \ge g_{\overline{\mathbf{C}}\setminus[-1,1]}(ir) > \frac{r}{2}, \quad 0<r<1,$$

therefore in this case the optimal smoothness for Green functions is Hölder 1 and we are interested in sets E satisfying

$$g_\Omega(z) \le C|z|, \qquad 0<|z|<1.$$

This is equivalent to

$$g_\Omega(ir) \le Cr, \qquad 0 < r < 1 \tag{1.0.5}$$

because $g_\Omega(x+iy)$ is monotone in y. As we will see, the necessary condition for the optimal smoothness can be generalized to this case, as well.

Let us consider now the more general setting when E is an arbitrary compact subset of $\mathbf{C}$. Assume that 0 is a boundary point of Ω. Several equivalent conditions are known for the regularity of 0 (see e.g. ([14, Appendix A2.]). One of them is due to Wiener. It characterizes the regularity with the capacity of the sets

$$E^n = E \cap (\overline{D}_{2^{-n+1}} \setminus D_{2^{-n}}) = \Big\{ z \in E \ : \ 2^{-n} \le |z| \le 2^{-n+1} \Big\}.$$

Theorem 1.0.2. $g_\Omega(0) = 0$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{\log(1/\mathrm{cap}(E^n))} = \infty, \tag{1.0.6}$$

where $\mathrm{cap}(E^n)$ *denotes the logarithmic capacity of* E^n*.*

L. Carleson and V. Totik (see [5]) characterized in a similar manner the stronger Hölder continuity:

$$g_\Omega(z, \infty) \le C|z|^\kappa \tag{1.0.7}$$

with some positive numbers C, κ.

For $\varepsilon > 0$ set

$$\mathcal{N}_E(\varepsilon) = \{ n \in \mathbf{N} \ : \ \ \mathrm{cap}(E^n) \ge \varepsilon 2^{-n} \}, \tag{1.0.8}$$

and we say that a subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ of the natural numbers is of positive lower density if

$$\liminf_{N\to\infty} \frac{|\mathcal{N} \cap \{0, 1, \ldots, N\}|}{N+1} > 0,$$

which is clearly the same condition as $n_k = O(k)$.

Theorem 1.0.3 (Carleson, Totik). *Suppose that the compact set* E *satisfies the cone condition. Then Green's function* g_Ω *is Hölder continuous at* 0 *if and only if* $\mathcal{N}_E(\varepsilon)$ *is of positive lower density for some* $\varepsilon > 0$*.*

The Hölder continuity of the Green function can be stated as an equivalent condition in terms of the harmonic and equilibrium measure and the solution to the corresponding Dirichlet problem as well (see [5, Proposition 1.4]). It is also strongly related to the Markov inequality.

Let Π_n denote the set of algebraic polynomials of degree $\leq n$. Markov's inequality is a basic result comparing the supremum norm of a polynomial $P_n \in \Pi_n$ to the supremum norm of its derivative:

$$||P_n'||_{[-1,1]} \leq n^2 ||P_n||_{[-1,1]}.$$

If $C_1(0)$ is the unit circle, then the corresponding inequality

$$||P_n'||_{C_1(0)} \leq n||P_n||_{C_1(0)}$$

is due to Bernstein. Let us also remark that this is in some sense the optimal case, for if E is any compact set on the complex plane then there are polynomials $P_n \in \Pi_n$, $n = 1, 2, \ldots$ for which

$$||P_n'||_E \geq cn||P_n||_E$$

with some constant $c > 0$. Indeed, let D be a disk containing E with the smallest possible radius. Then $\partial D \cap E$ is not empty, say z_0 is a point in this set. If a is the center of D and r is its radius, then for $P_n(z) = (z-a)^n$ we have

$$|P_n'(z_0)| = \frac{n}{r}\|P_n\|_E.$$

Let $E \subset \mathbf{C}$ be an arbitrary compact set with positive logarithmic capacity. We say that E satisfies the Markov inequality with a polynomial factor if there exist $C,\ k > 0$ such that

$$||P_n'||_E \leq Cn^k||P_n||_E \tag{1.0.9}$$

holds for every n and $P_n \in \Pi_n$.

Let Ω be the outer domain of E. Green's function g_Ω is Hölder continuous if there exist $C_1,\ \alpha > 0$ such that

$$g_\Omega(z) \leq C_1\Big(\mathrm{dist}(z, E)\Big)^\alpha. \tag{1.0.10}$$

for all $z \in \mathbf{C}$. It is known that in certain cases the Markov inequality is equivalent to the Hölder continuity of the Green function. Totik (see [18]) proved that this is true for Cantor-type sets, i.e. (1.0.9) is equivalent to (1.0.10) if E is Cantor-type. It is an open problem if (1.0.9) and (1.0.10) are equivalent for any compact set E. In Chapter 3 our aim is to show that in the optimal cases $k = 1$ and $\alpha = 1$ they are, indeed, equivalent.

Totik suggested that Theorem 1.0.3 could be extended to the higher dimensional case, i.e. when $E \subset \mathbf{R}^d$. For this case a Wiener type condition like in Theorem 1.0.3 was already defined by Maz'ja (see [8]- [11]). Maz'ja proved its sufficiency for the Hölder continuity of the solution to the Dirichlet problem and showed that in general it is not necessary. In Chapter 4 we will prove the sufficiency of this condition for the Hölder continuity of the Green function and show that it is also necessary provided E satisfies the cone condition. We also give an equivalent characterization in terms of the equilibrium measure. In other words, under the cone condition we completely characterize Hölder continuity, which has been a long standing open problem.

Chapter 2

Optimal Smoothness for $E \subset [0, 1]$

2.1 Notations, Definitions

We shall use $c, c_0, c_1, c_2, \ldots, C, C_0, C_1, C_2, \ldots$ and $d_1, d_2, \ldots$ to denote positive constants. These constants may be either absolute or they may depend on E depending on the context. We may use the same symbol for different constants if this does not lead to confusion.

$|F|$ denotes the linear Lebesgue measure of a measurable subset $F \subset \mathbf{R}$ of the real line $\mathbf{R}$.

$\mathbf{D} := \{z \ : \ |z| < 1\}$ is the unit disk, $\mathbf{T} = \partial\mathbf{D}$ is the unit circle and for $z_1, z_2 \in \mathbf{C}$, $z_1 \neq z_2$ let

$$[z_1, z_2] := \{tz_2 + (1-t)z_1 : 0 \le t \le 1\}$$

be the interval between these points.

For the notions of logarithmic potential theory see e.g. [13] or [14]. In what follows μ_E denotes the equilibrium measure of E,

$$U^\nu(z) := \int \log \frac{1}{|z-t|} d\nu(t)$$

the logarithmic potential of the measure ν, $g_G(z, a)$ the Green function of the domain G with pole at a, $\omega(x, H, G)$ the harmonic measure in G corresponding to the set $H \subseteq \partial G$. We shall frequently use the relation

$$g_{\mathbf{C}\setminus E}(z) = \log \frac{1}{\operatorname{cap}(E)} - U^{\mu_E}(z), \quad z \in \mathbf{C} \setminus E \tag{2.1.1}$$

valid for any compact set E of positive capacity.

Let G be a domain with compact boundary and with $\operatorname{cap}(\partial G) > 0$, and let ν be a measure supported on $\overline{G}$. We shall need the concept of balayage (or sweeping) of ν out of G (sometimes we say balayage onto

∂G), see e.g. [14, Sec. II.4]. It is the unique measure $\overline{\nu}$ supported on ∂G with the property that

$$U^{\overline{\nu}}(z) = U^{\nu}(z) + \text{const} \tag{2.1.2}$$

for $z \in \partial G$ with the exception of a set of capacity 0. For regular G the exceptional set is empty. If G is bounded, then the constant is 0 ([14, Ch. II, Theorem 4.1]), and if G is unbounded, then it is ([14, Ch. II, Theorem 4.4])

$$\text{const} = \int_G g_G(a, \infty)\, d\nu(a). \tag{2.1.3}$$

We shall use the notation $\mathrm{Bal}\Big(\nu, G\Big)$ for the balayage measure $\overline{\nu}$.

There is a connection between harmonic and balayage measures: if $K \subseteq \partial G$ are compact sets, then for $x \in G$ the equality

$$\mathrm{Bal}\Big(\delta_x, G\Big)(K) = \omega(x, K, G) \tag{2.1.4}$$

holds, where δ_x denotes the point mass (Dirac measure) placed at the point x (see e.g. [14, Appendix A3, (3.3)]). Therefore, in what follows we shall interchangeably use the harmonic measure and balayage notations.

We shall also use Harnack's inequality: if u is a positive harmonic function in the unit disk and $|z| < 1$, then

$$\frac{1 - |z|}{1 + |z|} u(0) \le u(z) \le \frac{1 + |z|}{1 - |z|} u(0) \tag{2.1.5}$$

It follows from this (see e.g. [4]) that if K is a compact subset of G, then there is a constant c such that for all positive harmonic functions u on G

$$cu(x) \le u(y) \le \frac{1}{c} u(x)$$

for all x and y in K.

2.2 Results

Let $E \subset [0, 1]$ be a compact set with positive (logarithmic) capacity and let $\Omega := \overline{\mathbf{C}} \setminus E$.

Recall the definition of $E_\varepsilon(t)$ in (1.0.3) from the previous Chapter and that $\mathrm{cap}(I) = |I|/4$ for any interval I, where $|I|$ denotes the length (Lebesgue measure) of I.

Our first result is

Theorem 2.2.1. *For any* $\varepsilon > 0$

$$\int_r^1 \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt < C_0 \frac{g_\Omega(-r)}{\sqrt{r}} \tag{2.2.6}$$

where C_0 *is independent of* r.

The integral variant of Carleson's theorem (Theorem 1.0.1) is a consequence of this result.

Theorem 2.2.2. *Let* $\varepsilon < 1/2$. E *satisfies* (1.0.1) *if and only if*

$$\int_0^1 \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt < \infty. \tag{2.2.7}$$

The method used in the proofs of Theorems 2.2.1 and 2.2.2 can be applied to the case $E \subset [-1,1]$ as well. The highest smoothness of the Green function at the origin (Lipschitz condition) is again equivalent to the highest density at 0. Namely, let $E \subset [-1,1]$ and set $E_\varepsilon(t)$ as in (1.0.3) and

$$E_\varepsilon(-t) = (E \cap [-t,0]) \cup [-t,(1-\varepsilon)(-t)] \cup [-\varepsilon t, 0].$$

Theorem 2.2.3. *If* $E \subseteq [-1,1]$ *and* $\varepsilon > 0$ *then*

$$\int_r^1 \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt < C_0 \frac{g_\Omega(ir)}{r} \tag{2.2.8}$$

The same is true for $E_\varepsilon(-t)$.

Theorem 2.2.4. *Let* $\varepsilon < 1/2$. E *satisfies*

$$g_\Omega(z) \le C|z|, \qquad 0 < |z| < 1, \tag{2.2.9}$$

if and only if (2.2.7) *holds for* $E_\varepsilon(t)$ *and* $E_\varepsilon(-t)$.

This is a variant of [5, Theorem 1.11].

Corollary 2.2.5. *If* E *satisfies* (2.2.9) *then*

$$\lim_{r\to 0} \frac{\operatorname{cap}(E \cap [-r,r])}{r} = \frac{1}{2}. \tag{2.2.10}$$

Corollary 2.2.6. (c.f. [5, Corollary 1.12]) g_Ω *is Hölder* 1 *continuous at 0 if and only if both* $g_{\overline{\mathbf{C}}\setminus(E\cap[0,1])}$ *and* $g_{\overline{\mathbf{C}}\setminus(E\cap[-1,0])}$ *are Hölder* 1/2 *continuous there.*

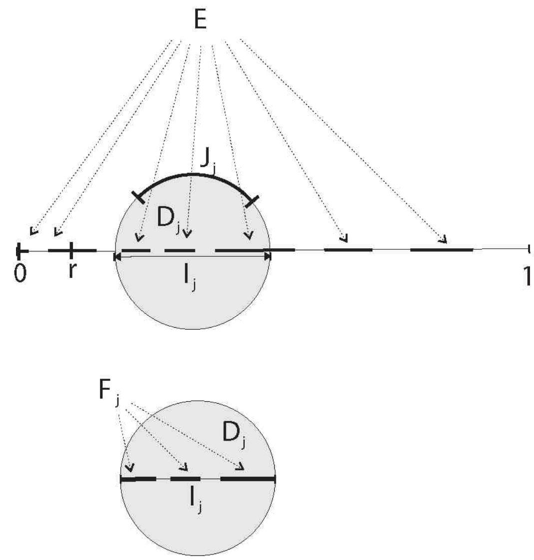

Figure 2.1: the disk D_j and the set F_j

2.3 Proof of Theorem 2.2.1

We divide the proof into three steps.

Step I. First we are going to verify the following: *let* $I_j = [a_j, b_j]$, $j \in \mathbf{N}$ *be disjoint closed subintervals of* $(0,1]$ *such that* $b_j \le C_1 |I_j|$, $j \in \mathbf{N}$ *for some* C_1, *and for* $\varepsilon > 0$ *set*

$$F_j = (I_j \cap E) \cup [a_j, a_j + (\varepsilon/2)|I_j|] \cup [b_j - (\varepsilon/2)|I_j|, b_j]. \qquad (2.3.11)$$

Then

$$\sum_{j:\ I_j \subseteq [r,1]} \left(\frac{1}{4} - \frac{\operatorname{cap}(F_j)}{|I_j|} \right) < c_0 \frac{g_\Omega(-r)}{\sqrt{r}}. \qquad (2.3.12)$$

For the proof first of all notice that

$$\mu_{E\cup[0,r]}([0,r]) \le C_2 g_\Omega(-r), \qquad 0 < r < 1, \qquad (2.3.13)$$

for some $C_2 > 0$ (recall that $\mu_{E\cup[0,r]}$ denotes the equilibrium measure of $E \cup [0,r]$). This is immediate, since (see (2.1.1))

$$\begin{aligned} g_\Omega(-r) &\ge g_{\overline{\mathbf{C}}\setminus(E\cup[0,r])}(-r) \\ &= \log \frac{1}{\operatorname{cap}(E \cup [0,r])} - U^{\mu_{E\cup[0,r]}}(-r) \\ &= U^{\mu_{E\cup[0,r]}}(0) - U^{\mu_{E\cup[0,r]}}(-r) = \int \log \frac{t+r}{t} d\mu_{E\cup[0,r]}(t) \\ &\ge (\log 2) \int_0^r d\mu_{E\cup[0,r]}(t) = (\log 2)\mu_{E\cup[0,r]}([0,r]). \end{aligned}$$

Let D_j resp. C_j be the open disk, resp. circle with diameter I_j, let J_j be the middle third part of the arc $\partial D_j \cap \{\Im z > 0\}$. If $I = (a,b)$ we use the notation $I(\varepsilon)=(a+\varepsilon(b-a),b-\varepsilon(b-a))$. Taking balayage of some measure supported in D_j onto $E\cup[0,r]$ can be done in two steps: first take balayage onto $\partial(D_j\setminus(E\cup[0,r]))$, and then onto $E\cup[0,r]$ (see Figure 2.1). Hence for $a\in I_j(\varepsilon/2)=[a_j+(\varepsilon/2)|I_j|,b_j-(\varepsilon/2)|I_j|]$, $I_j\subset[r,1]$

$$\begin{aligned}&\mathrm{Bal}\Big(\delta_a,\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\&\quad=\int_{C_j}\mathrm{Bal}\Big(\delta_b,\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])d\mathrm{Bal}\Big(\delta_a,D_j\setminus E\Big)(b).\end{aligned}$$

Next we use that $\mu_{E\cup[0,r]}$ is the balayage of $\mu_{[0,1]}$ onto $E\cup[0,r]$ ([14, Theorem IV.1.6, (e)]), and so

$$\begin{aligned}\mu_{E\cup[0,r]}([0,r]) &= \mathrm{Bal}\Big(\mu_{[0,1]},\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\
&\geq \mathrm{Bal}\Big(\mu_{[0,1]}\Big|_{[r,1]\setminus E},\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\
&= \int_{[r,1]\setminus E}\mathrm{Bal}\Big(\delta_a,\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])d\mu_{[0,1]}(a)\\
&\geq \sum_{j:\ I_j\subseteq[r,1]}\int_{I_j}\mathrm{Bal}\Big(\delta_a,\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])d\mu_{[0,1]}(a)\\
&= \sum_{j:\ I_j\subseteq[r,1]}S_j, \end{aligned}\tag{2.3.14}$$

and here

$$\begin{aligned}S_j &\geq \int_{I_j(\varepsilon/2)}\int_{C_j}\mathrm{Bal}\Big(\delta_b,\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\
&\qquad d\mathrm{Bal}\Big(\delta_a,D_j\setminus E\Big)(b)d\mu_{[0,1]}(a)\\
&\geq \left(\inf_{b\in J_j}\mathrm{Bal}\Big(\delta_b,\mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\right)\\
&\qquad\times\int_{I_j(\varepsilon/2)}\mathrm{Bal}\Big(\delta_a,D_j\setminus E\Big)(J_j)d\mu_{[0,1]}(a).\end{aligned}$$

Denote by $2C_j$, resp. $2D_j$ what we obtain from C_j resp. D_j by enlarging them twice from their center. Then $2D_j$ contains in its interior the interval I_j. For $a\in I_j(\varepsilon/2)$ Lemma 2.7.2 gives

$$\begin{aligned}\mathrm{Bal}\Big(\delta_a,D_j\setminus E\Big)(J_j) &\geq \mathrm{Bal}\Big(\delta_a,D_j\setminus F_j\Big)(J_j)\\
&\geq c_{\varepsilon/2}\mathrm{Bal}\Big(\delta_a,D_j\setminus F_j\Big)(C_j)\\
&\geq c_{\varepsilon/2}\mathrm{Bal}\Big(\delta_a,2D_j\setminus F_j\Big)(2C_j)\\
&= c_{\varepsilon/2}\omega(a,2C_j,2D_j\setminus F_j).\end{aligned}$$

Since $g_{\overline{\mathbf{C}}\setminus F_j}(z) \sim 1 = \omega(z, 2C_j, 2D_j \setminus F_j)$ for $z \in 2C_j$, and both functions $g_{\overline{\mathbf{C}}\setminus F_j}(z)$ and $\omega(z, 2C_j, 2D_j \setminus F_j)$ are harmonic in $2D_j \setminus F_j$ and vanish on F_j, these functions are comparable throughout $2D_j \setminus F_j$. Therefore the preceding estimate yields a constant $c > 0$ such that for $a \in I_j(\varepsilon/2)$ we have

$$\mathrm{Bal}\Big(\delta_a, D_j \setminus E\Big)(J_j) \geq c g_{\overline{\mathbf{C}}\setminus F_j}(a).$$

Therefore we can continue the inequality for S_j as

$$S_j \geq c\left(\inf_{b\in J_j} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r])\right) \\ \times \int_{I_j(\varepsilon/2)} g_{\overline{\mathbf{C}}\setminus F_j}(a) d\mu_{[0,1]}(a). \tag{2.3.15}$$

Here

$$d\mu_{[0,1]}(a) = \frac{1}{\pi\sqrt{a(1-a)}}da,$$

and hence for $a \in I_j = [a_j, b_j]$ we have

$$\frac{1}{\pi\sqrt{a(1-a)}} \geq \frac{1}{\pi\sqrt{b_j}} \geq \frac{1}{4C_1\sqrt{|I_j|}} \tag{2.3.16}$$

by the assumption $b_j \leq C_1|I_j|$. If

$$\theta_j = \frac{1}{4} - \frac{\mathrm{cap}(F_j)}{|I_j|} = \frac{1}{|I_j|}(\mathrm{cap}(I_j) - \mathrm{cap}(F_j))$$

then

$$\theta_j \sim \log\frac{\mathrm{cap}(I_j)}{\mathrm{cap}(F_j)} = \int_{I_j} g_{\overline{\mathbf{C}}\setminus F_j}(t) d\mu_{I_j}(t) \sim \frac{1}{|I_j|}\int_{I_j} g_{\overline{\mathbf{C}}\setminus F_j}(t)dt, \tag{2.3.17}$$

where the equality is known (see e.g. [5, (2.7)]) and the last relation is true because the integrals are actually integrals over $[a_j + (\varepsilon/2)|I_j|, b_j - (\varepsilon/2)|I_j|] \setminus E$ and $d\mu_{I_j}(t) = 1/(\pi\sqrt{(t-a_j)(b_j-t)}) \sim 1/|I_j|dt$ there. Thus we obtain

$$S_j \geq c_1\theta_j\sqrt{|I_j|}\left(\inf_{b\in J_j} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r])\right). \tag{2.3.18}$$

For $b \notin E \cup [0,r]$ the quantity

$$\mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r]) = \omega\Big(b, [0,r], \mathbf{C} \setminus (E \cup [0,r])\Big)$$

is a nonnegative harmonic function of b, hence by Harnack's inequality we have for $b \in J_j$ and $d_j = |I_j|$ the inequality

$$\mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r]) \geq c_2 \mathrm{Bal}\Big(\delta_{-d_j}, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r])$$

with some absolute constant $c_2 > 0$ because $\mathrm{dist}(J_j, 0) \sim \mathrm{dist}(J_j, [0,1]) \sim |I_j| = d_j$. By ([14, Ch. II, (4.47)]) we have

$$\begin{aligned} \mathrm{Bal}\Big(\delta_{-d_j}, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r]) &\geq \mathrm{Bal}\Big(\delta_{-d_j}, \mathbf{C} \setminus [0,1]\Big)([0,r]) \\ &= \frac{1}{\pi}\int_0^r \frac{\sqrt{d_j}\sqrt{1+d_j}}{\sqrt{t(1-t)(t+d_j)}}dt \\ &\geq \frac{1}{\pi}\frac{\sqrt{r}}{\sqrt{d_j}} = \frac{1}{\pi}\frac{\sqrt{r}}{\sqrt{|I_j|}}. \end{aligned} \tag{2.3.19}$$

This, the previous inequality, (2.3.18) and (2.3.14) give

$$\mu_{E\cup[0,r]}([0,r]) \geq c_3\sqrt{r} \sum_{j:\ I_j \subseteq [r,1]} \theta_j,$$

which together with (2.3.13) proves (2.3.12).

Step II. Let $E \subseteq [0,1]$ be compact and for $\varepsilon > 0$, $0 < t < 1$ set

$$E_\varepsilon^*(t) = (E \cap [\varepsilon t/2, t]) \cup [\varepsilon t/2, \varepsilon t] \cup [(1-\varepsilon/2)t, t]. \tag{2.3.20}$$

Then for $0 < q < 1$

$$\sum_{m:\ q^m > \frac{2r}{\varepsilon}} \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon^*(q^m))}{q^m(1-\varepsilon/2)}\right) < c_0\frac{g_\Omega(-r)}{\sqrt{r}}, \tag{2.3.21}$$

where c_0 depends only on ε and q.

To prove this let the integer M be so large that $q^M < \varepsilon/2$. Clearly, it is sufficient to show that for each $l = 1, \ldots, M$ the sum for the subsequence $m = jM + l$, $j \in \mathbf{N}$ satisfies

$$\sum_{j:\ q^{jM+l} > \frac{2r}{\varepsilon}} \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon^*(q^{jM+l}))}{q^{jM+l}(1-\varepsilon/2)}\right) < c_l\frac{g_\Omega(-r)}{\sqrt{r}}.$$

But this immediately follows from the result proved in Part 1, since the intervals $I_j = [\varepsilon q^{jM+l}/2, q^{jM+l}]$, $j \in \mathbf{N}$ are pairwise disjoint and the set F_j defined in (2.3.11) for these intervals is contained in $E_\varepsilon^*(q^{jM+l})$.

Step III. Finally, we complete the proof of Theorem 2.2.1. Let $\varepsilon > 0$ and $0 < u < 1$. If $u \leq t \leq u(1-\varepsilon/2)/(1-\varepsilon)$, then for the sets (1.0.3) and (2.3.20) the relation $E_\varepsilon(t) \cap [\varepsilon u/2, u] \supseteq E_\varepsilon^*(u)$ holds, and so

$$\frac{\mathrm{cap}(E_\varepsilon^*(u))}{u(1-\varepsilon/2)} \leq \frac{\mathrm{cap}(E_\varepsilon(t) \cap ([\varepsilon u/2, u]))}{u(1-\varepsilon/2)}. \tag{2.3.22}$$

But $E_\varepsilon(t) = [0, \varepsilon u/2] \cup (E_\varepsilon(u) \cap [\varepsilon u/2, u]) \cup [u, t]$, i.e. $E_\varepsilon(t)$ is obtained from $E_\varepsilon(t) \cap [\varepsilon u/2, u]$ by attaching one-one intervals to the right and to the left. Therefore, we can apply Lemma 2.7.4 below $\Big($(2.7.57),twice$\Big)$ to conclude

$$\frac{\operatorname{cap}(E_\varepsilon(u)\cap([\varepsilon u/2,u]))}{u(1-\varepsilon/2)}\le\frac{\operatorname{cap}(E_\varepsilon(t))}{t},$$

which, together with (2.3.22), gives

$$\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}\le\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon^*(u))}{u(1-\varepsilon/2)}. \tag{2.3.23}$$

This is true for all $u\le t\le u(1-\varepsilon/2)/(1-\varepsilon)$, therefore if we divide both sides by t and integrate with respect to t over the interval $[u,u(1-\varepsilon/2)/(1-\varepsilon)]$ then we obtain with $q=(1-\varepsilon)/(1-\varepsilon/2)$

$$\int_u^{u/q}\left(\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt\le\left(\log\frac{1-\varepsilon/2}{1-\varepsilon}\right)\left(\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon^*(u))}{u(1-\varepsilon/2)}\right). \tag{2.3.24}$$

Let k be the largest integer for which $q^k>\frac{2r}{\varepsilon}$. Summing up (2.3.24) for $u=q,q^2,q^3,\dots,q^k$ and making use of (2.3.21) we obtain

$$\int_{q^k}^1\left(\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt<C_3\frac{g_\Omega(-r)}{\sqrt{r}}.$$

Since

$$q^k\le\frac{2r}{\varepsilon}\frac{1}{q}=\frac{2r}{\varepsilon}\frac{1-\frac{\varepsilon}{2}}{1-\varepsilon}\le\frac{4r}{\varepsilon},$$

we can change the limit of the integral to $\frac{4r}{\varepsilon}$. Then, changing $\frac{4r}{\varepsilon}$ for r we can use Harnack's inequality to obtain

$$g_\Omega(-\frac{\varepsilon r}{4})\le C_4 g_\Omega(-r), \tag{2.3.25}$$

where C_4 depends only on ε. This completes the proof of Theorem 2.2.1.

■

2.4 Proof of Theorem 2.2.2

It follows from Theorem 2.2.1 that (1.0.1) implies (2.2.7). Therefore we only need to show that the converse is true. We divide the proof into two steps.

Step I. First we are going to verify the following: *If* $0<q<1$ *and* $\varepsilon<q/(q+1)$ *then*

$$\sum_{m=1}^{\infty}\left(\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(q^m))}{q^m}\right)<\infty \tag{2.4.26}$$

implies (1.0.1).

For the proof let

$$\theta_m^* = \frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(q^m))}{q^m}$$

and suppose $\sum \theta_m^* < \infty$. Let $\eta > 0$ be chosen below, choose M so that $\sum_{m \geq M} \theta_m^* < \eta$, and let $\tilde{E} = E \cup [\varepsilon q^{-M}, 1]$. By considering Green's functions $g_{\mathbf{C}\setminus E}$ and $g_{\mathbf{C}\setminus \tilde{E}}$ on a small circle about the origin not intersecting $\tilde{E}$, we can see that the Hölder 1/2 property at the origin for $g_{\mathbf{C}\setminus E}$ is the same as for $g_{\mathbf{C}\setminus \tilde{E}}$, therefore we may assume that $E = \tilde{E}$, i.e. $\sum_m \theta_m^* < \eta$ and (for sufficiently large M) $[(1-\alpha)q/2, 1] \subseteq E$, where $\alpha = q/(q+1)$.

It is sufficient to show that $\mu_E([0,\delta]) = O(\sqrt{\delta})$. In fact, then

$$\begin{aligned} g_\Omega(-r) &= g_\Omega(-r) - g_\Omega(0) = \int \log \frac{r+t}{t} d\mu_E(t) \\ &= \sum_{m=0}^{\infty} \int_{r/2^{m+1}}^{r/2^m} + \sum_{m:\ r \leq 2^m r < 1} \int_{2^m r}^{2^{m+1} r} \\ &= \sum_{m=0}^{\infty} O\left((m+1)\sqrt{\frac{r}{2^m}}\right) + \sum_{m=0}^{\infty} O\left(\frac{1}{2^m}\sqrt{2^m r}\right) = O(\sqrt{r}). \end{aligned}$$

This time let $I_m = [0, q^m]$ and C_m^* resp. D_m^* be the circle resp. disk with diameter $I_m(\varepsilon) = [\varepsilon q^m, (1-\varepsilon)q^m]$ (c.f. Figure 2.2). For $a \in I_m(\alpha) = [\alpha q^m, (1-\alpha)q^m]$ we obtain from Lemma 2.7.2

$$\omega(a, C_m^*, D_m^* \setminus E) \leq d\omega(a, J_m^*, D_m^* \setminus E),$$

where J_m^* is the middle third part of the arc $C_m^* \cap \{\Im z > 0\}$ (see Lemma 2.7.2) and $d = 1/c_{2(\alpha-\varepsilon)/(1-2\varepsilon)}$. Let $E_m = E_\varepsilon(q^m)$. Since for $z \in J_m^*$ we have $g_{\overline{\mathbf{C}}\setminus E_m}(z) \sim 1$ and $g_{\overline{\mathbf{C}}\setminus E_m}(z) \geq 0$ for $z \in C_m^* \setminus J_m^*$, it follows from the maximum principle and the previous inequality that with some constant d_1

$$\omega(a, C_m^*, D_m^* \setminus E) \leq d_1 g_{\overline{\mathbf{C}}\setminus E_m}(a) \qquad \text{for any } a \in I_m(\alpha). \tag{2.4.27}$$

Let n be large, $\mu_n = \mu_{E\cup I_n}$ the equilibrium measure of $E \cup I_n = E \cup [0, q^n]$, and define M_n as $\mu_n([0,q^n]) = M_n q^{n/2}$. Since $\mu_E([0,q^n]) \leq \mu_n([0,q^n])$ (note that μ_E is the balayage of μ_n out of Ω), it is sufficient to show that $M_n = O(1)$. μ_n is obtained by taking the balayage of $\mu_{[0,1]}$ onto $E \cup I_n$, hence

$$M_n q^{n/2} = \mu_{[0,1]}(I_n) + \int_{[q^n,1]\setminus E} \operatorname{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) d\mu_{[0,1]}(a). \tag{2.4.28}$$

Set p such that $q^n < \alpha q^p < q^{n-1}$. Then $\cup_{m=1}^{\infty} I_m(\alpha) \supset (0, (1-\alpha)q]$ and since $[(1-\alpha)q/2, 1] \subseteq E$, the last integral can be written as

$$\int_{q^n}^{\alpha q^p} \operatorname{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) d\mu_{[0,1]}(a)$$

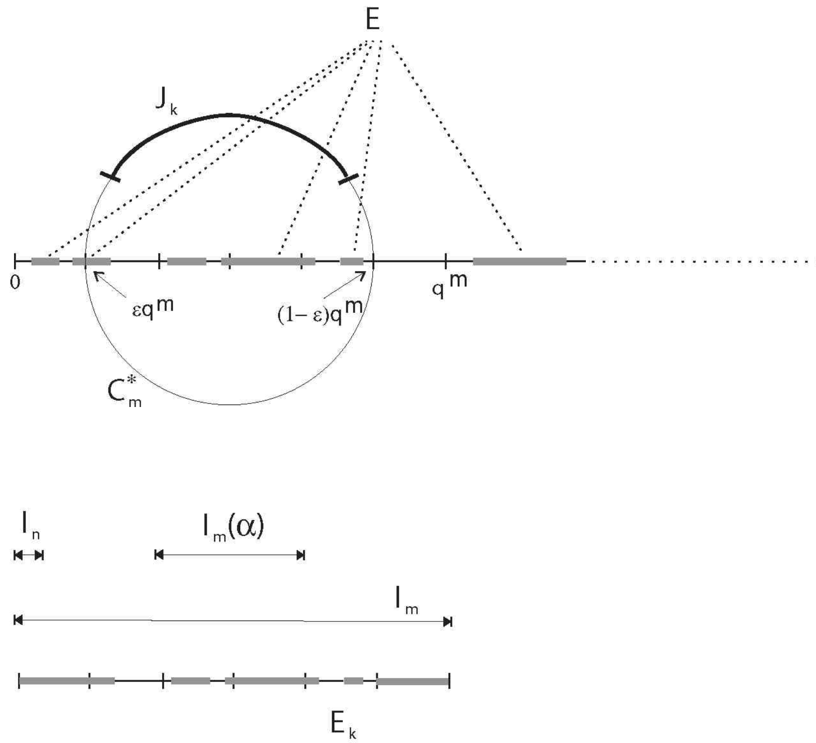

Figure 2.2: the disk D_m^* and the set E_m

$$
\begin{aligned}
&+ \sum_{m=1}^{p} \int_{I_m(\alpha)} \mathrm{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) d\mu_{[0,1]}(a) \\
&= S_0^* + \sum_{m=1}^{p} S_m^*. \qquad (2.4.29)
\end{aligned}
$$

Clearly

$$S_0^* + \mu_{[0,1]}(I_n) \leq \mu_{[0,1]}([0, q^{n-1}]) \leq q^{n/2}.$$

Therefore, if we can show that with some constant d_2 depending only on $\varepsilon < q/(q+1)$ we have $S_m^* \leq d_2 \theta_m^* M_n q^{n/2}$, then (2.4.28) and (2.4.29) give

$$M_n q^{n/2} \leq q^{n/2} + d_2 M_n q^{n/2} \sum_{m=1}^{p} \theta_m^* \leq q^{n/2} + d_2 M_n q^{n/2} \eta,$$

and for $\eta < 1/2d_2$ this yields $M_n \leq 2$.

Thus, it has left to prove $S_m^* \leq d_2 \theta_m^* M_n q^{n/2}$. Since the balayage measure $\mathrm{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup I_n)\Big)$ can be obtained by taking first balayage out of $D_m^* \setminus E$ and then taking the balayage of the so obtained measure out of $\mathbf{C} \setminus (E \cup I_n)$, we have the formula

$$
\begin{aligned}
&\mathrm{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) \\
&\quad = \int_{C_m^*} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) d\mathrm{Bal}\Big(\delta_a, D_m^* \setminus E\Big)(b).
\end{aligned}
$$

Here

$$\mathrm{Bal}\Big(\delta_a, D_m^* \setminus E\Big)(C_m^*) = \omega(a, C_m^*, D_m^* \setminus E),$$

hence

$$\begin{aligned} S_m^* &= \int_{I_m(\alpha)} \int_{C_m^*} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) \qquad (2.4.30) \\ &\quad d\mathrm{Bal}\Big(\delta_a, D_m^* \setminus E\Big)(b) d\mu_{[0,1]}(a) \\ &\le \left(\sup_{b \in C_m^*} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) \right) \\ &\quad \times \int_{I_m(\alpha)} \omega(a, C_m^*, D_m^* \setminus E) d\mu_{[0,1]}(a). \end{aligned}$$

Since for $a \in I_m(\alpha)$ the equilibrium measure $\mu_{[0,1]}(a)$ is given by the density $1/\left(\pi\sqrt{a(1-a)}\right) \sim q^{-m/2}$, we obtain from (2.4.27)

$$S_m^* \le \left(\sup_{b \in C_m^*} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) \right) \int_{I_m(\alpha)} g_{\mathbf{C} \setminus E_m}(a) d_3 q^{-m/2} da.$$

Now, just like in (2.3.17)

$$\theta_m^* \sim q^{-m} \int_{I_m} g_{\overline{\mathbf{C}} \setminus E_m}(t) dt,$$

therefore the last integral is at most $d_4 q^{m/2} \theta_m^*$, hence it is left to show

$$\sup_{b \in C_m^*} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) \le d_5 \mathrm{Bal}\Big(\delta_\infty, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) q^{-m/2},$$

because $\mathrm{Bal}\Big(\delta_\infty, \mathbf{C} \setminus (E \cup I_n)\Big)$ is the equilibrium measure of $E \cup I_n$ and $\mathrm{Bal}\Big(\delta_\infty, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n) = \mu_n(I_n) = M_n q^{n/2}$.

We estimate the harmonic measure $\mathrm{Bal}\Big(\delta_z, \mathbf{C} \setminus (E \cup I_n)\Big)(I_n)$ $= \omega(z, I_n, \overline{\mathbf{C}} \setminus (E \cup I_n))$ for $z \in C_m^*$ by taking the conformal map

$$w(z) = \left(q^{-n-1} z - 1\right) - \sqrt{\left(q^{-n-1} z - 1\right)^2 - 1}$$

of $\overline{\mathbf{C}} \setminus I_n$ onto the unit disk $\mathbf{D}$. This maps I_n onto $\mathbf{T}$, $E \setminus I_n$ into a subset E^* of $[q^{n+2}, 1]$, the point ∞ into the origin and the circle C_m^* into a closed curve γ such that all points of γ are of distance $\sim q^{n-m}$ from 0. Thus, there is a constant d_6 such that γ lies inside the circle $\mathbf{T}_{d_6 q^{n-m}}$ of radius $d_6 q^{n-m}$ about the origin. Now for $b \in C_m^*$ we obtain from the maximum principle

$$\begin{aligned}\omega(b, I_n, \overline{\mathbf{C}} \setminus (E \cup I_n)) &= \omega(w(b), \mathbf{T}, \mathbf{D} \setminus E^*)\\ &\leq \sup_{w \in \mathbf{T}_{d_6 q^{n-m}}} \omega(w, \mathbf{T}, \mathbf{D} \setminus E^*)\\ &= \omega(-d_6 q^{n-m}, \mathbf{T}, \mathbf{D} \setminus E^*),\end{aligned}$$

where the last equality follows from the solution to Milloux' problem (see [1, Section 3.3]). Hence, it is enough to prove

$$\omega(-d_6 q^{n-m}, \mathbf{T}, \mathbf{D} \setminus E^*) \leq d_7 q^{-m/2} \omega(0, \mathbf{T}, \mathbf{D} \setminus E^*). \tag{2.4.31}$$

Map now $\mathbf{D}$ onto the exterior of $[-1, 1]$ by $v(w) = (w + 1/w)/2$, and then the exterior of $[-1, q^{-n-2} - 1]$ onto $\mathbf{D}$ by

$$u(v) = \left(q^{n+1}(v+1) - 1\right) - \sqrt{\left(q^{n+1}(v+1) - 1\right)^2 - 1}.$$

Under these mappings the point $-d_6 q^{n-m}$ is mapped into $-1 + r_m$ with $r_m \sim q^{m/2}$, while 0 is mapped into 0, furthermore with $z = u(v(w)) = h(w)$ the function $\omega(h^{-1}(z), \mathbf{T}, \mathbf{D} \setminus E^*)$ is harmonic in $\mathbf{D}$ (note that the image of E^* under $w \mapsto h(w)$ is part of the unit circle). Hence Harnack's inequality gives

$$\omega(h^{-1}(-1 + r_m), \mathbf{T}, \mathbf{D} \setminus E^*) \leq d_7 q^{-m/2} \omega(h^{-1}(0), \mathbf{T}, \mathbf{D} \setminus E^*),$$

and this is (2.4.31).

Step II. *Let $\varepsilon' < 1/2$ and suppose (2.2.7) holds with ε' in place of ε. Then (2.4.26) holds for $\varepsilon' < \varepsilon < q/(q+1) \quad (q < 1)$.*

Let $0 < u < 1$. If $u(1-\varepsilon)/(1-\varepsilon') \leq t \leq u$, then for the sets (1.0.3) and $E_\varepsilon(u)$ the relation $E'_\varepsilon(t) \subset E_\varepsilon(u) \cap [0, t]$ holds, and so

$$\frac{\operatorname{cap}(E_{\varepsilon'}(t))}{t} \leq \frac{\operatorname{cap}(E_\varepsilon(u) \cap [0, t])}{t}. \tag{2.4.32}$$

Since $E_\varepsilon(u) = (E_\varepsilon(u) \cap [0, t]) \cup [t, u]$, applying Lemma 2.7.4 (like in the Proof of Theorem 2.2.1) we can conclude

$$\frac{\operatorname{cap}(E_{\varepsilon'}(t))}{t} \leq \frac{\operatorname{cap}(E_\varepsilon(u))}{u},$$

hence

$$\frac{1}{4} - \frac{\operatorname{cap}(E_{\varepsilon'}(t))}{t} \geq \frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(u))}{u}. \tag{2.4.33}$$

Dividing both sides by t and integrating with respect to t over the interval $[u(1-\varepsilon)/(1-\varepsilon'), u]$ we obtain with $q = (1-\varepsilon)/(1-\varepsilon')$

$$\int_{qu}^{u} \left(\frac{1}{4} - \frac{\operatorname{cap}(E_{\varepsilon'}(t))}{t}\right) \frac{1}{t} dt \geq \left(\log \frac{1-\varepsilon'}{1-\varepsilon}\right) \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(u))}{u}\right). \tag{2.4.34}$$

Summing up for $u = 1, q, q^2, q^3, \dots, q^m$ and making use of (2.2.7) we obtain

$$\sum_{m=1}^{\infty} \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(q^m))}{q^m} \right) < \infty,$$

and the proof of Theorem 2.2.2 is complete.

■

2.5 Proof of Theorem 2.2.3

First of all notice that in the proof of Theorem 2.2.1 we used the fact that $E \subset [0,1]$ only in Step I. Actually, we used it at two main steps: proving (2.3.18) (using the equilibrium measure of $[0,1]$) and establishing (2.3.19). Therefore, we will only mention the steps where the proof differs from that of Theorem 2.2.1.

We are going to use the notations of Step I. Instead of (2.3.13) now we have

$$\mu_{E \cup [0,r]}([0,r]) \le C_2 g_\Omega(ir), \qquad 0 < r < 1. \tag{2.5.35}$$

Indeed,

$$\begin{aligned} g_\Omega(ir) &\ge g_{\overline{\mathbf{C}} \setminus (E \cup [0,r])}(ir) \\ &= U^{\mu_{E\cup[0,r]}}(0) - U^{\mu_{E\cup[0,r]}}(ir) = \int \log \left| \frac{ir - t}{t} \right| d\mu_{E \cup [0,r]}(t) \\ &\ge \log \sqrt{2} \int_0^r d\mu_{E\cup[0,r]}(t) = \left(\log \sqrt{2} \right) \mu_{E \cup [0,r]}([0,r]). \end{aligned}$$

Replacing $\mu_{[0,1]}$ by $\mu_{[-1,1]}$ in the argument before (2.3.15) we have (c.f. (2.3.15))

$$\begin{aligned} S_j \ge c \left(\inf_{b \in J_j} \operatorname{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r]) \right) \\ \times \int_{I_j(\varepsilon)} g_{\overline{\mathbf{C}} \setminus F_j}(a) d\mu_{[-1,1]}(a). \end{aligned} \tag{2.5.36}$$

Now (c.f. the proof of (2.3.18)) we have

$$S_j \ge c_1 \theta_j |I_j| \left(\inf_{b \in J_j} \operatorname{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r]) \right), \tag{2.5.37}$$

since

$$d\mu_{[-1,1]}(t) = \frac{1}{\pi \sqrt{(1+t)(1-t)}} dt$$

and (c.f. (2.3.16))

$$\frac{1}{\pi \sqrt{(1+t)(1-t)}} \ge \frac{1}{\pi}.$$

In (2.3.19) we used δ_{-d_j}. Now, since $-d_j$ may be in E, let us change it for id_j. By Harnack's inequality we have for $b \in J_j$

$$\begin{aligned}\mathrm{Bal}\Big(\delta_b, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r]) &\geq c_1\mathrm{Bal}\Big(\delta_{id_j}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\ &\geq c_1\mathrm{Bal}\Big(\delta_{id_j}, \mathbf{C}\setminus[-1,1]\Big)([0,r])\\ &= c_1\omega\Big(id_j, [0,r], \mathbf{C}\setminus[-1,1]\Big). \quad (2.5.38)\end{aligned}$$

Applying the transformation $\varphi(z) = z - \sqrt{z^2-1}$ and using ([14, Ch. II,(4.8)]) we have

$$\begin{aligned}\omega\Big(id_j, [0,r], \mathbf{C}\setminus[-1,1]\Big) &= \omega\Big(i\Big(d_j - \sqrt{1+d_j^2}\Big), A, D\Big)\\ &= \frac{1}{2\pi}\Big(\int\limits_{\arccos r}^{\frac{\pi}{2}} P\Big(\zeta, i\Big(d_j - \sqrt{1+d_j^2}\Big)\Big)dt\\ &+ \int\limits_{\frac{3\pi}{2}}^{2\pi-\arccos r} P\Big(\zeta, i\Big(d_j - \sqrt{1+d_j^2}\Big)\Big)dt\Big),\end{aligned}$$

where $\zeta = e^{it}$, P is the Poisson kernel and A is the intersection $\mathbf{T}\cap\{z : 0 \leq \Re(z) \leq r\}$ consisting of two arcs on the unit circle. Thus,

$$\omega\Big(id_j, [0,r], \mathbf{C}\setminus[-1,1]\Big) \geq \frac{1}{2\pi}\int\limits_{\frac{3\pi}{2}}^{2\pi-\arccos r} f(t)dt,$$

where

$$f(t) = \frac{\Big(1 - \Big(1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2}\Big)\Big)}{1 - 2(\sqrt{1+d_j^2} - d_j)\cos\Big(t + \frac{\pi}{2}\Big) + (1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2})}.$$

It follows that

$$\begin{aligned}&\omega\Big(id_j, [0,r], \mathbf{C}\setminus[-1,1]\Big)\\ &\quad\geq \frac{r}{\pi}\frac{d_j(\sqrt{1+d_j^2} - d_j)}{1 - 2(\sqrt{1+d_j^2} - d_j)\sqrt{1-r^2} + (1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2})}.\end{aligned}$$

Assuming $d_j \geq r$ we get

$$\omega\Big(id_j, [0,r], \mathbf{C}\setminus[-1,1]\Big)$$

$$
\begin{aligned}
&\geq \frac{r}{\pi} \frac{d_j(\sqrt{1+d_j^2}-d_j)}{1-2(\sqrt{1+d_j^2}-d_j)\sqrt{1-d_j^2}+(1+2d_j^2-2d_j\sqrt{1+d_j^2})} \\
&= \frac{r}{2\pi} \frac{d_j(\sqrt{1+d_j^2}-d_j)}{1-(\sqrt{1+d_j^2}-d_j)(\sqrt{1-d_j^2}+d_j)} \\
&= \frac{r}{2\pi} g(d_j),
\end{aligned}
$$

where

$$
g(x) = \frac{x(\sqrt{1+x^2}-x)}{1-(\sqrt{1+x^2}-x)(\sqrt{1-x^2}+x)} = \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{2x}.
$$

$xg(x)$ is monotone decreasing on $[0,1]$, hence

$$
g(x) \geq \frac{g(1)}{x} = \frac{\sqrt{2}}{2}\left(\frac{1}{x}\right),
$$

which gives

$$
\omega\Big(id_j, [0,r], \mathbf{C} \setminus [-1,1]\Big) \geq \frac{\sqrt{2}}{4\pi}\left(\frac{r}{d_j}\right).
$$

This, (2.5.38) and (2.5.37) give

$$
S_j \geq c_3 \theta_j r
$$

for $d_j \geq r$, $I_j \subset [r,1]$, which proves

$$
\sum_{j:\ I_j \subseteq [r,1],\ |I_j| \geq r} \left(\frac{1}{4} - \frac{\operatorname{cap}(F_j)}{|I_j|}\right) < c_0 \frac{g_\Omega(ir)}{r},
$$

and the proof of Step I is complete.

The requirement $|I_j| \geq r$ in the summation doesn't affect the proofs in Steps II and III, because if $q^m > \frac{2r}{\varepsilon}$, then $r \leq q^m(1-\varepsilon/2) = |I_j|$. Finally, in (2.3.25) we can change the arguments for $i\varepsilon r/4$ and ir respectively.

■

2.6 Proof of Theorem 2.2.4 and Corollary 2.2.5

Again, we will only mention the steps where the proof differs from that of Theorem 2.2.2. We only need to show that if $E_\varepsilon(t)$ and $E_\varepsilon(-t)$ satisfy (2.2.7), then (1.0.5) holds. We need some changes in Step I.

It suffices to show that $\mu_E([-\delta,\delta]) = O(\delta)$. This time let $\mu_n = \mu_{E\cup[-q^n,q^n]}$ the equilibrium measure of $E \cup [-q^n, q^n]$, and define M_n as

$\mu_n([-q^n, q^n]) = M_n q^n$. Again it is sufficient to show that $M_n = O(1)$. μ_n is obtained by taking the balayage of $\mu_{[-1,1]}$ onto $E \cup [-q^n, q^n]$, hence

$$\begin{aligned} M_n q^n = {} & \mu_{[-1,1]}([-q^n, q^n]) \\ & + \int_{[q^n,1]\setminus E} \mathrm{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup [-q^n, q^n])\Big)([-q^n, q^n]) d\mu_{[-1,1]}(a) \\ & + \int_{[-1,-q^n]\setminus E} \mathrm{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup [-q^n, q^n])\Big)([-q^n, q^n]) d\mu_{[-1,1]}(a). \end{aligned} \tag{2.6.39}$$

Now

$$\mu_{[-1,1]}([-q^{n-1}, q^{n-1}]) \le c q^n$$

and we can write the two integrals as sums like before. We will only deal with the first integral, the other one can be handled similarly. Let α, $I_m(\alpha)$ and θ_m^* be as in the proof of Theorem 2.2.2. It suffices to show that with some constant d_2 depending only on ε we have $S_m^* \le d_2 \theta_m^* M_n q^n$, where

$$S_m^* = \int_{I_m(\alpha)} \mathrm{Bal}\Big(\delta_a, \mathbf{C} \setminus (E \cup [-s^n, s^n])\Big)([-q^n, q^n]) d\mu_{[-1,1]}(a)$$

and $I_m(\alpha) = [\alpha q^m, (1-\alpha) q^m]$.

The inequality in (2.4.30) and the equations before remain valid if we change I_n for $[-q^n, q^n]$ and $[0,1]$ for $[-1,1]$ (and C_m^* and D_m^* are the circle and disk with diameter I_m). This time for $a \in I_m(\alpha)$ the density of the equilibrium measure $\mu_{[-1,1]}(a)$ is $1/\Big(\pi\sqrt{(a+1)(1-a)}\Big) \sim 1$, hence it is left to show

$$\begin{aligned} & \sup_{b \in C_m^*} \mathrm{Bal}\Big(\delta_b, \mathbf{C} \setminus (E \cup [-q^n, q^n])\Big)([-q^n, q^n]) \\ & \qquad \le d_5 \mathrm{Bal}\Big(\delta_\infty, \mathbf{C} \setminus (E \cup [-q^n, q^n])\Big)([-q^n, q^n]) q^{-m}. \end{aligned}$$

The conformal map

$$w(z) = q^{-n} z - \sqrt{(q^{-n} z)^2 - 1}$$

takes $\overline{\mathbf{C}} \setminus [-q^n, q^n]$ onto the unit disk $\mathbf{D}$, $[-q^n, q^n]$ onto $\mathbf{T}$, $E \setminus [-q^n, q^n]$ into a subset E^* of $[-1, -q^{n+2}] \cup [q^{n+2}, 1]$, the point ∞ into the origin and the circle C_m^* into a closed curve γ such that all points of γ are of distance $\sim q^{n-m}$ from 0. Hence, it is enough to prove

$$\sup_{w \in \mathbf{T}_{d_6 q^{n-m}}} \omega(w, \mathbf{T}, \mathbf{D} \setminus E^*) \le d_7 q^{-m} \omega(0, \mathbf{T}, \mathbf{D} \setminus E^*). \tag{2.6.40}$$

Let $\delta < \varepsilon_0$, where ε_0 is defined in Lemma 2.7.5. Set $E^{**} = [-1 + \delta, 1 - \delta] \cap E^*$. The image of E^{**} under w^{-1} is $E \cap \Big([-1, -s^{-n}(2 - \delta + \delta^2)/(2-\delta)] \cup [s^{-n}(2 - \delta + \delta^2)/(2-\delta), 1]\Big)$, which has positive capacity for large n. Therefore we can assume that $\operatorname{cap}(E^{**}) > 0$. The left-hand side of (2.6.40) is increasing if we replace $\mathbf{D} \setminus E^{*}$ by $(\mathbf{D} \setminus E^{**})$. The function $\omega(z, \mathbf{T}, \mathbf{D} \setminus E^{**})$ is harmonic in $\mathbf{D} \setminus E^{**}$ and equals 1 on $\mathbf{T}$, therefore it is comparable with $g_{\overline{\mathbf{C}} \setminus E^{**}}$. More precisely, let $A = \inf_{\mathbf{T}} g_{\overline{\mathbf{C}} \setminus E^{**}}$. It follows from Harnack's principle that there is a constant c_δ depending only on δ such that $g_{\overline{\mathbf{C}} \setminus E^{**}}(z) \le c_\delta A$ for $z \in \mathbf{T}$. Thus

$$A\omega(z, \mathbf{T}, \mathbf{D} \setminus E^{**}) \le g_{\overline{\mathbf{C}} \setminus E^{**}}(z) \le c_\delta A \omega(z, \mathbf{T}, \mathbf{D} \setminus E^{**}) \tag{2.6.41}$$

holds true on $\mathbf{T}$ and an application of the maximum principle shows that it is true for all $z \in \mathbf{D} \setminus E^{**}$. This implies

$$\omega(w, \mathbf{T}, \mathbf{D} \setminus E^{*}) \le \frac{1}{A} g_{\overline{\mathbf{C}} \setminus E^{**}}(w) \tag{2.6.42}$$

for $w \in \mathbf{T}_{d_6 q^{n-m}}$. It follows from the definition of the Green function (2.1.1) that if $F \subset \mathbf{R}$ and $z = x + iy$, $x,\ y \in \mathbf{R}, y > 0$ then $g_{\overline{\mathbf{C}} \setminus F}(z)$ is monotone increasing in y. Using also the symmetry with respect to the real axis we have

$$\sup_{w \in \mathbf{T}_{d_6 q^{n-m}}} g_{\overline{\mathbf{C}} \setminus E^{**}}(w) \le \sup_{w \in L} g_{\overline{\mathbf{C}} \setminus E^{**}}(w), \tag{2.6.43}$$

where $L = \{z \ : \ z = x + i d_6 q^{n-m},\ |x| \le d_6 q^{n-m}\}$. By Harnack's inequality (2.1.5) and (2.6.41) we obtain

$$\begin{aligned} \sup_{w \in L} g_{\overline{\mathbf{C}} \setminus E^{**}}(w) &\le d_8 g_{\overline{\mathbf{C}} \setminus E^{**}}(i d_6 q^{n-m}) \\ &\le d_8 c_\delta A \omega(i d_6 q^{n-m}, \mathbf{T}, \mathbf{D} \setminus E^{**}). \end{aligned} \tag{2.6.44}$$

On the other hand, by Lemma 2.7.5 the right-hand side of (2.6.40) is not less than $d_9 q^{-m} \omega(0, \mathbf{T}, \mathbf{D} \setminus E^{**})$. In view of this and (2.6.41)-(2.6.44) it suffices to prove

$$\omega(i d_6 q^{n-m}, \mathbf{T}, \mathbf{D} \setminus E^{**}) \le d_{10} q^{-m} \omega(0, \mathbf{T}, \mathbf{D} \setminus E^{**}). \tag{2.6.45}$$

Now set $v(w) = (w + 1/w)/2$ and

$$u(v) = q^{n+2} v - \sqrt{(q^{n+2} v)^2 - 1}.$$

Under the mapping $z = u\Big(v(w)\Big)$ the origin is mapped into itself, $\mathbf{D} \setminus E^{**}$ is mapped into a set containing $\mathbf{D}$, E^{**} is mapped into a subset of the unit circle $\mathbf{T}$ and the point $i d_6 q^{n-m}$ is mapped into a point of distance $\ge d q^m$ from $\mathbf{T}$, where d is a constant. Thus with $z = u\Big(v(w)\Big) = h(w)$, using Harnack's inequality we get (2.6.45). This completes the proof of Theorem 2.2.4.

The proof of Corollary 2.2.5 is immediate from Lemmas 2.7.1 and 2.7.4. First of all, Lemma 2.7.1 implies (1.0.2) and

$$\lim_{r \to 0} \frac{\operatorname{cap}(E \cap [-r, 0])}{r} = \frac{1}{4}.$$

Then, taking $I=[-r,0]$, $J=[0,r]$ and $F=E\cap[-r,0]$ in (2.7.57) we get

$$\lim_{r\to 0}\frac{\operatorname{cap}\Big((E\cap[-r,0])\cup[0,r]\Big)}{r}=\frac{1}{2}. \tag{2.6.46}$$

Next, taking $I=[0,r]$, $J=[-r,0]$, $F=E\cap[0,r]$ and $G=E\cap[-r,0]$ in (2.7.56) we can infer

$$1\geq\frac{\operatorname{cap}(E\cap[-r,r])}{\operatorname{cap}\Big((E\cap[-r,0])\cup[0,r]\Big)}\geq\frac{4\operatorname{cap}(E\cap[0,r])}{r}\to 1. \tag{2.6.47}$$

Finally, (2.2.10) is a direct consequence of (2.6.46) and (2.6.47).

■

2.7 Lemmas

Lemma 2.7.1. (2.2.7) *for every* $\varepsilon>0$ *implies* (1.0.2).

Proof. Let $\eta>0$ be arbitrary such that $1+\eta\leq(1-\varepsilon/2)/(1-\varepsilon)$. For $t/(1+\eta)\leq u\leq t$ we have $E_{\varepsilon/2}(u)\subseteq E_\varepsilon(t)$, therefore

$$\begin{aligned}\frac{1}{4}-\frac{\operatorname{cap}(E_{\varepsilon/2}(u))}{u} &\geq \frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{u}\geq\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}(1+\eta)\\ &\geq \frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}-\eta.\end{aligned}$$

On adding η, dividing by u both sides and integrating with respect to u over the interval $t/(1+\eta)\leq u\leq t$ we obtain

$$\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}\leq\frac{1}{\log(1+\eta)}\int_{t/(1+\eta)}^{t}\left(\frac{1}{4}-\frac{\operatorname{cap}(E_{\varepsilon/2}(u))}{u}\right)\frac{1}{u}du+\eta.$$

Therefore, the finiteness of the integral in (2.2.7) (for $\varepsilon/2$ rather than for ε) gives

$$\limsup_{t\to 0}\left(\frac{1}{4}-\frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)\leq\eta,$$

and since here $\eta>0$ can be arbitrary small, it follows that

$$\lim_{t\to 0}\frac{\operatorname{cap}(E_\varepsilon(t))}{t}=\frac{1}{4}. \tag{2.7.48}$$

Now let $\{t_n\}$ be an arbitrary positive sequence tending to 0 and set

$$F_n=E_\varepsilon(t_n)/t_n,\qquad \nu_n=\frac{1}{\mu_{F_n}([\varepsilon,1-\varepsilon])}\mu_{F_n}\Big|_{[\varepsilon,1-\varepsilon]}.$$

We have just proved that $\operatorname{cap}(F_n)\to 1/4$ as $n\to\infty$, and below we verify that this implies the convergence $\mu_{F_n}\to\mu_{[0,1]}$ in the weak* topology. Since

$$\mu_{[0,1]}((\varepsilon,1-\varepsilon))>1-2\sqrt{\varepsilon},$$

there is an n_0 such that for $n \geq n_0$ we have $\mu_{F_n}((\varepsilon, 1-\varepsilon)) \geq 1 - 2\sqrt{\varepsilon}$. There is also an n_1 such that for $n \geq n_1$ the inequality

$$U^{\mu_{F_n}}(x) = \log \frac{1}{\mathrm{cap}(F_n)} \leq (1+\varepsilon)\log 4, \qquad x \in F_n,$$

holds, which implies for $n \geq \max(n_0, n_1)$

$$U^{\nu_n}(x) \leq \frac{1}{1-2\sqrt{\varepsilon}}(1+\varepsilon)\log 4, \qquad x \in F_n.$$

But the measure ν_n is supported on $F_n \cap [\varepsilon, 1-\varepsilon]$ and has mass 1, hence the preceding inequality gives

$$\log \frac{1}{\mathrm{cap}(F_n \cap ([\varepsilon, 1-\varepsilon])} \leq \int U^{\nu_n} d\nu_n \leq \frac{1}{1-2\sqrt{\varepsilon}}(1+\varepsilon)\log 4, \quad x \in F_n,$$

i.e.

$$\mathrm{cap}((E_n \cap [0, t_n])/t_n) \geq \mathrm{cap}(F_n \cap [\varepsilon, 1-\varepsilon]) \geq \left(\frac{1}{4}\right)^{(1+\varepsilon)/(1-2\sqrt{\varepsilon})}.$$

Since here $\varepsilon > 0$ is arbitrary, it follows that $\mathrm{cap}(E_n \cap [0, t_n])/t_n) \to 1/4$ as $n \to \infty$, and this is (1.0.2).

Above we used that as $n \to \infty$, we have $\mu_{F_n} \to \mu_{[0,1]}$ in the weak* topology on measures. In fact, let σ be a weak* limit of some subsequence, say $\mu_{F_{n_l}} \to \sigma$ as $l \to \infty$. Then σ is supported in $[0,1]$, has total mass 1, and all we have to show is that $\sigma = \mu_{[0,1]}$. We know that

$$U^{\mu_{F_n}}(x) = \log \frac{1}{\mathrm{cap}(F_n)} \tag{2.7.49}$$

for $x \in F_n$ with the exception of a set of capacity 0, and the same is true for $[0,1]$. Since $F_n \subset [0,1]$, it follows that

$$U^{\mu_{F_n}}(x) \leq U^{\mu_{[0,1]}}(x) + \log \frac{\mathrm{cap}([0,1])}{\mathrm{cap}(F_n)}$$

for $x \in F_n$ with the exception of a set of capacity 0, and since every set of zero capacity has zero μ_{F_n}-measure (see [14, Remark I.1.7, p. 28]), it follows that this inequality is true μ_{F_n}-almost everywhere. But then by the principle of domination [14, Theorem II.3.2] the same inequality is true for all $x \in \mathbf{C}$. Fixing such an $x \notin [0,1]$ and letting n tend to infinity through the subsequence $\{n_l\}$ it follows from $\mathrm{cap}(F_n) \to 1/4 = \mathrm{cap}(E)$ that

$$U^{\sigma}(x) \leq U^{\mu_{[0,1]}}(x).$$

Thus, this inequality is true for all $x \in \mathbf{C} \setminus [0,1]$.

However, the function

$$U^{\mu_{[0,1]}}(x) - U^{\sigma}(x)$$

vanishes at infinity, so it is harmonic there, and an appeal to the minimum principle on the domain $\overline{\mathbf{C}} \setminus [0,1]$ yields that we must have

$$U^\sigma(x) \equiv U^{\mu_{[0,1]}}(x), \qquad x \in \mathbf{C} \setminus [0,1].$$

Now we can conclude $\sigma = \mu_{[0,1]}$ from the unicity theorem [14, Theorem II.4.13].

■

Lemma 2.7.2. *Let $J = \{e^{i\varphi} \ : \ \pi/3 \leq \varphi \leq 2\pi/3\}$ be the middle third of the upper part of the unit circle. For every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ with the following property: if $F \subset [-1,1]$ is any compact set with $[-1,-1+\varepsilon] \cup [1-\varepsilon, 1] \subseteq F$, then for $x \in [-1,1] \setminus F$ the inequality*

$$\omega(x, J, \mathbf{D} \setminus F) \geq c_\varepsilon \omega(x, \mathbf{T}, \mathbf{D} \setminus F) \tag{2.7.50}$$

holds.

Remark 2.7.3. *The proof actually works the same way for any arc $J \subset \mathbf{T}$.*

Proof. First we verify the lemma in the special case when $[-1,1] \setminus F = I = (u,v)$ is an interval. Let $\alpha \in I$ be the point for which

$$-\frac{u-\alpha}{1-\alpha u} = \frac{v-\alpha}{1-\alpha v},$$

and apply the conformal map $\psi_1(z) = (z-\alpha)/(1-\alpha z)$. This maps the unit circle into itself, F into a set F' of type $[-1,-a] \cup [a,1]$, and J into some arc J' of the upper half circle $\mathbf{T}_+ = \{e^{i\varphi} \ : \ 0 \leq \varphi \leq \pi\}$ (see Figure 2.3). It is easy to see that there is constant $b_\varepsilon > 0$ depending only ε such that F' contains the intervals $[-1,-1+b_\varepsilon]$ and $[1-b_\varepsilon, 1]$ and the both the arc length of J' and the distance of J' from the points ± 1 is $\geq b_\varepsilon$. Map now $\mathbf{D} \setminus F'$ conformally onto $\mathbf{D}$ via the mapping ψ_2 normalized by $\psi_2(0) = 0$, $\psi_2'(0) > 0$. The image of $[-1,0] \cap F' = [-1,-a]$ is an arc on $\mathbf{T}$ symmetric about the point -1, and similarly the image of $[0,1] \cap F' = [a,1]$ is an arc on $\mathbf{T}$ symmetric about the point 1, furthermore the length of these arcs are bounded from below by some constant $d_\varepsilon > 0$. $\mathbf{T}$ is mapped into the complementary arcs of $\mathbf{T}$, and let us denote the complementary arc lying on the upper half plane by A'' (which is the image of the upper half circle $\mathbf{T}_+$ under ψ_2, i.e. $A'' = \psi_2(\mathbf{T}_+)$). The image J'' of J' is a subarc of A'', and its length is comparable to the length of the latter, i.e. with some $\delta_\varepsilon > 0$ we have

$$(\text{arc length of } J'') \geq \delta_\varepsilon (\text{arc length of } A'').$$

If $y = \psi_2(\psi_1(x)) \in (-1,1)$ is the image of x, then using the conformal invariance of harmonic measures, (2.7.50) takes the form

$$\omega(y, J'', \mathbf{D}) \geq c_\varepsilon \omega(y, \psi_2 \circ \psi_1(\mathbf{T}), \mathbf{D}),$$

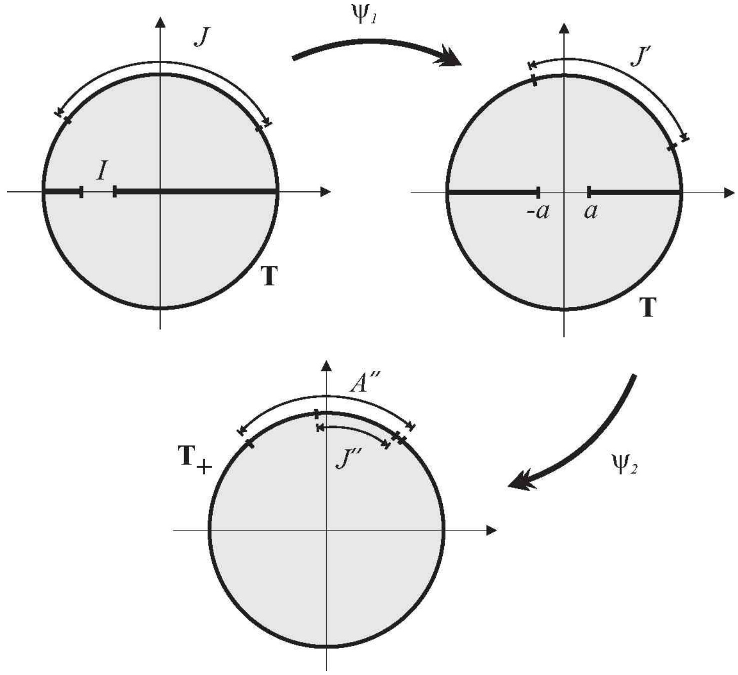

Figure 2.3: the mappings ψ_1 and ψ_2

which, using the symmetry of the image $\psi_2 \circ \psi_1(\mathbf{T}) = A'' \cup (-A'')$, is equivalent to

$$\omega(y, J'', \mathbf{D}) \geq 2c_\varepsilon \omega(y, A'', \mathbf{D}).$$

But in $\mathbf{D}$ harmonic measures are given by the Poisson kernel, so the preceding inequality is the same as

$$\frac{1}{2\pi}\int_{J''} \frac{1-y^2}{|\xi - y|^2}|d\xi| \geq 2c_\varepsilon \frac{1}{2\pi}\int_{A''} \frac{1-y^2}{|\xi - y|^2}|d\xi|, \tag{2.7.51}$$

which is clear with some $c_\varepsilon > 0$, since $y \in [-1,1]$ and on the two sides during integration ξ runs trough two arcs of comparable length both of which lie of distance $\geq d_\varepsilon/4$ from $[-1,1]$. Thus, (2.7.51) is true with some $c_\varepsilon > 0$, and this gives (2.7.50).

Next we turn to the general case, i.e. when $[-1,1]\backslash F = [-1+\varepsilon, 1-\varepsilon]\backslash F$ is an arbitrary open set. Since the constant c_ε should be independent of the set F (depending only on ε with $[-1,-1+\varepsilon] \cup [1-\varepsilon, 1] \subseteq F$), without loss of generality we may assume F to consist of finitely many intervals, in which case $[-1,1] \setminus F$ consists of finitely many open intervals, say $I_1, \ldots, I_m$.

According to (4.1.7), what we have to prove is that there is a constant $c_\varepsilon > 0$ such that for $x \in [-1,1] \setminus F$ we have

$$\mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)(J) \geq c_\varepsilon \mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)(\mathbf{T}). \tag{2.7.52}$$

We show that the constant c_ε verified above for the special case when $[-1,1] \setminus F$ was an interval, is appropriate. To this end, starting from

$\nu_0 = \delta_x$, we successively define the measures ν_n by

$$\nu_{n+1} = \mathrm{Bal}\Big(\nu_n, \mathbf{D} \setminus ([-1,1] \setminus I_{j_n})\Big),$$

where $j_n \in \{1, 2, \ldots, m\}$ is the index j for which $\nu_n(I_j)$ is maximal for $j = 1, \ldots, m$. Each ν_n is supported on $\mathbf{T} \cup [-1,1]$, and on $\mathbf{T} \cup F$ the measures ν_n form a monotone increasing sequence of measures. Note also that on $\mathbf{T} \cup F$ we have

$$\nu_{n+1} - \nu_n = \int_{I_{j_n}} \mathrm{Bal}\Big(\delta_y, \mathbf{D} \setminus ([-1,1] \setminus I_{j_n})\Big) d\nu_n(y),$$

and by the special case proved in the first part of this proof, here we have for all $x \in I_{j_n}$ the inequality

$$\mathrm{Bal}\Big(\delta_y, \mathbf{D} \setminus ([-1,1] \setminus I_{j_n})\Big)(J) \geq c_\varepsilon \mathrm{Bal}\Big(\delta_y, \mathbf{D} \setminus ([-1,1] \setminus I_{j_n})\Big)(\mathbf{T}).$$

Therefore the same is true of $\nu_{n+1} - \nu_n$, i.e. we have

$$\nu_{n+1}(J) - \nu_n(J) \geq c_\varepsilon(\nu_{n+1}(\mathbf{T}) - \nu_n(\mathbf{T})), \qquad n = 0, 1, \ldots.$$

Since $\nu_0(J) = \nu_0(\mathbf{T}) = 0$, induction gives

$$\nu_{n+1}(J) \geq c_\varepsilon \nu_{n+1}(\mathbf{T}), \qquad n = 0, 1, \ldots,$$

therefore (2.7.52) will follow from here if we show that $\nu_n \to \mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)$ as $n \to \infty$. As $\{\nu_n|_{\mathbf{T} \cup F}\}_{n=0}^\infty$ is an increasing sequence of measures on $\mathbf{T} \cup F$, it converges to some measure ν supported on $\mathbf{T} \cup F$, and to complete the proof we show that $\nu = \mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)$ and $\nu_n([-1,1] \setminus F) \to 0$ as $n \to 0$. Since the total mass of each ν_n is 1, it is clear that the total mass of ν is at most 1. Also, by the properties of balayage measures, for $z \in C \cup F$ and for all n we have the equality

$$U^{\nu_{n+1}}(z) = U^{\nu_n}(z) = \cdots = U^{\nu_0}(z) = \log \frac{1}{|z - x|},$$

and it is easy to see that then the same is true of ν, i.e.

$$U^\nu(z) = \log \frac{1}{|z - x|}, \qquad z \in \mathbf{T} \cup F. \tag{2.7.53}$$

Now $\mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)$ is the unique measure supported on $\mathbf{T} \cup F$ which has mass 1 and its logarithmic potential is $\log 1/|z - x|$, thus the proof will be complete if we show that ν has mass 1, i.e. $\nu(\mathbf{T} \cup F) = 1$, which is the same as

$$\lim_{n \to \infty} \nu_n([-1,1] \setminus F) = 0$$

which we wanted to prove anyway. This will be done by showing that in each step when going from ν_n to ν_{n+1} a fixed portion of the mass $\nu_n\Big|_{[-1,1]\setminus F}$ is moved to F, i.e. with some $\gamma<1$ we have

$$\nu_{n+1}([-1,1]\setminus F)\le \gamma\nu_n([-1,1]\setminus F). \tag{2.7.54}$$

Let $I_j=[a_j,b_j]$, and let $\tau>0$ be so small that all the intervals $[a_j-\tau,a_j]$ and $[b_j,b_j+\tau]$ are part of $(-1,1)$ and they are disjoint. For $I=I_{j_n}$ and $y\in I$ the value

$$\mathrm{Bal}\Big(\delta_y,\mathbf{D}\setminus([-1,1]\setminus I)\Big)([a_{j_n}-\tau,a_{j_n}]\cup[b_{j_n},b_{j_n}+\tau]),$$

which is the same as

$$\omega\Big(y,[a_{j_n}-\tau,a_{j_n}]\cup[b_{j_n},b_{j_n}+\tau],\mathbf{D}\setminus([-1,1]\setminus I)\Big)$$

is bounded from below by a constant ρ independent of n and $y\in I=I_{j_n}$. In fact, consider the conformal maps ψ_1,ψ_2 from the first part of the proof. Under $\psi_2\circ\psi_1$ the set $[a_{j_n}-\tau,a_{j_n}]\cup[b_{j_n},b_{j_n}+\tau]$ is mapped into the union of two arcs $A_\pm$, one-one around ±1, of length bounded from below by a positive constant depending only on ε and τ. Now the inequality

$$\omega\Big(y,[a_{j_n}-\tau,a_{j_n}]\cup[b_{j_n},b_{j_n}+\tau],\mathbf{D}\setminus([-1,1]\cup I)\Big)\ge\rho \tag{2.7.55}$$

with some positive constant ρ follows from the fact that here the left hand side is

$$\omega(z,A_-\cup A_+,\mathbf{D})=\frac{1}{2\pi}\int_{A_-\cup A_+}\frac{1-z^2}{|\xi-z|^2}|d\xi|,\qquad z=\psi_2(\psi_1(y)),$$

and the integral is bounded from below by a positive constant ρ for any point $z\in[-1,1]$ (and hence in particular also for the point $z=\psi_2(\psi_1(y))$).

We obtain from (2.7.55)

$$\begin{aligned}&\mathrm{Bal}\Big(\delta_y,\mathbf{D}\setminus([-1,1]\setminus I)\Big)(F)\ge\\&\qquad\mathrm{Bal}\Big(\delta_y,\mathbf{D}\setminus([-1,1]\setminus I)\Big)([a_{j_n}-\tau,a_{j_n}]\cup[b_{j_n},b_{j_n}+\tau])\ge\rho,\end{aligned}$$

which gives

$$\begin{aligned}&\mathrm{Bal}\Big(\nu_n\Big|_{I_{j_n}},\mathbf{D}\setminus([-1,1]\setminus I)\Big)(F)\\&\quad=\int_{I_{j_n}}\mathrm{Bal}\Big(\delta_y,\mathbf{D}\setminus([-1,1]\setminus I)\Big)(F)d\nu_n(y)\\&\quad\ge\int_{I_{j_n}}\rho\,d\nu_n(y)=\rho\nu_n(I_{j_n}),\end{aligned}$$

and here the right hand side is at least $\rho\nu_n([-1,1]\setminus F)/m$ by the choice of the interval I_{j_n}. Thus,

$$\begin{aligned}\nu_{n+1}([-1,1]\setminus F) &\leq \nu_n([-1,1]\setminus F)\\ &- \mathrm{Bal}\Big(\nu_n\Big|_{I_{j_n}}, \mathbf{D}\setminus([-1,1]\setminus I)\Big)(F)\\ &\leq \nu_n([-1,1]\setminus F)-\rho\nu_n([-1,1]\setminus F)/m\\ &= \nu_n([-1,1]\setminus F)(1-\rho/m).\end{aligned}$$

This proves (2.7.54) and the proof is complete.

■

Lemma 2.7.4. *Let I be a closed interval in* $\mathbf{R}$ *and let J be a closed interval that is attached either from the left or from the right to I. Let F and G be closed subsets of I and J, respectively. Then*

$$\frac{\mathrm{cap}(F)}{|I|}\leq\frac{\mathrm{cap}(F\cup G)}{4\mathrm{cap}(I\cup G)}. \tag{2.7.56}$$

In particular, if $G=J$ then

$$\frac{\mathrm{cap}(F)}{|I|}\leq\frac{\mathrm{cap}(F\cup J)}{|I|+|J|}. \tag{2.7.57}$$

Proof. Without loss of generality we may assume F and G to be regular compact sets (or to consist of finitely many closed intervals if we wish). The equilibrium measure μ_F is obtained from μ_I by adding to $\mu_I\Big|_F$ the balayage of $\nu:=\mu_I\Big|_{I\setminus F}$ out of $\mathbf{C}\setminus F$ (see [14, Theorem IV.1.6, (e)]), and in this balayage process the potential on F increases by a constant value. More precisely (see (4.1.6), (2.1.3)) for $x\in F$ and $\overline{\nu}:=\mathrm{Bal}\Big(\nu,\mathbf{C}\setminus F\Big)$ we have

$$U^{\overline{\nu}}(x)=U^{\nu}(x)+\int_{I\setminus F}g_{\overline{\mathbf{C}}\setminus F}(a)d\nu(a),$$

and this gives

$$U^{\mu_F}(x)=U^{\mu_I}(x)+\int_{I\setminus F}g_{\overline{\mathbf{C}}\setminus F}(a)d\mu_I(a).$$

Taking into account that for $x\in F$ the equilibrium potentials on the left and right hand sides are the constants $\log 1/\mathrm{cap}(F)$ and $\log 1/\mathrm{cap}(I)=\log 4/|I|$, respectively, we obtain the identity

$$\log\frac{1}{\mathrm{cap}(F)}-\log\frac{4}{|I|}=\int_{I\setminus F}g_{\overline{\mathbf{C}}\setminus F}(a)d\mu_I(a). \tag{2.7.58}$$

The analogous formula for $F \cup G$ and $I \cup G$ reads as

$$\log \frac{1}{\operatorname{cap}(F \cup G)} - \log \frac{1}{\operatorname{cap}(I \cup G)}$$
$$= \int_{(I\cup G)\setminus(F\cup G)} g_{\overline{\mathbf{C}}\setminus(F\cup G)}(a)d\mu_{I\cup G}(a)$$
$$= \int_{I\setminus F} g_{\overline{\mathbf{C}}\setminus(F\cup G)}(a)d\mu_{I\cup G}(a), \tag{2.7.59}$$

where we used that $(I \cup G) \setminus (F \cup G) = I \setminus F$, so the integration is over the same set on the right hand sides of (2.7.58) and (2.7.59). Since the measure μ_I is the balayage of $\mu_{I\cup G}$ onto I (see [14, Theorem IV.1.6, (e)]), we have on $I \setminus F$ the inequality $d\mu_{I\cup G}(a) \le d\mu_I(a)$. At the same time $g_{\overline{\mathbf{C}}\setminus(F\cup G)}(a) \le g_{\overline{\mathbf{C}}\setminus F}(a)$, and these show that the integral on the right hand side of (2.7.59) is not larger than the integral on the right hand side of (2.7.58). This gives

$$\log \frac{1}{\operatorname{cap}(F \cup G)} - \log \frac{1}{\operatorname{cap}(I \cup G)} \le \log \frac{1}{\operatorname{cap}(F)} - \log \frac{4}{|I|},$$

which is the same as (2.7.56).

■

Lemma 2.7.5. *There is an ε_0 such that for each $0 < \varepsilon < \varepsilon_0$ there exists a constant C_ε with the following property: if $F \subset [-1, 1]$ then*

$$\omega(0, \mathbf{T}, \mathbf{D} \setminus F) \ge C_\varepsilon \omega\Big(0, \mathbf{T}, (\mathbf{D} \setminus F) \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]\Big). \tag{2.7.60}$$

Proof. Clearly, for every $\varepsilon > 0$

$$\omega\Big(0, \mathbf{T}, (\mathbf{D} \setminus F) \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]\Big)$$
$$\le \omega\Big(0, \mathbf{T} \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1],$$
$$(\mathbf{D} \setminus F) \setminus ([-1, -1 - \varepsilon] \cup [1 - \varepsilon, 1])\Big). \tag{2.7.61}$$

There is a conformal mapping Φ of $\mathbf{D} \setminus ([-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1])$ onto $\mathbf{D}$ which maps $\mathbf{T} \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]$ onto $\mathbf{T}$, $[1 - \varepsilon, 1]$ into an arc of the form $L = \{e^{i\phi} \ : \ |\phi| \le \delta_\varepsilon$ (and symmetrically $[-1, -1 + \varepsilon]$ into an opposite arc L'), it takes 0 into 0 and $F \cap [-1 + \varepsilon, 1 - \varepsilon]$ into some subset $F^* \subset [-1, 1]$. Because of the conformal invariance of harmonic measures,

$$\omega\Big(0, \mathbf{T} \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1],$$
$$(\mathbf{D} \setminus F) \setminus ([-1, -1 - \varepsilon] \cup [1 - \varepsilon, 1])\Big)$$
$$= \omega(0, \mathbf{T}, \mathbf{D} \setminus F^*). \tag{2.7.62}$$

If ε is sufficiently small, then $\mathbf{T} \setminus (L \cup L')$ contains the pair of arcs $J = \{e^{i\phi} \ : \ \pi/4 \le |\phi| \le 3\pi/4\}$. According to [5, Lemma 7.2]

$$\frac{1}{2}\omega(0, \mathbf{T}, \mathbf{D} \setminus F^*) \le \omega(0, J, \mathbf{D} \setminus F^*), \tag{2.7.63}$$

and here the right hand side is at most $\omega(0, \mathbf{T} \setminus (L \cup L'), \mathbf{D} \setminus F^*)$. Now application of Φ^{-1} gives

$$\begin{aligned} &\omega(0, \mathbf{T} \setminus (L \cup L'), \mathbf{D} \setminus F^*) \\ &\quad = \omega\Big(0, \mathbf{T}, (\mathbf{D} \setminus F) \setminus ([-1, -1+\varepsilon] \cup ([1-\varepsilon, 1])\Big) \\ &\quad \le \omega(0, \mathbf{T}, \mathbf{D} \setminus F). \end{aligned}$$

This, along with (2.7.61)-(2.7.63) gives (2.7.60) and the proof is complete.

■

Chapter 3

Markov Inequality and Green Functions

3.1 Definitions and results

For notions in potential theory - Green's function, equilibrium measure etc. - we shall use the notations of Chapter 2. Let Π_n denote the set of algebraic polynomials of degree $\leq n$ and $E \subset \mathbf{C}$ be compact with positive logarithmic capacity. Recall from the Introduction that we say that E satisfies the Markov inequality with a polynomial factor if there exist $C,\ k > 0$ such that

$$||P_n'||_E \leq Cn^k ||P_n||_E \tag{3.1.1}$$

holds for every n and $P_n \in \Pi_n$.

Let Ω be the outer domain of E. Green's function g_Ω is said to be Hölder continuous if there exist $C_1,\ \alpha > 0$ such that

$$g_\Omega(z) \leq C_1 \Big(\mathrm{dist}(z, E)\Big)^\alpha. \tag{3.1.2}$$

for all $z \in \mathbf{C}$.

Theorem 3.1.1. *Let E be a compact subset of the plane such that the unbounded component Ω of $\overline{\mathbf{C}} \setminus E$ is regular. Then the following are pairwise equivalent.*

i) *Optimal Markov inequality holds on E, i.e. there exists a $C > 0$ such that*

$$||P_n'||_E \leq Cn ||P_n||_E \tag{3.1.3}$$

for every polynomial $P_n \in \Pi_n$, $n = 1, 2, \ldots$.

ii) *Green's function g_Ω is Lipschitz continuous, i.e. there exists a $C_1 > 0$ such that*

$$g_\Omega(z) \leq C_1 \mathrm{dist}(z,\ E) \tag{3.1.4}$$

for every $z \in \mathbf{C}$.

iii) *The equilibrium measure μ_E of E satisfies a Lipschitz type condition, i.e. there exists a $C_2 > 0$ such that*

$$\mu_E\Big(D_\delta(z)\Big) \le C_2\delta \tag{3.1.5}$$

for every $z \in E$ and $\delta > 0$.

If, in addition, Ω is simply connected, then **i)–iii)** *are also equivalent to*

iv) *The conformal mapping Φ from Ω onto the exterior of the unit disk is Lipschitz continuous, i.e.*

$$|\Phi(z_1) - \Phi(z_2)| \le C_3|z_1 - z_2|, \qquad z_1, z_2 \in \Omega.$$

We mention that each of **i), ii)** and **iv)** implies regularity, so in their equivalence the regularity assumption is not needed. However, **iii)** may be true without Ω being regular, in which case **iii)** is not equivalent to the other statements. Consider e.g. as E the unit disk together with the single point $\{2\}$. In this case μ_E is the normalized arc measure on the unit circle and the one point set $\{2\}$ does not carry any mass. Thus, **iii)** holds, but the other statements in the theorem are not true.

There is a local version of our theorem which we formulate now. We say that E has the optimal local Markov property at the point $z_0 \in \partial\Omega$ if there is a constant C such that

$$|P_n^{(k)}(z_0)| \le C^k n^k \|P_n\|_E, \qquad P_n \in \Pi_n,\ n = 1, 2, \ldots$$

for all $k = 1, 2, \ldots$.

Theorem 3.1.2. *Let E be a compact subset of the plane, Ω the unbounded component of $\overline{\mathbf{C}} \setminus E$, and suppose that $z_0 \in \partial\Omega$ is a regular boundary point of Ω (i.e. $g_\Omega(z_0) = 0$). Then the following are equivalent.*

i) *E has the optimal Markov property at z_0.*

ii) *Green's function g_Ω is Lipschitz continuous at z_0, i.e.*

$$g_\Omega(z) \le C_1|z - z_0|$$

with some constant C_1.

iii) *The equilibrium measure μ_E of E satisfies a Lipschitz type condition at z_0, i.e. there exists a $C_2 > 0$ such that*

$$\mu_E\Big(D_\delta(z_0)\Big) \le C_2\delta$$

for every $\delta > 0$.

If, in addition, Ω is simply connected, then **i)–iii)** *are also equivalent to*

iv) *The conformal mapping Φ from Ω onto the exterior of the unit disk is Lipschitz continuous at z_0.*

In the last statement we think of Φ as being extended continuously onto the boundary of Ω.

It is worth noticing that much more is true than the equivalence of **ii)** and **iii)**, namely we can give a very precise two sided estimate for Green's function in terms of the equilibrium measure.

Theorem 3.1.3. *Let E be a compact subset of the plane, Ω the unbounded component of $\overline{\mathbf{C}} \setminus E$, and suppose that $z_0 \in \partial\Omega$ is a regular boundary point of Ω. Then for every $0 < r < 1$ we have*

$$\int_0^r \frac{\mu_E(D_t(z_0))}{t} dt \leq \sup_{|z-z_0|=r} g_\Omega(z) \leq 3 \int_0^{4r} \frac{\mu_E(D_t(z_0))}{t} dt. \tag{3.1.6}$$

Let F be a connected component of E which is of positive distance from the set $E \setminus F$. Then on F the Lipschitz continuity of Green's functions g_Ω and $g_{\overline{\mathbf{C}}\setminus F}$ are equivalent, and for the latter one can use the conformal mapping characterization given in Theorem 3.1.1, **iv)**. In particular, if g_Ω is Lipschitz, then so is every $g_{\overline{\mathbf{C}}\setminus F}$ for every component F which is of positive distance from F. In Section 3.4 (Example 2) we shall show that this need not be true for components of F that are not of positive distance from $E \setminus F$, even if they are consisting of more than one point. This will be based on a construction in Example 1, which exhibits a set E with infinitely many connected components and Lipschitz-continuous Green function, which is an interesting fact in itself (note that the simplest example of a set E satisfying Theorem 3.1.1 is any finite union of disjoint smooth simple closed curves, and one is tempted to think that sets appearing in Theorem 3.1.1 can have only finitely many connected components).

In what follows $C_r(a)$ resp. $D_r(a)$ denote the circle resp. open disk centered at a with radius r. We shall need the Bernstein-Walsh inequality

$$|P_n(z)| \leq e^{ng_\Omega(z)} \|P_n\|_E \tag{3.1.7}$$

valid for all polynomials P_n of degree $n = 1, 2, \ldots$, as well as its sharpness:

$$e^{g_\Omega(z)} = \sup_{P_n \in \Pi_n,\ \|P_n\|_E \leq 1} |P_n(z)|^{1/n} \tag{3.1.8}$$

valid for any $z \in \Omega$.

3.2 Proof of Theorems 3.1.1 and 3.1.2

We shall only prove Theorem 3.1.1, the proof of the local version (Theorem 3.1.2) is similar.

First we show that **(i)** is equivalent to **(ii)**. Suppose (3.1.4) holds true. Let $z \in \mathbf{C}$ and apply Cauchy's formula to $P_n \in \Pi_n$ on $C_r(z)$:

$$P_n'(z) = \frac{1}{2\pi i}\int_{C_r(z)} \frac{P_n(\xi)}{|\xi - z|^2}d\xi = \frac{1}{2\pi}\int_0^{2\pi} \frac{P_n(z+re^{it})}{re^{it}}dt.$$

Taking absolute value on both sides and setting $r = 1/n$ we get

$$|P_n'(z)| \leq \frac{n}{2\pi}\int_0^{2\pi} |P_n(z + \frac{1}{n}e^{it})|dt. \tag{3.2.9}$$

For $z \in E$ the Bernstein-Walsh inequality (3.1.7) and (3.1.4) give

$$\begin{aligned} |P_n(z+\frac{1}{n}e^{it})| &\leq ||P_n||_E e^{ng_\Omega(z+\frac{1}{n}e^{it})} \\ &\leq ||P_n||_E e^{nC_1\frac{1}{n}} = ||P_n||_E e^{C_1}. \end{aligned}$$

This and (3.2.9) prove **(i)**.

Conversely, suppose (3.1.3). Then it follows by induction that

$$||P_n^{(m)}||_E \leq C^m n^m ||P_n||_E, \qquad m = 1, 2, \ldots. \tag{3.2.10}$$

(3.1.4) is obviously true for $z \in E$, therefore we can assume $z \in \mathbf{C} \setminus E$. Choose $z_0 \in E$ such that $\operatorname{dist}(z,\ E) = |z - z_0|$. Suppose $P_n \in \Pi_n$ and $||P_n||_E \leq 1$. We can use the (finite) Taylor-expansion of P_n around z_0 and (3.2.10) to obtain:

$$\begin{aligned} |P_n(z)| &= |P_n(z_0)| + \Big|\sum_{m=1}^{\infty} \frac{P_n^{(m)}(z_0)}{m!}(z-z_0)^m\Big| \\ &\leq 1 + \sum_{m=1}^{\infty}\frac{C^m n^m}{m!}|z-z_0|^m = e^{Cn|z-z_0|}. \end{aligned}$$

Thus $|P_n(z)|^{1/n} \leq e^{C|z-z_0|}$ and this, using the representation (3.1.8) and the choice of z_0, proves **(ii)**.

The equivalence of **ii)** and **iii)** follows from Theorem 3.1.3.

Finally, suppose that Ω is simply connected, and Φ is the conformal map of Ω onto the complement of the unit disk. Then $g_\Omega(z)$ is the real part of $\log \Phi(z)$ (this follows e.g. from the defining properties of g_Ω), hence the Lipschitz property of Φ implies the Lipschitz property of g_Ω. Conversely, suppose that g_Ω is Lipschitz continuous. Since g_Ω is infinitely differentiable, this happens precisely if the partial derivatives $\partial g_\Omega/\partial x$ and $\partial g_\Omega/\partial y$ are bounded in Ω. But then using the Cauchy-Riemann equations it follows that the partial derivatives of $\Im \log \Phi(z)$ are also bounded, hence $\log \Phi$ is a Lipschitz function. But then so is Φ in any bounded set. On the other hand, around infinity the derivative of Φ tends to a constant (recall that $\lim_{z\to\infty} F(z)/z$ exists and apply Cauchy's formula for the derivative), and the proof is complete.

■

3.3 Proof of Theorem 3.1.3

Without loss of generality let $z_0 = 0$, and $r > 0$. It follows from the representation (2.1.1) and the assumed regularity ($g_\Omega(0) = 0$) that

$$g_\Omega(re^{i\varphi}) = g_\Omega(re^{i\varphi}) - g_\Omega(0) = \int \log \frac{|re^{i\varphi} - t|}{|t|} d\mu_E(t).$$

Since (see [13, p. 29])

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{|re^{i\varphi} - t|}{|t|} d\varphi = \log^+ \frac{r}{|t|},$$

we get with Fubini's Theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g_\Omega(re^{i\varphi}) d\varphi \quad = \quad \int \left(\log^+ \frac{r}{|t|} \right) d\mu_E(t).$$

With $S(t) = \mu_E(D_t(0))$ the last integral can be written as

$$\int_0^r \log \frac{r}{t} dS(t) = \int_0^r \frac{S(t)}{t} dt,$$

where the equality follows by integration by parts. Thus,

$$\int_0^r \frac{S(t)}{t} dt \leq \sup_\varphi g_\Omega(re^{i\varphi}),$$

which is the left inequality in (3.1.6).

To prove the right inequality we write again for $|z| \leq 2r$

$$\begin{aligned} 0 \leq g_\Omega(z) &= g_\Omega(z) - g_\Omega(0) = \int \log \frac{|z-t|}{|t|} d\mu_E(t) \\ &= \int_{D_{2r}(0)} + \int_{E \setminus D_{2r}(0)} = h_1(z) + h_2(z) \end{aligned}$$

where both functions h_1 and h_2 vanish at 0. Clearly, for $|z| \leq 2r$ we have

$$\begin{aligned} h_1(z) = \int_{D_{2r}(0)} \log \frac{|z-t|}{|t|} d\mu_E(t) &\leq \int_{D_{2r}(0)} \log \frac{4r}{|t|} d\mu_E(t) \\ &\leq \int_{D_{4r}(0)} \log \frac{4r}{|t|} d\mu_E(t), \end{aligned}$$

and with the function $S(t)$ defined above this can be written again as $h_1(z) \leq m$ where

$$m = \int_0^{4r} \frac{S(t)}{t} dt.$$

Therefore, we have for all $|z| \le 2r$

$$0 \le h_1(z) + h_2(z) \le m + h_2(z).$$

But the function on the right hand side is harmonic in $D_{2r}(0)$ and takes the value $m + h_2(0) = m$ at the origin, therefore we obtain from Harnack's inequality (2.1.5) (applied to the function $u(w) = m + h_2(w2r)$ and to the point $z/2r$) that for $|z| = r$ the inequality $m + h_2(z) \le 3(m + h_2(0)) = 3m$ holds. Together with this we have for $|z| = r$

$$g(z) = h_1(z) + h_2(z) \le m + 2m = 3m,$$

and this is the right inequality in (3.1.6).

■

3.4 Constructions of Examples 1 and 2

Example 1 We are going to construct the first connected component E_0 of E in the following way. Let $0 < \theta < \pi$ be arbitrary. Set $J_\theta = \{e^{it} : \theta \le |t| \le \pi\}$ and

$$A_\theta = \bigcup_{a \in J_\theta} \overline{D}_{1/10}(a).$$

Thus, A_θ is a "thickened arc" of the unit circle $\mathbf{T}$. There is a largest θ^* such that the complement of A_{θ^*} is not connected (the two "arms" of A_{θ^*} touch each other).

Let $\omega(z, J, G)$ be the harmonic measure corresponding to the set $J \subseteq \partial G$ in the domain G. Clearly $\omega(0, \mathbf{T} \setminus A_\theta, \mathbf{D}) \to 0$ if $\theta \searrow \theta^*$. Fix $\theta_0 > \theta^*$ such that $\omega(0, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/9$.

Set $E_0 = A_{\theta_0}$, for every integer $n \ge 1$ define $E_n = (1/3)^n E_0$, and let $E = \bigcup_{n=1}^{\infty} E_n \cup \{0\}$. This is a compact set consisting of infinitely many components such that $\Omega = \overline{\mathbf{C}} \setminus E$ is connected. Below we show that g_Ω is Lipschitz.

Since $\omega(0, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/9$, it follows from Harnack's inequality ((2.1.5) with $|z| = 1/2$) that $\omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/3$ for all $z \in D_{1/2}(0)$, in particular this is true for $z \in C_{1/3}(0)$. Both $\omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D})$ and $g_{\overline{\mathbf{C}} \setminus E}(z)$ are positive harmonic functions in $\mathbf{D} \setminus E$,

$$g_{\overline{\mathbf{C}} \setminus E}(z) \le g_{\overline{\mathbf{C}} \setminus E_0}(z) \le \omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D})$$

on $\mathbf{T} \setminus A_{\theta_0}$ and $g_{\overline{\mathbf{C}} \setminus E}$ vanishes on E, while $\omega(z, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D})$ is nonnegative there. Thus, the same inequality holds true everywhere in $\mathbf{D} \setminus E$ by the maximum principle for harmonic functions. Hence

$$g_{\overline{\mathbf{C}} \setminus E}(z) \le 1/3, \qquad z \in D_{1/3}(0) \setminus E. \tag{3.4.11}$$

Now consider the domain $D_{1/3}(0) \setminus E$ and the positive harmonic functions $\omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big)$ and $3g_{\overline{\mathbf{C}}\setminus E}(z)$. Since $3g_{\overline{\mathbf{C}}\setminus E}(z)$ vanishes on E and, in view of (3.4.11), it is at most 1 on $C_{1/3}(0) \setminus E$, where $\omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big)$ takes the (boundary) value 1, we have

$$3g_{\overline{\mathbf{C}}\setminus E}(z) \le \omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big)$$

in $D_{1/3}(0) \setminus E$. Because of similarity,

$$\omega\Big(0, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big) = \omega(0, \mathbf{T} \setminus A_{\theta_0}, \mathbf{D}) < 1/9,$$

which implies via Harnack's inequality as before that

$$\omega\Big(z, C_{1/3}(0) \setminus E_1, D_{1/3}(0)\Big) < 1/3, \qquad z \in D_{1/3}(0).$$

Thus, we can conclude $3g_{\overline{\mathbf{C}}\setminus E}(z) \le 1/3$ if $z \in C_{1/9}(0)$, i.e.

$$g_{\overline{\mathbf{C}}\setminus E}(z) \le 1/9, \qquad z \in D_{1/9}(0) \setminus E.$$

In a similar manner it follows by induction that

$$g_{\overline{\mathbf{C}}\setminus E}(z) \le \left(\frac{1}{3}\right)^n, \qquad z \in D_{(1/3)^n} \setminus E. \tag{3.4.12}$$

We are going to show that Green's function $g_{\overline{\mathbf{C}}\setminus E}$ is Lipschitz continuous. Let $z_0 \in \mathbf{C} \setminus E$ be arbitrarily chosen. Suppose the component closest to z_0 is E_{n_0}, i.e. $\mathrm{dist}(z_0,\ E) = \mathrm{dist}(z_0,\ E_{n_0})$. We will compare $g_{\overline{\mathbf{C}}\setminus E_{n_0}}$ and $g_{\overline{\mathbf{C}}\setminus E}$. It follows from the construction of E that E_{n_0} is included in the disk $D_{1/(2\cdot 3^{n_0-1})}(0)$. Thus the monotonicity of Green functions gives for $|z| = (1/3)^{n_0-1}$

$$g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z) \ge g_{\overline{\mathbf{C}}\setminus \overline{D}_{1/(2\cdot 3^{n_0-1})}(0)}(z) = \log \frac{(1/3)^{n_0-1}}{1/(2\cdot 3^{n_0-1})} = \log 2. \tag{3.4.13}$$

On the other hand, according to (3.4.12), $g_{\overline{\mathbf{C}}\setminus E}(z) \le (1/3)^{n_0-1}$ holds for $z \in C_{(1/3)^{n_0-1}}$. Hence

$$g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z) \ge \frac{3^{n_0-1}}{\log 2} g_{\overline{\mathbf{C}}\setminus E}(z), \qquad z \in C_{(1/3)^{n_0-1}}. \tag{3.4.14}$$

Since both sides are positive and harmonic in $D_{(1/3)^{n_0-1}} \setminus E$, inequality (3.4.14) remains true throughout this domain, and by the definition of n_0 the point z_0 is included in this set.

Let C denote the Lipschitz-constant of E_0, i.e.

$$g_{\overline{\mathbf{C}}\setminus E_0}(z) \le C\mathrm{dist}(z,\ E_0).$$

Using the fact $g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z) = g_{\overline{\mathbf{C}}\setminus E_0}(3^{n_0}z)$ we obtain

$$\begin{aligned}
g_{\overline{\mathbf{C}}\setminus E}(z_0) &\leq \frac{\log 2}{3^{n_0-1}} g_{\overline{\mathbf{C}}\setminus E_{n_0}}(z_0) = \frac{\log 2}{3^{n_0-1}} g_{\overline{\mathbf{C}}\setminus E_0}(3^{n_0}z_0) \\
&\leq \frac{\log 2}{3^{n_0-1}} C \mathrm{dist}(3^{n_0}z_0,\ E_0) \\
&= \frac{\log 2}{3^{n_0-1}} C 3^{n_0} \mathrm{dist}(z_0,\ E_{n_0}) \\
&= 3C\log 2 \cdot \mathrm{dist}(z_0,\ E).
\end{aligned}$$

Therefore $g_{\overline{\mathbf{C}}\setminus E}(z)$ is Lipschitz continuous with Lipschitz constant $3C\log 2$.

■

Example 2 Let E be the set from the preceding section (Example 1), and let

$$E^* = E \cup [0,2] \cup \overline{D_1(3)},$$

i.e. we add to E the segment $[0,2]$ and to that attach the disk $\overline{D_1(2)}$. Then $F = [0,2] \cup \overline{D_1(3)}$ is a connected component of E^* for which $g_{\overline{\mathbf{C}}\setminus F}$ is not Lipschitz (in fact, around 0 this behaves like $g_{\mathbf{C}\setminus[0,2]}$, which is only Lipschitz $1/2$ smooth at 0). Thus, it is left to show that $g_{\overline{\mathbf{C}}\setminus E^*}$ is a Lipschitz function.

Since $g_{\overline{\mathbf{C}}\setminus E^*}$ is bounded by either of $g_{\overline{C}\setminus E}$, $g_{\overline{\mathbf{C}}\setminus[0,3]}$ and $g_{\overline{\mathbf{C}}\setminus\overline{D_1(3)}}$, the Lipschitz property is clear on E (Example 1), on $[1,2]$ and on $\overline{D_1(3)}$.

Thus, we have to worry about points close to the segment $[0,1]$. Let $z_0 \notin E^*$ be an arbitrary point, and let x_0 be a point in E that is closest to z_0. If $x_0 \in E \cup [1,2] \cup \overline{D_1(3)}$, then according to what we have just said, $g_{\overline{\mathbf{C}}\setminus E^*}(z) \leq C|z - x_0|$ with some constant C. Thus, let us assume that $x_0 \in (0,1)$. Choose n_0 so that $3^{-n_0-1} \leq x_0 < 3^{-n_0}$. Since $g_{\overline{\mathbf{C}}\setminus[-1,1]}(z) \geq 1$ for $|z| = 2$, we have

$$g_{\overline{\mathbf{C}}\setminus[0,2\cdot 3^{-n_0}]}(z) \geq 1, \qquad |z| = 3^{-n_0+1}.$$

(3.4.12) implies

$$3^{n_0-1} g_{\overline{\mathbf{C}}\setminus E^*}(z) \leq 3^{n_0-1} g_{\overline{\mathbf{C}}\setminus E}(z) \leq 1, \qquad |z| = 3^{-n_0+1},$$

hence by the comparison technique applied several times before we get that

$$3^{n_0-1} g_{\overline{\mathbf{C}}\setminus E^*}(z) \leq g_{\overline{\mathbf{C}}\setminus[0,2\cdot 3^{-n_0}]}(z)$$

for all $|z| \leq 3^{-n_0+1}$. In particular,

$$g_{\overline{\mathbf{C}}\setminus E^*}(z_0) \leq 3^{-n_0+1} g_{[0,2\cdot 3^{-n_0}]}(z_0) = 3^{-n_0+1} g_{\overline{\mathbf{C}}\setminus[0,2]}(3^{n_0}z_0).$$

Here $3^{n_0}z_0$ lies in the annulus $1/3 \leq |w| \leq 1$, on which $g_{\overline{\mathbf{C}}\setminus[0,2]}$ is Lipschitz continuous, and the closest point in $[0,2]$ to $3^{n_0}z_0$ is $3^{n_0}x_0$. Hence we get with some constant C

$$g_{\overline{\mathbf{C}}\setminus E^*}(z_0) \leq 3^{-n_0+1}C|3^{n_0}z_0 - 3^{n_0}x_0| = 3C|z_0 - x_0|$$

and the proof is over.

■

Chapter 4

A Wiener-type Condition in $\mathbf{R}^d$

4.1 Preliminaries

We shall use $c, c_0, c_1, c_2, \ldots$ to denote positive constants. $B_r(x)$ resp. $\overline{B}_r(x)$ denote the open resp. closed ball about the point x of radius r, and $S_r(x)$ is the bounding surface of these balls. $||\mu||$ denotes the total mass of the measure μ.

For the notions of classical potential theory in $\mathbf{R}^d$ see e.g. [7]. The Newtonian potential of the measure ν is defined as

$$U^\nu(x) := \int \frac{1}{|x-t|^{d-2}} d\nu(t),$$

and the energy integral is

$$I(\nu) := \int\int \frac{1}{|x-t|^{d-2}} d\nu(t) d\nu(x).$$

The capacity of a compact set E is the number

$$\mathrm{cap}(E) := \frac{1}{\inf I(\nu)},$$

where the infimum is taken over all probability measures on E. There is a unique measure λ for which the infimum (minimum) is attained. $\mu_E = \mathrm{cap}(E)\lambda$ is called the equilibrium measure of E. E.g. the equilibrium measure of $\overline{B}_r$ (and S_r) is

$$\mu_{\overline{B}_r} = r^{d-2}\sigma_{S_r}, \tag{4.1.1}$$

where σ_{S_r} is the $(d-1)$-dimensional normalized surface area measure on S_r.

If the compact set E has positive capacity then for the Newtonian potential of the equilibrium measure we have

$$U^{\mu_E}(z) = 1, \qquad \text{for q.e. } x \in E, \tag{4.1.2}$$

where q.e. means "quasi-everywhere", i.e. with the exception of a set of zero capacity.

If E is of positive capacity, then μ_E has finite energy. Hence a set of zero capacity has zero μ_E-measure, and so if a property holds quasi-everywhere, i.e. with the exception of a set of zero capacity, then it also holds μ_E-almost everywhere.

If σ is a measure supported on the compact set F and $U^\sigma(x) \leq 1$ for all $x \in \mathbf{R}^d$, then the set

$$K := \{x \ : \ U^\sigma(x) \geq \gamma\} \tag{4.1.3}$$

has capacity at most $(1/\gamma)\mathrm{cap}(F)$. In fact, if K is of positive capacity, then the inequality

$$\frac{U^\sigma(x)}{\mathrm{cap}(F)} \geq \frac{U^{\mu_K}(x)}{\mathrm{cap}(K)} + \frac{\gamma}{\mathrm{cap}(F)} - \frac{1}{\mathrm{cap}(K)}$$

holds true for quasi-every $x \in K$. Hence this is true for μ_K-almost all x, and then the principle of domination ([7, Theorem 1.27]) gives the same inequality for all $x \in \mathbf{R}^d$. Now

$$\mathrm{cap}(K) \leq \frac{1}{\gamma}\mathrm{cap}(F) \tag{4.1.4}$$

follows if we let x tend to infinity.

We shall also need the following result. There is a positive constant c such that if $A \subseteq S_1$ and $\beta(A)$ denotes the $(d-1)$-dimensional surface area measure of A then

$$\beta(A) \leq c\sqrt{\mathrm{cap}(A)}. \tag{4.1.5}$$

Indeed, if λ denotes the normalized surface area measure on S_1 then based on the definition of capacity:

$$\begin{aligned}\frac{1}{\mathrm{cap}(A)} &\leq \frac{1}{\beta(A)^2}\int_A\int_A \frac{1}{|x-t|^{d-2}}d\lambda(x)d\lambda(t)\\ &\leq \frac{1}{\beta(A)^2}\int_{S_1}\int_{S_1} \frac{1}{|x-t|^{d-2}}d\lambda(x)d\lambda(t).\end{aligned}$$

Hence, (4.1.5) follows with

$$c = \sqrt{\int_{S_1}\int_{S_1} \frac{1}{|x-t|^{d-2}}d\lambda(x)d\lambda(t)}.$$

Let G be a domain with compact boundary and with $\mathrm{cap}(\partial G) > 0$, and let ν be a Borel measure supported on G (by which we mean that $\nu(\mathbf{R}^d \setminus G) = 0$). We shall again need the concept of balayage of ν out of G, see e.g. [14, Sec. II.4] or [7, Chapter IV]. The definition is slightly different from the two dimensional case. It is the unique Borel measure $\overline{\nu}$ supported on ∂G with the properties:

- $\|\overline{\nu}\| \leq \|\nu\|$, where $\|\nu\|$ denotes the total mass of ν,
- for all $x \in \partial G$ with the exception of a set of capacity 0

$$U^{\overline{\nu}}(x) = U^{\nu}(x), \tag{4.1.6}$$

- $\overline{\nu}$ is so called C-continuous, i.e. the $\overline{\nu}$-measure of any set of zero capacity is zero.

For regular G the exceptional set is empty. If G is bounded, then $\overline{\nu}$ has the same total mass as ν. If ν is not supported on G, then taking its balayage out of G is understood in the sense that we take the balayage of $\nu\Big|_G$ and leave the rest of ν unchanged. In this sense if $G_1 \subseteq G_2$, then taking balayage out of G_2 can be done in two steps: first take balayage out of G_1, and then take the balayage of the resulting measure out of G_2.

Perhaps the most important connection between equilibrium and balayage measures is the fact that if $E \subseteq F$ are compact sets of positive capacity, then μ_E is the balayage of μ_F onto E (i.e. out of the unbounded component of $\overline{\mathbf{R}}^d \setminus E$).

If $K \subseteq \partial G$ are compact sets of positive capacity, then the harmonic measure
$\omega(x, K, G)$ is the unique solution of the generalized Dirichlet-problem in G corresponding to the characteristic function of K in ∂G. There is a connection between harmonic and balayage measures: for $a \in G$ the equality

$$\overline{\delta_a}(K) = \omega(a, K, G) \tag{4.1.7}$$

holds, where δ_a denotes the point mass (Dirac measure) placed at the point a and $\overline{\delta_a}$ denotes its balayage out of G (see e.g. [14, Appendix A3, (3.3)] or [7, IV.3]).

Green's function of G with pole at $y \in G$ is defined as

$$g_G(x, y) = U^{\delta_y}(x) - U^{\overline{\delta_y}}(x).$$

Let $0 < r < R$, $x \in S_R$ and let $\overline{\delta_x}$ be the balayage of δ_x out of $\mathbf{R}^d \setminus \overline{B}_r$. This measure is given by the formula

$$\frac{d\overline{\delta_x}(y)}{d\sigma_{S_r}} = r^{d-2}\frac{R^2 - r^2}{|x-y|^d}, \tag{4.1.8}$$

where $y \in S_r$ and σ_{S_r} is the normalized surface area measure on S_r. Indeed, Poisson's formula (see e.g. [3, Section 1.3, (1.3.1)]) gives

$$\frac{d\overline{\delta_x}(y)}{d\sigma} = \frac{1}{\omega_n r}\frac{R^2 - r^2}{|x-y|^d},$$

where σ is the surface area measure (not normalized) on S_r and $\omega_n = \sigma(S_1)$. Multiplying by $d\sigma/d\sigma_{S_r} = \omega_n r^{d-1}$ we obtain (4.1.8). Thus, for the density of $\overline{\delta_x}$ with respect to σ_{S_r} we have the inequalities

$$r^{d-2}\frac{R-r}{(R+r)^{d-1}} \leq \frac{d\overline{\delta_x}(y)}{d\sigma_{S_r}} \leq r^{d-2}\frac{R+r}{(R-r)^{d-1}}. \tag{4.1.9}$$

Multiplying by R^{d-2} and letting $R \to \infty$ we get that $\overline{\delta_\infty}$ can be understood as r^{d-2} -times the normalized surface area measure on S_r, which is the equilibrium measure of S_r ($\overline{B}_r$). On applying this for a large r containing the set E of positive capacity we can see that if $\widehat{\cdot}$ denotes balayage onto E, then $\mu_E = \widehat{\delta_\infty}$. It also follows that $r^{d-2}\widehat{\sigma_{S_r}} = \mu_E$. But $\widehat{\sigma_{S_r}} = \int \widehat{\delta_a} d\sigma_{S_r}(a)$, so it follows from Harnack's inequality that there are constants c_r, C_r such that for $a \in S_r$ we have $c_r\mu_E \leq \widehat{\delta_a} \leq C_r\mu_E$. Another application of Harnack's inequality gives

$$c_a\mu_E \leq \widehat{\delta_a} \leq C_a\mu_E \tag{4.1.10}$$

for any a lying in the unbounded component of $\mathbf{R}^d \setminus E$ with some constants c_a, C_a.

Let μ be a measure on S_r. The lower Radon-Nikodym derivative (density) of μ with respect to normalized surface area measure on S_r is defined as follows (see e.g. [6, Chapter 3] or [12, Chapter VII]). Let $x_0 \in S_r$ and $0 < \tau < 1$. Then the cone

$$C(x_0, \tau) := \{x \in \mathbf{R}^d \ : \ \frac{\langle x, x_0\rangle}{r\|x\|} \geq 1 - \tau\}$$

determines a closed polar cap $K(x_0, \tau) = C(x_0, \tau) \cap S_r$ centered at x_0. The lower derivative of μ with respect to σ_{S_r} at x_0 is

$$v(x_0) := \liminf_{\sigma(K)\to 0} \mu(K)/\sigma(K),$$

where K is an arbitrary closed polar cap containing $x_0 \in S_r$. Wherever the ordinary Radon-Nikodym derivative exists, it agrees with v. Therefore, $v(y)d\sigma_{S_r}(y) \leq d\mu(y)$.

Finally, let us recall that the Newtonian capacity is subadditive: if $F = \cup_{i=1}^k F_i$, then

$$\mathrm{cap}(F) \leq \sum_{i=1}^k \mathrm{cap}(F_i). \tag{4.1.11}$$

In particular, one of the sets F_i must have capacity $\geq \mathrm{cap}(F)/k$. On the other hand, if the distance between the sets F_1 and F_2 is at least l, then

$$\mathrm{cap}(F_1 \cup F_2) \geq \frac{\mathrm{cap}(F_1) + \mathrm{cap}(F_2)}{1 + 2\frac{\mathrm{cap}(F_1)\mathrm{cap}(F_2)}{l^{d-2}(\mathrm{cap}(F_1)+\mathrm{cap}(F_2))}}. \tag{4.1.12}$$

Indeed, set

$$\nu = \frac{1-t}{\operatorname{cap}(F_1)}\mu_{F_1} + \frac{t}{\operatorname{cap}(F_2)}\mu_{F_2},$$

where t is between 0 and 1. Then ν is a probability measure and

$$I(\nu) \le \frac{(1-t)^2}{\operatorname{cap}(F_1)} + \frac{t^2}{\operatorname{cap}(F_2)} + \frac{2t(1-t)}{l^{d-2}\operatorname{cap}(F_1)\operatorname{cap}(F_2)}||\mu_{F_1}||||\mu_{F_2}||.$$

This yields with $||\mu_{F_1}|| = \operatorname{cap}(F_1)$ and $||\mu_{F_2}|| = \operatorname{cap}(F_2)$

$$\operatorname{cap}(F_1 \cup F_2) \ge \frac{1}{\frac{(1-t)^2}{\operatorname{cap}(F_1)} + \frac{t^2}{\operatorname{cap}(F_2)} + \frac{2t(1-t)}{l^{d-2}}}.$$

Now $t = \operatorname{cap}(F_2)/(\operatorname{cap}(F_1) + \operatorname{cap}(F_2))$ gives (4.1.12).

4.2 Results

Let $E \subset \mathbf{R}^d$ be a compact set of positive Newtonian capacity, Ω the unbounded component of $\mathbf{R}^d \setminus E$ and $g_\Omega(x,a)$ the Green's function of Ω with pole at $a \in \Omega$. We extend g_Ω to $\partial\Omega$ by

$$g_\Omega(x,a) = \limsup_{w \to x,\ w \in \Omega} g_\Omega(w,a),$$

and to $\mathbf{R}^d \setminus \overline{\Omega}$ by setting $g_\Omega(x,a) = 0$ there. We are interested in the behavior of g_Ω at a boundary point of Ω, which we assume to be 0, i.e. let $0 \in \partial\Omega$.

Let $B_r = B_r(0)$ be the ball of radius r about the origin, and we shall denote its closure by $\overline{B}_r$ and its boundary (the sphere of center 0 and radius r) by S_r. With

$$E^n = E \cap (\overline{B}_{2^{-n+1}} \setminus B_{2^{-n}}) = \left\{ x \in E \ :\ 2^{-n} \le |x| \le 2^{-n+1} \right\}$$

the regularity of the boundary point 0 was characterized by Wiener (see e.g. [7, Theorem 5.2]): Green's function $g_G(x,a)$ ($a \in \Omega$) is continuous at $0 \in \partial\Omega$ (i.e. 0 is a regular boundary point of E) if and only if

$$\sum_{n=1}^{\infty} \operatorname{cap}(E^n) 2^{n(d-2)} = \infty, \tag{4.2.13}$$

where $\operatorname{cap}(E^n)$ denotes the (d-dimensional) Newtonian capacity of E^n. We would like to characterize in a similar manner the stronger Hölder continuity:

$$g_\Omega(x,a) \le C|x|^\kappa \tag{4.2.14}$$

with some positive numbers C, κ.

Following the definitions in [5], for $\varepsilon > 0$ set

$$\mathcal{N}_E(\varepsilon) = \{n \in \mathbf{N} \ : \ \operatorname{cap}(E^n) \geq \varepsilon 2^{-n(d-2)}\}, \tag{4.2.15}$$

and we say that a subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ of the natural numbers is of positive lower density if

$$\liminf_{N\to\infty} \frac{|\mathcal{N} \cap \{0, 1, \ldots, N\}|}{N+1} > 0.$$

Let $x_0 \in S_1$, $0 < \tau < 1$, $\ell > 0$ and set

$$C(x_0, \tau, \ell) := \{x \in B_\ell \ : \ \frac{\langle x, x_0 \rangle}{\|x\|} \geq 1 - \tau\}. \tag{4.2.16}$$

This is a cone with vertex at 0 and x_0 as the direction of its axis. We say that E satisfies the cone condition if

$$C(x_0, \tau, \ell) \subset \Omega \tag{4.2.17}$$

with some $x_0 \in S_1$, τ and $\ell > 0$, which means that Ω contains a cone with vertex at 0.

Theorem 4.2.1. **a)** *If $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$ then Green's function g_Ω is Hölder continuous at 0.*

b) *If Green's function g_Ω is Hölder continuous at 0 and E satisfies the cone condition then $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$.*

The importance of the Hölder property is explained by the following result. Let G be a domain in $\mathbf{R}^d$ with compact boundary such that 0 is on the boundary of G. We may assume that $G \not\subseteq B_1$, and set $E = \overline{B}_1 \setminus G$. Then $\Omega := \mathbf{R}^d \setminus E = G \cup (\mathbf{R}^d \setminus \overline{B}_1)$ is a domain larger than G and 0 is on the boundary of Ω. If f is a bounded Borel function on the boundary of G, then let u_f denote the Perron-Wiener-Brelot solution of the Dirichlet problem in G with boundary function f. We think u_f to be extended to ∂G as $u_f = f$ there.

Lemma 4.2.2. *Suppose that 0 is a regular boundary point of G. Then the following are equivalent.*

1) *$g_G(\cdot, a)$ is Hölder continuous at 0 for $a \in G$.*

2) *$\mu_E(\overline{B}_r) \leq C r^{d-2+\kappa}$ for some C, $\kappa > 0$ and all $r < 1$.*

If, in addition, G satisfies the cone condition at 0, then **1)** - **2)** *are also equivalent to*

3) *If f is Hölder continuous at 0, then so is u_f.*

Note also that it is indifferent if "for $a \in G$" in **1)** is understood as "for some $a \in G$" or as "for all $a \in G$".

4.3 Proof of Theorem 4.2.1

Proof of a) in Theorem 4.2.1

Let us suppose that $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$. Clearly then 0 is a regular boundary point of Ω, hence by the equivalence of **1)** and **2)** in Lemma 4.2.2 it is sufficient to verify $\mu_E(\overline{B}_r) \le r^{d-2+\kappa}$ for some $\kappa > 0$ and sufficiently small r.

Let F be a compact set such that 0 is on the boundary of the unbounded component of $\mathbf{R}^d \setminus F$, and let $\widehat{\nu}$ denote the balayage of some measure ν out of $\mathbf{R}^d \setminus (F \cup \overline{B}_1)$. First we verify that if $\mathrm{cap}(F \cap (\overline{B}_8 \setminus B_4)) \ge 4\varepsilon$, with some $\varepsilon \le 1/8$, then

$$\widehat{\sigma_{S_8}}\Big|_{\overline{B}_1} \le \frac{1}{8^{d-2}}\left(1 - \frac{\varepsilon}{9^d}\right)\sigma_{S_1}. \tag{4.3.18}$$

In fact, let $F_1 = F \cap (\overline{B}_8 \setminus B_4)$, and $F_2 = F_1 \cup \overline{B}_1$. We enlarge the balayage measure on the left in (4.3.18) if we replace the domain $\mathbf{R}^d \setminus (F \cup \overline{B}_1)$ with $\mathbf{R}^d \setminus (F_1 \cup \overline{B}_1)$, hence we may suppose $F = F_1$, $F_2 = F \cup \overline{B}_1$. Let $\overline{\nu}$ be the balayage of some measure ν out of $\mathbf{R}^d \setminus \overline{B}_1$. Then $\overline{\nu} = \overline{\widehat{\nu}}$, i.e.

$$\overline{\nu} = \widehat{\nu}\Big|_{\overline{B}_1} + \overline{\widehat{\nu}\Big|_F},$$

and we apply this with $\nu = \sigma_{S_8}$. Thus,

$$\widehat{\sigma_{S_8}}\Big|_{\overline{B}_1} = \overline{\sigma_{S_8}} - \overline{\widehat{\sigma_{S_8}}\Big|_F}. \tag{4.3.19}$$

The left hand side is what is on the left of (4.3.18), and since $\sigma_{S_8} = \mu_{S_8}/8^{d-2}$, and $\overline{\mu_{S_8}} = \mu_{S_1} = \sigma_{S_1}$, the first term on the right hand side is $\sigma_{S_1}/8^{d-2}$. Therefore, it has left to estimate from below the second measure on the right of (4.3.19).

For every $a \in S_8$ (4.1.9) with $r = 1$ and $R = 8$ shows that

$$\overline{\delta_a} \ge \frac{7}{9^{d-1}}\sigma_{S_1} > \frac{1}{9^d}\sigma_{S_1},$$

and so

$$\overline{\widehat{\sigma_{S_8}}\Big|_F} \ge \frac{\widehat{\sigma_{S_8}}(F)}{9^d}\sigma_{S_1}, \tag{4.3.20}$$

and we have to estimate how much of $\widehat{\sigma_{S_8}}$ goes on to F. Since we assumed $F_2 = F \cup \overline{B}_1 \subseteq \overline{B}_8$, and, as we have just remarked, $\sigma_{S_8} = \mu_{S_8}/8^{d-2}$, it follows that

$$\widehat{\sigma_{S_8}}(F) = \frac{1}{8^{d-2}}\mu_{F_2}(F). \tag{4.3.21}$$

The distance of the sets $F \subseteq \overline{B}_8 \setminus B_4$ and $\overline{B}_1$ is at least 3, so (4.1.12) yields for the capacities of F, $\overline{B}_1$ and $F_2 = F \cup \overline{B}_1$ the inequality

$$\mathrm{cap}(F_2) \ge \frac{\mathrm{cap}(F) + \mathrm{cap}(\overline{B}_1)}{1 + \frac{2\mathrm{cap}(F)\mathrm{cap}(\overline{B}_1)}{3^{d-2}(\mathrm{cap}(F)+\mathrm{cap}(\overline{B}_1))}} \ge \frac{1+\mathrm{cap}(F)}{1 + \frac{2}{3^{d-2}}\mathrm{cap}(F)},$$

because $\operatorname{cap}(\overline{B}_1) = 1$. The latter expression is monotone increasing in $\operatorname{cap}(F)$, and the assumption gives $\operatorname{cap}(F) \geq 4\varepsilon$ and $\varepsilon \leq 1/8$, thus

$$\operatorname{cap}(F_2) \geq \frac{1 + 4\varepsilon}{1 + \frac{2}{3^{d-2}}4\varepsilon} \geq 1 + \varepsilon.$$

Therefore

$$\mu_{F_2}(F) + \mu_{F_2}(\overline{B}_1) = ||\mu_{F_2}|| = \operatorname{cap}(F_2) \geq 1 + \varepsilon.$$

Here $\mu_{F_2}(\overline{B}_1) \leq 1$ because $\mu_{\overline{B}_1}(B_1) = 1$ and $\mu_{\overline{B}_1}$ is obtained by taking the balayage of μ_{F_2} onto $\overline{B}_1$. Hence $\mu_{F_2}(F) \geq \varepsilon$ follows. This and (4.3.20)–(4.3.21) give

$$\overline{\widehat{\sigma_{S_8}}\Big|_F} \geq \frac{1}{8^{d-2}}\frac{\varepsilon}{9^d}\sigma_{S_1}.$$

Now all from (4.3.19) imply

$$\widehat{\sigma_{S_8}}\Big|_{\overline{B}_1} \leq \frac{1}{8^{d-2}}\sigma_{S_1} - \frac{1}{8^{d-2}}\frac{\varepsilon}{9^d}\sigma_{S_1} = \frac{1}{8^{d-2}}\left(1 - \frac{\varepsilon}{9^d}\right)\sigma_{S_1},$$

and (4.3.18) has been verified.

We shall use (4.3.18) in a scaled form, namely if E is compact, 0 is on the boundary of the unbounded component of $\mathbf{R}^d \setminus E$ and

$$\operatorname{cap}(E \cap (\overline{B}_{2^{-n}} \setminus B_{2^{-n-1}})) \geq 4\varepsilon 2^{(-n-3)(d-2)}, \tag{4.3.22}$$

then we have

$$\widehat{\sigma_{S_{2^{-n}}}}\Big|_{\overline{B}_{2^{-n-3}}} \leq \frac{1}{8^{d-2}}\left(1 - \frac{\varepsilon}{9^d}\right)\sigma_{S_{2^{-n-3}}}, \tag{4.3.23}$$

where now $\widehat{\cdot}$ denotes balayage out of $\mathbf{R}^d \setminus (E \cup \overline{B}_{2^{-n-3}})$.

After this preparation let us return to the set $\mathcal{N}_E(\varepsilon)$ which was assumed to be of positive lower density. Then there is an $\eta > 0$ such that for large N the set $\mathcal{N}_E(\varepsilon)$ has at least ηN elements smaller than N. For large N then we can select a subset

$$K \subseteq \mathcal{N}_E(\varepsilon) \cap \{2, \ldots, N-2\}$$

such that it has $k \geq \eta(N+1)/5$ elements, and if $n_1, n_2, \ldots, n_k$ is the increasing enumeration of K, then $n_{i+1} > n_i + 3$ for all $i < k$.

We set

$$E_n = E \cup \overline{B}_{2^{-n}}, \qquad \mu_n = \mu_{E_n}.$$

Our aim is to estimate the quantity $\mu_E(\overline{B}_{2^{-N}})$, which is at most as large as $\mu_N(\overline{B}_{2^{-N}})$ (recall that μ_E is the balayage of μ_N onto E, and during this we sweep out of $\mathbf{R}^d \setminus E$ the portion of μ_N sitting on $\overline{B}_{2^{-N}} \setminus E$, so the measure of $\overline{B}_{2^{-N}}$ is not increasing during this sweeping process). We shall recursively estimate $\mu_n\big|_{S_{2^{-n}}}$ by $\sigma_{S_{2^{-n}}}$ and the $n = N$ case will give the desired result.

First note that $\mu_0\big|_{S_1} \le \sigma_{S_1}$ ($\sigma_{S_1} = \mu_{S_1}$ is the balayage of μ_0 onto $\overline{B}_1$). Suppose $\mu_n\big|_{S_{2^{-n}}} \le c\sigma_{S_{2^{-n}}}$ holds true with some constant c. The measure μ_{n+1} is the balayage of μ_n onto E_{n+1} and during this balayage we sweep out only $\mu_n\big|_{S_{2^{-n}}}$ onto $S_{2^{-n-1}} \cup \left(E \cap (\overline{B}_{2^{-n}} \setminus B_{2^{-n-1}})\right)$. This balayage measure is not less than the balayage of $\mu_n\big|_{S_{2^{-n}}}$ onto $S_{2^{-n-1}}$. Therefore if $\overline{\cdot}$ denotes the balayage out of $\mathbf{R}^d \setminus \overline{B}_{2^{-n-1}}$ then we have (see (4.1.1))

$$\mu_{n+1}\big|_{S_{2^{-n-1}}} \le \overline{\mu_n\big|_{S_{2^{-n}}}} \le c\overline{\sigma_{S_{2^{-n}}}} = c\frac{1}{2^{d-2}}\sigma_{S_{2^{-n-1}}}. \tag{4.3.24}$$

On the other hand, if $n = n_i - 1$, with $n_i \in K$, then (4.3.22) is true, hence for such n we have (4.3.23). Again, the measure μ_{n+3} is the balayage of μ_n onto E_{n+3}, and in taking this balayage we sweep out only the part of μ_n that is sitting on $S_{2^{-n}} \setminus E$. Thus, if $\widehat{\cdot}$ denotes the balayage onto E_{n+3}, then

$$\begin{aligned}
\mu_{n+3}\big|_{S_{2^{-n-3}}} &= \widehat{\mu_n\big|_{S_{2^{-n}}}}\Big|_{S_{2^{-n-3}}} \le c\widehat{\sigma_{S_{2^{-n}}}}\Big|_{S_{2^{-n-3}}} \\
&\le c\frac{1}{8^{d-2}}\left(1 - \frac{\varepsilon}{9^d}\right)\sigma_{S_{2^{-n-3}}}.
\end{aligned}$$

This estimate holds for all n with $n + 1 \in K$, and consecutive numbers in K differ by at least 3, hence this estimate for going from n to $n + 3$ can be applied at least $k \ge (N + 1)\eta/5$ times. For other n we just use (4.3.24) ($N - 3k$ times altogether). Thus, we eventually obtain

$$\begin{aligned}
\mu_N(\overline{B}_N) &= \mu_N(S_{2^{-N}}) \\
&\le \left(\left(\frac{1}{8}\right)^{d-2}\left(1 - \frac{\varepsilon}{9^d}\right)\right)^k \left(\frac{1}{2^{d-2}}\right)^{N-3k} \sigma_{S_{2^{-N}}}(S_{2^{-N}}) \\
&\le \left(\frac{1}{2}\right)^{N(d-2)}\left(1 - \frac{\varepsilon}{9^d}\right)^k \le \left(\frac{1}{2}\right)^{N(d-2)}\left(1 - \frac{\varepsilon}{9^d}\right)^{\eta(N+1)/5}.
\end{aligned}$$

This is the desired inequality, for it immediately implies for $2^{-N-1} < r \le 2^{-N}$ that

$$\begin{aligned}
\mu_E(\overline{B}_r) &\le \mu_E(\overline{B}_{2^{-N}}) \le \mu_N(\overline{B}_{2^{-N}}) \\
&\le \left(\frac{1}{2}\right)^{N(d-2)}\left(1 - \frac{\varepsilon}{9^d}\right)^{\eta(N+1)/5} \le r^{d-2+\kappa},
\end{aligned}$$

provided

$$N > \frac{-10(d-2)\log 2}{\eta \log(1 - \varepsilon/9^d)} - 1$$

and κ is defined by the equation

$$2^{-\kappa} = \left(1 - \frac{\varepsilon}{9^d}\right)^{\eta/10}.$$

■

Remark 4.3.1. Note that the previous proof was effective in the sense that *if $\varepsilon > 0$ and $\eta > 0$ are given, then there are an N_0 and a $\kappa > 0$ such that if for a particular $M \geq N_0$ we have $|\mathcal{N}_E(\varepsilon) \cap \{0, 1, \ldots, M\}| \geq \eta M$, then*

$$\mu_E(\overline{B}_{2^{-M}}) \leq \mu_M(\overline{B}_{2^{-M}}) \leq (2^{-M})^{d-2+\kappa}. \tag{4.3.25}$$

Proof of b) in Theorem 4.2.1

The proof is rather long, therefore we break it into several steps.

Step I

Let $L \geq 2$ be a fixed natural number, $F \subseteq \overline{B}_{2^{-1}} \setminus B_{2^{-L-1}}$ a compact set such that

$$\mathrm{cap}(F \cap (\overline{B}_{2^{-j+1}} \setminus B_{2^{-j}})) \leq \varepsilon 2^{-j(d-2)}, \qquad j = 2, \ldots, L+1, \tag{4.3.26}$$

and let $\widehat{\delta_a}$ be the balayage of δ_a out of the domain $(B_{2^{-1}} \setminus \overline{B}_{2^{-L-1}}) \setminus F$. We shall estimate from below this balayage measure on $S_{2^{-L-1}}$ for $a \in S_{2^{-L}}$; namely we shall show that for large L and small $\varepsilon > 0$, disregarding a small subset of $S_{2^{-L}}$, for $a \in S_{2^{-L}}$ the measure $\widehat{\delta_a}\Big|_{S_{2^{-L-1}}}$ has almost full density (i.e. as in the case $F = \emptyset$) on a large (almost full) subset of $S_{2^{-L-1}}$.

For notational convenience let $\Delta_1 = B_{2^{-1}}$, $\Delta_L = B_{2^{-L}}$, $\Delta_{L+1} = B_{2^{-L-1}}$, $\Delta_{3/2} = B_{3 \cdot 2^{-L-2}}$, and let the bounding surfaces of these balls be denoted by T_1, T_L, T_{L+1} and $T_{3/2}$, respectively. Set also $F_{3/2} = F \cap \overline{\Delta}_{3/2}$. We shall take the balayage out of different sets, and for the convenience of the reader we list them here:

- $\widehat{\cdot}$ is the balayage out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$,
- $\widetilde{\cdot}$ is the balayage out of $(\mathbf{R}^d \setminus \overline{\Delta}_{L+1}) \setminus F_{3/2}$,
- $\overline{\cdot}$ is the balayage out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$.

We start from the representation

$$F = \bigcup_{j=2}^{L+1} F \cap (\overline{B}_{2^{-j+1}} \setminus B_{2^{-j}}),$$

hence (4.1.11) gives

$$\operatorname{cap}(F) \leq \sum_{j=2}^{L+1} \varepsilon 2^{-j(d-2)} \leq \varepsilon. \tag{4.3.27}$$

Now let $a \in \Delta_1 \setminus \overline{\Delta}_{L+1}$, and let $\nu = \nu_a = \widehat{\delta_a}$ be the balayage measure of δ_a out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$. This measure has total mass 1 and it is supported on $T_{L+1} \cup F \cup T_1$. First we verify that it has small mass on F.

Without loss of generality we may assume that F is of positive capacity (otherwise enlarge it), and then we can write

$$\nu(F) = \int_F U^{\mu_F} d\nu = \int U^{\mu_F} d\nu_{|F} = \int U^{\nu_{|F}} d\mu_F \leq \int U^{\nu} d\mu_F. \tag{4.3.28}$$

The potential $U^{\nu}(x)$ agrees with

$$U^{\delta_a}(x) = \frac{1}{|z-a|^{d-2}}$$

for quasi-every $x \in F$ (see (4.1.6)) and hence for μ_F-almost all x, therefore the last integral on the right of (4.3.28) is $U^{\mu_F}(a)$. This gives that if

$$U^{\mu_F}(a) \leq \frac{1}{L},$$

then

$$\nu_a(F) \leq \frac{1}{L}. \tag{4.3.29}$$

We shall need a similar reasoning for the balayage $\nu^* = \nu_a^* := \widetilde{\delta_a}$ of δ_a out of $(\mathbf{R}^d \setminus \overline{\Delta}_{L+1}) \setminus F_{3/2}$. In fact, replace in (4.3.28) F by $F_{3/2}$ and ν by ν^*. This gives that if

$$U^{\mu_{F_{3/2}}}(a) \leq \frac{1}{L},$$

then

$$\nu_a^*(F_{3/2}) \leq \frac{1}{L}. \tag{4.3.30}$$

Thus, if

$$K := \left\{ a \ : \ U^{\mu_F}(a) \geq \frac{1}{L} \right\}, \tag{4.3.31}$$

then for $a \notin K$ we have (4.3.29), and if

$$K_{3/2} := \left\{ a \ : \ U^{\mu_{F_{3/2}}}(a) \geq \frac{1}{L} \right\}, \tag{4.3.32}$$

then for $a \in T_L \setminus K_{3/2}$ we have (4.3.30). For the capacity of K we get from (4.1.3)–(4.1.4) and (4.3.27) the inequality

$$\operatorname{cap}(K) \leq L\operatorname{cap}(F) \leq \varepsilon L, \tag{4.3.33}$$

and similarly we get

$$\operatorname{cap}(K_{3/2}) \leq L\operatorname{cap}(F_{3/2}) \leq \varepsilon L. \tag{4.3.34}$$

If σ_L denotes the $(d-1)$-dimensional normalized surface area measure on T_L, then by (4.1.5) we have

$$\begin{aligned}\sigma_L(K \cap T_L) \leq c\sqrt{2^L \operatorname{cap}(K \cap T_L)} &\leq c 2^{L/2}\sqrt{\operatorname{cap}(K)} \\ &\leq c2^{L/2}\sqrt{\varepsilon L}. \end{aligned} \tag{4.3.35}$$

An identical inequality is true for $K_{3/2}$ (c.f. (4.3.34)):

$$\sigma_L(K_{3/2} \cap T_L) \leq c2^{L/2}\sqrt{\varepsilon L}. \tag{4.3.36}$$

Let $a, b \in T_L$, and let $\widetilde{\delta}_a$, $\widetilde{\delta}_b$ be the balayage of δ_a, δ_b out of the domain $\mathbf{R}^d \setminus (F_{3/2} \cup \overline{\Delta}_{L+1})$. This balayage is obtained by first taking balayage of δ_a, δ_b out of $\mathbf{R}^d \setminus \Delta_{3/2}$, and if these balayage measures are denoted by α_a and α_b, then take the balayage of α_a and α_b (which are supported on $T_{3/2}$) out of $\mathbf{R}^d \setminus (F_{3/2} \cup \overline{\Delta}_{L+1})$. The measures α_a and α_b are given by the formula (4.1.8) with $r = 3 \cdot 2^{-L-2}$ and $R = 2^{-L}$, hence (4.1.9) gives the inequality

$$\alpha_a \leq \left(\frac{1+3/4}{1-3/4}\right)^d \alpha_b = 7^d \alpha_b,$$

therefore we also have $\widetilde{\delta}_a \leq 7^d \widetilde{\delta}_b$. Now $\widehat{\delta}_a$ is the balayage out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$, while $\widetilde{\delta}_a$ is the balayage out of the larger domain $(\mathbf{R}^d \setminus \overline{\Delta}_{L+1}) \setminus F_{3/2}$, hence

$$\widehat{\delta}_a \Big|_{\overline{\Delta}_{L+1} \cup F_{3/2}} \leq \widetilde{\delta}_a.$$

These give for all $a, b \in T_L$ the inequality

$$\widehat{\delta}_a \Big|_{\overline{\Delta}_{L+1} \cup F_{3/2}} \leq 7^d \widetilde{\delta}_b. \tag{4.3.37}$$

Choose and fix a $b \in T_L \setminus K_{3/2}$ (see (4.3.32)). By (4.3.36) if ε is sufficiently small compared to L, then there is such a b. In this case (4.3.30) gives $\widetilde{\delta}_b(F_{3/2}) \leq 1/L$, hence the balayage $\tau := \overline{\widetilde{\delta}_b \Big|_{F_{3/2}}}$ of $\widetilde{\delta}_b \Big|_{F_{3/2}}$ out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$ also has total mass at most $1/L$. Therefore, if we define

$$H^* = \left\{ y \in T_{L+1} \ : \ \frac{d\tau(y)}{d\sigma_{L+1}} \geq \frac{1}{\sqrt{L}} \right\}, \tag{4.3.38}$$

then

$$\sigma_{L+1}(H^*) \leq \frac{1}{\sqrt{L}}.$$

Taking into account (4.3.37) we obtain for the measures $\rho_a := \overline{\widehat{\delta_a}\Big|_{F_{3/2}}}$ the inequality

$$\frac{d\rho_a(x)}{d\sigma_{L+1}} \le \frac{7^d}{\sqrt{L}} \tag{4.3.39}$$

for all $a \in T_L$ and all $x \in T_{L+1} \setminus H^*$.

Next consider the balayage $\rho_a^* := \overline{\widehat{\delta_a}\Big|_{F \setminus F_{3/2}}}$ of the restriction $\widehat{\delta_a}\Big|_{F \setminus F_{3/2}}$ out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$. The set $F \setminus F_{3/2}$ lies outside $\Delta_{3/2}$, and for each c outside $\Delta_{3/2}$ the inequality (4.1.9) gives for the density of the balayage $\overline{\delta_c}$ of δ_c out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$ the estimate

$$\frac{d\overline{\delta_c}}{d\sigma_{L+1}} \le c_0 \left(\frac{1}{2^{L+1}}\right)^{d-2} \frac{3/2^{L+2} + 1/2^{L+1}}{(3/2^{L+2} - 1/2^{L+1})^{d-1}} = 5c_0 2^{d-2}.$$

Hence for $a \in T_L \setminus K$ we get

$$\begin{aligned} \frac{d\rho_a^*}{d\sigma_{L+1}} &= \int_{F \setminus F_{3/2}} \frac{d\overline{\delta_c}}{d\sigma_{L+1}} d\widehat{\delta_a}(c) \le 5c_0 2^{d-2} \cdot \widehat{\delta_a}(F \setminus F_{3/2}) \\ &\le 5c_0 2^{d-2} \cdot \widehat{\delta_a}(F) \le \frac{5c_0 2^{d-2}}{L}, \end{aligned} \tag{4.3.40}$$

where we used (4.3.29) which is valid for $a \notin K$.

In a similar fashion we obtain for $\rho_a^{**} := \overline{\widehat{\delta_a}\Big|_{T_1}}$ the estimate

$$\frac{d\rho_a^{**}}{d\sigma_{L+1}} \le \frac{c_0(2^L+1)}{(2^L-1)^{d-1}} \cdot \widehat{\delta_a}(T_1) \le \frac{c_0(2^L+1)}{(2^L-1)^{d-1}}. \tag{4.3.41}$$

Now note that

$$\rho_a + \rho_a^* + \rho_a^{**} + \overline{\widehat{\delta_a}\Big|_{T_{L+1}}} = \overline{\widehat{\delta_a}} = \overline{\delta_a},$$

and the last term on the left hand side is actually $\widehat{\delta_a}\Big|_{T_{L+1}}$. Thus, (4.3.39), (4.3.40) and (4.3.41) give that for all $a \in T_L \setminus K$ and $y \in T_{L+1} \setminus H^*$ we have

$$\frac{d\widehat{\delta_a}(y)}{d\sigma_{L+1}} \ge \frac{d\overline{\delta_a}(y)}{d\sigma_{L+1}} - \frac{7^d}{\sqrt{L}} - \frac{5c_0 2^{d-2}}{L} - \frac{c_0(2^L+1)}{(2^L-1)^{d-1}},$$

which, in view of (4.1.8) gives for $a \in T_L \setminus K$ and for $y \in T_{L+1} \setminus H^*$

$$\frac{d\widehat{\delta_a}(y)}{d\sigma_{L+1}} \ge \frac{3}{2^{d(L+1)}|a-y|^d} - \frac{c_1}{\sqrt{L}}. \tag{4.3.42}$$

This derivation used the existence of $b \in T_L \setminus K_{3/2}$, and it is valid if ε is sufficiently small compared to L (see (4.3.36)).

Step II

We follow the notations from Step I, in particular suppose that F is a compact set with (4.3.26).

Let $\delta > 0$. Suppose that μ is a measure on T_L such that

$$\frac{d\mu(y)}{d\sigma_L} \geq 1 \qquad \text{for } y \in T_L \setminus H, \tag{4.3.43}$$

where $H \subseteq T_L$ is of (normalized surface area) measure at most δ. Let $\widehat{\mu}$ be the balayage of μ out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$. We are going to show that for large L and small $\varepsilon > 0$ the measure $\widehat{\mu}$ satisfies a similar condition as (4.3.43) but on T_{L+1}, namely we verify

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq \frac{1}{2^{d-2}}(1 - c_2\delta) \qquad \text{for } y \in T_{L+1} \setminus H^*, \tag{4.3.44}$$

where H^* is a set of (normalized surface area) measure at most δ and $c_2 > 0$ is a constant depending only on d.

First of all note that we have (4.3.42) for $a \in T_L \setminus K$ and $y \in T_{L+1} \setminus H^*$, where H^* is the fixed set defined in (4.3.38), and also note that the integral over T_{L+1} of the first term on the right of (4.3.42) with respect to $d\sigma_L$ is

$$\begin{aligned}
\int_{T_L} \frac{3}{2^{d(L+1)}|a-y|^d} d\sigma_L(a) &= \int_{T_L} \frac{d\overline{\delta_a}(y)}{d\sigma_{L+1}} d\sigma_L(a) \\
&= \frac{d\overline{\sigma_L}(y)}{d\sigma_{L+1}} = \frac{(2^L)^{d-2} d\overline{\mu_{T_L}}(y)}{d\sigma_{L+1}} \\
&= \frac{2^{L(d-2)} d\mu_{T_{L+1}}(y)}{2^{(L+1)(d-2)} d\mu_{T_{L+1}}} = \left(\frac{1}{2}\right)^{d-2},
\end{aligned}$$

where μ_{T_L} denotes the equilibrium measure of T_L. Therefore we have

$$\int_{T_L} \left(\frac{3}{2^{d(L+1)}|a-y|^d} - \frac{c_1}{\sqrt{L}} \right) d\sigma_L(a) \geq \frac{1}{2^{d-2}} - \frac{c_1}{\sqrt{L}}. \tag{4.3.45}$$

We write with $a \in T_L$ and $y \in T_{L+1}$

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} = \int_{T_L} \frac{d\widehat{\delta_a}(y)}{d\sigma_{L+1}} d\mu(a).$$

The integral element $d\mu(a)$ is at least as large as $(d\mu(y)/d\sigma_L)d\sigma_L$, and here
$d\mu(y)/d\sigma_L \geq 1$ if $y \in T_L \setminus H$. Furthermore, as we have just mentioned, for $a \notin K$ the integrand is at least as large as the integrand in (4.3.45), and these give for $y \in T_{L+1} \setminus H^*$

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq \frac{1}{2^{d-2}} - \frac{c_1}{\sqrt{L}} - A$$

where

$$A = \left(\int_H + \int_K \right) \frac{3}{2^{d(L+1)}|a-y|^d} d\sigma_L(a).$$

The integrand on the right hand side is at most 3, hence the integral is bounded by 3 times the normalized surface area measure of H and those $a \in T_L$ for which $a \in K$, which is at most $\sigma_L(H) + \sigma_L(K \cap T_L)$. Thus, the assumption $\sigma_L(H) \leq \delta$ and the inequality (4.3.35) give

$$A \leq 3\Big(\sigma_L(H) + \sigma_L(K \cap T_L)\Big) \leq 3\Big(\delta + c2^{L/2}\sqrt{\varepsilon L}\Big).$$

Thus, if $y \in T_{L+1} \setminus H^*$ then

$$\begin{aligned} \frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} &\geq \frac{1}{2^{d-2}} - \frac{c_1}{\sqrt{L}} - 3\Big(\delta + c2^{L/2}\sqrt{\varepsilon L}\Big) \\ &\geq \frac{1}{2^{d-2}}(1 - 2^d\delta), \end{aligned} \tag{4.3.46}$$

provided

$$\frac{c_1}{\sqrt{L}} + 3c2^{L/2}\sqrt{\varepsilon L} \leq \delta. \tag{4.3.47}$$

This condition should be understood in the sense that first we choose L large enough, then for fixed L choose $\varepsilon > 0$ small to satisfy (4.3.47). Furthermore, assuming this condition, H^* has measure at most

$$\sigma_{L+1}(H^*) \leq \frac{1}{\sqrt{L}} \leq \delta. \tag{4.3.48}$$

Thus, with such a choice for L and ε the estimate (4.3.44) holds with $c_2 = 2^d$.

Step III

We follow the notations from steps I and II, and assume that F is a compact set with (4.3.26).

Let $c_3/4 > \delta > 0$, where c_3 is a constant to be chosen later, and suppose that μ is a measure on T_L such that

$$\frac{d\mu(y)}{d\sigma_L} \geq 1 \qquad \text{for } y \in H', \tag{4.3.49}$$

where H' is of (normalized surface area) measure at least $c_3 - \delta$. Thus, we consider the same situation as in step II, but there the assumption on the density of μ with respect to σ_L was on a large set (namely on $T_L \setminus H$ of measure $\geq 1 - \delta$), while here the assumption is on a set H' of measure at least $c_3 - \delta$.

Let, as before, $\widehat{\mu}$ be the balayage of μ out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$. We are going to show that for large L and small $\varepsilon > 0$ the measure $\widehat{\mu}$ satisfies

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq c_4 \qquad \text{for } y \in T_{L+1} \setminus H^*, \tag{4.3.50}$$

where H^* is a set of (normalized surface area) measure at most δ and c_4 depends only on d.

For a proof just follow the proof in step II. We have (4.3.42) for $a \in T_L \setminus K$ with H^* given in (4.3.38), and note that

$$\frac{3}{2^{d(L+1)}|a-y|^d} \geq \frac{3}{2^{2d}}.$$

Therefore, if $a \in T_L \setminus K$ and $y \in T_{L+1} \setminus H^*$ then (4.3.42) yields

$$\frac{d\widehat{\delta_a}(y)}{d\sigma_{L+1}} \geq \frac{3}{2^{2d}} - \frac{c_1}{\sqrt{L}} \geq c_5$$

provided L is large enough. Integrating this inequality with respect to $\mu(a)$ for $a \in H' \setminus K$, we obtain for $y \in T_{L+1}$ as in (4.3.46)

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq c_5 \left(c_3 - \delta - c2^{L/2}\sqrt{\varepsilon L} \right),$$

where we used (4.3.35) and the fact that the measure of H' is at least $c_3 - \delta$. Since $\delta < c_3/4$, we get that if $c2^{L/2}\sqrt{\varepsilon L} < \delta < c_3/4$ then (4.3.50) follows for all $y \in T_{L+1} \setminus H^*$ where H^* is the set defined in (4.3.38) of measure at most $1/\sqrt{L}$. If, in addition, $1/\sqrt{L} < \delta$, then $\sigma_{L+1}(H^*) \leq \delta$, as was claimed in (4.3.50).

Note that both of these conditions are satisfied if L is sufficiently large and ε is sufficiently small.

■

Step IV

The estimate below will be used when our compact set omits the cone $C_{2\tau}$, where

$$C_\tau := \{x \in \mathbf{R}^d \ : \ \frac{\langle x, x_0 \rangle}{\|x\|} \geq 1 - \tau\}. \tag{4.3.51}$$

Consider the domain $(B_2 \setminus \overline{B}_{1/2}) \cap C_{2\tau}$, and let $a \in S_1 \cap C_\tau$. It is clear (by Harnack's inequality) that there is a positive constant c_τ depending only on τ such that if $\breve{\delta}_a$ is the balayage of δ_a out of $(B_2 \setminus \overline{B}_{1/2}) \cap C_{2\tau}$, then on $S_{1/2} \cap C_\tau$ this balayage has density at least c_τ, i.e.

$$\frac{d\breve{\delta}_a(y)}{d\sigma_{1/2}} \geq c_\tau, \qquad y \in S_{1/2} \cap C_\tau, \quad a \in C_\tau \cap S_1.$$

Thus, if $c_3 = \sigma_1(C_\tau \cap S_1)$, $\delta < c_3/2$ and μ is a measure on S_1 such that $d\mu(y)/d\sigma_1 \geq 1$ on a set $H'' \subseteq C_\tau \cap S_1$ of measure at least $c_3 - \delta$, then

$$\frac{d\breve{\mu}(y)}{d\sigma_{1/2}} \geq \frac{c_3 c_\tau}{2}, \qquad y \in S_{1/2} \cap C_\tau. \tag{4.3.52}$$

Step V

Now we can complete the proof of the necessity direction part (b) in Theorem 4.2.1. Let us suppose that $\mathcal{N}_E(\varepsilon)$ is of zero lower density for every $\varepsilon > 0$ and that E satisfies the cone condition. We may assume that the cone that E omits is $C_{2\tau} \cap B_1$ with C_τ defined in (4.3.51), and first let us suppose that E is contained in the unit ball. Then $E \cap C_{2\tau} = \emptyset$.

Let $\delta < c_3/2 < 1/2$, and select the integer L and $\varepsilon > 0$ in such a way that all the estimates in steps II–IV hold.

Let $E_n = E \cup \overline{B}_{2^{-n}}$, $\mu_n = \mu_{E_n}$, and let

$$v_n(y) = \frac{d\mu_n(y)}{d\sigma_n}$$

be the lower Radon-Nikodym derivative (density) of μ_n on $S_{2^{-n}}$ with respect to normalized surface area measure on $S_{2^{-n}}$. Thus, $v_n(y)d\sigma_n(y) \leq d\mu_n(y)$. Note that μ_0 is the normalized surface area measure on S_1, hence $v_0(y) \equiv 1$.

Let

- $\Sigma_0 = \{n \geq L \ : \ \ n+1, n, \ldots, n-L+1 \notin \mathcal{N}_E(\varepsilon)\}$,
- $\Sigma_1 = \{n \geq L \ : \ \ n+1, n, \ldots, n-L+2 \notin \mathcal{N}_E(\varepsilon),\ n-L+1 \in \mathcal{N}_E(\varepsilon)\}$,
- $\Sigma_2 = \{n \in \mathbf{N} \ : \ n < L$ or
  one of $n+1, n, \ldots, n-L+2$ belongs to $\mathcal{N}_E(\varepsilon)\}$.

These give a partition of the integers. For every natural number n we define a number A_n as follows. If $n > L$ and $n, n-1, \ldots, n-L+1 \notin \mathcal{N}_E(\varepsilon)$, then let A_n be the largest number with the property that $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}}$ with the exception of a set of normalized surface measure $\leq \delta$. Let us call this case for A_n of the first type. If, however, $n \leq L$ or one of $n, n-1, \ldots, n-L+1$ belongs to $\mathcal{N}_E(\varepsilon)$, then let A_n be the largest number with the property that $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}} \cap C_\tau$ with the exception of a set of normalized surface measure $\leq \delta$. Let us call this case for A_n of the second type.

We want to compare A_{n+1} with A_n for $n \geq L$. Let $\overline{\cdot}$ denote the balayage out of $\overline{\mathbf{R}}^d \setminus (E \cup \overline{B}_{2^{-n-1}})$. Then $\mu_{n+1} = \overline{\mu_n}$ and $\mu_{n+1}\big|_{S_{2^{-n-1}}} = \overline{\mu_n\big|_{S_{2^{-n}}}}\Big|_{S_{2^{-n-1}}}$. Thus, v_{n+1} is at least as large as the density (on $S_{2^{-n-1}}$) of the balayage $\overline{v_n(y)d\sigma_n}$ of $v_n(y)d\sigma_n$. If $\widehat{\cdot}$ denotes balayage out of the narrower domain $B_{2^{n-L+1}} \setminus (E \cup \overline{B}_{2^{-n-1}})$, then the density of $\widehat{v_n(y)d\sigma_n}$ is not larger on $S_{2^{-n-1}}$ than the density of the balayage $\overline{v_n(y)d\sigma_n}$, which (as we have just seen) is not larger than v_{n+1}. Now if $n \in \Sigma_0$, then both A_n and A_{n+1} are of first type (i.e. $v_n(y) \geq A_n$ and $v_{n+1}(y) \geq A_{n+1}$ for all $y \in S_{2^{-n}}$ and $y \in S_{2^{-n-1}}$, respectively, with the exception of a set of measure $\leq \delta$), hence (4.3.44) can be applied (in a scaled form) for the measure $d\mu(y) = \widehat{v_n(y)d\sigma_n}$ to conclude that $A_{n+1} \geq (1/2^{d-2})(1 - c_2\delta)A_n$.

In a completely similar manner, if $n \in \Sigma_1$, then A_n is of the second type while A_{n+1} is of the first type, i.e. $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}} \cap C_\tau$ with the exception of a set of measure $\leq \delta$ and $v_{n+1}(y) \geq A_{n+1}$ for all $y \in S_{2^{-n-1}}$ with the exception of a set of measure $\leq \delta$. Now instead of (4.3.44) we apply (4.3.50) to conclude that $A_{n+1} \geq c_4 A_n$.

Finally, if $n \in \Sigma_2$, $n \geq L$, then A_{n+1} is definitely of the second type, but A_n may be of the first or second type (of the first type only if $n+1 \in \mathcal{N}_E(\varepsilon)$, but $n, n-1, \ldots n-L+2 \notin \mathcal{N}_E(\varepsilon)$). In either case $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}} \cap C_\tau$ with the exception of a set of measure $\leq \delta$, and hence we can apply (4.3.52) to conclude $A_{n+1} \geq (c_3 c_\tau/2)A_n$. This is also the estimate we use for all $n < L$.

In summary, we have $A_{n+1} \geq (1/2^{d-2})(1 - c_2\delta)A_n$ for $n \in \Sigma_0$, $A_{n+1} \geq c_4 A_n$ for $n \in \Sigma_1$ and $A_{n+1} \geq (c_3 c_\tau/2)A_n$ for $n \in \Sigma_2$. If $s = s_N$ denotes the number of elements of $\mathcal{N}_E(\varepsilon) \cup \{0\}$ not larger than N, then there are at most s elements of Σ_1 and at most sL elements of Σ_2 not larger than N. Thus, we can conclude

$$A_{N+1} \geq \left(\frac{1}{2^{d-s}}\right)^N (1 - c_2\delta)^N (c_4)^{s_N} \left(\frac{c_3 c_\tau}{2}\right)^{s_N L} A_0.$$

Since $\mathcal{N}_E(\varepsilon)$ is of zero lower density, the limit inferior of s_N/N is zero, hence there are infinitely many N's for which

$$(c_4)^{s_N} \left(\frac{c_3 c_\tau}{2}\right)^{s_N L} A_0 \geq \frac{2}{c_3}(1 - c_2\delta)^N. \tag{4.3.53}$$

For all such N we can conclude that

$$A_{N+1} \geq (2/c_3)(1/2^{d-2})^N (1 - c_2\delta)^{2N},$$

which implies

$$\mu_{N+1}(S_{N+1}) \geq \left(\frac{1}{2^{d-2}}\right)^N (1 - c_2\delta)^{2N} \tag{4.3.54}$$

because, independently if A_N is of the first or second type, we have $v_N(y) \geq A_N$ on a set of measure at least $c_3 - \delta \geq c_3/2$.

Now we can easily complete the proof. Let Ω_{N+1} be the unbounded component of $\mathbf{R}^d \setminus E_{N+1}$. Consider Green's function with pole at $y_0 \in \Omega_{N+1}$ and integrate it over the sphere S_r with $r = r_N = 2^{-N}$.

$$\int_{S_r} g_{\Omega_{N+1}}(x, y_0) d\sigma_{S_r}(x) = \int_{S_r} \left(g_{\Omega_{N+1}}(x, y_0) - g_{\Omega_{N+1}}(0, y_0)\right) d\sigma_{S_r}(x)$$

$$
\begin{aligned}
&= \int_{S_r} \left(U^{\delta_{y_0}}(x) - U^{\delta_{y_0}}(0)\right) d\sigma_{S_r}(x) \\
&+ \int_{S_r} \left(U^{\widetilde{\delta_{y_0}}}(0) - U^{\widetilde{\delta_{y_0}}}(x)\right) d\sigma_{S_r}(x), \quad (4.3.55)
\end{aligned}
$$

where $\widetilde{\cdot}$ denotes the balayage out of Ω_{N+1}.

Here the first integrand is

$$
\frac{1}{|x-y_0|^{d-2}} - \frac{1}{|y_0|^{d-2}} \leq c_6|x|, \tag{4.3.56}
$$

where c_6 depends only on y_0 and d. For the second integral we have

$$
\begin{aligned}
&\int_{S_r} \left(U^{\widetilde{\delta_{y_0}}}(0) - U^{\widetilde{\delta_{y_0}}}(x)\right) d\sigma_{S_r}(x) \qquad (4.3.57) \\
&= \int_{E_{N+1}} \frac{1}{|t|^{d-2}} d\widetilde{\delta_{y_0}}(t) - \int_{S_r}\int_{E_{N+1}} \frac{1}{|x-t|^{d-2}} d\widetilde{\delta_{y_0}}(t) d\sigma_{S_r}(x).
\end{aligned}
$$

Since

$$
\int_{S_r} \frac{1}{|x-t|^{d-2}} d\sigma_{S_r}(x) = \min\left(\frac{1}{|t|^{d-2}}, \frac{1}{r^{d-2}}\right)
$$

(see e.g. [3, Example 4.2.9]) and there exists c_7 depending only on y_0 and d such that $d\widetilde{\delta_{y_0}} \geq c_7 d\mu_{N+1}$ (see (4.1.10)), we get from (4.3.55), (4.3.56) and (4.3.57)

$$
\begin{aligned}
&\int_{S_r} g_{\Omega_{N+1}}(x, y_0) d\sigma_{S_r}(x) \\
&\geq \int_{E_{N+1}} \left(\frac{1}{|t|^{d-2}} - \min\left(\frac{1}{|t|^{d-2}}, \frac{1}{r^{d-2}}\right)\right) d\widetilde{\delta_{y_0}}(t) - c_6 r \\
&\geq c_7 \int_{\overline{B}_{2^{-N-1}}} \left(\frac{1}{|t|^{d-2}} - \frac{1}{r^{d-2}}\right) d\mu_{N+1}(t) - c_6 r \\
&= (2^{d-2}-1)c_7 \frac{1}{r^{d-2}} \mu_{N+1}(S_{N+1}) - c_6 r \\
&\geq (2^{d-2}-1)c_7(1-c_2\delta)^{2N} - c_6 r \geq r^{\kappa}, \qquad (4.3.58)
\end{aligned}
$$

provided $\delta < (1-\sqrt{2}/2)/c_2$,

$$
\begin{aligned}
N \geq \max\Bigg(&\frac{\log(2c_6) - \log\left((2^{d-2}-1)c_7\right)}{\log\left(2(1-c_2\delta)^2\right)}, \\
&\frac{1}{\delta \log 2}\left(\log 2 - \log\left((2^{d-2}-1)c_7\right)\right)\Bigg) \qquad (4.3.59)
\end{aligned}
$$

and

$$
\kappa = \delta + \frac{2}{\log 2} \log \frac{1}{1-c_2\delta}.
$$

Hence, there is an x_N such that $g_{\Omega_{N+1}}(x_N, y_0) \geq r^\kappa$, and this implies

$$g_\Omega(x, y_0) \geq g_{\Omega_{N+1}}(x, y_0) \geq r_N^\kappa.$$

Here $\kappa > 0$ can be arbitrarily small since $\delta > 0$ is as close to 0 as we wish, and this inequality is true for a sequence $r_N = 2^{-N} \to 0$ (for which s_N satisfies (4.3.53) and (4.3.59)). Therefore Green's function g_Ω is not Hölder continuous at 0.

The proof above used that E is contained in the unit ball and omits the cone $C_{2\tau}$. The general case can be similarly handled. In fact, let Ω contain the cone $C = C(x_0, 2\tau, \ell)$. Select a sphere S_{r_0}, $r_0 < l/2$, that intersects C. Without loss of generality (use a dilation) we may assume that $S_{r_0} = S_1$, and let $J = S_1 \cap C_\tau$ be the middle part of $S_1 \cap C$. Then $\mu_{E\cup\overline{B}_1}$ has strictly positive density on J, say (with the notations of the preceding proof) $v_0(y) \geq c_9 > 0$ for $y \in J$. Now the preceding proof can be repeated, the only difference is that in this case the starting value of A_0 is c_9 (note that for $n = 0$ the number A_n is of the first type).

∎

Remark 4.3.2. The proof above was effective in the following sense. *Let y_0, τ, ℓ, c, r_0, $\kappa > 0$, $r_0 < \ell/2$ be given. Then there are $\varepsilon > 0$, $\eta > 0$ and M that depend only on $d, y_0, \tau, \ell, c, r_0, \kappa$, with the following property. Let E be a compact set of positive capacity, Ω the unbounded component of $\mathbf{R}^d \setminus E$, $0 \in \partial\Omega$, and assume that Ω contains a cone $C(x_0, 2\tau, \ell)$. If for the measure $\mu_0 = \mu_{E\cup\overline{B}_{r_0}}$ the condition*

$$\frac{d\mu_0(y)}{d\sigma_{S_{r_0}}} \geq c, \qquad y \in S_{r_0} \cap C(x_0, \tau, \ell)$$

holds, and if for a particular $N \geq M$ we have $|\mathcal{N}_E(\varepsilon) \cap \{0, 1, \ldots, N\}| \leq \eta N$, then there is an $x \in S_{2^{-N}}$ such that

$$g_\Omega(x, y_0) \geq (2^{-N})^\kappa.$$

4.4 Proof of Lemma 4.2.2

First we show that **1)** is equivalent to **2)**. As at the end of the proof of Theorem 4.2.1 in Section 4.3 (see (4.3.58)), we can write

$$\int_{S_r} g_G(x, a) d\sigma_{S_r}(x) \geq \frac{c_1}{r^{d-2}} \mu_E(\overline{B}_r) - c_2 r$$

with some constants c_1 and c_2. If $g_G(\cdot, a)$ is Hölder continuous at 0, then the left-hand side is less than $c_3 r^\kappa$ for some positive constants c_3 and $\kappa < 1$. Therefore it follows for $r < 1$ that

$$\mu_E(\overline{B}_r) \le c_4 r^{d-2+\kappa}$$

for some constant c_4 and this shows that **1)** implies **2)**.

Conversely, suppose **2)**. Let $|x| = r$ be small, and $E^* = E \setminus B_{2r}$. Let $\overline{\cdot}$ and $\widehat{\cdot}$ denote the balayage out of $\mathbf{R}^d \setminus E^*$ and $\mathbf{R}^d \setminus E$, respectively. Since 0 is a regular point, $g_\Omega(0, a) = 0$. Therefore

$$\begin{aligned} g_{\mathbf{R}^d\setminus E^*}(0, a) &= g_{\mathbf{R}^d\setminus E^*}(0, a) - g_\Omega(0, a) \\ &= (U^{\delta_a}(0) - U^{\overline{\delta_a}}(0)) - (U^{\delta_a}(0) - U^{\widehat{\delta_a}}(0)) \\ &= U^{\widehat{\delta_a}}(0) - U^{\overline{\delta_a}}(0) \\ &= \int \frac{1}{|y|^{d-2}}(d\widehat{\delta_a}(y) - d\overline{\delta_a}(y)). \end{aligned}$$

Now, $\overline{\delta_a}$ is the balayage of $\widehat{\delta_a}$ onto E^*, and so $\widehat{\delta_a}\Big|_{E^*} \le \overline{\delta_a}$. Thus, we do not decrease the integral by integrating only over B_{2r} with respect to $\widehat{\delta_a}$, i.e.

$$g_{\mathbf{R}^d\setminus E^*}(0, a) \le \int_{B_{2r}} \frac{1}{|y|^{d-2}} d\widehat{\delta_a}(y). \tag{4.4.60}$$

Furthermore, using **2)** we obtain

$$\begin{aligned} \int_{B_{2r}} \frac{1}{|y|^{d-2}} d\mu_E(y) &= \sum_{i=0}^{\infty} \int_{B_{2r/2^i}\setminus \overline{B}_{r/2^i}} \frac{1}{|y|^{d-2}} d\mu_E(y) \\ &\le \sum_{i=0}^{\infty} \left(\frac{2^i}{r}\right)^{d-2} \mu_E(B_{2r/2^i}) \\ &\le \sum_{i=0}^{\infty} \frac{2^{(d-2)i}}{r^{d-2}} C \left(\frac{2r}{2^i}\right)^{d-2+\kappa} \\ &= \frac{C 2^{d-2+2\kappa}}{2^\kappa - 1} r^\kappa. \end{aligned}$$

This, (4.4.60) and (4.1.10) give $g_{\mathbf{R}^d\setminus E^*}(0, a) \le c_5 r^\kappa$ for all small $r > 0$ with some constant c_5. Now the ball B_{2r} is contained in $\mathbf{R}^d \setminus E^*$, hence Harnack's inequality gives $g_{\mathbf{R}^d\setminus E^*}(x, a) \le c_6 g_{\mathbf{R}^d\setminus E^*}(0, a) \le c_6 c_5 r^\kappa$ for all $|x| = r$. Since here $g_{\mathbf{R}^d\setminus E^*}(x, a) \ge g_\Omega(x, a) \ge g_G(x, a)$, the Hölder continuity of $g_G(x, a)$ follows, and this proves **2)** $\Rightarrow$ **1)**.

Remark 4.4.1. The proof just given is effective in the following sense. *If for some $r > 0$ we have $\mu_E(\overline{B}_t) \le C t^{d-2+\kappa}$ for $t \le 2r$, then for $|x| = r$*

$$g_G(x, a) \le C_1 r^\kappa, \qquad C_1 = \frac{C_a C c_6 2^{d-2+2\kappa}}{2^\kappa - 1}. \tag{4.4.61}$$

Next we show that **3)** implies **1)**. Let R be so large that $\partial G \subset B_R$ and construct a domain T in the following way. If $a \notin \overline{B}_R$ then set $T = B_R \cap G$. Otherwise take a small ball $B_a \subset G$ centered at a and set $T = (B_R \setminus \overline{B}_a) \cap G$. Let r be small and set $f = 0$ on $\overline{B}_r$ and $f = 1$ on $\partial G \setminus \overline{B}_r$. Compare u_f and g_G in the region T. Both are harmonic in T and positive on S_R and ∂B_a. Hence an application of the maximum principle shows that $g_G(x,a) \le c_7 u_f(x)$ in T, and this proves **3)** $\Rightarrow$ **1)**.

It is left to show **1)** $\Rightarrow$ **3** under the cone condition. Under the cone condition **1)** implies the positive lower density of $\mathcal{N}_E(\varepsilon)$ for some $\varepsilon > 0$, i.e. there is an η and an N_1 such that $|\mathcal{N}_E(\varepsilon) \cap \{0, 1, \ldots, N\}| \ge 4\eta N$ for $N \ge N_1$. Then for large N, say for $N \ge N_2$, we also have

$$|\mathcal{N}_E(\varepsilon) \cap \{[(2\eta)N] + 1, [(2\eta)N] + 2, \ldots, M\}| \ge \eta M$$

for any $M \ge N$. Set $r_N = 2^{-[2\eta N]}$ and $F = \overline{B}_{r_N} \cap (\mathbf{R}^d \setminus G)$. For this F the preceding inequality gives

$$|\mathcal{N}_F(\varepsilon) \cap \{0, 1, \ldots, M\}| \ge \eta M, \qquad \text{for } M \ge N \ge N_2,$$

hence, by the proof of **a)** in Theorem 4.2.1, see in particular Remark 4.3.1, there are a $\kappa > 0$ and an $N_0 \ge N_2$ (depending only on ε and η) such that for all $M \ge N \ge N_0$ the inequality $\mu_F(\overline{B}_{2^{-M+1}}) \le (2^{-M+1})^{d-2+\kappa}$ is true. This implies $\mu_F(\overline{B}_t) \le 2^{d-2+\kappa} t^{d-2+\kappa}$ for $t \le 2 \cdot 2^{-N}$. Hence, by the effective form of the implication **2)** $\Rightarrow$ **1)** given in Remark 4.4.1, we can conclude for $|x| = r = 2^{-N}$ the inequality $g_{\mathbf{R}^d \setminus F}(x,a) \le C_1 (2^{-N})^{\kappa}$ with $C_1 = C_a c_6 2^{2(d-2+2\kappa)}/(2^{\kappa} - 1)$.

∎

Summary

This dissertation investigates local smoothness properties of the Green function of the complement of a compact set E. The continuity of Green's functions at boundary points has been extensively studied for a long time. The aim of this research is to give conditions for the stronger Hölder continuity in terms of the geometry of the set. We consider both the planar and the higher dimensional case. The dissertation consists of 3 parts based on 3 papers: [15], [16] and [17].

Optimal Smoothness for $E \subset [0,1]$

Suppose that $E \subset \mathbf{C}$ is a compact set with positive logarithmic capacity $\mathrm{cap}(E) > 0$. Let $\Omega := \overline{\mathbf{C}} \setminus E$, where $\overline{\mathbf{C}} := \{\infty\} \cup \mathbf{C}$ is the extended complex plane. Denote by $g_\Omega(z) = g_\Omega(z, \infty)$, $z \in \Omega$, the Green function of Ω with pole at ∞. We are interested in the behavior of g_Ω at a regular boundary point.

Suppose that 0 is a regular point of E, i.e., $g_\Omega(z)$ is continuous at 0 and $g_\Omega(0) = 0$. First consider the case $E \subset [0,1]$. The monotonicity of the Green function yields

$$g_\Omega(z) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(z), \quad z \in \mathbf{C} \setminus [0,1],$$

that is, if E has the "highest density" at 0, then g_Ω has the "highest smoothness" at the origin. In particular

$$g_\Omega(-r) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(-r) > \sqrt{r}, \quad 0 < r < 1.$$

In this regard, we would like to explore properties of E whose Green function has the "highest smoothness" at 0, that is, E conforming to the following condition

$$g_\Omega(z) \leq C|z|^{1/2}, \qquad z \in \mathbf{C},$$

which is known to be the same as

$$g_\Omega(-r) \leq C r^{1/2}, \qquad 0 < r < 1 \tag{1}$$

(c.f. [1, Theorem 3.6]).

For $0 < \varepsilon < 1/2$ we set (see [5])

$$E_\varepsilon(t) = (E \cap [0,t]) \cup [0, \varepsilon t] \cup [(1-\varepsilon)t, t].$$

Extending the results of V. Andrievskii, L. Carleson and V. Totik we prove the following main theorems:

Theorem 1. *For any $\varepsilon > 0$*

$$\int_r^1 \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < C_0 \frac{g_\Omega(-r)}{\sqrt{r}}$$

where C_0 is independent of r.

Theorem 2. *Let $\varepsilon < 1/2$. E satisfies* (1) *if and only if*

$$\int_0^1 \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < \infty.$$

The method used in the proofs of Theorems 1 and 2 can be applied to the case $E \subset [-1,1]$ as well (c.f. [5, Theorem 1.11]). In this case

$$g_\Omega(ir) \geq g_{\overline{\mathbf{C}} \setminus [-1,1]}(ir) > \frac{r}{2}, \quad 0 < r < 1,$$

therefore in this case the optimal smoothness for Green functions is Hölder 1 and we are interested in the sets E satisfying

$$g_\Omega(z) \leq C|z|, \qquad 0 < |z| < 1.$$

The highest smoothness of the Green function at the origin (Lipschitz condition) is again equivalent to the highest density at 0 and the corresponding theorems, similar to Theorems 1 and 2 hold as well.

Markov Inequality and Green Functions

This part of the dissertation is joint work with Vilmos Totik.

Let Π_n denote the set of algebraic polynomials of degree $\leq n$. Let $E \subset \mathbf{C}$ be compact with positive logarithmic capacity. We say that E satisfies the Markov inequality with a polynomial factor if there exist $C,\ k > 0$ such that

$$||P_n'||_E \leq C n^k ||P_n||_E \tag{2}$$

holds for every n and $P_n \in \Pi_n$.

Inequality (2) is strongly related to the smoothness properties of the Green function belonging to E. Let Ω be the outer domain of E, i.e. the unbounded component of $\overline{\mathbf{C}} \setminus E$, and let $g_\Omega(z)$ denote Green's function of Ω with pole at infinity. g_Ω is said to be Hölder continuous if there exist $C_1,\ \alpha > 0$ such that

$$g_\Omega(z) \leq C_1 \Big(\mathrm{dist}(z, E) \Big)^\alpha \tag{3}$$

for all $z \in \mathbf{C}$. It is an open problem if (2) and (3) are equivalent for any compact set E. In Chapter 3 our aim is to show that in the optimal cases $k = 1$ and $\alpha = 1$ they are, indeed, equivalent. Our main result is:

Theorem. *Let E be a compact subset of the plane such that the unbounded component Ω of $\overline{\mathbf{C}} \setminus E$ is regular. Then the following are pairwise equivalent.*

i) *Optimal Markov inequality holds on E, i.e. there exists a $C > 0$ such that*

$$||P_n'||_E \le Cn||P_n||_E$$

for every polynomial $P_n \in \Pi_n$, $n = 1, 2, \ldots$.

ii) *Green's function g_Ω is Lipschitz continuous, i.e. there exists a $C_1 > 0$ such that*

$$g_\Omega(z) \le C_1 \mathrm{dist}(z,\ E)$$

for every $z \in \mathbf{C}$.

iii) *The equilibrium measure μ_E of E satisfies a Lipschitz type condition, i.e. there exists a $C_2 > 0$ such that*

$$\mu_E\Big(D_\delta(z)\Big) \le C_2\delta$$

for every $z \in E$ and $\delta > 0$.

If, in addition, Ω is simply connected, then **i)–iii)** *are also equivalent to*

iv) *The conformal mapping Φ from Ω onto the exterior of the unit disk is Lipschitz continuous, i.e.*

$$|\Phi(z_1) - \Phi(z_2)| \le C_3|z_1 - z_2|, \qquad z_1, z_2 \in \Omega.$$

In Chapter 3 we also state a local version of the theorem.

A Wiener-type Condition in $\mathbf{R}^d$

Let $E \subset \mathbf{R}^d$ be a compact set of positive Newtonian capacity, Ω the unbounded component of $\mathbf{R}^d \setminus E$ and $g_\Omega(x, a)$ the Green's function of Ω with pole at $a \in \Omega$. We are interested in the behavior of g_Ω at a boundary point of Ω, which we assume to be 0, i.e. let $0 \in \partial\Omega$.

Let $B_r = \{x \ : \ |x| < r\}$ be the ball of radius r about the origin, and we shall denote its closure by $\overline{B}_r$ and its boundary (the sphere of center 0 and radius r) by S_r. With

$$E^n = E \cap (\overline{B}_{2^{-n+1}} \setminus B_{2^{-n}}) = \Big\{x \in E \ : \ 2^{-n} \le |x| \le 2^{-n+1}\Big\}$$

the regularity of the boundary point 0 was characterized by Wiener (see e.g. [7, Theorem 5.2]): Green's function $g_G(x,a)$ $(a \in \Omega)$ is continuous at $0 \in \partial\Omega$ (i.e. 0 is a regular boundary point of E) if and only if

$$\sum_{n=1}^{\infty} \mathrm{cap}(E^n) 2^{n(d-2)} = \infty,$$

where $\mathrm{cap}(E^n)$ denotes the (d-dimensional) Newtonian capacity of E^n. Our aim is to characterize in a similar manner the stronger Hölder continuity:

$$g_\Omega(x,a) \le C|x|^\kappa$$

with some positive numbers C, κ.

Following the definitions in [5], for $\varepsilon > 0$ set

$$\mathcal{N}_E(\varepsilon) = \{n \in \mathbf{N} \ : \ \ \mathrm{cap}(E^n) \ge \varepsilon 2^{-n(d-2)}\},$$

and we say that a subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ of the natural numbers is of positive lower density if

$$\liminf_{N\to\infty} \frac{|\mathcal{N} \cap \{0,1,\ldots,N\}|}{N+1} > 0,$$

which is clearly the same condition as $n_k = O(k)$.

Let $x_0 \in S_1$, $0 < \tau < 1$, $\ell > 0$ and set

$$C(x_0,\tau,\ell) := \{x \in B_\ell \ : \ \ \frac{\langle x, x_0\rangle}{\|x\|} \ge 1-\tau\}.$$

This is a cone with vertex at 0 and x_0 as the direction of its axis. We say that E satisfies the cone condition if

$$C(x_0,\tau,\ell) \subset \Omega$$

with some $x_0 \in S_1$, τ and $\ell > 0$, which means that Ω contains a cone with vertex at 0. Our main result in Chapter 4 is

Theorem. a) *If $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$ then Green's function g_Ω is Hölder continuous at 0.*

b) *If Green's function g_Ω is Hölder continuous at 0 and E satisfies the cone condition then $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$.*

The sufficiency of the density condition for Hölder continuity of the solution to Dirichlet's problems and various elliptic equations was proved by Maz'ja in [8]- [11]. Maz'ja used the condition

$$\sum_{n=1}^{N} 2^{n(d-2)} \mathrm{cap}(E \cap \overline{D}_{2^{-n}}) \ge \delta N, \quad N = 1, 2, \ldots \tag{4}$$

for some $\delta > 0$, which is equivalent to the positive density of $\mathcal{N}_E(\varepsilon)$. It was also shown in [11] that in general this condition is not necessary. The problem to find conditions under which (4) is necessary was raised in [10]. Thus, the above theorem solves a long standing open problem under the simple cone condition.

Összefoglalás

A disszertáció egy E kompakt halmaz komplementerének Green-függvényének lokális tulajdonságait vizsgálja. A Green-függvények folytonosságával határpontokban sok tanulmány foglalkozik. Ezen munka célja, hogy a halmaz geometriáján alapuló feltételeket adjon az erősebb Hölder folytonosságra. Egyaránt tárgyaljuk a síkbeli és a magasabb dimenziós esetet. A disszertáció 3 részből áll, melyek egy-egy cikken alapulnak: [15], [16] és [17].

Optimális simaság $E \subset [0,1]$-re

Tegyük fel, hogy $E \subset \mathbf{C}$ egy kompakt halmaz $\mathrm{cap}(E) > 0$ logaritmikus kapacitással. Legyen $\Omega := \overline{\mathbf{C}} \setminus E$, ahol $\overline{\mathbf{C}} := \{\infty\} \cup \mathbf{C}$ a kibővített komplex sík. Jelölje $g_\Omega(z) = g_\Omega(z, \infty)$, $z \in \Omega$ az Ω Green-függvényét ∞ pólussal. A g_Ω viselkedését tanulmányozzuk egy reguláris határpontban.

Tegyük fel, hogy a 0 az E-nek egy reguláris pontja, vagyis, hogy $g_\Omega(z)$ folytonos 0-ban, és $g_\Omega(0) = 0$. Először tekintsük az $E \subset [0,1]$ esetet. A Green-függvény monotonitása miatt

$$g_\Omega(z) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(z), \quad z \in \mathbf{C} \setminus [0,1],$$

azaz, ha E-nek a "legnagyobb a sűrűsége" a 0-ban, akkor g_Ω-nak a "legnagyobb a simasága" az origóban. Ezért

$$g_\Omega(-r) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(-r) > \sqrt{r}, \quad 0 < r < 1.$$

Szeretnénk jellemezni azokat az E halmazokat, amelyek Green-függvényének a "legnagyobb a simasága" 0-ban, vagyis azokat az E-ket, amelyek eleget tesznek a következő feltételnek:

$$g_\Omega(z) \leq C|z|^{1/2}, \qquad z \in \mathbf{C},$$

amely ekvivalens a

$$g_\Omega(-r) \leq Cr^{1/2}, \qquad 0 < r < 1 \tag{1}$$

egyenlőtlenséggel (c.f. [1, Theorem 3.6]).

$0 < \varepsilon < 1/2$-re legyen (lásd [5])

$$E_\varepsilon(t) = (E \cap [0,t]) \cup [0, \varepsilon t] \cup [(1-\varepsilon)t, t].$$

V. Andrievskii, L. Carleson és Totik Vilmos eredményeit kibővítve a következő tételeket bizonyítjuk:

1. Tétel. *Bármely $\varepsilon > 0$-ra*

$$\int_r^1 \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt < C_0 \frac{g_\Omega(-r)}{\sqrt{r}}$$

ahol C_0 független r-től.

2. Tétel. *Legyen $\varepsilon < 1/2$. E akkor és csak akkor elégíti ki* (1)*-et, ha*

$$\int_0^1 \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon(t))}{t}\right)\frac{1}{t}dt < \infty.$$

Az 1. és 2. Tétel bizonyításánál használt módszer az $E \subset [-1,1]$ esetre is alkalmazható (c.f. [5, Theorem 1.11]). Ebben az esetben

$$g_\Omega(ir) \geq g_{\overline{\mathbf{C}}\setminus[-1,1]}(ir) > \frac{r}{2}, \quad 0 < r < 1,$$

tehát a Green-függvény optimális simasága itt Hölder 1, és most azokat az E halmazokat keressük, melyekre

$$g_\Omega(z) \leq C|z|, \qquad 0 < |z| < 1.$$

A Green-függvény legnagyobb simasága az origónál (Lipschitz-feltétel) ismét ekvivalens a legnagyobb sűrűséggel a 0-ban, és az 1. és 2. Tételhez hasonló megfelelő tételek is igazak.

Markov-egyenlőtlenség és Green-függvények

A disszertáció ezen része Totik Vilmossal közös munka.

Jelölje Π_n a legfeljebb n-edfokú algebrai polinomok halmazát. Legyen $E \subset \mathbf{C}$ kompakt pozitív logaritmikus kapacitással. Azt mondjuk, hogy E eleget tesz a Markov-egyenlőtlenségnek polinomiális faktorral, ha létezik $C,\ k > 0$ úgy, hogy

$$||P_n'||_E \leq Cn^k||P_n||_E \tag{2}$$

igaz minden n-re és $P_n \in \Pi_n$-re.

A (2) egyenlőtlenség szoros kapcsolatban áll E Green-függvényének simasági tulajdonságaival. Legyen Ω az E külső tartománya, azaz a $\overline{\mathbf{C}} \setminus E$ nemkorlátos komponense, és jelölje $g_\Omega(z)$ az Ω Green-függvényét végtelen pólussal. g_Ω-t Hölder-folytonosnak nevezzük, ha létezik $C_1,\ \alpha > 0$ úgy, hogy

$$g_\Omega(z) \leq C_1\Big(\mathrm{dist}(z,E)\Big)^\alpha. \tag{3}$$

minden $z \in \mathbf{C}$-re. Nyitott probléma, hogy (2) és (3) ekvivalensek-e bármely E kompakt halmazra. A disszertáció 3. fejezetében az a célunk, hogy megmutassuk, hogy a $k = 1$ és $\alpha = 1$ optimális esetek valóban ekvivalensek. Fő eredményünk:

Tétel. *Legyen E a sík egy kompakt részhalmaza, és tegyük fel, hogy Ω, a $\overline{\mathbf{C}} \setminus E$ nemkorlátos komponense reguláris. Ekkor a következők páronként ekvivalensek.*

i) *E-n optimális Markov-egyenlőtlenség igaz, azaz létezik $C > 0$ úgy, hogy*

$$||P_n'||_E \leq Cn||P_n||_E$$

minden $P_n \in \Pi_n$, $n = 1, 2, \ldots$ polinomra.

ii) *A g_Ω Green-függvény Lipschitz-folytonos, azaz létezik $C_1 > 0$ úgy, hogy*

$$g_\Omega(z) \leq C_1 \mathrm{dist}(z,\ E)$$

minden $z \in \mathbf{C}$-re.

iii) *Az E halmaz μ_E egyensúlyi mértéke kielégít egy Lipschitz-féle feltételt, azaz létezik $C_2 > 0$ úgy, hogy*

$$\mu_E\Big(D_\delta(z)\Big) \leq C_2\delta$$

minden $z \in E$-re és $\delta > 0$-ra.

Továbbá, ha Ω egyszeresen összefüggő, akkor az **i)–iii)** *feltételekkel szintén ekvivalens a következő*

iv) *Az Ω-t az egységkör külsejére képező Φ konformis leképezés Lipschitz-folytonos, azaz*

$$|\Phi(z_1) - \Phi(z_2)| \leq C_3|z_1 - z_2|, \qquad z_1, z_2 \in \Omega.$$

A 3. fejezetben ezen tétel egy lokális változatát is kimondjuk.

Egy Wiener-tipusú feltétel $\mathbf{R}^d$-ben

Legyen $E \subset \mathbf{R}^d$ egy kompakt halmaz pozitív Newton-féle kapacitással, Ω az $\mathbf{R}^d \setminus E$ nemkorlátos komponense, és $g_\Omega(x, a)$ az Ω Green-függvénye $a \in \Omega$ pólussal. g_Ω viselkedését vizsgáljuk Ω egy határpontjában, melyről az általánosság megszorítása nélkül feltehetjük, hogy a 0-ban van, vagyis hogy $0 \in \partial\Omega$.

Jelölje $B_r = \{x\ :\ |x| < r\}$ az origó körüli r sugarú nyílt gömböt, $\overline{B}_r$ a lezártját, és S_r a határát (a gömbhéjat). Legyen továbbá

$$E^n = E \cap (\overline{B}_{2^{-n+1}} \setminus B_{2^{-n}}) = \Big\{x \in E\ :\ 2^{-n} \leq |x| \leq 2^{-n+1}\Big\}.$$

Wienertől származik a 0 határpont regularitásának karakterizációja (lásd pl. [7, Theorem 5.2]): A $g_G(x, a)$ ($a \in \Omega$) Green-függvény akkor és csak akkor folytonos $0 \in \partial\Omega$-ban (vagyis 0 az E reguláris határpontja), ha

$$\sum_{n=1}^{\infty} \mathrm{cap}(E^n) 2^{n(d-2)} = \infty,$$

ahol $\text{cap}(E^n)$ jelöli az E^n halmaz (d-dimenziós) Newton-féle kapacitását. Célunk egy hasonló jellemzést adni az erősebb Hölder-folytonosságra:

$$g_\Omega(x,a) \le C|x|^\kappa$$

valamely pozitív C, κ konstansokkal.

[5] definícióit követve tetszőleges $\varepsilon > 0$-ra vezessük be a következő jelölést:

$$\mathcal{N}_E(\varepsilon) = \{n \in \mathbf{N} \ : \ \text{cap}(E^n) \ge \varepsilon 2^{-n(d-2)}\}.$$

Azt mondjuk, hogy a természetes számok egy $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ részsorozata pozitív alsó sűrűségű, ha

$$\liminf_{N\to\infty} \frac{|\mathcal{N} \cap \{0,1,\ldots,N\}|}{N+1} > 0,$$

ami nyilvánvalóan ekvivalens $n_k = O(k)$-val.

Legyen $x_0 \in S_1$, $0 < \tau < 1$, $\ell > 0$, továbbá

$$C(x_0,\tau,\ell) := \{x \in B_\ell \ : \ \frac{\langle x, x_0\rangle}{\|x\|} \ge 1-\tau\}.$$

Ez egy 0 csúcsú és x_0 tengelyirányú kúp. Azt mondjuk, hogy E eleget tesz a kúpfeltételnek, ha

$$C(x_0,\tau,\ell) \subset \Omega$$

valamilyen $x_0 \in S_1$-re, τ-ra és $\ell > 0$-ra, tehát ha Ω tartalmaz egy 0 csúcsú kúpot. A 4. fejezet fő eredménye:

Tétel. a) *Ha $\mathcal{N}_E(\varepsilon)$ pozitív alsó sűrűségű valamilyen $\varepsilon > 0$-ra, akkor a g_Ω Green-függvény Hölder-folytonos 0-ban.*

b) *Ha a g_Ω Green-függvény Hölder-folytonos 0-ban, és E eleget tesz a kúpfeltételnek, akkor $\mathcal{N}_E(\varepsilon)$ pozitív alsó sűrűségű valamilyen $\varepsilon > 0$-ra.*

A tétel első részét, a sűrűségi feltétel elégségességét a Dirichlet-probléma és más elliptikus egyenletek megoldásának Hölder-folytonosságához Maz'ja már a '60-as években belátta (lásd [8]- [11]). Maz'ja ezt a feltételt használta:

$$\sum_{n=1}^{N} 2^{n(d-2)}\text{cap}(E \cap \overline{D}_{2^{-n}}) \ge \delta N, \quad N = 1,2,... \tag{4}$$

valamely $\delta > 0$-ra, ami ekvivalens $\mathcal{N}_E(\varepsilon)$ pozitív sűrűségével. Maz'ja [11]-ben azt is megmutatta, hogy általánosságban ez a feltétel nem szükséges. [10]-ben vetette föl a problémát olyan feltételek keresésére, melyek mellett (4) szükséges is lenne. Tehát a fenti tétel egy régi nyílt problémát old meg az egyszerű kúpfeltétel mellett.

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