

# **Smoothness of Green's Functions and Density of Sets**

**Thesis**

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Behavior of Green's functions around boundary points is a fundamental question of harmonic analysis, which has applications in different areas such as smoothness properties of solutions to Dirichlet problems or bounds for polynomials and polynomial inequalities. The continuity of Green's functions at boundary points has been extensively studied for a long time. The aim of this research is to give conditions for the stronger Hölder continuity in terms of the geometry of the set. We consider both the planar and the higher dimensional case. The dissertation consists of 3 parts based on 3 papers: [9], [10] and [11].

## 1 Optimal Smoothness for $E \subset [0, 1]$

Suppose that  $E \subset \mathbf{C}$  is a compact set with positive logarithmic capacity  $\text{cap}(E) > 0$ . Let  $\Omega := \overline{\mathbf{C}} \setminus E$ , where  $\overline{\mathbf{C}} := \{\infty\} \cup \mathbf{C}$  is the extended complex plane. Denote by  $g_\Omega(z) = g_\Omega(z, \infty)$ ,  $z \in \Omega$ , the Green function of  $\Omega$  with pole at  $\infty$ . We are interested in the behavior of  $g_\Omega$  at a regular boundary point.

Suppose that 0 is a regular point of  $E$ , i.e.,  $g_\Omega(z)$  is continuous at 0 and  $g_\Omega(0) = 0$ . First consider the case  $E \subset [0, 1]$ . The monotonicity of the Green function yields

$$g_\Omega(z) \geq g_{\overline{\mathbf{C}} \setminus [0, 1]}(z), \quad z \in \mathbf{C} \setminus [0, 1],$$

that is, if  $E$  has the "highest density" at 0, then  $g_\Omega$  has the "highest smoothness" at the origin. In particular

$$g_\Omega(-r) \geq g_{\overline{\mathbf{C}} \setminus [0, 1]}(-r) > \sqrt{r}, \quad 0 < r < 1.$$

In this regard, we would like to explore properties of  $E$  whose Green function has the “highest smoothness” at 0, that is,  $E$  conforming to the following condition

$$g_{\Omega}(z) \leq C|z|^{1/2}, \quad z \in \mathbf{C},$$

which is known to be the same as

$$g_{\Omega}(-r) \leq Cr^{1/2}, \quad 0 < r < 1 \quad (1)$$

(c.f. [1, Theorem 3.6]). V. Andrievskii [2] proved that if  $E$  satisfies (1) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrary close to the density of  $[0, 1]$  in that neighborhood, i.e. (1) implies

$$\lim_{r \rightarrow 0} \frac{\text{cap}(E \cap [0, r])}{r} = \frac{1}{4}. \quad (2)$$

For  $0 < \varepsilon < 1/2$  we set (see [8])

$$E_{\varepsilon}(t) = (E \cap [0, t]) \cup [0, \varepsilon t] \cup [(1 - \varepsilon)t, t]. \quad (3)$$

Our first result is

**Theorem 1** *For any  $\varepsilon > 0$*

$$\int_r^1 \left( \frac{1}{4} - \frac{\text{cap}(E_{\varepsilon}(t))}{t} \right) \frac{1}{t} dt < C_0 \frac{g_{\Omega}(-r)}{\sqrt{r}} \quad (4)$$

where  $C_0$  is independent of  $r$ .

L. Carleson and V. Totik [8] have characterized the optimal smoothness in terms of a Wiener type condition. They proved

**Theorem 2 (Carleson, Totik)** *Let  $\varepsilon < 1/3$ .  $E$  satisfies (1) if and only if*

$$\sum_k \left( \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(2^{-k}))}{2^{-k}} \right) < \infty.$$

This theorem plays the same role for Lip 1/2 smoothness as Wiener's theorem for continuity. The proof of Theorem 2 in [8], due to L. Carleson, was based on Poisson's formula. There is an alternative approach: using the technique of balayage; and with it we prove the following variant of Theorem 2.

**Theorem 3** *Let  $\varepsilon < 1/2$ .  $E$  satisfies (1) if and only if*

$$\int_0^1 \left( \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < \infty. \quad (5)$$

Andrievskii's theorem is a consequence of Theorem 3.

The method used in the proofs of Theorems 1 and 3 can be applied to the case  $E \subset [-1, 1]$  as well (c.f. [8, Theorem 1.11]). In this case

$$g_\Omega(ir) \geq g_{\overline{\mathbb{C}} \setminus [-1, 1]}(ir) > \frac{r}{2}, \quad 0 < r < 1,$$

therefore in this case the optimal smoothness for Green functions is Hölder 1 and we are interested in the sets  $E$  satisfying

$$g_\Omega(z) \leq C|z|, \quad 0 < |z| < 1.$$

This is equivalent to

$$g_\Omega(ir) \leq Cr, \quad 0 < r < 1 \quad (6)$$

because  $g_\Omega(x + iy)$  is monotone in  $y$ . The highest smoothness of the Green function at the origin (Lipschitz condition) is again equivalent to the highest density at 0. Namely, let  $E \subset [-1, 1]$  and set  $E_\varepsilon(t)$  as in (3) and

$$E_\varepsilon(-t) = (E \cap [-t, 0]) \cup [-t, (1 - \varepsilon)(-t)] \cup [-\varepsilon t, 0].$$

**Theorem 4** *If  $E \subseteq [-1, 1]$  and  $\varepsilon > 0$  then*

$$\int_r^1 \left( \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < C_0 \frac{g_\Omega(ir)}{r} \quad (7)$$

*The same is true for  $E_\varepsilon(-t)$ .*

**Theorem 5** *Let  $\varepsilon < 1/2$ .  $E$  satisfies*

$$g_\Omega(z) \leq C|z|, \quad 0 < |z| < 1, \quad (8)$$

*if and only if (5) holds for  $E_\varepsilon(t)$  and  $E_\varepsilon(-t)$ .*

This is a variant of [8, Theorem 1.11].

**Corollary 1** *If  $E$  satisfies (8) then*

$$\lim_{r \rightarrow 0} \frac{\text{cap}(E \cap [-r, r])}{r} = \frac{1}{2}. \quad (9)$$

**Corollary 2** (c.f. [8, Corollary 1.12])  *$g_\Omega$  is Hölder 1 continuous at 0 if and only if both  $g_{\overline{\mathbb{C}} \setminus (E \cap [0, 1])}$  and  $g_{\overline{\mathbb{C}} \setminus (E \cap [-1, 0])}$  are Hölder 1/2 continuous there.*

## 2 Markov Inequality and Green Functions

This part of the dissertation is joint work with Vilmos Totik.

Let  $\Pi_n$  denote the set of algebraic polynomials of degree  $\leq n$ . Markov's inequality is a basic result comparing the supremum norm of a polynomial  $P_n \in \Pi_n$  to the supremum norm of its derivative:

$$\|P'_n\|_{[-1,1]} \leq n^2 \|P_n\|_{[-1,1]}.$$

If  $C_1(0)$  is the unit circle, then the corresponding inequality

$$\|P'_n\|_{C_1(0)} \leq n \|P_n\|_{C_1(0)}$$

is due to Bernstein. Let us also remark that this is in some sense the optimal case, for if  $E$  is any compact set on the complex plane then there are polynomials  $P_n \in \Pi_n$ ,  $n = 1, 2, \dots$  for which

$$\|P'_n\|_E \geq cn \|P_n\|_E$$

with some constant  $c > 0$ .

Let  $E \subset \mathbf{C}$  be compact with positive logarithmic capacity. We say that  $E$  satisfies the Markov inequality with a polynomial factor if there exist  $C$ ,  $k > 0$  such that

$$\|P'_n\|_E \leq Cn^k \|P_n\|_E \tag{10}$$

holds for every  $n$  and  $P_n \in \Pi_n$ .

Inequality (10) is strongly related to the smoothness properties of the Green function belonging to  $E$ . Let  $\Omega$  be the outer

domain of  $E$ , i.e. the unbounded component of  $\overline{\mathbf{C}} \setminus E$ , and let  $g_\Omega(z)$  denote Green's function of  $\Omega$  with pole at infinity.  $g_\Omega$  is said to be Hölder continuous if there exist  $C_1$ ,  $\alpha > 0$  such that

$$g_\Omega(z) \leq C_1 \left( \text{dist}(z, E) \right)^\alpha. \quad (11)$$

for all  $z \in \mathbf{C}$ . It is known that in certain cases the Markov inequality is equivalent to the Hölder continuity of the Green function. Totik (see [12]) proved that this is true for Cantor-type sets, i.e. (10) is equivalent to (11) if  $E$  is Cantor-type. It is an open problem if (10) and (11) are equivalent for any compact set  $E$ . In this work our aim is to show that in the optimal cases  $k = 1$  and  $\alpha = 1$  they are, indeed, equivalent.

**Theorem 6** *Let  $E$  be a compact subset of the plane such that the unbounded component  $\Omega$  of  $\overline{\mathbf{C}} \setminus E$  is regular. Then the following are pairwise equivalent.*

- i) *Optimal Markov inequality holds on  $E$ , i.e. there exists a  $C > 0$  such that*

$$\|P'_n\|_E \leq Cn\|P_n\|_E \quad (12)$$

*for every polynomial  $P_n \in \Pi_n$ ,  $n = 1, 2, \dots$*

- ii) *Green's function  $g_\Omega$  is Lipschitz continuous, i.e. there exists a  $C_1 > 0$  such that*

$$g_\Omega(z) \leq C_1 \text{dist}(z, E) \quad (13)$$

*for every  $z \in \mathbf{C}$ .*

iii) *The equilibrium measure  $\mu_E$  of  $E$  satisfies a Lipschitz type condition, i.e. there exists a  $C_2 > 0$  such that*

$$\mu_E\left(D_\delta(z)\right) \leq C_2\delta \quad (14)$$

*for every  $z \in E$  and  $\delta > 0$ .*

*If, in addition,  $\Omega$  is simply connected, then i)–iii) are also equivalent to*

iv) *The conformal mapping  $\Phi$  from  $\Omega$  onto the exterior of the unit disk is Lipschitz continuous, i.e.*

$$|\Phi(z_1) - \Phi(z_2)| \leq C_3|z_1 - z_2|, \quad z_1, z_2 \in \Omega.$$

We mention that each of i), ii) and iv) implies regularity, so in their equivalence the regularity assumption is not needed.

There is also a local version of our theorem. We say that  $E$  has the optimal local Markov property at the point  $z_0 \in \partial\Omega$  if there is a constant  $C$  such that

$$|P_n^{(k)}(z_0)| \leq C^k n^k \|P_n\|_E, \quad P_n \in \Pi_n, \quad n = 1, 2, \dots$$

for all  $k = 1, 2, \dots$

**Theorem 7** *Let  $E$  be a compact subset of the plane,  $\Omega$  the unbounded component of  $\overline{\mathbb{C}} \setminus E$ , and suppose that  $z_0 \in \partial\Omega$  is a regular boundary point of  $\Omega$  (i.e.  $g_\Omega(z_0) = 0$ ). Then the following are equivalent.*

i)  *$E$  has the optimal Markov property at  $z_0$ .*



ii) Green's function  $g_\Omega$  is Lipschitz continuous at  $z_0$ , i.e.

$$g_\Omega(z) \leq C_1|z - z_0|$$

with some constant  $C_1$ .

iii) The equilibrium measure  $\mu_E$  of  $E$  satisfies a Lipschitz type condition at  $z_0$ , i.e. there exists a  $C_2 > 0$  such that

$$\mu_E\left(D_\delta(z_0)\right) \leq C_2\delta$$

for every  $\delta > 0$ .

If, in addition,  $\Omega$  is simply connected, then i)–iii) are also equivalent to

iv) The conformal mapping  $\Phi$  from  $\Omega$  onto the exterior of the unit disk is Lipschitz continuous at  $z_0$ .

It is worth noticing that much more is true than the equivalence of ii) and iii), namely we can give a very precise two sided estimate for Green's function in terms of the equilibrium measure.

**Theorem 8** *Let  $E$  be a compact subset of the plane,  $\Omega$  the unbounded component of  $\overline{\mathbb{C}} \setminus E$ , and suppose that  $z_0 \in \partial\Omega$  is a regular boundary point of  $\Omega$  (i.e.  $g_\Omega(z_0) = 0$ ). Then for every  $0 < r < 1$  we have*

$$\int_0^r \frac{\mu_E(D_t(z_0))}{t} dt \leq \sup_{|z-z_0|=r} g_\Omega(z) \leq 3 \int_0^{4r} \frac{\mu_E(D_t(z_0))}{t} dt. \quad (15)$$

### 3 A Wiener-type Condition in $\mathbf{R}^d$

Let  $E \subset \mathbf{R}^d$  be a compact set of positive Newtonian capacity,  $\Omega$  the unbounded component of  $\mathbf{R}^d \setminus E$  and  $g_\Omega(x, a)$  the Green's function of  $\Omega$  with pole at  $a \in \Omega$ . We are interested in the behavior of  $g_\Omega$  at a boundary point of  $\Omega$ , which we assume to be 0, i.e. let  $0 \in \partial\Omega$ .

Let  $B_r = \{x \mid |x| < r\}$  be the ball of radius  $r$  about the origin, and we shall denote its closure by  $\overline{B}_r$  and its boundary (the sphere of center 0 and radius  $r$ ) by  $S_r$ . With

$$E^n = E \cap (\overline{B}_{2^{-n+1}} \setminus B_{2^{-n}}) = \left\{ x \in E \mid 2^{-n} \leq |x| \leq 2^{-n+1} \right\}$$

the regularity of the boundary point 0 was characterized by Wiener (see e.g. [3, Theorem 5.2]): Green's function  $g_G(x, a)$  ( $a \in \Omega$ ) is continuous at  $0 \in \partial\Omega$  (i.e. 0 is a regular boundary point of  $E$ ) if and only if

$$\sum_{n=1}^{\infty} \text{cap}(E^n) 2^{n(d-2)} = \infty, \quad (16)$$

where  $\text{cap}(E^n)$  denotes the ( $d$ -dimensional) Newtonian capacity of  $E^n$ . Our aim is to characterize in a similar manner the stronger Hölder continuity:

$$g_\Omega(x, a) \leq C|x|^\kappa \quad (17)$$

with some positive numbers  $C, \kappa$ .

Following the definitions in [8], for  $\varepsilon > 0$  set

$$\mathcal{N}_E(\varepsilon) = \{n \in \mathbf{N} \mid \text{cap}(E^n) \geq \varepsilon 2^{-n(d-2)}\}, \quad (18)$$

and we say that a subsequence  $\mathcal{N} = \{n_1 < n_2 < \dots\}$  of the natural numbers is of positive lower density if

$$\liminf_{N \rightarrow \infty} \frac{|\mathcal{N} \cap \{0, 1, \dots, N\}|}{N+1} > 0,$$

which is clearly the same condition as  $n_k = O(k)$ .

Let  $x_0 \in S_1$ ,  $0 < \tau < 1$ ,  $\ell > 0$  and set

$$C(x_0, \tau, \ell) := \left\{ x \in B_\ell \left| \frac{\langle x, x_0 \rangle}{\|x\|} \geq 1 - \tau \right. \right\}. \quad (19)$$

This is a cone with vertex at 0 and  $x_0$  as the direction of its axis. We say that  $E$  satisfies the cone condition if

$$C(x_0, \tau, \ell) \subset \Omega \quad (20)$$

with some  $x_0 \in S_1$ ,  $\tau$  and  $\ell > 0$ , which means that  $\Omega$  contains a cone with vertex at 0.

**Theorem 9 a)** *If  $\mathcal{N}_E(\varepsilon)$  is of positive lower density for some  $\varepsilon > 0$  then Green's function  $g_\Omega$  is Hölder continuous at 0.*

**b)** *If Green's function  $g_\Omega$  is Hölder continuous at 0 and  $E$  satisfies the cone condition then  $\mathcal{N}_E(\varepsilon)$  is of positive lower density for some  $\varepsilon > 0$ .*

The sufficiency of the density condition for Hölder continuity of the solution to Dirichlet's problems and various elliptic equations was proved by Maz'ja in [4]- [7]. Maz'ja used the condition

$$\sum_{n=1}^N 2^{n(d-2)} \text{cap}(E \cap \overline{D}_{2^{-n}}) \geq \delta N, \quad N = 1, 2, \dots \quad (21)$$

for some  $\delta > 0$ , which is equivalent to the positive density of  $\mathcal{N}_E(\varepsilon)$ . It was also shown in [7] that in general this condition is not necessary. The problem to find conditions under which (21) is necessary was raised in [6]. Thus, the above theorem solves a long standing open problem under the simple cone condition.

The importance of the Hölder property is explained by the following result. Let  $G$  be a domain in  $\mathbf{R}^d$  with compact boundary such that 0 is on the boundary of  $G$ . We may assume that  $G \not\subseteq B_1$ , and set  $E = \overline{B}_1 \setminus G$ . Then  $\Omega := \mathbf{R}^d \setminus E = G \cup (\mathbf{R}^d \setminus \overline{B}_1)$  is a domain larger than  $G$  and 0 is on the boundary of  $\Omega$ . If  $f$  is a bounded Borel function on the boundary of  $G$ , then let  $u_f$  denote the Perron-Wiener-Brelot solution of the Dirichlet problem in  $G$  with boundary function  $f$ . We think  $u_f$  to be extended to  $\partial G$  as  $u_f = f$  there.

**Lemma 1** *Suppose that 0 is a regular boundary point of  $G$ . Then the following are equivalent.*

- 1)  $g_G(\cdot, a)$  is Hölder continuous at 0 for  $a \in G$ .
- 2)  $\mu_E(\overline{B}_r) \leq Cr^{d-2+\kappa}$  for some  $C, \kappa > 0$  and all  $r < 1$ , where  $\mu_E$  denotes the equilibrium measure of  $E$ .

*If, in addition,  $G$  satisfies the cone condition at 0, then 1) - 2) are also equivalent to*

- 3) *If  $f$  is Hölder continuous at 0, then so is  $u_f$ .*

Note also that it is indifferent if "for  $a \in G$ " in 1) is understood as "for some  $a \in G$ " or as "for all  $a \in G$ ".

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