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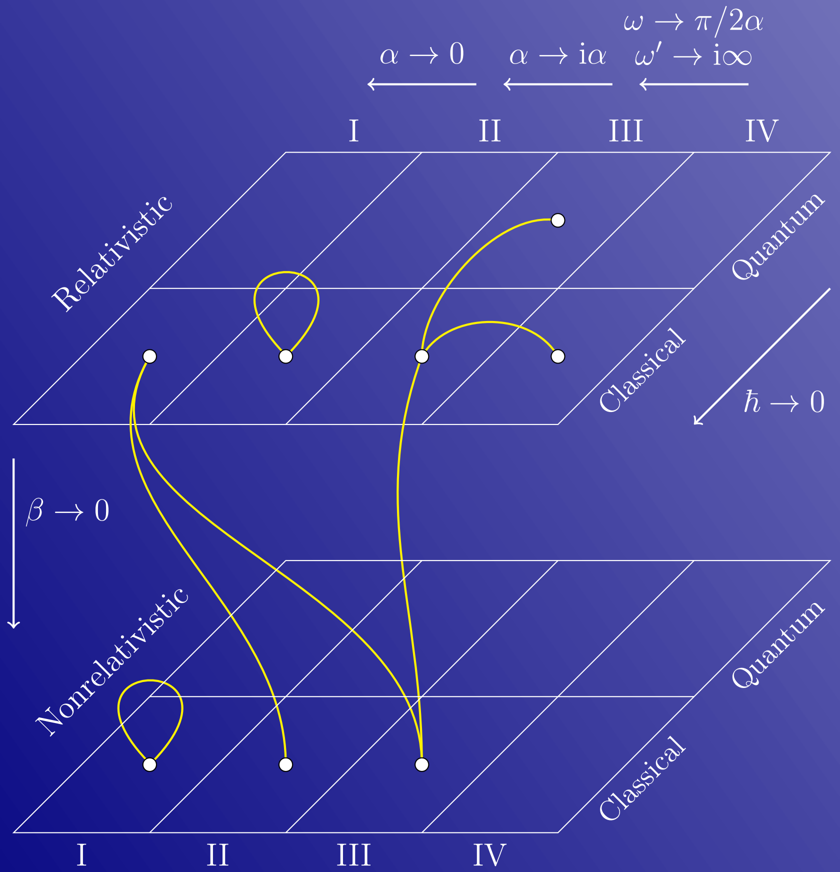

INTEGRABLE MANY-BODY SYSTEMS OF CALOGERO-RUIJSENAARS TYPE

Ph.D. thesis

Tamás F. Görbe

Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
website: www.staff.u-szeged.hu/∼tfgorbe
e-mail: tfgorbe@physx.u-szeged.hu

Supervisor: Prof. László Fehér
Department of Theoretical Physics, University of Szeged

Doctoral School of Physics
Department of Theoretical Physics
Faculty of Science and Informatics
University of Szeged
Szeged, Hungary
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Author's declaration

This thesis is submitted in accordance with the regulations for the Doctor of Philosophy degree at the University of Szeged. The results presented in the thesis are the author's original work (see Publications) with the exceptions of Section 1.1, which reviews some pre-existing material, and Subsections 4.2.2, 4.3.1, 4.3.3, 4.3.4, 4.3.5 that contain results obtained by B.G. Pusztai. These are included to make the exposition self-contained.

The research was carried out within the Ph.D. programme "Geometric and field-theoretic aspects of integrable systems" at the Department of Theoretical Physics, University of Szeged between September 2013 and August 2016.

On the cover

A schematic diagram of the various versions of Calogero-Ruijsenaars type integrable systems with dots and lines indicating the ones studied in the thesis.

Eprint

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Contents

Introduction

Integrable Systems is a broad area of research that joins seemingly unrelated problems of natural sciences amenable to exact mathematical treatment[1]. It serves as a busy crossroad of many subjects ranging from pure mathematics to experimental physics. As a result, the notion of 'integrability' is hard to pinpoint as, depending on context, it can refer to different phenomena, and "where you have two scientists you have (at least) three different definitions of integrability"[2]. Fortunately, the systems of our interest are integrable in the Liouville sense, which has a precise definition (see below). Loosely speaking, in such systems an abundance of conservation laws restricts the motion and allows the solutions to be exactly expressed with integrals, hence the name.

0.1 The golden age of integrable systems

Studying integrable systems is by no means a new activity as its origins can be traced back to the early days of modern science, when Newton solved the gravitational two-body problem and derived Kepler's laws of planetary motion (for more, see [128]). With hindsight, one might say that the solution of the Kepler problem was possible due to the existence of many conserved quantities, such as energy, angular momentum, and the Laplace-Runge-Lenz vector. In fact, the Kepler problem is a prime example of a (super)integrable system (also to be defined). As the mathematical foundations of Newtonian mechanics were established through work of Euler, Lagrange, and Hamilton, more and more examples of integrable/solvable mechanical problems were discovered. Just to name a few, these systems include the harmonic oscillator, the "spinning tops"/ rigid bodies [8] of Euler (1758), Lagrange (1788), and Kovalevskaya (1888), the geodesic motion on the ellipsoid solved by Jacobi (1839), and Neumann's oscillator model (1859). This golden age of integrable systems was ended abruptly in the late 1800s, when Poincaré, while trying to correct his flawed work on the three-body problem, realized that integrability is a fragile property, that even small perturbations can destroy [28]. This subsided scientific interest and the subject went into a dormant state for more than half a century.

[1]For those who are unfamiliar with Integrable Systems, we recommend reading the survey [121].
[2]A quote from another good read, the article *Integrability – and how to detect it* [74, pp. 31-94].

0.2 Definition of Liouville integrability

In the Hamiltonian formulation of Classical Mechanics the state of a physical system, which has n degrees of freedom, is encoded by $2n$ real numbers. These numbers consist of (generalised) positions $q = (q_1, \dots, q_n)$ and (generalised) momenta $p = (p_1, \dots, p_n)$ and are collectively called canonical coordinates of the space of states, the phase space. The time evolution of an initial state $(q_0, p_0) \in \mathbb{R}^{2n}$ is governed by Hamilton's equations of motion, a first-order system of ordinary differential equations that can be written as

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \dots, n,$$

where H is the Hamiltonian, i.e. the total energy of the system. In modern terminology, a Hamiltonian system is a triple (M, ω, H), where the phase space (M, ω) is a $2n$-dimensional symplectic manifold[3] and H is a sufficiently smooth real-valued function on M. An initial state $x_0 \in M$ evolves along integral curves of the Hamiltonian vector field X_H of H defined via $\omega(X_H, \cdot) = dH$. Darboux's theorem [2, 3.2.2 Theorem] guarantees the existence of canonical coordinates[4] (q, p) locally, in which by definition the symplectic form ω can be written as

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j,$$

and the equations of motion take the canonical form displayed above. The symplectic form ω gives rise to a Poisson structure on M, which is a handy device that takes two observables $f, g \colon M \to \mathbb{R}$ and turns them into a third one $\{f, g\}$, the Poisson bracket of f and g given by $\{f, g\} = \omega(X_f, X_g)$. In canonical coordinates, we have

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right).$$

It is bilinear, skew-symmetric, satisfies the Jacobi identity and the Leibniz rule. The equations of motion, for any $f \colon M \to \mathbb{R}$, can be rephrased using the Poisson bracket

$$\dot{f} = \{f, H\}.$$

Consequently, if $\{f, H\} = 0$, that is f Poisson commutes with the Hamiltonian H, then f is a constant of motion. In fact, this relation is symmetric, since $\{f, H\} = 0$ ensures that H is constant along the integral curves of the Hamiltonian vector field X_f.

[3] A symplectic manifold (M, ω) is a manifold M equipped with a non-degenerate, closed 2-form ω.

[4] Notice the slight and customary abuse of notation as we use the symbols q_j, p_j for representing real numbers as well as coordinate functions on M. Hopefully, this does not cause any confusion.

Having conserved quantities can simplify things, since it restricts the motion to the intersection of their level surfaces, selected by the initial conditions. Thus one should aim at finding as many independent Poisson commuting functions as possible. By independence we mean that at generic points (on a dense open subset) of the phase space the functions have linearly independent derivatives. Of course, the non-degeneracy of the Poisson bracket limits the maximum number of independent functions in involution to n. If this maximum is reached, we found a Liouville integrable system.

Definition. A Hamiltonian system (M, ω, H), with n degrees of freedom, is called *Liouville integrable*, if there exists a family of independent functions $H_1, \ldots, H_n$ in involution, i.e. $\{H_j, H_k\} = 0$ for all j, k, and H is a function of $H_1, \ldots, H_n$.

The most prominent feature of Liouville integrable systems is the existence of action-angle variables. This is a system of canonical coordinates $I = (I_1, \ldots, I_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, in which the (transformed) Hamiltonians $H_1, \ldots, H_n$ depend only on the action variables I, which are themselves first integrals, while the angle variables φ evolve linearly in time. An important result is the following

Liouville-Arnold theorem. [2, 5.2.24 Theorem] *Consider (M, ω, H) to be a Liouville integrable system with the Poisson commuting functions $H_1, \ldots, H_n$. Then the level set*

$$M_c = \{x \in M \mid H_j(x) = c_j,\ j = 1, \ldots, n\}$$

is a smooth n-dimensional submanifold of M, which is invariant under the Hamiltonian flow of the system. Moreover, if M_c is compact and connected, then it is diffeomorphic to an n-torus $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$, and the Hamiltonian flow is linear on M_c, i.e. the angle variables φ on M_c satisfy $\dot{\varphi}_j = \nu_j$, for some constants ν_j, $j = 1, \ldots, n$.

The action variables I are also encoded in the level set M_c. Roughly speaking, they determine the size of M_c, since I_j is obtained by integrating the canonical 1-form the phase space over the j-th cycle of the torus M_c.

Another relevant notion is superintegrability, which requires the existence of extra constants of motion.

Definition. A Liouville integrable system is called *superintegrable*, if in addition to the Hamiltonians $H_1, \ldots, H_n$ there exist independent first integrals $f_1, \ldots, f_k$ $(1 \leq k < n)$. If $k = n - 1$, then the system is *maximally superintegrable*.

Examples of maximally superintegrable systems include the Kepler problem, the harmonic oscillator with rational frequencies, and the rational Calogero-Moser system considered in Chapter 1. For more on the theory of integrable systems, see [11].

Remark. It should be noted that, although there is no generally accepted notion of integrability at the quantum level, there are quantum mechanical systems that are called *integrable*.

0.3 Solitary splendor: The renascence of integrability

About fifty years ago a revival has taken place in the field of Integrable Systems, when Zabusky and Kruskal [150] conducted a numerical study of the Korteweg-de Vries (KdV) equation[5], that is the nonlinear $(1+1)$-dimensional partial differential equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

and re-discovered its stable solitary wave solutions[6], whose interaction resembled that of colliding particles, hence they gave them the name *solitons*. Subsequently, Kruskal et al. [50] started a detailed investigation of the KdV equation and found an infinite number of conservation laws associated to it. More explicitly, they showed that the eigenvalues of the Schrödinger operator

$$L = \partial_x^2 + u$$

are invariant in time if the 'potential' u is a solution of the KdV equation. Moreover, they used the Inverse Scattering Method to reconstruct the potential from scattering data. Lax showed [78] that the KdV equation is equivalent to an equation involving a pair of operators, now called *Lax pair*, of the form

$$\dot{L} = [B, L],$$

where L is the Schrödinger operator above, and B is a skew-symmetric operator. The commutator form of the Lax equation explains the isospectral nature of the operator L. The connection to integrable systems was made by Faddeev and Zakharov [151], who showed that the KdV equation can be viewed as a completely integrable Hamiltonian system with infinitely many degrees of freedom. These initial findings renewed interest in integrable systems and their applications. For example, Lax pairs associated to other integrable systems were found and used to generate conserved quantities.

The ideas and developments presented so far were all about the KdV equation. However, there are other physically relevant nonlinear PDEs with soliton solutions, which have been solved using the Inverse Scattering Method. For example, the sine-Gordon equation [1]

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0,$$

which can be interpreted as the equation that describes the twisting of a continuous

[5]The motivation for Zabusky and Kruskal's work was to understand the recurrent behaviour in the Fermi-Pasta-Ulam-Tsingou problem [45], which turns into the KdV equation in the continuum limit.

[6]Korteweg and de Vries [73] devised their equation to reproduce such stable travelling waves, that were first observed by Russell [122] in the canals of Edinburgh.

chain of needles attached to an elastic band. It has different kinds of soliton solutions, called *kink*, *antikink*, and *breather*, that can interact with one another. It is a relativistic equation, since its solutions are invariant under the action of the Poincaré group of $(1+1)$-dimensional space-time.

The nonlinear Schrödinger equation [153] is another famous example. It reads

$$\mathrm{i}\psi_t + \frac{1}{2}\psi_{xx} - \kappa|\psi|^2\psi = 0,$$

where ψ is a complex-valued wave function and κ is constant. It is also an exactly solvable Hamiltonian system [152]. The equation is nonrelativistic (Galilei invariant).

Now let us list some applications of these soliton equations. The Korteweg-de Vries equation can be applied to describe shallow-water waves with weakly non-linear restoring forces and long internal waves in a density-stratified ocean. It is also useful in modelling ion acoustic waves in a plasma and acoustic waves on a crystal lattice. The kinks and breathers of the sine-Gordon equation can used as models of nonlinear excitations in complex systems in physics and even in cellular structures. The nonlinear Schrödinger equation is of central importance in fluid dynamics, plasma physics, and nonlinear optics as it appears in the Manakov system, a model of wave propagation in fibre optics.

Parallel to soliton theory, various exactly solvable quantum many-body systems appeared, that describe the interaction of quantum particles in one spatial dimension. These models proved to be a fruitful source of ideas and a great influence on the development of mathematical physics. Earlier important milestones include Bethe's solution of the one-dimensional Heisenberg model (Bethe Ansatz, 1931), Pauling's work on the 6-vertex model (1935), Onsager's solution of the planar Ising model (1944), and the delta Bose gas of Lieb-Liniger (1963). At the level of classical mechanics, a crucial step was Toda's discovery of a nonlinear, one-dimensional lattice model [137] with soliton solutions. The Toda lattice is an infinite chain of particles interacting via exponential nearest neighbour potential. The nonperiodic and periodic Toda chains are n particles with such interaction put on a line and a ring, and have the Hamiltonians

$$H_{\mathrm{np}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_{j+1}-q_j)}, \quad \text{and} \quad H_{\mathrm{per}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_{j+1}-q_j)} + g^2 e^{2(q_1-q_n)},$$

respectively. Hénon [60] found n conserved quantities for both of these systems, and Flashka [46, 47] and Manakov [81] found Lax pairs giving rise to these first integrals and proved them to be in involution. Therefore the Toda lattices are completely integrable. The scattering theory of the nonperiodic Toda lattice was examined by Moser [89]. Bogoyavlensky [17] generalised the Toda lattice to root systems of simple Lie algebras. Olshanetsky, Perelomov [96, 97] and Kostant [75] initiated group-theoretic treatments.

0.4 Calogero-Ruijsenaars type systems

In the early 1970s further exactly solvable quantum many-body systems were found by Calogero [20, 21] and Sutherland [132, 133]. Calogero considered particles on a line in harmonic confinement with a pairwise interaction inversely proportional to the square of their relative distances (rational case). Sutherland solved the corresponding problem of particles on a ring, i.e. interacting via a periodic pair-potential (trigonometric case). The classical versions were examined by Moser [88], who provided Lax pairs, analysed the particle scattering in the rational case, which he proved to be Liouville integrable[7]. Models with short-range interaction (hyperbolic case) [26] and with elliptic potentials (elliptic case) [22] were also formulated (see Figures 1 and 2).

We give a short description of the classical systems. Let the number of particles n be fixed, $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ collect the particle-positions and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ the conjugate momenta. The configuration space is usually some open domain $C \subseteq \mathbb{R}^n$, and the phase space M is its cotangent bundle

$$M = T^*C = \{(q,p) \mid q \in C,\ p \in \mathbb{R}^n\},$$

equipped with the canonical symplectic form

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

The Hamiltonian of the models can be written in the general form

$$H_{\mathrm{nr}} = \frac{1}{2m}\sum_{j=1}^{n} p_j^2 + \frac{g^2}{m}\sum_{j<k} V(q_j - q_k),$$

where $m > 0$ denotes the mass of particles, g is a positive coupling constant regulating the strength of particle repulsion[8], and the pair-potential V can be one of four types:

$$V(q) = \begin{cases} 1/q^2, & \text{rational (I)}, \\ \alpha^2/\sinh^2(\alpha q), & \text{hyperbolic (II)}, \\ \alpha^2/\sin^2(\alpha q), & \text{trigonometric (III)}, \\ \wp(q;\omega,\omega'), & \text{elliptic (IV)}. \end{cases}$$

Here $\wp$ stands for Weierstrass's elliptic function with half-periods $(\omega,\omega') \in \mathbb{R}_+ \times \mathrm{i}\mathbb{R}_+$. By taking the parameter $\alpha \to \mathrm{i}\alpha$, II and III are exchanged, while $\alpha \to 0$ produces I.

[7] The rational three-body system was treated by Marchioro [82] and to some extent by Jacobi [65].
[8] The interaction is attractive, if $g^2 < 0$. Setting $g = 0$ yields free particles.

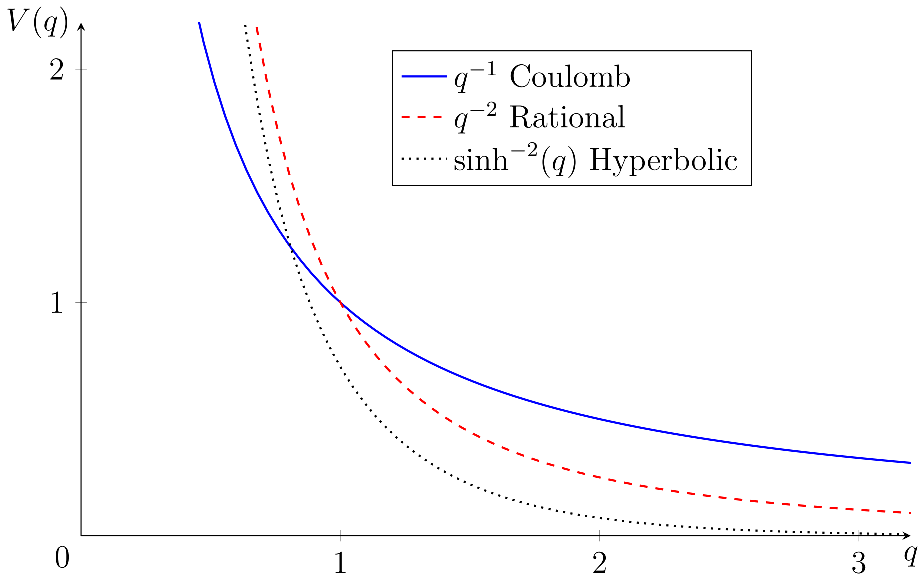

Figure 1: Three repulsive potential functions. The Coulomb potential $V(q) = q^{-1}$ (solid blue) and rational potential $V(q) = q^{-2}$ (dashed red) express long-range interaction in comparison to the hyperbolic potential $V(q) = \sinh^{-2}(q)$ (dotted black).

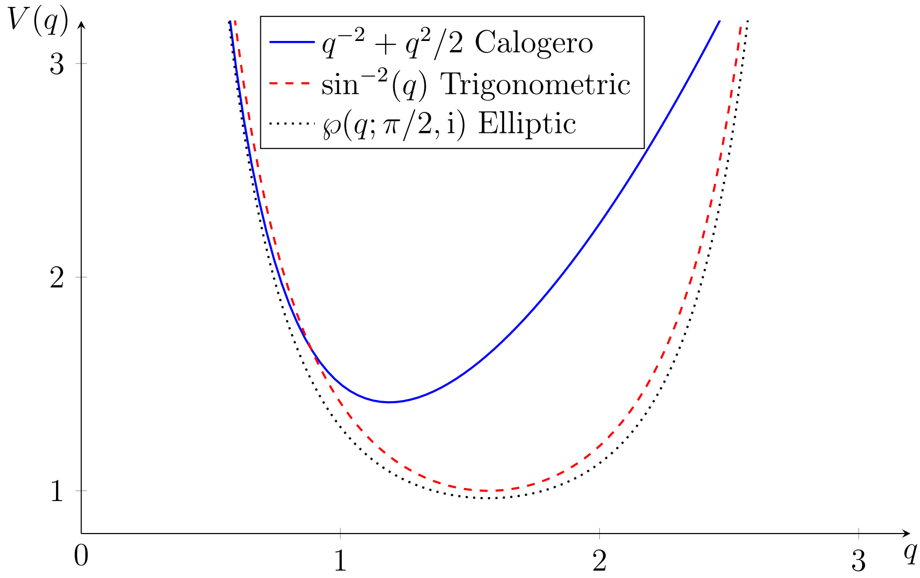

Figure 2: Three confining potential functions. Calogero potential $V(q) = q^{-2} + q^2/2$ (solid blue), trigonometric potential $V(q) = \sin^{-2}(q)$ (dashed red), and elliptic potential $V(q) = \wp(q; \omega, \omega')$ (dotted black) with half-periods $\omega = \pi/2$, $\omega' = \mathrm{i}$.

The elliptic potential degenerates to the other ones in various limits[9]

$$\wp(q; \omega, \omega') \to \begin{cases} 1/q^2, & \text{if } \omega \to \infty,\ \omega' \to \mathrm{i}\infty, \\ \alpha^2/3 + \alpha^2/\sinh^2(\alpha q), & \text{if } \omega \to \infty,\ \omega' \to \mathrm{i}\pi/2\alpha, \\ -\alpha^2/3 + \alpha^2/\sin^2(\alpha q), & \text{if } \omega \to \pi/2\alpha,\ \omega' \to \mathrm{i}\infty. \end{cases}$$

[9]It is worth mentioning that the Toda lattices (both periodic and nonperiodic) can be also obtained from the elliptic model. For details, see [64, 115, 118].

These models are nonrelativistic, that is invariant under the Galilei group of $(1+1)$-dimensional space-time. Relativistic (i.e. Poincaré-invariant) integrable deformations were constructed[10] by Ruijsenaars and Schneider [111], and Ruijsenaars [112]. The Hamiltonians of the relativistic systems read

$$H_{\mathrm{rel}} = \frac{1}{\beta^2 m}\sum_{j=1}^{n}\cosh(\beta p_j)\prod_{k\neq j} f(q_j - q_k),$$

where $\beta = 1/mc > 0$ is the deformation parameter (c can be interpreted as the speed of light), and the function f can be one of the following

$$f(q) = \begin{cases} (1+\beta^2 g^2/q^2)^{1/2}, & \text{rational (I)},\\ (1+\sin^2(\alpha\beta g)/\sinh^2(\alpha q))^{1/2}, & \text{hyperbolic (II)},\\ (1+\sinh^2(\alpha\beta g)/\sin^2(\alpha q))^{1/2}, & \text{trigonometric (III)},\\ (\sigma^2(\mathrm{i}\beta g;\omega,\omega')[\wp(\mathrm{i}\beta g;\omega,\omega') - \wp(q;\omega,\omega')])^{1/2}, & \text{elliptic (IV)}. \end{cases}$$

Here σ is the Weierstrass sigma function. In the nonrelativistic limit $\beta \to 0$ we get

$$\lim_{\beta\to 0}(H_{\mathrm{rel}} - \frac{n}{\beta^2 m}) = H_{\mathrm{nr}}.$$

The quantum Hamiltonians at the nonrelativistic level consist of commuting partial differential operators, obtained from classical Hamiltonians by canonical quantization. For example, the Hamiltonian operator can be written as

$$\hat{H}_{\mathrm{nr}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\frac{\partial^2}{\partial q_j^2} + \frac{g(g-\hbar)}{m}\sum_{j<k} V(\hat{q}_j - \hat{q}_k).$$

The corresponding Hilbert space, on which these operators act, is the space $L^2(C, dq)$ of square integrable complex-valued functions over the classical configuration space C. In contrast, the relativistic quantum Hamiltonians have an exponential dependence on the momentum operators, resulting in analytic differential operators, such as

$$\hat{H}_{\mathrm{rel}} = \frac{1}{2\beta^2 m}(S_1 + S_{-1}), \quad \text{with} \quad S_{\pm 1} = \sum_{j=1}^{n}\Big[\prod_{k\neq j} f_{\mp}(\hat{q}_j - \hat{q}_k)\Big] e^{\mp \mathrm{i}\hbar\beta\partial_j}\Big[\prod_{k\neq j} f_{\pm}(\hat{q}_j - \hat{q}_k)\Big].$$

In the elliptic case $f_{\pm}(q) = \sigma(\mathrm{i}\beta g + q)/\sigma(q)$ and the other cases are obtained as limits. Therefore these operators act on functions that have an analytic continuation to an at least $2\hbar\beta$ wide strip in the complex plane. For more details on these models, the reader is referred to the articles [25, 119, 120] or the exhaustive surveys [115, 118].

[10] With the motivation to reproduce the scattering of sine-Gordon solitons using interacting particles.

A scheme of the Calogero-Ruijsenaars type systems is depicted in Figure 3.

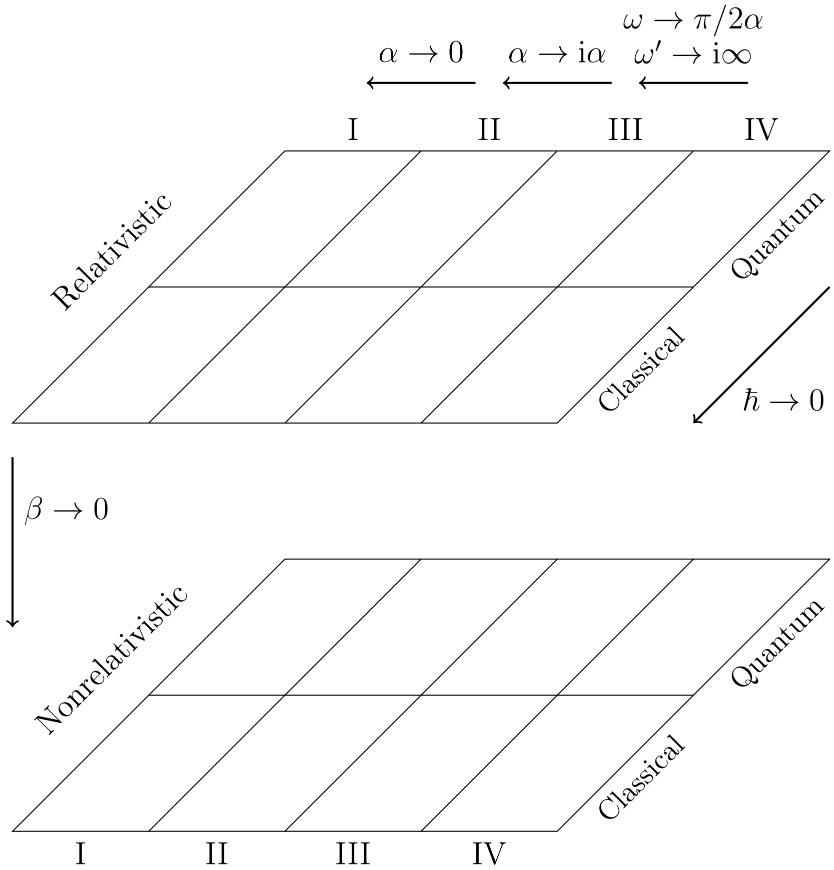

Figure 3: Schematics of Calogero-Ruijsenaars type systems.

The above-mentioned models have generalisations formulated using root systems[11]. To this end, notice that in the Hamiltonians presented above $q_j - q_k = a \cdot q$ are the inner product of q and the root vectors $a \in \mathrm{A}_{n-1}$ of the simple Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. It turns out that if A_{n-1} is replaced with any root system the resulting system is still integrable. Such root system generalisations were introduced by Olshanetsky and Perelomov [98, 99], who found Lax pairs and proved integrability for models attached to the classical root systems $\mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ (and BC_n). For arbitrary root systems, the integrability of non-elliptic quantum systems was showed by Heckman and Opdam [57], and Sasaki et al. [69], and for classical systems (including the elliptic case) by Khastgir and Sasaki [70]. Integrable Ruijsenaars-Schneider models attached to non-A type root systems were found by van Diejen [140, 141, 142, 143]. It is a remarkable fact that the eigenfunctions of these generalised Calogero-Ruijsenaars type operators are multivariate orthogonal polynomials, and the equilibrium positions of the classical systems are given by the zeros of classical orthogonal polynomials [23, 94].

There are other ways to generalise the Calogero-Ruijsenaars type systems, e.g. by allowing internal degrees of freedom (spins) [51, 110] or supersymmetry [127, 19, 14].

[11]A short summary of facts about root systems can be found in [123]. For more details, see [61].

0.5 Basic idea of Hamiltonian reduction

In their pioneering work, Kazhdan, Kostant, and Sternberg [68] offered a key insight into the origin of the Poisson commuting first integrals of Calogero-Moser-Sutherland models. In a nutshell, they derived the complicated motion of these many-body systems by applying Marsden-Weinstein reduction [85] to a higher dimensional free particle. The reduction framework and its application to Hamiltonian systems have undergone considerable development since then [101, 83]. Here we only present a description of the reduction machinery that is tailored to our purposes. Part I of the thesis contains specific implementations of this approach.

The reduction procedure starts with choosing a 'big phase space' of group-theoretic origin. This might be, say, the cotangent bundle $P = T^*X$ of a matrix Lie group or algebra X. The natural symplectic structure Ω of the cotangent bundle P permits one to define a Hamiltonian system $(P, \Omega, \mathcal{H})$ by specifying a Hamiltonian $\mathcal{H}\colon P \to \mathbb{R}$. If $\mathcal{H}$ is simple enough, then the equations of motion can be solved, or even better, a family of Poisson commuting functions $\{\mathcal{H}_j\}$ be found, which $\mathcal{H}$ is a member/function of. Then by choosing an appropriate group action (of some group G) on X (hence P), under which $\mathcal{H}_j$ are invariant[12], one can construct the momentum map $\Phi\colon P \to \mathfrak{g}^*$ corresponding to this action. Fixing the value μ of the momentum map Φ produces a level surface $\Phi^{-1}(\mu)$ in the 'big phase space'. This constraint surface is foliated by the orbits of the isotropy/gauge group $G_\mu \subset G$ of the momentum value. The reduced phase space $(P_{\text{red}}, \omega_{\text{red}})$ consists of these orbits. The point is that the flows of the commuting 'free' Hamiltonians $\{\mathcal{H}_j\}$ preserve the momentum surface and are constant along obits. Therefore they admit reduced versions $H_j\colon P_{\text{red}} \to \mathbb{R}$, which still Poisson commute[13] and the resulting Hamiltonian system $(P_{\text{red}}, \omega_{\text{red}}, H)$ is Liouville integrable. In practice, we model the reduced phase space by a smooth slice S of the gauge orbits (see Figure 4). This slice S is obtained by solving the momentum equation $\Phi = \mu$. Systems in action-angle duality (see below) can emerge in this picture if one has two sets of invariant functions and two models $S, \tilde{S}$ of the reduced phase space.

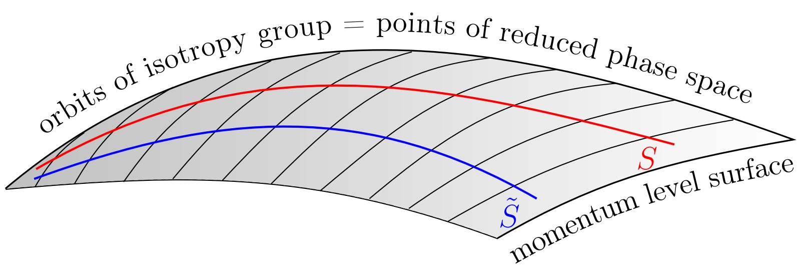

Figure 4: The geometry of reduction and action-angle duality.

[12]It can go the other way around, that is have a group action first, then find invariant functions.
[13]With respect to the Poisson bracket induced by the reduced symplectic form ω_{red}.

0.6 Action-angle dualities

Action-angle duality is a relation between two Liouville integrable systems, say (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$, requiring the existence of canonical coordinates (q, p) on M and $(\tilde{q}, \tilde{p})$ on $\tilde{M}$ (or on dense open submanifolds of M and $\tilde{M}$) and a global symplectomorphism $\mathcal{R}\colon M \to \tilde{M}$, the *action-angle map*, such that $(\tilde{q}, \tilde{p}) \circ \mathcal{R}$ are action-angle variables for the Hamiltonian H and $(q, p) \circ \mathcal{R}^{-1}$ are action-angle variables for the Hamiltonian $\tilde{H}$. This means that $H \circ \mathcal{R}^{-1}$ depends only on $\tilde{q}$ and $\tilde{H} \circ \mathcal{R}$ only on q. Then one says that the systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ are in *action-angle duality*. In addition, for the systems of our interest it also happens that the Hamiltonian H, when expressed in the coordinates (q, p), admits interpretation in terms of interacting 'particles' with position variables q, and similarly, $\tilde{H}$ expressed in $(\tilde{q}, \tilde{p})$ describes the interacting points with positions $\tilde{q}$. Thus q are particle positions for H and action variables for $\tilde{H}$, and the $\tilde{q}$ are positions for $\tilde{H}$ and actions for H. The significance of this curious property is clear for instance from the fact that it persists at the quantum mechanical level as the bispectral character of the wave functions [30, 114], which are important special functions.

$$\begin{tikzcd}
\boxed{\text{Hyperbolic Ruijsenaars-Schneider}} \arrow[r, leftrightarrow, "\mathcal{R}"] \arrow[d, "\beta \to 0"'] & \boxed{\text{Hyperbolic Ruijsenaars-Schneider}} \arrow[d, "\alpha \to 0"] \\
\boxed{\text{Hyperbolic Calogero-Moser}} \arrow[r, leftrightarrow, "\mathcal{R}"] \arrow[d, "\alpha \to 0"'] & \boxed{\text{Rational Ruijsenaars-Schneider}} \arrow[d, "\beta \to 0"] \\
\boxed{\text{Rational Calogero-Moser}} \arrow[r, leftrightarrow, "\mathcal{R}"] & \boxed{\text{Rational Calogero-Moser}}
\end{tikzcd}$$

Figure 5: Action-angle dualities among Calogero-Ruijsenaars type systems.

Dual pairs of many-body systems were exhibited by Ruijsenaars (see Figure 5) in the course of his direct construction [113, 115, 117, 118] of action-angle variables for the many-body systems (of non-elliptic Calogero-Ruijsenaars type and non-periodic Toda type) associated with the root system A_{n-1}. The idea that dualities can be interpreted in terms of Hamiltonian reduction can be distilled from [68] and was put forward explicitly in several papers in the 1990s, e.g. [48, 54]. These papers contain a wealth of interesting ideas and results, but often stated without full proofs. In the last decade or so, Fehér and collaborators undertook the systematic study of these dualities within the framework of reduction [36, 35, 9, 37, 38, 34, 40]. It seems natural to expect that action-angle dualities exist for many-body systems associated with other root systems. Substantial evidence in favour of this expectation was given by Pusztai [105, 106, 107, 108, 109]. This thesis presents results (see Publications) that were obtained in connection to these earlier developments.

0.7 Outline of the thesis

The main content of the thesis is divided into two parts with a total of five chapters.

Part I takes the reduction approach to Calogero-Ruijsenaars type systems. In each of its chapters the basic idea of reduction that we just sketched is put into practice, only at an increasing level of complexity. In particular, Chapter 1 presents a streamlined derivation of the rational Calogero-Moser system using reduction. Section 1.2 exhibits the utility of the reduction perspective, as we give a simple proof of a formula providing action-angle coordinates. Chapter 2 is a study of the trigonometric BC_n Sutherland system. We provide a physical interpretation of the model in Section 2.1 and prepare the ingredients of reduction in Section 2.2. In Section 2.3, we solve the momentum equations and obtain the action-angle dual of the BC_n Sutherland system. In Section 2.4, we apply our duality map to various problems, such as equilibrium configurations, proving superintegrability, and showing the equivalence of two sets of Hamiltonians. Chapter 3 generalises certain results of the previous chapter as it derives a 1-parameter deformation of the trigonometric BC_n Sutherland system using Hamiltonian reduction of the Heisenberg double of $\mathrm{SU}(2n)$. We define the pertinent reduction in Section 3.1, solve the momentum constraints in Section 3.2, and characterize the reduced system in Section 3.3. In Section 3.4, we complete a recent derivation of the hyperbolic analogue.

Part II is a collection of work motivated by, but not involving reduction techniques. Chapter 4 reports our discovery of a Lax pair for the hyperbolic van Diejen system with two independent coupling parameters. The preparatory Section 4.1 is followed by the explicit formulation of our Lax matrix in Section 4.2. In Section 4.3, we show that the dynamics can be solved by a projection method, which in turn allows us to initiate the study of the scattering properties. We prove the equivalence between the first integrals provided by the eigenvalues of the Lax matrix and the family of van Diejen's commuting Hamiltonians in Section 4.4. Chapter 5 is concerned with the explicit construction of compactified versions of trigonometric and elliptic Ruijsenaars-Schneider systems. In Section 5.1, we embed the local phase space of the model into the complex projective space $\mathbb{CP}^{n-1}$. Section 5.2 contains our proof of the global extension of the trigonometric Lax matrix to $\mathbb{CP}^{n-1}$. We use our direct construction to introduce new compactified elliptic systems in Section 5.3.

The chapters are complemented by Appendices collecting supplementary material (alternative proofs, detailed derivations, etc.). A Summary presents the most important results in a concise form. A list of Publications, on which this thesis is based, and a Bibliography are also included.

Part I

Reduction approach, action-angle duality, applications

1 A pivotal example

We start this chapter by describing the rational Calogero-Moser system and recalling how it originates from Hamiltonian reduction [68]. Then we use reduction treatment to simplify Falqui and Mencattini's recent proof [33] of Sklyanin's expression [129] providing spectral Darboux coordinates of the rational Calogero-Moser system.

1.1 Rational Calogero-Moser system

The Hamiltonian H (1.1) with rational potential models equally massive interacting particles moving along a line with a pair potential inversely proportional to the square of the distance. The model was introduced and solved at the quantum level by Calogero [21]. The complete integrability of its classical version was established by Moser [88], who employed the Lax formalism [78] to identify a complete set of commuting integrals as coefficients of the characteristic polynomial of a certain Hermitian matrix function, called the Lax matrix.

These developments might prompt one to consider the Poisson commuting eigenvalues of the Lax matrix and be interested in searching for an expression of conjugate variables. Such an expression was indeed formulated by Sklyanin [129] in his work on bispectrality, and worked out in detail for the open Toda chain [130]. Sklyanin's formula for the rational Calogero-Moser model was recently confirmed within the framework of bi-Hamiltonian geometry by Falqui and Mencattini [33] in a somewhat circuitous way, although a short-cut was pointed out in the form of a conjecture. The purpose of this chapter is to prove this conjecture and offer an alternative simple proof of Sklyanin's formula using results of Hamiltonian reduction.

1.1.1 Description of the model

For n particles, let the n-tuples $q = (q_1, \dots, q_n)$ and $p = (p_1, \dots, p_n)$ collect their coordinates and momenta, respectively. Then the Hamiltonian of the model reads

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{\substack{j,k=1 \\ (j<k)}}^{n} \frac{g^2}{(q_j - q_k)^2}, \tag{1.1}$$

where g is a real coupling constant tuning the strength of particle interaction. The pair potential is singular at $q_j = q_k$ $(j \neq k)$, hence any initial ordering of the particles remains unchanged during time-evolution. The configuration space is chosen to be the domain $\mathcal{C} = \{q \in \mathbb{R}^n \mid q_1 > \cdots > q_n\}$, and the phase space is its cotangent bundle

$$T^*\mathcal{C} = \{(q,p) \mid q \in \mathcal{C},\ p \in \mathbb{R}^n\}, \tag{1.2}$$

endowed with the standard symplectic form

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j. \tag{1.3}$$

1.1.2 Calogero particles from free matrix dynamics

The Hamiltonian system $(T^*\mathcal{C}, \omega, H)$, called the rational Calogero-Moser system, can be obtained as an appropriate Marsden-Weinstein reduction of the free particle moving in the space of $n \times n$ Hermitian matrices as follows.

Consider the manifold of pairs of $n \times n$ Hermitian matrices

$$M = \{(X,P) \mid X,P \in \mathfrak{gl}(n,\mathbb{C}),\ X^\dagger = X,\ P^\dagger = P\}, \tag{1.4}$$

equipped with the symplectic form

$$\Omega = \operatorname{tr}(dX \wedge dP). \tag{1.5}$$

The Hamiltonian of the analogue of a free particle reads

$$\mathcal{H}(X,P) = \frac{1}{2}\operatorname{tr}(P^2). \tag{1.6}$$

The equations of motion can be solved explicitly for this Hamiltonian system $(M, \Omega, \mathcal{H})$, and the general solution is given by $X(t) = tP_0 + X_0$, $P(t) = P_0$. Moreover, the functions $\mathcal{H}_k(X,P) = \frac{1}{k}\operatorname{tr}(P^k)$, $k = 1,\dots,n$ form an independent set of commuting first integrals.

The group of $n \times n$ unitary matrices $\mathrm{U}(n)$ acts on M (1.4) by conjugation

$$(X,P) \to (UXU^\dagger, UPU^\dagger), \quad U \in \mathrm{U}(n), \tag{1.7}$$

leaves both the symplectic form Ω (1.5) and the Hamiltonians $\mathcal{H}_k$ invariant, and the matrix commutator $(X,P) \to [X,P]$ is a momentum map for this $\mathrm{U}(n)$-action. Consider the Hamiltonian reduction performed by factorizing the momentum constraint surface

$$[X, P] = \mathrm{i}g(vv^\dagger - \mathbf{1}_n) = \mu, \quad v = (1\dots1)^\dagger \in \mathbb{R}^n, \quad g \in \mathbb{R}, \tag{1.8}$$

with the stabilizer subgroup $G_\mu \subset \mathrm{U}(n)$ of μ, e.g. by diagonalization of the X component. This yields the gauge slice $S = \{(Q(q,p), L(q,p)) \mid q \in \mathcal{C},\ p \in \mathbb{R}^n\}$, where

$$Q_{jk} = (UXU^\dagger)_{jk} = q_j\delta_{jk}, \quad L_{jk} = (UPU^\dagger)_{jk} = p_j\delta_{jk} + \mathrm{i}g\frac{1-\delta_{jk}}{q_j - q_k}, \quad j,k = 1,\dots,n. \tag{1.9}$$

This S is symplectomorphic to the reduced phase space and to $T^*\mathcal{C}$ (1.2) since it inherits the reduced symplectic form ω (1.3). The unreduced Hamiltonians project to a commuting set of independent integrals $H_k = \frac{1}{k}\mathrm{tr}(L^k)$, $k = 1,\dots,n$, such that $H_2 = H$ (1.1) and what's more, the completeness of Hamiltonian flows follows automatically from the reduction. Therefore the rational Calogero-Moser system is completely integrable.

The similar role of matrices X and P in the derivation above can be exploited to construct action-angle variables for the rational Calogero-Moser system. This is done by switching to the gauge, where the P component is diagonalized by some matrix $\tilde{U} \in G_\mu$, and it boils down to the gauge slice $\tilde{S} = \{(\tilde{Q}(\phi,\lambda), \tilde{L}(\phi,\lambda)) \mid \phi \in \mathbb{R}^n,\ \lambda \in \mathcal{C}\}$, where

$$\tilde{Q}_{jk} = (\tilde{U}X\tilde{U}^\dagger)_{jk} = \phi_j\delta_{jk} - \mathrm{i}g\frac{1-\delta_{jk}}{\lambda_j - \lambda_k}, \quad \tilde{L}_{jk} = (\tilde{U}P\tilde{U}^\dagger)_{jk} = \lambda_j\delta_{jk}, \quad j,k = 1,\dots,n. \tag{1.10}$$

By construction, $\tilde{S}$ with the symplectic form $\tilde{\omega} = \sum_{j=1}^n d\phi_j \wedge d\lambda_j$ is also symplectomorphic to the reduced phase space, thus a canonical transformation $(q,p) \to (\phi,\lambda)$ is obtained, where the reduced Hamiltonians depend only on λ, viz. $H_k = \frac{1}{k}(\lambda_1^k + \dots + \lambda_n^k)$, $k = 1,\dots,n$.

1.2 Application: Canonical spectral coordinates

Now, we turn to the question of variables conjugate to the Poisson commuting eigenvalues $\lambda_1,\dots,\lambda_n$ of L (1.9), i.e. such functions $\theta_1,\dots,\theta_n$ in involution that

$$\{\theta_j, \lambda_k\} = \delta_{jk}, \quad j,k = 1,\dots,n. \tag{1.11}$$

At the end of Subsection 1.1.2 we saw that the variables $\phi_1,\dots,\phi_n$ are such functions. These action-angle variables λ, ϕ were already obtained by Moser [88] using scattering theory, and also appear in Ruijsenaars's proof of the self-duality of the rational Calogero-Moser system [113].

Let us define the following functions over the phase space $T^*\mathcal{C}$ (1.2) with dependence on an additional variable z:

$$A(z) = \det(z\mathbf{1}_n - L), \quad C(z) = \operatorname{tr}(Q \operatorname{adj}(z\mathbf{1}_n - L)vv^\dagger), \quad D(z) = \operatorname{tr}(Q \operatorname{adj}(z\mathbf{1}_n - L)), \tag{1.12}$$

where Q and L are given by (1.9), $v = (1 \dots 1)^\dagger \in \mathbb{R}^n$ and adj denotes the adjugate matrix, i.e. the transpose of the cofactor matrix. Sklyanin's formula [129] for $\theta_1, \dots, \theta_n$ then reads

$$\theta_k = \frac{C(\lambda_k)}{A'(\lambda_k)}, \quad k = 1, \dots, n. \tag{1.13}$$

In [33] Falqui and Mencattini have shown that

$$\mu_k = \frac{D(\lambda_k)}{A'(\lambda_k)}, \quad k = 1, \dots, n \tag{1.14}$$

are conjugate variables to $\lambda_1, \dots, \lambda_n$, and

$$\theta_k = \mu_k + f_k(\lambda_1, \dots, \lambda_n), \quad k = 1, \dots, n, \tag{1.15}$$

with such λ-dependent functions $f_1, \dots, f_n$ that

$$\frac{\partial f_j}{\partial \lambda_k} = \frac{\partial f_k}{\partial \lambda_j}, \quad j, k = 1, \dots, n \tag{1.16}$$

thus $\theta_1, \dots, \theta_n$ given by Sklyanin's formula (1.13) are conjugate to $\lambda_1, \dots, \lambda_n$. This was done in a roundabout way, although the explicit form of relation (1.15) was conjectured.

Here we take a different route by making use of the reduction viewpoint of Subsection 1.1.2. From this perspective, the problem becomes transparent and can be solved effortlessly. First, we show that $\mu_1, \dots, \mu_n$ (1.14) are nothing else than the angle variables $\phi_1, \dots, \phi_n$.

Lemma 1.1. *The variables $\mu_1, \dots, \mu_n$ defined in (1.14) are the angle variables $\phi_1, \dots, \phi_n$ of the rational Calogero-Moser system.*

Proof. Notice that, by definition, $\mu_1, \dots, \mu_n$ are gauge invariant, thus by working in the gauge, where the P component is diagonal, that is with the matrices $\tilde{Q}$, $\tilde{L}$ (1.10), we get

$$\frac{D(z)}{A'(z)} = \frac{\sum_{j=1}^n \phi_j \prod_{\substack{\ell=1 \\ (\ell \neq j)}}^n (z - \lambda_\ell)}{\sum_{j=1}^n \prod_{\substack{\ell=1 \\ (\ell \neq j)}}^n (z - \lambda_\ell)}. \tag{1.17}$$

Substituting $z = \lambda_k$ into (1.17) yields $\mu_k = \phi_k$, for each $k = 1, \dots, n$. □

Next, we prove the relation of functions A, C, D (1.12), that was conjectured in [33].

Theorem 1.2. *For any $n \in \mathbb{N}$, $(q,p) \in T^*\mathcal{C}$ (1.2), and $z \in \mathbb{C}$ we have*

$$C(z) = D(z) + \frac{\mathrm{i}g}{2} A''(z). \tag{1.18}$$

Proof. Pick any point (q,p) in the phase space $T^*\mathcal{C}$ and consider the corresponding point (λ, ϕ) in the space of action-angle variables. Since $A(z) = (z-\lambda_1)\dots(z-\lambda_n)$ we have

$$\frac{\mathrm{i}g}{2} A''(z) = \mathrm{i}g \sum_{\substack{j,k=1\\(j<k)}}^{n} \prod_{\substack{\ell=1\\(\ell\neq j,k)}}^{n} (z-\lambda_\ell). \tag{1.19}$$

The difference of functions C and D (1.12) reads

$$C(z) - D(z) = \operatorname{tr}\big(Q \operatorname{adj}(z\mathbf{1}_n - L)(vv^\dagger - \mathbf{1}_n)\big). \tag{1.20}$$

Since this is a gauge invariant function, we are allowed to work with $\tilde{Q}, \tilde{L}$ (1.10) instead of Q, L (1.9). Therefore (1.20) can be written as the sum of all off-diagonal components of $\tilde{Q} \operatorname{adj}(z\mathbf{1}_n - \tilde{L})$, that is

$$C(z) - D(z) = \mathrm{i}g \sum_{\substack{j,k=1\\(j\neq k)}}^{n} \frac{-1}{\lambda_j - \lambda_k} \prod_{\substack{\ell=1\\(\ell\neq k)}}^{n} (z-\lambda_\ell) = \mathrm{i}g \sum_{\substack{j,k=1\\(j<k)}}^{n} \prod_{\substack{\ell=1\\(\ell\neq j,k)}}^{n} (z-\lambda_\ell). \tag{1.21}$$

This concludes the proof. □

Our theorem confirms that indeed relation (1.15) is valid with

$$f_k(\lambda_1,\dots,\lambda_n) = \frac{\mathrm{i}g}{2}\frac{A''(\lambda_k)}{A'(\lambda_k)} = \mathrm{i}g \sum_{\substack{\ell=1\\(\ell\neq k)}}^{n} \frac{1}{\lambda_k - \lambda_\ell}, \quad k = 1,\dots,n, \tag{1.22}$$

for which (1.16) clearly holds. An immediate consequence, as we indicated before, is that $\theta_1,\dots,\theta_n$ (1.13) are conjugate variables to $\lambda_1,\dots,\lambda_n$, thus Sklyanin's formula is verified.

Corollary 1.3 (Sklyanin's formula)**.** *The variables $\theta_1,\dots,\theta_n$ defined by*

$$\theta_k = \frac{C(\lambda_k)}{A'(\lambda_k)}, \quad k = 1,\dots,n \tag{1.23}$$

are conjugate to the eigenvalues $\lambda_1,\dots,\lambda_n$ of the Lax matrix L.

1.3 Discussion

There seem to be several ways for generalisation. For example, one might consider rational Calogero-Moser models associated to root systems other than type A. The hyperbolic Calogero-Moser systems as well as, the 'relativistic' Calogero-Moser systems, also known as Ruijsenaars-Schneider systems, are also of considerable interest.

In Appendix A.1, we give another proof for Theorem 1.2 based on the scattering theory of particles in the rational Calogero-Moser system.

2 Trigonometric BC_n Sutherland system

In this chapter, we present a new case of action-angle duality between integrable many-body systems of Calogero-Ruijsenaars type. This chapter contains our results reported in [P1, P8, P5].

The two systems live on the action-angle phase spaces of each other in such a way that the action variables of each system serve as the particle positions of the other one. Our investigation utilizes an idea that was exploited previously to provide group-theoretic interpretation for several dualities discovered originally by Ruijsenaars. In the group-theoretic framework one applies Hamiltonian reduction to two Abelian Poisson algebras of invariants on a higher dimensional phase space and identifies their reductions as action and position variables of two integrable systems living on two different models of the single reduced phase space. Taking the cotangent bundle of $\mathrm{U}(2n)$ as the upstairs space, we demonstrate how this mechanism leads to a new dual pair involving the BC_n trigonometric Sutherland system. Thereby we generalise earlier results pertaining to the A_{n-1} trigonometric Sutherland system [35] as well as a recent work by Pusztai [107] on the hyperbolic BC_n Sutherland system.

The specific goal in this chapter is to find out how this result can be generalised if one replaces the hyperbolic BC_n system with its trigonometric analogue. A similar problem has been studied previously in the A_{n-1} case, where it was found that the dual of the trigonometric Sutherland system possesses intricate global structure [35, 117]. The global description of the duality necessitates a separate investigation also in the BC_n case, since it cannot be derived by naive analytic continuation between trigonometric and hyperbolic functions. This problem turns out to be considerably more complicated than those studied in [35, 107].

The trigonometric BC_n Sutherland system is defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n}\left[\frac{\gamma}{\sin^2(q_j-q_k)} + \frac{\gamma}{\sin^2(q_j+q_k)}\right] + \sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)} + \sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)}. \tag{2.1}$$

Here (q,p) varies in the cotangent bundle $M = T^*C_1 = C_1\times\mathbb{R}^n$ of the domain

$$C_1 = \left\{ q\in\mathbb{R}^n \,\middle|\, \frac{\pi}{2} > q_1 > \cdots > q_n > 0\right\}, \tag{2.2}$$

and the three independent real coupling constants $\gamma, \gamma_1, \gamma_2$ are supposed to satisfy

$$\gamma > 0, \quad \gamma_2 > 0, \quad 4\gamma_1 + \gamma_2 > 0. \tag{2.3}$$

The inequalities in (2.3) guarantee that the n particles with coordinates q_j cannot leave the open interval $(0, \pi/2)$ and they cannot collide. At a 'semi-global' level, the dual system will be shown to have the Hamiltonian

$$\begin{aligned}\tilde{H}^0 = &\sum_{j=1}^{n} \cos(\vartheta_j) \left[1 - \frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}} \left[1 - \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} \prod_{\substack{k=1\\(k\neq j)}}^{n} \left[1 - \frac{4\mu^2}{(\lambda_j - \lambda_k)^2}\right]^{\frac{1}{2}} \left[1 - \frac{4\mu^2}{(\lambda_j + \lambda_k)^2}\right]^{\frac{1}{2}} \\ &- \frac{\nu\kappa}{4\mu^2} \prod_{j=1}^{n} \left[1 - \frac{4\mu^2}{\lambda_j^2}\right] + \frac{\nu\kappa}{4\mu^2}.\end{aligned} \tag{2.4}$$

Here $\mu > 0, \nu, \kappa$ are real constants, $\vartheta_1, \ldots, \vartheta_n$ are angular variables, and λ varies in the Weyl chamber with thick walls:

$$C_2 = \left\{ \lambda \in \mathbb{R}^n \,\middle|\, \begin{matrix} \lambda_a - \lambda_{a+1} > 2\mu, \\ (a = 1, \ldots, n-1) \end{matrix} \quad \text{and} \quad \lambda_n > \max\{|\nu|, |\kappa|\} \right\}. \tag{2.5}$$

The inequalities defining C_2 ensure the reality and the smoothness of $\tilde{H}^0$ on the phase space $\tilde{M}^0 = C_2 \times \mathbb{T}^n$, which is equipped with the symplectic form

$$\tilde{\omega}^0 = \sum_{k=1}^{n} d\lambda_k \wedge d\vartheta_k. \tag{2.6}$$

Duality will be established under the following relation between the couplings,

$$\gamma = \mu^2, \quad \gamma_1 = \frac{\nu\kappa}{2}, \quad \gamma_2 = \frac{(\nu - \kappa)^2}{2}, \tag{2.7}$$

where in addition to $\mu > 0$ we also adopt the condition

$$\nu > |\kappa| \geq 0. \tag{2.8}$$

This entails that equation (2.7) gives a one-to-one correspondence of the parameters $(\gamma, \gamma_1, \gamma_2)$ subject to (2.3) and (μ, ν, κ), and also simplifies our analysis. In the above, the qualification 'semi-global' indicates that $\tilde{M}^0$ represents a dense open submanifold of the full dual phase space $\tilde{M}$. The completion of $\tilde{M}^0$ into $\tilde{M}$ guarantees both the completeness of the Hamiltonian flows of the dual system and the global nature of the symplectomorphism between M and $\tilde{M}$. The structure of $\tilde{M}$ will also be clarified. For example, we shall see that the action variables of the Sutherland system fill the closure of the domain C_2, with the boundary points corresponding to degenerate Liouville tori.

The integrable systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ as well as their duality relation will emerge from an appropriate Hamiltonian reduction. Specifically, we will reduce the cotangent bundle $T^*\mathrm{U}(2n)$ with respect to the symmetry group $G_+ \times G_+$, where $G_+ \cong \mathrm{U}(n) \times \mathrm{U}(n)$ is the fix-point subgroup of an involution of $\mathrm{U}(2n)$. This enlarges the range of the reduction approach to action-angle dualities [48, 53, 92].

2.1 Physical interpretation

The trigonometric BC_n Sutherland model has the following physical interpretation. Consider a circle of radius $1/2$ with centre O. First, put one particle on the circle to an arbitrary point Q_0, hence creating reference direction $\overrightarrow{OQ_0}$, which coordinates a point Q on the circle with the angle $\phi(Q) = \angle QOQ_0 \in (-\pi, \pi]$, i.e. $\phi(Q_0) = 0$. Next, place n particles on the circle at some points $Q_1, \dots, Q_n$, such that their angles $\phi_j = \phi(Q_j)$ $(j = 1, \dots, n)$ satisfy $\pi > \phi_1 > \dots > \phi_n > 0$. Put n additional particles on the circle at 'mirror images' Q_{-j} of Q_j with respect to the point Q_0, that is $\phi(Q_j) = -\phi(Q_{-j})$.

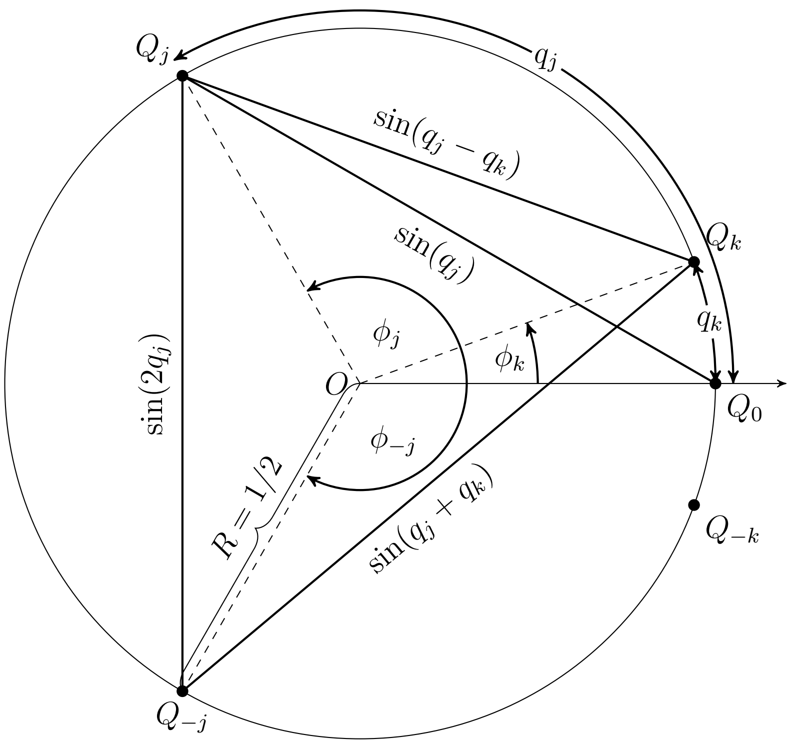

Figure 6: The schematics of trigonometric BC_n Sutherland model.

Now, let these particles interact via a pair-potential that is inversely proportional to the square of the chord-distance. This interaction clearly preserves the initial symmetric configuration. Therefore Q_0 is fixed and acts as a boundary. Let us use the arc lengths $q_j = \phi_j/2$ instead of the angles. Due to the symmetry, the configuration is specified by $q_1, \dots, q_n$, which satisfy the inequalities in C_1 (2.2). Let $\gamma_1, \gamma_2, \gamma$ be particle-boundary, particle-mirror particle, and bulk interaction couplings, respectively.

One can distinguish four types of chord-distances corresponding to these couplings (see Figure 6), namely

$$\gamma_1 : \ \sin(q_j), \quad \gamma_2 : \ \sin(2q_j), \quad \gamma : \ \sin(q_j - q_k), \quad \sin(q_j + q_k). \tag{2.9}$$

Let $p_1, \dots, p_n$ stand for the generalised momenta of the particles at $q_1, \dots, q_n$. Then the total energy of the system is given by the Hamiltonian H (2.1), which exhibits symmetry under the Weyl group of the BC_n root system.

2.2 Definition of the Hamiltonian reduction

Next we describe the starting data which will lead to integrable many-body systems in duality by means of the mechanism outlined in the Introduction. We also collect some group-theoretic facts that will be used in the demonstration of this claim.

Our investigation requires the unitary group of degree $2n$, i.e.

$$G = \mathrm{U}(2n) = \{y \in \mathrm{GL}(2n, \mathbb{C}) \mid y^\dagger y = \mathbf{1}_{2n}\}, \tag{2.10}$$

and its Lie algebra

$$\mathcal{G} = \mathfrak{u}(2n) = \{Y \in \mathfrak{gl}(2n, \mathbb{C}) \mid Y^\dagger + Y = \mathbf{0}_{2n}\}, \tag{2.11}$$

where $\mathbf{1}_{2n}$ and $\mathbf{0}_{2n}$ denote the identity and null matrices of size $2n$, respectively. We endow the Lie algebra $\mathcal{G}$ with the Ad-invariant bilinear form

$$\langle \cdot, \cdot \rangle \colon \mathcal{G} \times \mathcal{G} \to \mathbb{R}, \quad (Y_1, Y_2) \mapsto \langle Y_1, Y_2 \rangle = \mathrm{tr}(Y_1 Y_2), \tag{2.12}$$

and identify $\mathcal{G}$ with the dual space $\mathcal{G}^*$ in the usual manner. By using left-translations to trivialize the cotangent bundle T^*G, we also adopt the identification

$$T^*G \cong G \times \mathcal{G}^* \cong G \times \mathcal{G} = \{(y, Y) \mid y \in G, \ Y \in \mathcal{G}\}. \tag{2.13}$$

Then the canonical symplectic form of T^*G can be written as

$$\Omega^{T^*G} = -d\langle y^{-1} dy, Y \rangle. \tag{2.14}$$

It can be evaluated according to the formula

$$\Omega^{T^*G}_{(y,Y)}(\Delta y \oplus \Delta Y, \Delta' y \oplus \Delta' Y) = \langle y^{-1}\Delta y, \Delta' Y\rangle - \langle y^{-1}\Delta' y, \Delta Y\rangle + \langle [y^{-1}\Delta y, y^{-1}\Delta' y], Y\rangle, \tag{2.15}$$

where $\Delta y \oplus \Delta Y, \Delta' y \oplus \Delta' Y \in T_{(y,Y)}T^*G$ are tangent vectors at a point $(y, Y) \in T^*G$.

We introduce the $2n \times 2n$ Hermitian, unitary matrix partitioned into four $n \times n$ blocks

$$C = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix} \in G, \tag{2.16}$$

and the involutive automorphism of G defined as conjugation with C

$$\Gamma\colon G \to G, \quad y \mapsto \Gamma(y) = CyC^{-1}. \tag{2.17}$$

The set of fix-points of Γ forms the subgroup of G consisting of $2n \times 2n$ unitary matrices with centro-symmetric block structure,

$$G_+ = \{y \in G \mid \Gamma(y) = y\} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in G \right\} \cong \mathrm{U}(n) \times \mathrm{U}(n). \tag{2.18}$$

We also introduce the closed submanifold G_- of G by the definition

$$G_- = \{y \in G \mid \Gamma(y) = y^{-1}\} = \left\{ \begin{bmatrix} a & b \\ c & a^\dagger \end{bmatrix} \in G \,\middle|\, b, c \in \mathrm{i}\mathfrak{u}(n) \right\}. \tag{2.19}$$

By slight abuse of notation, we let Γ stand for the induced involution of the Lie algebra $\mathcal{G}$, too. We can decompose $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-, \quad Y = Y_+ + Y_-, \tag{2.20}$$

where $\mathcal{G}_\pm$ are the eigenspaces of Γ corresponding to the eigenvalues ± 1, respectively, i.e.

$$\begin{aligned} \mathcal{G}_+ &= \ker(\Gamma - \mathrm{id}) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \,\middle|\, A, B \in \mathfrak{u}(n) \right\}, \\ \mathcal{G}_- &= \ker(\Gamma + \mathrm{id}) = \left\{ \begin{bmatrix} A & B \\ -B & -A \end{bmatrix} \,\middle|\, A \in \mathfrak{u}(n),\ B \in \mathrm{i}\mathfrak{u}(n) \right\}. \end{aligned} \tag{2.21}$$

We are interested in a reduction of T^*G based on the symmetry group $G_+ \times G_+$. We shall use the shifting trick of symplectic reduction [101], and thus we first prepare a coadjoint orbit of the symmetry group. To do this, we take any vector $V \in \mathbb{C}^{2n}$ that satisfies $CV + V = 0$, and associate to it the element $\upsilon^\ell_{\mu,\nu}(V)$ of $\mathcal{G}_+$ by the definition

$$\upsilon^\ell_{\mu,\nu}(V) = \mathrm{i}\mu\big(VV^\dagger - \mathbf{1}_{2n}\big) + \mathrm{i}(\mu - \nu)C, \tag{2.22}$$

where $\mu, \nu \in \mathbb{R}$ are real parameters. The set

$$\mathcal{O}^\ell = \big\{\upsilon^\ell \in \mathcal{G}_+ \mid \exists\, V \in \mathbb{C}^{2n},\ V^\dagger V = 2n,\ CV + V = 0,\ \upsilon^\ell = \upsilon^\ell_{\mu,\nu}(V)\big\} \tag{2.23}$$

represents a coadjoint orbit of G_+ of dimension $2(n-1)$. We let $\mathcal{O}^r = \{\upsilon^r\}$ denote the one-point coadjoint orbit of G_+ containing the element

$$\upsilon^r = -\mathrm{i}\kappa C \quad \text{with some constant } \kappa \in \mathbb{R}, \tag{2.24}$$

and consider

$$\mathcal{O} = \mathcal{O}^\ell \oplus \mathcal{O}^r \subset \mathcal{G}_+ \oplus \mathcal{G}_+ \cong (\mathcal{G}_+ \oplus \mathcal{G}_+)^*, \tag{2.25}$$

which is a coadjoint orbit[1] of $G_+ \times G_+$. Our starting point for symplectic reduction will be the phase space (P, Ω) with

$$P = T^*G \times \mathcal{O} \quad \text{and} \quad \Omega = \Omega^{T^*G} + \Omega^{\mathcal{O}}, \tag{2.26}$$

where $\Omega^{\mathcal{O}}$ denotes the Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}$. The natural symplectic action of $G_+ \times G_+$ on P is defined by

$$\Phi_{(g_L, g_R)}(y, Y, \upsilon^\ell \oplus \upsilon^r) = \big(g_L y g_R^{-1}, g_R Y g_R^{-1}, g_L \upsilon^\ell g_L^{-1} \oplus \upsilon^r\big). \tag{2.27}$$

The corresponding momentum map $J\colon P \to \mathcal{G}_+ \oplus \mathcal{G}_+$ is given by the formula

$$J(y, Y, \upsilon^\ell \oplus \upsilon^r) = \big((yYy^{-1})_+ + \upsilon^\ell\big) \oplus \big(-Y_+ + \upsilon^r\big). \tag{2.28}$$

We shall see that the reduced phase space

$$P_{\mathrm{red}} = P_0/(G_+ \times G_+), \quad P_0 = J^{-1}(0), \tag{2.29}$$

is a smooth symplectic manifold, which inherits two Abelian Poisson algebras from P.

Using the identification $\mathcal{G}^* \cong \mathcal{G}$, the invariant functions $C^\infty(\mathcal{G})^G$ form the center of the Lie-Poisson bracket. Denote by $C^\infty(G)^{G_+ \times G_+}$ the set of smooth functions on G that are invariant under the $(G_+ \times G_+)$-action on G that appears in the first component of (2.27). Let us also introduce the maps

$$\pi_1\colon P \to G, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto y, \tag{2.30}$$

and

$$\pi_2\colon P \to \mathcal{G}, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto Y. \tag{2.31}$$

It is clear that

$$\mathfrak{Q}^1 = \pi_1^*(C^\infty(G)^{G_+ \times G_+}) \quad \text{and} \quad \mathfrak{Q}^2 = \pi_2^*(C^\infty(\mathcal{G})^G) \tag{2.32}$$

[1] The same coadjoint orbit was used in [107].

are two Abelian subalgebras in the Poisson algebra of smooth functions on (P, Ω) and these Abelian Poisson algebras descend to the reduced phase space $P_{\rm red}$.

Later we shall construct two models of $P_{\rm red}$ by exhibiting two global cross-sections for the action of $G_+ \times G_+$ on P_0. For this, we shall apply two different methods for solving the constraint equations that, according to (2.28), define the level surface $P_0 \subset P$:

$$(yYy^{-1})_+ + \upsilon^\ell = \mathbf{0}_{2n} \quad \text{and} \quad -Y_+ + \upsilon^r = \mathbf{0}_{2n}, \tag{2.33}$$

where $\upsilon^\ell = \upsilon^\ell_{\mu,\lambda}(V)$ (2.22) for some vector $V \in \mathbb{C}^{2n}$ subject to $CV + V = 0$, $V^\dagger V = 2n$ and $\upsilon^r = -\mathrm{i}\kappa C$. We below collect the group-theoretic results needed for our constructions. To start, let us associate the diagonal $2n \times 2n$ matrix

$$Q(q) = \mathrm{diag}(q, -q) \tag{2.34}$$

with any $q \in \mathbb{R}^n$. Notice that the set

$$\mathcal{A} = \{\mathrm{i}Q(q) \mid q \in \mathbb{R}^n\} \subset \mathcal{G}_- \tag{2.35}$$

is a maximal Abelian subalgebra in $\mathcal{G}_-$. The corresponding subgroup of G has the form

$$\exp(\mathcal{A}) = \big\{e^{\mathrm{i}Q(q)} = \mathrm{diag}\big(e^{\mathrm{i}q_1}, \dots, e^{\mathrm{i}q_n}, e^{-\mathrm{i}q_1}, \dots, e^{-\mathrm{i}q_n}\big) \mid q \in \mathbb{R}^n\big\}. \tag{2.36}$$

The centralizer of $\mathcal{A}$ inside G_+ (2.18) (with respect to conjugation) is the Abelian subgroup

$$Z = Z_{G_+}(\mathcal{A}) = \big\{e^{\mathrm{i}\xi} = \mathrm{diag}\big(e^{\mathrm{i}x_1}, \dots, e^{\mathrm{i}x_n}, e^{\mathrm{i}x_1}, \dots, e^{\mathrm{i}x_n}\big) \mid x \in \mathbb{R}^n\big\} < G_+. \tag{2.37}$$

The Lie algebra of Z is

$$\mathcal{Z} = \{\mathrm{i}\xi = \mathrm{i}\,\mathrm{diag}(x, x) \mid x \in \mathbb{R}^n\} < \mathcal{G}_+. \tag{2.38}$$

The results that we now recall (see e.g. [59, 87, 124]) will be used later. First, for any $y \in G$ there exist elements y_L, y_R from G_+ and unique $q \in \mathbb{R}^n$ satisfying

$$\frac{\pi}{2} \geq q_1 \geq \cdots \geq q_n \geq 0 \tag{2.39}$$

such that

$$y = y_L e^{\mathrm{i}Q(q)} y_R^{-1}. \tag{2.40}$$

If all components of q satisfy strict inequalities, then the pair y_L, y_R is unique precisely up to the replacements $(y_L, y_R) \to (y_L\zeta, y_R\zeta)$ with arbitrary $\zeta \in Z$. The decomposition (2.40) is referred to as the generalised Cartan decomposition corresponding to the involution Γ.

Second, every element $g \in G_-$ can be written in the form

$$g = \eta e^{2\mathrm{i}Q(q)} \eta^{-1} \tag{2.41}$$

with some $\eta \in G_+$ and uniquely determined $q \in \mathbb{R}^n$ subject to (2.39). In the case of strict inequalities for q, the freedom in η is given precisely by the replacements $\eta \to \eta\zeta$, $\forall\, \zeta \in Z$.

Third, every element $Y_- \in \mathcal{G}_-$ can be written in the form

$$Y_- = g_R \mathrm{i} D g_R^{-1}, \quad D = \mathrm{diag}(d_1, \dots, d_n, -d_1, \dots, -d_n), \tag{2.42}$$

with $g_R \in G_+$ and uniquely determined real d_i satisfying

$$d_1 \geq \cdots \geq d_n \geq 0. \tag{2.43}$$

If the d_j $(j = 1, \dots, n)$ satisfy strict inequalities, then the freedom in g_R is exhausted by the replacements $g_R \to g_R\zeta$, $\forall\, \zeta \in Z$.

The first and the second statements are essentially equivalent since the map

$$G \to G_-, \quad y \mapsto y^{-1} C y C \tag{2.44}$$

descends to a diffeomorphism from

$$G/G_+ = \{G_+ g \mid g \in G\} \tag{2.45}$$

onto G_- [59].

2.3 Action-angle duality

2.3.1 The Sutherland gauge

We here exhibit a symplectomorphism between the reduced phase space $(P_{\mathrm{red}}, \Omega_{\mathrm{red}})$ and the Sutherland phase space

$$M = T^* C_1 = C_1 \times \mathbb{R}^n \tag{2.46}$$

equipped with its canonical symplectic form, where C_1 was defined in (2.2). As preparation, we associate with any $(q, p) \in M$ the $\mathcal{G}$-element

$$Y(q, p) = K(q, p) - \mathrm{i}\kappa C, \tag{2.47}$$

where $K(q,p)$ is the $2n \times 2n$ matrix

$$\begin{aligned} K_{j,k} = -K_{n+j,n+k} &= \mathrm{i} p_j \delta_{j,k} - \mu(1-\delta_{j,k})/\sin(q_j - q_k), \\ K_{j,n+k} = -K_{n+j,k} &= (\nu/\sin(2q_j) + \kappa\cot(2q_j))\delta_{j,k} + \mu(1-\delta_{j,k})/\sin(q_j+q_k), \end{aligned} \tag{2.48}$$

with $j,k = 1,\dots,n$. We also introduce the $2n$-component vector

$$V_{\mathbb{R}} = (\underbrace{1,\dots,1}_{n \text{ times}}, \underbrace{-1,\dots,-1}_{n \text{ times}})^{\top}. \tag{2.49}$$

Notice from (2.21) that $K(q,p) \in \mathcal{G}_-$.

Throughout the chapter we adopt the conditions (2.8) and take $\mu > 0$, although the next result requires only that the real parameters μ, ν, κ satisfy

$$\mu \neq 0 \quad \text{and} \quad |\nu| \neq |\kappa|. \tag{2.50}$$

Theorem 2.1. *Using the notations introduced in (2.22), (2.34) and (2.47), the subset S of the phase space P (2.26) given by*

$$S = \left\{ (e^{\mathrm{i}Q(q)}, Y(q,p), \upsilon^{\ell}_{\mu,\nu}(V_{\mathbb{R}}), \upsilon^r) \mid (q,p) \in M \right\}, \tag{2.51}$$

is a global cross-section for the action of $G_+ \times G_+$ on $P_0 = J^{-1}(0)$. Identifying P_{red} with S, the reduced symplectic form is equal to the Darboux form $\omega = \sum_{k=1}^n dq_k \wedge dp_k$. Thus the obvious identification between S and M provides a symplectomorphism

$$(P_{\mathrm{red}}, \Omega_{\mathrm{red}}) \simeq (M, \omega). \tag{2.52}$$

Proof. We saw in Section 2.2 that the points of the level surface P_0 satisfy the equations

$$(yYy^{-1})_+ + \upsilon^{\ell}_{\mu,\nu}(V) = \mathbf{0}_{2n} \quad \text{and} \quad -Y_+ - \mathrm{i}\kappa C = \mathbf{0}_{2n}, \tag{2.53}$$

for some vector $V \in \mathbb{C}^{2n}$ subject to $CV + V = 0$, $V^{\dagger}V = 2n$. Remember that the block-form of any Lie algebra element $Y \in \mathcal{G}$ is

$$Y = \begin{bmatrix} A & B \\ -B^{\dagger} & D \end{bmatrix} \quad \text{with} \quad A + A^{\dagger} = \mathbf{0}_n = D + D^{\dagger}, \quad B \in \mathbb{C}^{n\times n}. \tag{2.54}$$

Now the second constraint equation in (2.53) can be written as

$$2Y_+ = \begin{bmatrix} A+D & B-B^{\dagger} \\ B-B^{\dagger} & A+D \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n & -2\mathrm{i}\kappa\mathbf{1}_n \\ -2\mathrm{i}\kappa\mathbf{1}_n & \mathbf{0}_n \end{bmatrix} = -2\mathrm{i}\kappa C, \tag{2.55}$$

which implies that

$$D = -A \quad \text{and} \quad B^\dagger = B + 2\mathrm{i}\kappa\mathbf{1}_n. \tag{2.56}$$

Thus every point of P_0 has $\mathcal{G}$-component Y of the form

$$Y = \begin{bmatrix} A & B \\ -B - 2\mathrm{i}\kappa\mathbf{1}_n & -A \end{bmatrix} \quad \text{with} \quad A + A^\dagger = \mathbf{0}_n, \quad B \in \mathbb{C}^{n\times n}. \tag{2.57}$$

By using the generalised Cartan decomposition (2.40) and applying a gauge transformation (the action of $G_+ \times G_+$ on P_0), we may assume that $y = e^{\mathrm{i}Q(q)}$ with some q satisfying (2.38). Then the first equation of the momentum map constraint (2.53) yields the matrix equation

$$\frac{1}{2\mathrm{i}}\big(e^{\mathrm{i}Q(q)}Ye^{-\mathrm{i}Q(q)} + e^{-\mathrm{i}Q(q)}CYCe^{\mathrm{i}Q(q)}\big) + \mu(VV^\dagger - \mathbf{1}_{2n}) + (\mu - \nu)C = \mathbf{0}_{2n}. \tag{2.58}$$

If we introduce the notation $V = (u, -u)^\top$, $u \in \mathbb{C}^n$, and assume that Y has the form (2.57) then (2.58) turns into the following equations for A and B

$$\frac{1}{2\mathrm{i}}\big(e^{\mathrm{i}q}Ae^{-\mathrm{i}q} - e^{-\mathrm{i}q}Ae^{\mathrm{i}q}\big) + \mu(uu^\dagger - \mathbf{1}_n) = \mathbf{0}_n, \tag{2.59}$$

and

$$\frac{1}{2\mathrm{i}}\big(e^{\mathrm{i}q}Be^{\mathrm{i}q} - e^{-\mathrm{i}q}Be^{-\mathrm{i}q}\big) - \kappa e^{-2\mathrm{i}q} - \mu uu^\dagger + (\mu - \nu)\mathbf{1}_n = \mathbf{0}_n. \tag{2.60}$$

Since $\mu \neq 0$, equation (2.59) implies that $|u_j|^2 = 1$ for all $j = 1, \dots, n$. Therefore we can apply a 'residual' gauge transformation by an element $(g_L, g_R) = (e^{\mathrm{i}\xi(x)}, e^{\mathrm{i}\xi(x)})$, with suitable $e^{\mathrm{i}\xi(x)} \in Z$ (2.37) to transform $\upsilon^\ell_{\mu,\nu}(V)$ into $\upsilon^\ell_{\mu,\nu}(V_\mathbb{R})$. This amounts to setting $u_j = 1$ for all $j = 1, \dots, n$. After having done this, we return to equations (2.59) and (2.60). By writing out the equations entry-wise, we obtain that the diagonal components of A are arbitrary imaginary numbers (which we denote by $\mathrm{i}p_1, \dots, \mathrm{i}p_n$) and we also obtain the following system of equations

$$\begin{aligned} A_{j,k}\sin(q_j - q_k) &= -\mu = -B_{j,k}\sin(q_j + q_k), \quad j \neq k, \\ B_{j,j}\sin(2q_j) &= \nu + \kappa\cos(2q_j) - \mathrm{i}\kappa\sin(2q_j), \quad j, k = 1, \dots, n. \end{aligned} \tag{2.61}$$

So far we only knew that q satisfies $\pi/2 \geq q_1 \geq \dots \geq q_n \geq 0$. By virtue of the conditions (2.50), the system (2.61) can be solved if and only if $\pi/2 > q_1 > \dots > q_n > 0$. Substituting the unique solution for A and B back into (2.57) gives the formula $Y = Y(q,p)$ as displayed in (2.47).

The above arguments show that every gauge orbit in P_0 contains a point of S (2.51), and it is immediate by turning the equations backwards that every point of S belongs to P_0. By using that q satisfies strict inequalities and that all components of $V_\mathbb{R}$ are non-zero, it is also readily seen that no two different points of S are gauge equivalent. Moreover, the effectively acting symmetry group, which is given by

$$(G_+ \times G_+)/\mathrm{U}(1)_{\mathrm{diag}} \tag{2.62}$$

where $\mathrm{U}(1)$ contains the scalar unitary matrices, acts *freely* on P_0.

It follows from the above that P_{red} is a smooth manifold diffeomorphic to M. Now the proof is finished by direct computation of the pull-back of the symplectic form Ω of P (2.26) onto the global cross-section S. □

Let us recall that the Abelian Poisson algebras $\mathfrak{Q}^1$ and $\mathfrak{Q}^2$ (2.32) consist of $(G_+ \times G_+)$-invariant functions on P, and thus descend to Abelian Poisson algebras on the reduced phase space P_{red}. In terms of the model $M \simeq S \simeq P_{\mathrm{red}}$, the Poisson algebra $\mathfrak{Q}^2_{\mathrm{red}}$ is obviously generated by the functions $(q,p) \mapsto \mathrm{tr}((-\mathrm{i}Y(q,p)))^m$ for $m = 1, \ldots, 2n$. It will be shown in the following section[2] that these functions vanish identically for the odd integers, and functionally independent generators of $\mathfrak{Q}^2_{\mathrm{red}}$ are provided by the functions

$$H_k(q,p) = \frac{1}{4k}\mathrm{tr}(-\mathrm{i}Y(q,p))^{2k}, \quad k = 1, \ldots, n. \tag{2.63}$$

The first of these functions reads

$$\begin{aligned} H_1(q,p) = \frac{1}{4}\mathrm{tr}(-\mathrm{i}Y(q,p))^2 =& \frac{1}{2}\sum_{j=1}^n p_j^2 + \sum_{1\leq j<k\leq n}\left(\frac{\mu^2}{\sin^2(q_j - q_k)} + \frac{\mu^2}{\sin^2(q_j + q_k)}\right) \\ &+ \frac{1}{2}\sum_{j=1}^n \frac{\nu\kappa}{\sin^2(q_j)} + \frac{1}{2}\sum_{j=1}^n \frac{(\nu-\kappa)^2}{\sin^2(2q_j)}. \end{aligned} \tag{2.64}$$

That is, upon the identification (2.7) it coincides with the Sutherland Hamiltonian (2.1). This implies the Liouville integrability of the Hamiltonian (2.1). Since its spectral invariants yield a commuting family of n independent functions in involution that include the Sutherland Hamiltonian, the Hermitian matrix function $-\mathrm{i}Y(q,p)$ (2.47) serves as a Lax matrix for the Sutherland system (M, ω, H).

As for the reduced Abelian Poisson algebra $\mathfrak{Q}^1_{\mathrm{red}}$, we notice that the cross-section S permits to identify it with the Abelian Poisson algebra of the smooth functions of the variables $q_1, \ldots, q_n$. This is so since the level set P_0 lies completely in the 'regular part' of the phase space P, where the G-component y of (y, Y, v^ℓ, v^r) is such that $Q(q)$ in its decomposition (2.40) satisfies strict inequalities $\pi/2 > q_1 > \cdots > q_n > 0$. It is a well-known fact that in the regular part the components of q are smooth (actually real-analytic) functions of y (while globally they are only continuous functions). To

[2] In fact, we shall see that $Y(q,p)$ is conjugate to a diagonal matrix $\mathrm{i}\Lambda$ of the form in equation (2.71).

see that every smooth function depending on $q \in C_1$ is contained in $\mathfrak{Q}^1_{\rm red}$, one may further use that every $(G_+ \times G_+)$-invariant smooth function on P_0 can be extended to an invariant smooth function on P. Indeed, this holds since $G_+ \times G_+$ is compact and $P_0 \subset P$ is a regular submanifold, which itself follows from the free action property established in the course of the proof of Theorem 2.1.

We can summarize the outcome of the foregoing discussion as follows. Below, the generators of Poisson algebras are understood in the functional sense, i.e. if some $f_1, \dots, f_n$ are generators then all smooth functions of them belong to the Poisson algebra.

Corollary 2.2. *By using the model (M, ω) of the reduced phase space $(P_{\rm red}, \Omega_{\rm red})$ provided by Theorem 2.1, the Abelian Poisson algebra $\mathfrak{Q}^2_{\rm red}$ (2.31) can be identified with the Poisson algebra generated by the spectral invariants (2.62) of the 'Sutherland Lax matrix' $-{\rm i}Y(q,p)$ (2.47), which according to (2.64) include the many-body Hamiltonian $H(q,p)$ (2.1), and $\mathfrak{Q}^1_{\rm red}$ can be identified with the algebra generated by the corresponding position variables q_j $(j = 1, \dots, n)$.*

2.3.2 The Ruijsenaars gauge

It follows from the group-theoretic results quoted in Section 2.2 that the Abelian Poisson algebra $\mathfrak{Q}^1$ is generated by the functions

$$\tilde{\mathcal{H}}_k(y, Y, v^\ell, v^r) = \frac{(-1)^k}{2k} \mathrm{tr}\big(y^{-1}CyC\big)^k, \quad k = 1, \dots, n, \tag{2.65}$$

and thus the unitary and Hermitian matrix

$$L = -y^{-1}CyC \tag{2.66}$$

serves as an 'unreduced Lax matrix'. It is readily seen in the Sutherland gauge (2.51) that these n functions remain functionally independent after reduction. Here, we shall prove that the evaluation of the invariant function $\tilde{\mathcal{H}}_1$ in another gauge reproduces the dual Hamiltonian (2.4). The reduction of the matrix function L will provide a Lax matrix for the corresponding integrable system. Before turning to details, we advance the group-theoretic interpretation of the dual position variable λ that features in the Hamiltonian (2.4), and sketch the plan of this section.

To begin, recall that on the constraint surface $Y = Y_- - {\rm i}\kappa C$, and for any $Y_- \in \mathcal{G}_-$ there is an element $g_R \in G_+$ such that

$$g_R^{-1} Y_- g_R = \mathrm{diag}({\rm i}d_1, \dots, {\rm i}d_n, -{\rm i}d_1, \dots, -{\rm i}d_n) = {\rm i}D \in \mathcal{A} \quad \text{with} \quad d_1 \geq \dots \geq d_n \geq 0. \tag{2.67}$$

Then introduce the real matrix $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ whose diagonal components are[3]

$$\lambda_j = \sqrt{d_j^2 + \kappa^2}, \quad j \in \mathbb{N}_n. \tag{2.68}$$

One can diagonalize the matrix $D - \kappa C$ by conjugation with the unitary matrix

$$h(\lambda) = \begin{bmatrix} \alpha(\boldsymbol{\lambda}) & \beta(\boldsymbol{\lambda}) \\ -\beta(\boldsymbol{\lambda}) & \alpha(\boldsymbol{\lambda}) \end{bmatrix}, \tag{2.69}$$

where the real functions $\alpha(x), \beta(x)$ are defined on the interval $[|\kappa|, \infty) \subset \mathbb{R}$ by the formulae

$$\alpha(x) = \frac{\sqrt{x + \sqrt{x^2 - \kappa^2}}}{\sqrt{2x}}, \quad \beta(x) = \kappa \frac{1}{\sqrt{2x}} \frac{1}{\sqrt{x + \sqrt{x^2 - \kappa^2}}}, \tag{2.70}$$

at least if $\kappa \neq 0$. If $\kappa = 0$, then we set $\alpha(x) = 1$ and $\beta(x) = 0$. Indeed, it is easy to check that

$$h(\lambda) \Lambda h(\lambda)^{-1} = D - \kappa C \quad \text{with} \quad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n). \tag{2.71}$$

Note that $h(\lambda)$ belongs to the subset G_- of G (2.19).

The above diagonalization procedure can be used to define the map

$$\mathfrak{L}\colon P_0 \to \mathbb{R}^n, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto \lambda. \tag{2.72}$$

This is clearly a continuous map, which descends to a continuous map $\mathfrak{L}_{\mathrm{red}}\colon P_{\mathrm{red}} \to \mathbb{R}^n$. One readily sees also that these maps are smooth (even real-analytic) on the open submanifolds $P_0^{\mathrm{reg}} \subset P_0$ and $P_{\mathrm{red}}^{\mathrm{reg}} \subset P_{\mathrm{red}}$, where the $2n$ eigenvalues of Y_- are pairwise different.

The image of the constraint surface P_0 under the map $\mathfrak{L}$ will turn out to be the closure of the domain

$$C_2 = \left\{ \lambda \in \mathbb{R}^n \,\middle|\, \begin{matrix} \lambda_a - \lambda_{a+1} > 2\mu, \\ (a = 1, \dots, n-1) \end{matrix} \quad \text{and} \quad \lambda_n > \nu \right\}. \tag{2.73}$$

By solving the constrains through the diagonalization of Y, we shall construct a model of the open submanifold of P_{red} corresponding to the open submanifold $\mathfrak{L}^{-1}(C_2) \subset P_0$. This model will be symplectomorphic to the semi-global phase-space $C_2 \times \mathbb{T}^n$ of the dual Hamiltonian (2.4).

In the rest of this section, we present the construction of the aforementioned model of $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$. We demonstrate that $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2)$ is a dense subset of P_{red} and present the global characterization of the dual model of P_{red}.

[3] From now on we frequently use the notations $\mathbb{N}_n = \{1, \dots, n\}$ and $\mathbb{N}_{2n} = \{1, \dots, 2n\}$.

Many of the local formulae that appear in this section have analogues in [105, 106, 107], which inspired our considerations. However, the global structure is different.

The dual model of the open subset $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$

We first prepare some functions on $C_2 \times \mathbb{T}^n$. Denoting the elements of this domain as pairs

$$(\lambda, e^{\mathrm{i}\vartheta}) \quad \text{with} \quad \lambda = (\lambda_1, \dots, \lambda_n) \in C_2, \quad e^{\mathrm{i}\vartheta} = (e^{\mathrm{i}\vartheta_1}, \dots, e^{\mathrm{i}\vartheta_n}) \in \mathbb{T}^n, \tag{2.74}$$

we let

$$\begin{aligned} f_c &= \left[1 - \frac{\nu}{\lambda_c}\right]^{\frac{1}{2}} \prod_{\substack{a=1\\(a\neq c)}}^{n} \left[1 - \frac{2\mu}{\lambda_c - \lambda_a}\right]^{\frac{1}{2}} \left[1 - \frac{2\mu}{\lambda_c + \lambda_a}\right]^{\frac{1}{2}}, \quad \forall c \in \mathbb{N}_n, \\ f_{n+c} &= e^{\mathrm{i}\vartheta_c} \left[1 + \frac{\nu}{\lambda_c}\right]^{\frac{1}{2}} \prod_{\substack{a=1\\(a\neq c)}}^{n} \left[1 + \frac{2\mu}{\lambda_c - \lambda_a}\right]^{\frac{1}{2}} \left[1 + \frac{2\mu}{\lambda_c + \lambda_a}\right]^{\frac{1}{2}}. \end{aligned} \tag{2.75}$$

For $\lambda \in C_2$ (2.73), all factors under the square roots are positive. Using the column vector $f = (f_1, \dots, f_{2n})^\top$ together with $\Lambda_c = \lambda_c$ and $\Lambda_{c+n} = -\lambda_c$ for $c \in \mathbb{N}_n$, we define the $2n \times 2n$ matrices $\check{A}(\lambda, \vartheta)$ and $B(\lambda, \vartheta)$ by

$$\check{A}_{j,k} = \frac{2\mu f_j \overline{(Cf)}_k - 2(\mu - \nu) C_{j,k}}{2\mu + \Lambda_k - \Lambda_j}, \quad j, k \in \mathbb{N}_{2n}, \tag{2.76}$$

and

$$B(\lambda, \vartheta) = -\big(h(\lambda) \check{A}(\lambda, \vartheta) h(\lambda)\big)^\dagger. \tag{2.77}$$

We shall see that these are unitary matrices from $G_- \subset G$ (2.19). Then we write B in the form

$$B = \eta e^{2\mathrm{i}Q(q)} \eta^{-1} \tag{2.78}$$

with some $\eta \in G_+$ and unique $q = q(\lambda, \vartheta)$ subject to (2.39). (It turns out that $q(\lambda, \vartheta) \in C_1$ (2.2) and thus η is unique up to replacements $\eta \to \eta\zeta$ with arbitrary $\zeta \in Z$ (2.37).) Relying on (2.78), we set

$$y(\lambda, \vartheta) = \eta e^{\mathrm{i}Q(q(\lambda, \vartheta))} \eta^{-1} \tag{2.79}$$

and introduce the vector $V(\lambda, \vartheta) \in \mathbb{C}^{2n}$ by

$$V(\lambda, \vartheta) = y(\lambda, \vartheta) h(\lambda) f(\lambda, \vartheta). \tag{2.80}$$

It will be shown that $V + CV = 0$ and $|V|^2 = 2n$, which ensures that $\upsilon^\ell_{\mu,\nu}(V) \in \mathcal{O}^\ell$ (2.23).

Note that $\check{A}$, y and V given above depend on ϑ only through $e^{\mathrm{i}\vartheta}$ and are C^∞ functions on $C_2 \times \mathbb{T}^n$. It should be remarked that although the matrix element $\check{A}_{n,2n}$ (2.76) has an apparent singularity at $\lambda_n = \mu$, the zero of the denominator cancels. Thus $\check{A}$ extends by continuity to $\lambda_n = \mu$ and remains smooth there, which then also implies the smoothness of y and V.

Theorem 2.3. *By using the above notations, consider the set*

$$\tilde{S}^0 = \{(y(\lambda,\vartheta), \mathrm{i}h(\lambda)\Lambda(\lambda)h(\lambda)^{-1}, \upsilon^\ell_{\mu,\nu}(V(\lambda,\vartheta)), \upsilon^r) \mid (\lambda, e^{\mathrm{i}\vartheta}) \in C_2 \times \mathbb{T}^n\}. \tag{2.81}$$

This set is contained in the constraint surface $P_0 = J^{-1}(0)$ and it provides a cross-section for the $G_+ \times G_+$-action restricted to $\mathfrak{L}^{-1}(C_2) \subset P_0$. In particular, $C_2 \subset \mathfrak{L}(P_0)$ and $\tilde{S}^0$ intersects every gauge orbit in $\mathfrak{L}^{-1}(C_2)$ precisely in one point. Since the elements of $\tilde{S}^0$ are parametrized by $C_2 \times \mathbb{T}^n$ in a smooth and bijective manner, we obtain the identifications

$$\mathfrak{L}^{-1}_{\mathrm{red}}(C_2) \simeq \tilde{S}^0 \simeq C_2 \times \mathbb{T}^n. \tag{2.82}$$

Letting $\tilde{\sigma}_0 \colon \tilde{S}^0 \to P$ denote the tautological injection, the pull-backs of the symplectic form Ω (2.26) and the function $\tilde{\mathcal{H}}_1$ (2.65) obey

$$\tilde{\sigma}_0^*(\Omega) = \sum_{c=1}^n d\lambda_c \wedge d\vartheta_c, \quad (\tilde{\mathcal{H}}_1 \circ \tilde{\sigma}_0)(\lambda,\vartheta) = \frac{1}{2}\mathrm{tr}\big(h(\lambda)\check{A}(\lambda,\vartheta)h(\lambda)\big) = \tilde{H}^0(\lambda,\vartheta) \tag{2.83}$$

with the RSvD type Hamiltonian $\tilde{H}^0$ in (2.4). Consequently, the Hamiltonian reduction of the system $(P, \Omega, \tilde{\mathcal{H}}_1)$ followed by restriction to the open submanifold $\mathfrak{L}^{-1}_{\mathrm{red}}(C_2) \subset P_{\mathrm{red}}$ reproduces the system $(\tilde{M}^0, \tilde{\omega}^0, \tilde{H}^0)$ defined in (2.4)-(2.5).

Remark 2.4. Referring to (2.66), we have the Lax matrix

$$L(y(\lambda,\vartheta)) = h(\lambda)\check{A}(\lambda,\vartheta)h(\lambda). \tag{2.84}$$

Later we shall also prove that $\mathfrak{L}^{-1}_{\mathrm{red}}(C_2)$ is a dense subset of P_{red}, whereby the reduction of $(P, \Omega, \tilde{\mathcal{H}}_1)$ may be viewed as a completion of $(\tilde{M}^0, \tilde{\omega}^0, \tilde{H}^0)$.

Proof of Theorem 2.3

The proof will emerge from a series of lemmas. Our immediate aim is to construct gauge invariant functions that will be used for parametrizing the orbits of $G_+ \times G_+$ in (an open submanifold of) P_0. For introducing gauge invariants we can restrict ourselves to the submanifold $P_1 \subset P_0$ where Y in $(y, Y, \upsilon^\ell, \upsilon^r)$ has the form

$$Y = h(\lambda)\mathrm{i}\Lambda(\lambda)h(\lambda)^{-1} \tag{2.85}$$

with some $\lambda \in \mathbb{R}^n$ for which

$$\lambda_1 \geq \cdots \geq \lambda_n \geq |\kappa|. \tag{2.86}$$

Indeed, every element of P_0 can be gauge transformed into P_1. It will be advantageous to further restrict attention to $P_1^{\mathrm{reg}} \subset P_1$ where we have

$$\lambda_1 > \cdots > \lambda_n > |\kappa|. \tag{2.87}$$

The residual gauge transformations that map P_1^{reg} to itself belong to the group $G_+ \times Z < G_+ \times G_+$ with Z defined in (2.37). Since υ^r is constant and $\upsilon^\ell = \upsilon^\ell_{\mu,\nu}(V)$, we may label the elements of P_1 by triples (y, Y, V), with the understanding that V matters up to phase. Then the gauge action of $(g_L, \zeta) \in G_+ \times Z$ operates by

$$(y, V) \mapsto (g_L y \zeta^{-1}, g_L V), \tag{2.88}$$

while Y is already invariant. Now we can factor out the residual G_+-action by introducing the G_--valued function

$$\check{A}(y, Y, V) = h(\lambda)^{-1} L(y) h(\lambda)^{-1} \tag{2.89}$$

and the $\mathbb{C}^{2n}$-valued function

$$F(y, Y, V) = h(\lambda)^{-1} y^{-1} V. \tag{2.90}$$

Here $\lambda = \mathfrak{L}(y, Y, V)$, which means that (2.85) holds, and we used $L(y)$ in (2.66). Like V, F is defined only up to a $\mathrm{U}(1)$ phase. We obtain the transformation rules

$$\check{A}(g_L y \zeta^{-1}, Y, g_L V) = \zeta \check{A}(y, Y, V)\zeta^{-1}, \tag{2.91}$$

$$F(g_L y \zeta^{-1}, Y, g_L V) = \zeta F(y, Y, V), \tag{2.92}$$

and therefore the functions

$$\mathcal{F}_k(y, Y, V) = |F_k(y, Y, V)|^2, \quad k = 1, \ldots, 2n \tag{2.93}$$

are well-defined, gauge invariant, smooth functions on P_1^{reg}. They represent $(G_+ \times G_+)$-invariant smooth functions on P_0^{reg}. We shall see shortly that the functions $\mathcal{F}_k$ depend only on $\lambda = \mathfrak{L}(y, Y, V)$ and shall derive explicit formulae for this dependence. Then the non-negativity of $\mathcal{F}_k$ will be used to gain information about the set $\mathfrak{L}(P_0)$ of λ values that actually occurs.

Before turning to the inspection of the functions $\mathcal{F}_k$, we present a crucial lemma.

Lemma 2.5. *Fix $\lambda \in \mathbb{R}^n$ subject to (2.86) and set $\Lambda = \mathrm{diag}(\lambda, -\lambda)$ and $Y = hi\Lambda h^{-1}$. If $y \in G$ and $\upsilon^\ell_{\mu,\nu}(V) \in \mathcal{O}^\ell$ solve the momentum map constraint given according to the first equation in (2.53) by*

$$yYy^{-1} + CyYy^{-1}C + 2\upsilon^\ell_{\mu,\nu}(V) = 0, \tag{2.94}$$

then $\check{A} \in G_-$ and $F \in \mathbb{C}^{2n}$ defined by (2.89) and (2.90) solve the following equation:

$$2\mu\check{A} + \check{A}\Lambda - \Lambda\check{A} = 2\mu F(CF)^\dagger - 2(\mu - \nu)C. \tag{2.95}$$

Conversely, for any $\check{A} \in G_-$, $F \in \mathbb{C}^{2n}$ that satisfy $|F|^2 = 2n$ and equation (2.95), pick $y \in G$ such that $L(y) = h(\lambda)\check{A}h(\lambda)$ and define $V = yh(\lambda)F$. Then $CV + V = 0$ and $(y, Y, \upsilon^\ell_{\mu,\nu}(V))$ solve the momentum map constraint (2.94).

Proof. If eq. (2.94) holds, then we multiply it by $h(\lambda)^{-1}y^{-1}$ on the left and by $CyCh(\lambda)^{-1}$ on the right. Using (2.58), with $CV + V = 0$ and $|V|^2 = 2n$, and the notations (2.89) and (2.90), this immediately gives (2.95). Conversely, suppose that (2.95) holds for some $\check{A} \in G_-$ and $F \in \mathbb{C}^{2n}$ with $|F|^2 = 2n$. Since $h(\lambda)\check{A}h(\lambda)$ belongs to G_-, there exists $y \in G$ such that

$$h(\lambda)\check{A}h(\lambda) = L(y). \tag{2.96}$$

Such y is unique up to left-multiplication by an arbitrary element of G_+ (whereby one may bring y into G_- if one wishes to do so). Picking y according to (2.96), and then setting

$$V = yh(\lambda)F, \tag{2.97}$$

it is an elementary matter to show that (2.95) implies the following equation:

$$yYy^{-1} + CyYy^{-1}C + 2\mathrm{i}\mu(-V(CV)^\dagger - \mathbf{1}_{2n}) + 2\mathrm{i}(\mu - \nu)C = 0. \tag{2.98}$$

It is a consequence of this equation that

$$(V(CV)^\dagger)^\dagger = (CV)V^\dagger = V(CV)^\dagger. \tag{2.99}$$

This entails that $CV = \alpha V$ for some $\alpha \in \mathrm{U}(1)$. Then $V^\dagger = \alpha(CV)^\dagger$ also holds, and thus we must have $\alpha^2 = 1$. Hence α is either $+1$ or -1. Taking the trace of the equality (2.98), and using that $|V|^2 = 2n$ on account of $|F|^2 = 2n$, we obtain that $\alpha = -1$, i.e. $CV + V = 0$. This means that equation (2.98) reproduces (2.94). $\square$

To make progress, now we restrict our attention to the subset of $P_1^{\rm reg}$, where the eigenvalue-parameter λ of Y verifies in addition to (2.87) also the conditions

$$|\lambda_a \pm \lambda_b| \neq 2\mu \quad \text{and} \quad (\lambda_a - \nu)(\lambda_a - |2\mu - \nu|) \neq 0, \quad \forall a, b \in \mathbb{N}_n. \tag{2.100}$$

We call such λ values 'strongly regular', and let $P_1^{\rm vreg} \subset P_1$ and $P_0^{\rm vreg} \subset P_0$ denote the corresponding open subsets. Later we shall prove that $P_0^{\rm vreg}$ is *dense* in P_0. The above conditions will enable us to perform calculations that will lead to a description of a dense subset of the reduced phase space. They ensure that we never divide by zero in relevant steps of our arguments. The first such step is the derivation of the following consequence of equation (2.95).

Lemma 2.6. *The restriction of the matrix function $\check{A}$ (2.89) to $P_1^{\rm vreg}$ has the form*

$$\check{A}_{j,k} = \frac{2\mu F_j \overline{(CF)}_k - 2(\mu - \nu) C_{j,k}}{2\mu + \Lambda_k - \Lambda_j}, \quad j, k \in \mathbb{N}_2 n, \tag{2.101}$$

where $F \in \mathbb{C}^{2n}$ satisfies $|F|^2 = 2n$ and $\Lambda = \mathrm{diag}(\lambda, -\lambda)$ varies on $P_1^{\rm vreg}$ according to (2.85).

Lemma 2.7. *For any strongly regular λ and $a \in \mathbb{N}_n$ define*

$$w_a = \prod_{\substack{b=1 \\ (b \neq a)}}^{n} \frac{(\lambda_a - \lambda_b)(\lambda_a + \lambda_b)}{(2\mu - (\lambda_a - \lambda_b))(2\mu - (\lambda_a + \lambda_b))}, \quad w_{a+n} = \prod_{\substack{b=1 \\ (b \neq a)}}^{n} \frac{(\lambda_a - \lambda_b)(\lambda_a + \lambda_b)}{(2\mu + \lambda_a - \lambda_b)(2\mu + \lambda_a + \lambda_b)}, \tag{2.102}$$

and set $W_k = w_k \mathcal{F}_k$ with $\mathcal{F}_k = |F_k|^2$. Then the unitarity of the matrix $\check{A}$ as given by (2.101) *implies the following system of equations for the pairs of functions W_c and W_{c+n} for any $c \in \mathbb{N}_n$:*

$$(\mu + \lambda_c) W_c + (\mu - \lambda_c) W_{n+c} - 2(\mu - \nu) = 0, \tag{2.103}$$

$$\lambda_c^2 W_c W_{n+c} - \mu(\mu - \nu)(W_c + W_{n+c}) + (\mu - \nu)^2 + \mu^2 - \lambda_c^2 = 0. \tag{2.104}$$

For fixed $c \in \mathbb{N}_n$ and strongly regular λ, this system of equations admits two solutions, which are given by

$$(W_c, W_{n+c}) = (W_c^+, W_{n+c}^+) = (w_c \mathcal{F}_c^+, w_{c+n} \mathcal{F}_{c+n}^+) = (1 - \frac{\nu}{\lambda_c}, 1 + \frac{\nu}{\lambda_c}), \tag{2.105}$$

and by

$$(W_c, W_{n+c}) = (W_c^-, W_{n+c}^-) = (w_c \mathcal{F}_c^-, w_{c+n} \mathcal{F}_{c+n}^-) = (-1 + \frac{2\mu - \nu}{\lambda_c}, -1 - \frac{2\mu - \nu}{\lambda_c}). \tag{2.106}$$

The functions $\mathcal{F}_k^{\pm}$ satisfy the identities

$$\sum_{k=1}^{2n} \mathcal{F}_k^+(\lambda) = 2n \quad \textit{and} \quad \sum_{k=1}^{2n} \mathcal{F}_k^-(\lambda) = -2n. \tag{2.107}$$

Proof. The derivation of equations (2.103), (2.104) follows a similar derivation due to Pusztai [105], and is summarized in the appendix. We then solve the linear equation (2.103) say for W_{c+n} and substitute it into (2.104). This gives a quadratic equation for W_c whose two solutions we can write down. We note that the derivation of the equations (2.103) and (2.104) presented in the appendix utilizes the full set of the conditions (2.100).

To verify the identities (2.107), we first extend λ to vary in the open subset of $\mathbb{C}^n$ subject to the conditions $\lambda_a^2 \neq \lambda_b^2$ and $\lambda_c \neq 0$, and then consider the sums that appear in (2.107) as functions of a chosen component of λ with the other components fixed. These explicitly given sums are meromorphic functions having only first order poles, and one may check that all residues at the apparent poles vanish. Hence the sums are constant over $\mathbb{C}^n$, and the values of the constants can be established by looking at a suitable asymptotic limit in the domain C_2 (2.73), whereby all w_k tend to 1 and the pre-factors in (2.105) and (2.106) tend to 1 and -1, respectively. □

Observe that neither any w_k nor any $\mathcal{F}_k^{\pm}$ ($k \in \mathbb{N}_{2n}$) can vanish if λ is strongly regular. We know that the value of $\mathcal{F}_k$ (2.93) is uniquely defined at every point of P_1^{reg}. Therefore only one of the solutions $(\mathcal{F}_c^{\pm}, \mathcal{F}_{c+n}^{\pm})$ can be acceptable at any $\lambda \in \mathfrak{L}(P_1^{\text{vreg}})$. The identities in (2.107) and analyticity arguments strongly suggest that the acceptable solutions are provided by $\mathcal{F}_k^+$. The first statement of the following lemma confirms that this is the case for $\lambda \in C_2$ (2.73).

Lemma 2.8. *The formulae (2.105) and (2.106) can be used to define $\mathcal{F}_k^{\pm}$ as smooth real functions on the domain C_2, and none of these functions vanishes at any $\lambda \in C_2$. Then for any $\lambda \in C_2$ and $c \in \mathbb{N}_n$ at least one out of $\mathcal{F}_c^-$ and $\mathcal{F}_{c+n}^-$ is negative, while $\mathcal{F}_k^+ > 0$ for all $k \in \mathbb{N}_{2n}$. Hence for $\lambda \in C_2 \cap \mathfrak{L}(P_0)$ only $\mathcal{F}_k^+(\lambda)$ can give the value of the function $\mathcal{F}_k$ as defined in (2.93). Taking any $\lambda \in C_2$ and any $F \in \mathbb{C}^{2n}$ satisfying $|F_k|^2 = \mathcal{F}_k^+(\lambda)$, the formula (2.101) yields a unitary matrix that belongs to G_- (2.19). This matrix $\breve{A}$ and vector $F \in \mathbb{C}^{2n}$ solve equation (2.95).*

Proof. It is easily seen that $w_k(\lambda) > 0$ for all $\lambda \in C_2$ and $k \in \mathbb{N}_{2n}$. The statement about the negativity of either $\mathcal{F}_c^-$ or $\mathcal{F}_{c+n}^-$ thus follows from the identity $W_c^- + W_{n+c}^- = -2$. The positivity of $\mathcal{F}_k^+$ is easily checked. It is also readily verified that $\breve{A}^\dagger = C\breve{A}C$, which entails that $\breve{A} \in G_-$ once we know that $\breve{A}$ is unitary. For $\lambda \in C_2$ and $|F_k|^2 = \mathcal{F}_k^+(\lambda)$, the unitarity of $\breve{A}$ (2.101) can be shown by almost verbatim adaptation of the arguments proving Proposition 6 in [106].

If $\lambda \in C_2$ is such that the denominators in (2.101) do not vanish, then the formula (2.101) is plainly equivalent to (2.95). Observe that only those elements $\lambda \in C_2$ for which $\lambda_n = \mu$ fail to satisfy this condition. At such λ the matrix element $\breve{A}_{n,2n}$ has an apparent 'first order pole', but one can check by inspection of the formula (2.76) that $\breve{A}_{n,2n}$ actually remains finite and smooth even at such exceptional points, and thus solves also (2.95) because of continuity. $\square$

Before presenting the proof of Theorem 2.3, note that at the point of $\tilde{S}^0$ labelled by $(\lambda, e^{\mathrm{i}\vartheta})$ the value of the function F (2.90) is equal to $f(\lambda, e^{\mathrm{i}\vartheta})$ given in (2.75).

Proof of Theorem 2.3. It follows from Lemma 2.5 and Lemma 2.8 that $\tilde{S}^0$ is a subset of P_1^{reg} and $\mathfrak{L}(\tilde{S}^0) = C_2$. Taking into account Theorem 2.1, this implies that $y(\lambda, \vartheta)$ (2.79) and $V(\lambda, \vartheta)$ (2.80) are well-defined smooth functions on $C_2 \times \mathbb{T}^n$. We next show that $\tilde{S}^0$ is a cross-section for the residual gauge action on $\mathfrak{L}^{-1}(C_2) \cap P_1$. To do this, pick an arbitrary element

$$(\tilde{y}, h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}, \upsilon^{\ell}_{\mu,\nu}(\tilde{V}), \upsilon^r) \in \mathfrak{L}^{-1}(C_2) \cap P_1. \tag{2.108}$$

Because $\mathcal{F}_k(\lambda) \neq 0$, we can find a unique element $e^{\mathrm{i}\vartheta} \in \mathbb{T}^n$ and an element $\zeta \in Z$ (2.37) (which is unique up to scalar multiple) such that

$$F_k(\tilde{y}\zeta^{-1}, h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}, \tilde{V}) = f_k(\lambda, e^{\mathrm{i}\vartheta}), \quad \forall k \in \mathbb{N}_{2n}, \tag{2.109}$$

up to a k-independent phase. We then see from (2.95) that $L(\tilde{y}\zeta^{-1}) = L(y(\lambda, \vartheta))$, which in turn implies the existence of some (unique after ζ was chosen) $\eta_+ \in G_+$ for which

$$\eta_+ \tilde{y}\zeta^{-1} = y(\lambda, \vartheta). \tag{2.110}$$

Using also that $\zeta^{-1}h(\lambda)\zeta = h(\lambda)$, we conclude from the last two equations that

$$\eta_+ \tilde{V} = \eta_+ \tilde{y} h(\lambda) F(\tilde{y}, h(\lambda)\mathrm{i}\Lambda h(\lambda)^{-1}, \tilde{V}) = y(\lambda, \vartheta) h(\lambda) f(\lambda, \vartheta) = V(\lambda, e^{\mathrm{i}\vartheta}). \tag{2.111}$$

Thus we have shown that the element (2.108) can be gauge transformed into a point of $\tilde{S}^0$, and this point is uniquely determined since (2.109) fixes $e^{\mathrm{i}\vartheta}$ uniquely. In other words, $\tilde{S}^0$ intersects every orbit of the residual gauge action on $\mathfrak{L}^{-1}(C_2) \cap P_1$ in precisely one point.

The map from C_2 into P, given by the parametrization of $\tilde{S}^0$, is obviously smooth, and hence we obtain the identifications

$$C_2 \simeq \tilde{S}^0 \simeq (\mathfrak{L}^{-1}(C_2) \cap P_1)/(G_+ \times Z) \simeq \mathfrak{L}^{-1}(C_2)/(G_+ \times G_+) = \mathfrak{L}^{-1}_{\mathrm{red}}(C_2). \tag{2.112}$$

To establish the formula (2.83) of the reduced symplectic structure, we proceed as follows. We define $G_+ \times G_+$ invariant real functions on P by

$$\varphi_m(y, Y, V) = \frac{1}{m}\mathrm{Re}\big(\mathrm{tr}(Y^m)\big), \quad m \in \mathbb{N}, \tag{2.113}$$

and

$$\chi_k(y, Y, \upsilon) = \mathrm{Re}\big(\mathrm{tr}(Y^k y^{-1} V V^\dagger y C)\big), \quad k \in \mathbb{N} \cup \{0\}. \tag{2.114}$$

The restrictions of these functions to $\tilde{S}^0$ are the respective functions φ_m^{red} and χ_k^{red}:

$$\varphi_m^{\mathrm{red}}(\lambda, \vartheta) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (-1)^{\frac{m}{2}} \dfrac{2}{m} \displaystyle\sum_{j=1}^{n} \lambda_j^m, & \text{if } m \text{ is even,} \end{cases} \tag{2.115}$$

and

$$\chi_k^{\mathrm{red}}(\lambda, \vartheta) = \begin{cases} -2(-1)^{\frac{k-1}{2}} \displaystyle\sum_{j=1}^{n} \lambda_j^k \left[1 - \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} X_j \sin(\vartheta_j), & \text{if } k \text{ is odd,} \\ 2(-1)^{\frac{k}{2}} \displaystyle\sum_{j=1}^{n} \lambda_j^k \left[1 - \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} X_j \cos(\vartheta_j) - \kappa \lambda_j^{k-1}\big(\mathcal{F}_j - \mathcal{F}_{n+j}\big), & \text{if } k \text{ is even,} \end{cases} \tag{2.116}$$

where

$$X_j = \sqrt{\mathcal{F}_j \mathcal{F}_{n+j}} = e^{-\mathrm{i}\vartheta_j} \left[1 - \frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}} \prod_{\substack{k=1 \\ (k \neq j)}}^{n} \left[1 - \frac{4\mu^2}{(\lambda_j - \lambda_k)^2}\right]^{\frac{1}{2}} \left[1 - \frac{4\mu^2}{(\lambda_j + \lambda_k)^2}\right]^{\frac{1}{2}}. \tag{2.117}$$

Then we calculate the pairwise Poisson brackets of the set of functions φ_m, χ_k on P and restrict the results to $\tilde{S}^0$. This must coincide with the results of the direct calculation of the Poisson brackets of the reduced functions φ_m^{red}, χ_k^{red} based on the pull-back of the symplectic form Ω onto $\tilde{S}^0 \subset P$. Inspection shows that the required equalities hold if and only if we have the formula in (2.83) for the pull-back in question. This reasoning is very similar to that used in [106] to find the corresponding reduced symplectic form. Since the underlying calculations are rather laborious, we break them up into smaller pieces and only detail them following this proof. As for the formula for the restriction of $\tilde{\mathcal{H}}_1$ to $\tilde{S}^0$ displayed in (2.83), this is a matter of direct verification. □

The following line of thought is an appropriate adaptation of an argument presented by Pusztai in [106] which since has been applied in the simpler case of A_n root system in [P9]. Differences between these earlier results and the calculations below are highlighted in the Discussion.

Consider the families of real-valued smooth functions φ_m (2.113), χ_k (2.114) on the phase space P (2.26), and the corresponding reduced functions φ_m^{red} (2.115), χ_k^{red}

(2.116) on $\tilde{S}^0$ (2.81). Now let us take an arbitrary point $x = (y, Y, \upsilon^\ell, \upsilon^r) \in P$ and an arbitrary tangent vector $\delta x = \delta y \oplus \delta Y \oplus \delta\upsilon^\ell \oplus 0 \in T_x P$. The derivative of φ_m can be easily obtained and has the form

$$(d\varphi_m)_x(\delta x) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \langle Y^{m-1}, \delta Y\rangle, & \text{if } m \text{ is even.} \end{cases} \tag{2.118}$$

The derivative of χ_k can be written as

$$\begin{aligned} (d\chi_k)_x(\delta x) =& \Big\langle \frac{\big[[Y^k, C]_\pm, y^{-1}Z(\upsilon^\ell)y\big]}{2}, y^{-1}\delta y \Big\rangle \\ &+ \Big\langle \sum_{j=0}^{k-1} \frac{Y^{k-j-1}[y^{-1}Z(\upsilon^\ell)y, C]_\pm Y^j}{2}, \delta Y \Big\rangle \\ &+ \Big\langle \frac{y[C, Y^k]_\pm y^{-1} + Cy[C, Y^k]_\pm y^{-1}C}{4\mathrm{i}\mu}, \delta\upsilon^\ell \Big\rangle, \end{aligned} \tag{2.119}$$

where $[A, B]_\pm = AB \pm BA$ with the sign of $(-1)^k$. The Hamiltonian vector field of φ_m is

$$(\boldsymbol{X}_{\varphi_m})_x = \Delta y \oplus \Delta Y \oplus \Delta\upsilon^\ell \oplus 0 = yY^{m-1} \oplus 0 \oplus 0 \oplus 0, \tag{2.120}$$

while the Hamiltonian vector field corresponding to χ_k is

$$(\boldsymbol{X}_{\chi_k})_x = \Delta' y \oplus \Delta' Y \oplus \Delta' \upsilon^\ell \oplus 0, \tag{2.121}$$

where

$$\Delta' y = \frac{y}{2} \sum_{j=0}^{k-1} Y^{k-j-1}[y^{-1}Z(\upsilon^\ell)y, C]_\pm Y^j, \tag{2.122}$$

$$\Delta' Y = \frac{1}{2}\big[[Y^k, y^{-1}Z(\upsilon^\ell)y]_\pm, C\big], \tag{2.123}$$

$$\Delta' \upsilon^\ell = \frac{1}{4\mathrm{i}\mu}\big[\big(y[C, Y^k]_\pm y^{-1} + Cy[C, Y^k]_\pm y^{-1}C\big), \upsilon^\ell\big]. \tag{2.124}$$

Lemma 2.9. $\{\lambda_a, \lambda_b\} = 0$ *for any* $a, b \in \{1, \ldots, n\}$.

Proof. Using (2.118) one has $\{\varphi_m, \varphi_l\} \equiv 0$ for any $m, l \in \mathbb{N}$ which implies that $\{\varphi_m^{\text{red}}, \varphi_l^{\text{red}}\} \equiv 0$. Let $m, l \in \mathbb{N}$ be arbitrary even numbers. Direct calculation of the Poisson bracket $\{\varphi_m^{\text{red}}, \varphi_l^{\text{red}}\}$ using (2.115) and the Leibniz rule results in the formula

$$\{\varphi_m^{\text{red}}, \varphi_l^{\text{red}}\} = (-1)^{\frac{m+l}{2}} 4 \sum_{a,b=1}^{n} \lambda_a^{m-1}\{\lambda_a, \lambda_b\}\lambda_b^{l-1}. \tag{2.125}$$

By introducing the $n \times n$ matrices

$$\boldsymbol{P}_{a,b} = \{\lambda_a, \lambda_b\} \quad \text{and} \quad \boldsymbol{U}_{a,b} = \lambda_a^{2b-1}, \qquad a, b \in \{1, \dots, n\} \tag{2.126}$$

and choosing m and l from the set $\{1, \dots, 2n\}$, the equation $\{\varphi_m^{\text{red}}, \varphi_l^{\text{red}}\} \equiv 0$ can be cast into the matrix equation

$$(-1)^{\frac{m+l}{2}} \boldsymbol{U}^\dagger \boldsymbol{P} \boldsymbol{U} = \boldsymbol{0}_n. \tag{2.127}$$

Since $\boldsymbol{U}$ is an invertible Vandermonde-type matrix it follows from (2.127) that $\boldsymbol{P} = \boldsymbol{0}_n$ which reads as $\{\lambda_a, \lambda_b\} = 0$ for all $a, b \in \{1, \dots, n\}$. □

Lemma 2.10. *$\{\lambda_a, \vartheta_b\} = \delta_{a,b}$ for any $a, b \in \{1, \dots, n\}$.*

Proof. By choosing two even numbers, k and m, and calculating the Poisson bracket $\{\chi_k, \varphi_m\}$ at an arbitrary point $x = (y, Y, \upsilon^\ell, \upsilon^r) \in P$ the results (2.120)-(2.124) imply that

$$\{\chi_k, \varphi_m\}(x) = \chi_{k+m-1}(x) + \frac{1}{2}\operatorname{tr}\big((Y^k C Y^{m-1} - Y^{m-1} C Y^k) y^{-1} Z(\upsilon^\ell) y\big). \tag{2.128}$$

The computation of the reduced form of (2.128) shows that

$$\{\chi_k^{\text{red}}, \varphi_m^{\text{red}}\} = 2\chi_{k+m-1}^{\text{red}}. \tag{2.129}$$

By utilizing (2.115), (2.116) and the result of the previous lemma one can write the l.h.s. of (2.129) as

$$\{\chi_k^{\text{red}}, \varphi_m^{\text{red}}\} = (-1)^{\frac{k+m}{2}} 4 \sum_{b=1}^n \lambda_b^k \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_b(\lambda)| \sin(\vartheta_b) \sum_{a=1}^n \{\lambda_a, \vartheta_b\} \lambda_a^{m-1}. \tag{2.130}$$

Now, returning to equation (2.129) together with (2.130) one can obtain the following equivalent form

$$\sum_{b=1}^n \lambda_b^k \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_b(\lambda)| \sin(\vartheta_b) \left(\sum_{a=1}^n \{\lambda_a, \vartheta_b\} \lambda_a^{m-1} - \lambda_b^{m-1} \right) = 0. \tag{2.131}$$

By introducing the $n \times n$ matrices

$$\boldsymbol{V}_{b,d} = \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_b(\lambda)| \sin(\vartheta_b) \left(\sum_{a=1}^n \{\lambda_a, \vartheta_b\} \lambda_a^{2d-1} - \lambda_b^{2d-1} \right), \quad b, d \in \{1, \dots, n\} \tag{2.132}$$

and using the Vandermonde-type matrix $\boldsymbol{U}$ defined in (2.126) one is able to write (2.131) into the matrix equation $\boldsymbol{U}^\dagger \boldsymbol{V} = \boldsymbol{0}_n$. Since $\boldsymbol{U}$ is invertible $\boldsymbol{V} = \boldsymbol{0}_n$ and therefore in the dense subset of $C_2 \times \mathbb{T}^n$ where $\sin(\vartheta_b) \neq 0$ the following holds

$$\sum_{a=1}^{n} \{\lambda_a, \vartheta_b\} \lambda_a^{m-1} - \lambda_b^{m-1} = 0, \quad \forall\, b \in \{1, \dots, n\}. \tag{2.133}$$

With the matrices $\boldsymbol{U}$ and

$$\boldsymbol{Q}_{b,a} = \{\lambda_a, \vartheta_b\}, \quad a, b \in \{1, \dots, n\} \tag{2.134}$$

equation (2.133) can be written equivalently as $\boldsymbol{Q}\boldsymbol{U} - \boldsymbol{U} = \mathbf{0}_n$, which immediately implies that $\boldsymbol{Q} = \mathbf{1}_n$. Due to the continuity of Poisson bracket $\boldsymbol{Q} = \mathbf{1}_n$ must hold for every point in $C_2 \times \mathbb{T}^n$, therefore one has $\{\lambda_a, \vartheta_b\} = \delta_{a,b}$ for all $a, b \in \{1, \dots, n\}$. ☐

Lemma 2.11. $\{\vartheta_a, \vartheta_b\} = 0$ *for any* $a, b \in \{1, \dots, n\}$.

Proof. Let k and l be two arbitrary odd integers, and calculate the Poisson bracket $\{\chi_k^{\text{red}}, \chi_l^{\text{red}}\}$ indirectly, that is, work out the Poisson bracket $\{\chi_k, \chi_l\} = \Omega(\boldsymbol{X}_{\chi_l}, \boldsymbol{X}_{\chi_k})$ explicitly and restrict it to the gauge (2.81). The first term $\langle y^{-1}\Delta y, \Delta' Y\rangle$ in (2.15) reads

$$\begin{aligned}
\langle y^{-1}\Delta y, \Delta' Y\rangle =& (-1)^{\frac{k+l+2}{2}} 2\, l \sum_{a=1}^{n} \lambda_a^{k+l-1} \left[1 - \frac{\kappa}{\lambda_a^2}\right] |X_a(\lambda)|^2 \sin(2\vartheta_a) \\
& (-1)^{\frac{k+l+2}{2}} 2 \sum_{\substack{a,b=1 \\ (a \neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1 - \frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a - \vartheta_b)}{\lambda_a + \lambda_b} \\
& (-1)^{\frac{k-l+2}{2}} 2 \sum_{\substack{a,b=1 \\ (a \neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1 - \frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a + \vartheta_b)}{\lambda_a - \lambda_b}.
\end{aligned} \tag{2.135}$$

Due to antisymmetry in the indices $\langle y^{-1}\Delta' y, \Delta Y\rangle$ in (2.15) is obtained by interchanging k and l

$$\begin{aligned}
\langle y^{-1}\Delta' y, \Delta Y\rangle =& (-1)^{\frac{k+l+2}{2}} 2\, k \sum_{a=1}^{n} \lambda_a^{k+l-1} \left[1 - \frac{\kappa}{\lambda_a^2}\right] |X_a(\lambda)|^2 \sin(2\vartheta_a) \\
& (-1)^{\frac{k-l+2}{2}} 2 \sum_{\substack{a,b=1 \\ (a \neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1 - \frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a - \vartheta_b)}{\lambda_a + \lambda_b} \\
& (-1)^{\frac{k+l+2}{2}} 2 \sum_{\substack{a,b=1 \\ (a \neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1 - \frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a + \vartheta_b)}{\lambda_a - \lambda_b}.
\end{aligned} \tag{2.136}$$

One can easily check that the third term in (2.15) vanishes. The last term takes the form

$$\begin{aligned}\langle [D_\upsilon, D'_\upsilon], \upsilon\rangle =&(-1)^{\frac{k+l+2}{2}} 4 \sum_{\substack{a,b=1\\(a\neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a-\vartheta_b)}{\big(4\mu^2-(\lambda_a+\lambda_b)^2\big)(\lambda_a+\lambda_b)}\\ &(-1)^{\frac{k-l+2}{2}} 4 \sum_{\substack{a,b=1\\(a\neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a+\vartheta_b)}{\big(4\mu^2-(\lambda_a-\lambda_b)^2\big)(\lambda_a-\lambda_b)}.\end{aligned} \tag{2.137}$$

As a result of this indirect calculation one obtains the following expression for $\{\chi_k^{\rm red}, \chi_l^{\rm red}\}$

$$\begin{aligned}\{\chi_k^{\rm red}, \chi_l^{\rm red}\} = (-1)^{\frac{k-l+2}{2}} 2(k-l) &\sum_{a=1}^{n} \lambda_a^{k+l-1} \left[1-\frac{\kappa^2}{\lambda_a^2}\right] |X_a|^2 \sin(2\vartheta_a)\\ (-1)^{\frac{k+l+2}{2}} 16\mu^2 &\sum_{\substack{a,b=1\\(a\neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a-\vartheta_b)}{\big(4\mu^2-(\lambda_a+\lambda_b)^2\big)(\lambda_a+\lambda_b)}\\ (-1)^{\frac{k-l+2}{2}} 16\mu^2 &\sum_{\substack{a,b=1\\(a\neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a+\vartheta_b)}{\big(4\mu^2-(\lambda_a-\lambda_b)^2\big)(\lambda_a-\lambda_b)}.\end{aligned} \tag{2.138}$$

One can also carry out a direct computation of $\{\chi_k^{\rm red}, \chi_l^{\rm red}\}$ by using basic properties of the Poisson bracket and the previous two lemmas

$$\begin{aligned}\{\chi_k^{\rm red}, \chi_l^{\rm red}\} = (-1)^{\frac{k-l+2}{2}} 2(k-l) &\sum_{a=1}^{n} \lambda_a^{k+l-1} \left[1-\frac{\kappa^2}{\lambda_a^2}\right] |X_a|^2 \sin(2\vartheta_a)\\ (-1)^{\frac{k+l+2}{2}} 16\mu^2 &\sum_{\substack{a,b=1\\(a\neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a-\vartheta_b)}{\big(4\mu^2-(\lambda_a+\lambda_b)^2\big)(\lambda_a+\lambda_b)}\\ (-1)^{\frac{k-l+2}{2}} 16\mu^2 &\sum_{\substack{a,b=1\\(a\neq b)}}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \frac{\sin(\vartheta_a+\vartheta_b)}{\big(4\mu^2-(\lambda_a-\lambda_b)^2\big)(\lambda_a-\lambda_b)}\\ (-1)^{\frac{k-l}{2}} 4 &\sum_{a,b=1}^{n} \lambda_a^k \lambda_b^l \left[1-\frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1-\frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \cos(\vartheta_a)\cos(\vartheta_b)\{\vartheta_a, \vartheta_b\}.\end{aligned} \tag{2.139}$$

Now it is obvious that (2.138) and (2.139) must be equal therefore the extra term must vanish

$$\sum_{a,b=1}^{n} \lambda_a^k \lambda_b^l \left[1 - \frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} \left[1 - \frac{\kappa^2}{\lambda_b^2}\right]^{\frac{1}{2}} |X_a||X_b| \cos(\vartheta_a)\cos(\vartheta_b)\{\vartheta_a, \vartheta_b\} = 0. \tag{2.140}$$

By utilizing the $n \times n$ matrices

$$\boldsymbol{W}_{a,b} = \lambda_a^b \left[1 - \frac{\kappa^2}{\lambda_a^2}\right]^{\frac{1}{2}} |X_a(\lambda)| \cos(\vartheta_a), \quad \boldsymbol{R}_{a,b} = \{\vartheta_a, \vartheta_b\}, \quad a, b \in \{1, \dots, n\} \tag{2.141}$$

one can reformulate (2.140) as the matrix equation

$$\boldsymbol{W}^\dagger \boldsymbol{R} \boldsymbol{W} = \boldsymbol{0}_n. \tag{2.142}$$

Since $\boldsymbol{W}$ is easily seen to be invertible in a dense subset of the phase space $C_2 \times \mathbb{T}^n$, eq. (2.142) and the continuity of Poisson bracket imply $\boldsymbol{R} = \boldsymbol{0}_n$ for the full phase space, i.e. $\{\vartheta_a, \vartheta_b\} = 0$ for all $a, b \in \{1, \dots, n\}$. □

Lemmas 2.9, 2.10, and 2.11 together imply the following (claimed in Theorem 2.3)

Theorem 2.12. *The reduced symplectic structure on $\tilde{S}^0$ (2.81), given by the pull-back of Ω (2.26) by the tautological injection $\tilde{\sigma}_0 \colon \tilde{S}^0 \to P$, has the canonical form $\tilde{\sigma}_0^*(\Omega) = \sum_{a=1}^n d\lambda_a \wedge d\vartheta_a$.*

Density properties

So far we dealt with the open subset $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2)$ of the reduced phase space. Here we show that Theorem 2.3 contains 'almost all' information about the dual system since $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$ is a *dense* subset. This key result will be proved by combining two lemmas.

Lemma 2.13. *The subset $P_0^{\mathrm{vreg}} \subset P_0$ of the constraint surface where the range of the eigenvalue map $\mathfrak{L}$ (2.72) satisfies the conditions (2.87) and (2.100) is dense.*

Proof. Let us first of all note that P_0 is a connected regular analytic submanifold of P. In fact, it is a regular (embedded) analytic submanifold of the analytic manifold P since the momentum map is analytic and zero is its regular value (because the effectively acting gauge group (2.62) acts freely on P_0). The connectedness follows from Theorem 2.1, which implies that P_0 is diffeomorphic to the product of S (2.50) and the group (2.62), and both are connected.

For any $Y \in \mathcal{G}$ denote by $\{\mathrm{i}\Lambda_a\}_{a=1}^{2n}$ the set of its eigenvalues counted with multiplicities. Then the following formulae

$$\mathcal{R}(y,Y,V) = \prod_{\substack{a,b=1\\(a\neq b)}}^{2n} (\Lambda_a - \Lambda_b) \prod_{a=1}^{2n} (\Lambda_a^2 - \kappa^2), \tag{2.143}$$

$$\mathcal{S}(y,Y,V) = \prod_{\substack{a,b=1\\(a\neq b)}}^{2n} [(\Lambda_a - \Lambda_b)^2 - 4\mu^2] \prod_{a=1}^{2n} \left[(\Lambda_a^2 - \mu^2)(\Lambda_a^2 - \nu^2)(\Lambda_a^2 - (2\mu-\nu)^2)\right]. \tag{2.144}$$

define analytic functions on P_0. Indeed, $\mathcal{R}$ and $\mathcal{S}$ are symmetric polynomials in the eigenvalues of Y, and hence can be expressed as polynomials in the coefficients of the characteristic polynomial of Y, which are polynomials in the matrix elements of Y. The product $\mathcal{RS}$ is also an analytic function on P_0, and the subset P_0^{vreg}, can be characterized as

$$P_0^{\text{vreg}} = \{x \in P_0 \mid \mathcal{R}(x)\mathcal{S}(x) \neq 0\}. \tag{2.145}$$

It is clear from Theorem 2.3 that $\mathcal{RS}$ does not vanish identically on P_0. Since the zero set of a non-zero analytic function on a connected analytic manifold cannot contain any open set, equation (2.145) implies that P_0^{vreg} is a dense subset of P_0. □

Let $\overline{C}_2$ be the closure of the domain $C_2 \subset \mathbb{R}^n$. Eventually, it will turn out that $\mathfrak{L}(P_0) = \overline{C}_2$. For now, we wish to prove the following.

Lemma 2.14. *For every boundary point $\lambda^0 \in \partial\overline{C}_2$ there exist an open ball $B(\lambda^0) \subset \mathbb{R}^n$ around λ^0 that does not contain any strongly regular λ which lies outside $\overline{C}_2$ and belongs to $\mathfrak{L}(P_0)$.*

Proof. We start by noticing that for any boundary point $\lambda^0 \in \partial\overline{C}_2$ there is a ball $B(\lambda^0)$ centred at λ^0 such that any strongly regular $\lambda \in B(\lambda^0) \setminus \overline{C}_2$ is subject to either of the following: (i) there is an index $a \in \{1, \ldots, n-1\}$ such that

$$\lambda_a - \lambda_{a+1} < 2\mu \quad \text{and} \quad \lambda_b - \lambda_{b+1} > 2\mu \quad \forall\, b < a, \tag{2.146}$$

or (ii) we have

$$\lambda_a - \lambda_{a+1} > 2\mu, \quad a = 1, \ldots, n-1 \quad \text{and} \quad \lambda_n < \nu. \tag{2.147}$$

Let us consider a strongly regular $\lambda \in B(\lambda^0)$ that falls into case (i) (2.146) and is so close to C_2 that we still have

$$\lambda_k - \lambda_{k+1} > \mu, \quad \forall\, k \in \{1, \ldots, n-1\}. \tag{2.148}$$

It then follows that

$$\lambda_a - \lambda_b > 2\mu, \quad \forall\, b > a+1, \tag{2.149}$$

and

$$\lambda_a + \lambda_b > 2\mu, \quad \forall\, b \in \{1, \dots, n\}. \tag{2.150}$$

Inspection of the signs of $w_a(\lambda)$ and $w_{a+n}(\lambda)$ in (2.102) gives

$$w_a(\lambda) < 0 < w_{a+n}(\lambda). \tag{2.151}$$

Since every boundary point $\lambda^0 \in \partial \overline{C}_2$ satisfies $\lambda_a^0 > \lambda_n^0 \geq \nu$ for all $a \in \{1, \dots, n-1\}$, we may choose a small enough ball centred at λ^0 to ensure that for λ inside that ball the above inequalities as well as $\lambda_a > \nu$ hold. On account of $\lambda_a > \nu > 0$ and $\mu > 0$ we then have

$$1 - \frac{\nu}{\lambda_a} > 0 \quad \text{and} \quad -1 - \frac{2\mu - \nu}{\lambda_a} < 0. \tag{2.152}$$

By combining (2.104) and (2.105) with (2.151) and (2.152) we conclude that

$$\mathcal{F}_a^+(\lambda) < 0 \quad \text{and} \quad \mathcal{F}_{a+n}^-(\lambda) < 0. \tag{2.153}$$

By Lemma 2.7, these inequalities imply that $\mathcal{F}_a(\lambda)$ and $\mathcal{F}_{a+n}(\lambda)$ cannot be both non-negative, which contradicts the defining equation (2.93). This proves the claim in the case (i) (2.146).

Let us consider a strongly regular λ satisfying (ii) (2.147). In this case we can verify that

$$1 - \frac{\nu}{\lambda_n} < 0, \quad w_n(\lambda) > 0, \quad w_{n+a}(\lambda) > 0. \tag{2.154}$$

Thus we see from (2.105) that $\mathcal{F}_{2n}^+(\lambda) < 0$. Since the sum of the two components on the right hand side of (2.106) is negative, we also see that at least one out of $\mathcal{F}_n^-(\lambda)$ and $\mathcal{F}_{2n}^-(\lambda)$ is negative. Therefore equations (2.103) and (2.104) exclude the unitarity of $\check{A}$ (2.101) in the case (ii) (2.147) as well. □

Proposition 2.15. *The λ-image of the constraint surface is contained in $\overline{C}_2$, i.e. we have*

$$\mathfrak{L}(P_0) \subseteq \overline{C}_2. \tag{2.155}$$

As a consequence, $\mathfrak{L}_{\text{red}}^{-1}(C_2)$ is dense in P_{red}.

Proof. Since $P_0^{\text{vreg}} \subset P_0$ is dense and $\mathfrak{L}\colon P_0 \to \mathbb{R}^n$ (2.72) is continuous, $\mathfrak{L}(P_0^{\text{vreg}}) \subset \mathfrak{L}(P_0)$ is dense. Thus it follows from Lemma 2.14 that for any $\lambda^0 \in \partial C_2$ there exists a ball around λ^0 that does not contain *any* element of $\mathfrak{L}(P_0)$ lying outside $\overline{C}_2$.

Suppose that (2.155) is not true, which means that there exists some $\lambda^* \in \mathfrak{L}(P_0)\backslash\overline{C}_2$. Taking any element $\hat{\lambda} \in \mathfrak{L}(P_0)$ that lies in C_2, it is must be possible to connect λ^* to $\hat{\lambda}$ by a continuous curve in $\mathfrak{L}(P_0)$, since P_0 is connected. Starting from the point λ^*, any such continuous curve must pass through some point of the boundary ∂C_2. However, this is impossible since we know that $\mathfrak{L}(P_0) \setminus \overline{C}_2$ does not contain any series that converges to a point of ∂C_2. This contradiction shows that (2.155) holds.

By (2.155) we have $P_0^{\rm vreg} \subset \mathfrak{L}^{-1}(C_2)$, and we know from Lemma 2.13 that $P_0^{\rm vreg} \subset P_0$ is dense. These together entail that $\mathfrak{L}_{\rm red}^{-1}(C_2) \subset P_{\rm red}$ is dense. $\square$

Global characterization of the dual system

We have seen that

$$P_0^{\rm vreg} \subset \mathfrak{L}^{-1}(C_2) \subset P_0 \tag{2.156}$$

is a chain of dense open submanifolds. These project onto dense open submanifolds of $P_{\rm red}$ and their images under the map $\mathfrak{L}$ (2.72) are dense subsets of $\mathfrak{L}(P_0) = \mathfrak{L}_{\rm red}(P_{\rm red})$:

$$\mathfrak{L}(P_0^{\rm vreg}) \subset C_2 \subset \mathfrak{L}(P_0). \tag{2.157}$$

Now introduce the set

$$\mathbb{C}^n_{\neq} = \{z \in \mathbb{C}^n \mid \prod_{k=1}^n z_k \neq 0\}. \tag{2.158}$$

The parametrization

$$z_j = \sqrt{\lambda_j - \lambda_{j+1} - 2\mu} \prod_{a=1}^j e^{\mathrm{i}\vartheta_a},\ \ j = 1, \dots, n-1, \qquad z_n = \sqrt{\lambda_n - \nu} \prod_{a=1}^n e^{\mathrm{i}\vartheta_a} \tag{2.159}$$

provides a diffeomorphism between $C_2 \times \mathbb{T}^n$ and $\mathbb{C}^n_{\neq}$. Thus we can view $z \in \mathbb{C}^n_{\neq}$ as a variable parametrizing $C_2 \times \mathbb{T}^n$ that corresponds to the semi-global cross-section $\tilde{S}^0$ by Theorem 2.3. Below, we shall exhibit a *global cross-section* in P_0, which will be diffeomorphic to $\mathbb{C}^n$. In other words, the 'semi-global' model of the dual systems will be completed into a global model by allowing the zero value for the complex variables z_k. This completion results from the symplectic reduction automatically.

First of all, let us note that the inverse of the parametrization (2.159) gives

$$\lambda_k(z) = \nu + 2(n-k)\mu + \sum_{j=k}^n z_j \bar{z}_j, \quad k = 1, \dots, n, \tag{2.160}$$

which extend to smooth functions over $\mathbb{C}^n$. The range of the extended map $z \mapsto (\lambda_1, \dots, \lambda_n)$ is the closure $\overline{C}_2$ of the polyhedron C_2. The variables $e^{\mathrm{i}\vartheta_k}$ are well-defined only over $\mathbb{C}^n_{\neq}$, where the parametrization (2.159) entails the equality

$$\sum_{k=1}^n d\lambda_k \wedge d\vartheta_k = \mathrm{i} \sum_{k=1}^n dz_k \wedge d\bar{z}_k. \tag{2.161}$$

An easy inspection of the formulae (2.75) shows that the functions f_a can be recast as

$$f_k(\lambda, e^{\mathrm{i}\vartheta}) = |z_k| g_k(z), \quad f_{n+k}(\lambda, e^{\mathrm{i}\vartheta}) = e^{\mathrm{i}\vartheta_k} |z_{k-1}| g_{n+k}(z), \qquad k = 1, \ldots, n, \;\; z_0 = 1, \tag{2.162}$$

with uniquely defined functions $g_1(z), \ldots, g_{2n}(z)$ that extend to smooth (actually real-analytic) positive functions on $\mathbb{C}^n$. Note that these functions depend on z only through $\lambda(z)$, i.e. one has

$$g_a(z) = \eta_a(\lambda(z)), \quad a = 1, \ldots, 2n, \tag{2.163}$$

with suitable functions η_a that one could display explicitly. The absolute values $|z_k|$ that appear in (2.162) are not smooth at $z_k = 0$, and the phases $e^{\mathrm{i}\vartheta_k}$ are not well-defined there. The crux is that both of these 'troublesome features' can be removed by applying suitable gauge transformations to the elements of the cross-section $\tilde{S}^0$ (2.81). To demonstrate this, we define $m = m(e^{\mathrm{i}\vartheta}) \in Z_{G_+}(\mathcal{A})$ by

$$m_k(e^{\mathrm{i}\vartheta}) = \prod_{j=1}^{k} e^{-\mathrm{i}\vartheta_j}, \quad k = 1, \ldots, n. \tag{2.164}$$

Conforming with (2.37), we also set $m_{k+n} = m_k$. Then the gauge transformation by $(m, m) \in G_+ \times G_+$ operates on the $\mathbb{C}^{2n}$-valued vector $f(\lambda, e^{\mathrm{i}\vartheta})$ and on the matrix $\breve{A}(\lambda, e^{\mathrm{i}\vartheta})$ according to

$$f(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) f(\lambda, e^{\mathrm{i}\vartheta}) \equiv \phi(z), \qquad \breve{A}(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) \breve{A}(\lambda, e^{\mathrm{i}\vartheta}) m(e^{\mathrm{i}\vartheta})^{-1} \equiv \tilde{A}(z), \tag{2.165}$$

which defines the functions $\phi(z)$ and $\tilde{A}(z)$ over $\mathbb{C}^n_{\neq}$. The resulting functions have the form

$$\phi_k(z) = \bar{z}_k g_k(z), \quad \phi_{n+k}(z) = \bar{z}_{k-1} g_{n+k}(z), \quad k = 1, \ldots, n, \tag{2.166}$$

and

$$\tilde{A}_{a,b}(z) = -\frac{2\mu \bar{z}_a z_{b-1} g_a(z) g_{n+b}(z)}{\lambda_a(z) - \lambda_b(z) - 2\mu}, \quad 1 \leq a, b \leq n, \tag{2.167}$$

$$\tilde{A}_{a,n+b}(z) = -\frac{2\mu \bar{z}_a z_b g_a(z) g_b(z)}{\lambda_a(z) + \lambda_b(z) - 2\mu} + \delta_{a,b} \frac{\mu - \nu}{\lambda_a(z) - \mu}, \tag{2.168}$$

$$\tilde{A}_{n+a,b}(z) = \frac{2\mu \bar{z}_{a-1} z_{b-1} g_{n+a}(z) g_{n+b}(z)}{\lambda_a(z) + \lambda_b(z) + 2\mu} - \delta_{a,b} \frac{\mu - \nu}{\lambda_a(z) + \mu}, \tag{2.169}$$

$$\tilde{A}_{n+a,n+b}(z) = \frac{2\mu \bar{z}_{a-1} z_b g_{n+a}(z) g_b(z)}{\lambda_a(z) - \lambda_b(z) + 2\mu}. \tag{2.170}$$

Now the important point is that, as is easily verified, the apparent singularities coming from vanishing denominators in $\tilde{A}$ all cancel, and both $\phi(z)$ and $\tilde{A}(z)$ extend to smooth (actually real-analytic) functions on the whole of $\mathbb{C}^n$. In particular, note the relation

$$\tilde{A}_{k,k+1}(z) = \tilde{A}_{k+n+1,k+n}(z) = -2\mu g_k(z) g_{k+n+1}(z), \quad k = 1, \dots, n-1. \tag{2.171}$$

Corresponding to (2.77), we also have the matrix $\tilde{B}(z) \equiv -(h(\lambda(z))\tilde{A}(z)h(\lambda(z)))^\dagger$. This is smooth over $\mathbb{C}^n$ since both $\tilde{A}(z)$ and $h(\lambda(z))$ (2.69) are smooth. It follows from their defining equations that the induced gauge transformations of $y(\lambda, e^{\mathrm{i}\vartheta})$ (2.79) and $V(\lambda, e^{\mathrm{i}\vartheta})$ (2.80) are given by

$$y(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) y(\lambda, e^{\mathrm{i}\vartheta}) m(e^{\mathrm{i}\vartheta})^{-1} \equiv \tilde{y}(z), \tag{2.172}$$

and

$$V(\lambda, e^{\mathrm{i}\vartheta}) \to m(e^{\mathrm{i}\vartheta}) V(\lambda, e^{\mathrm{i}\vartheta}) = \tilde{y}(z) h(\lambda(z)) \phi(z) \equiv \tilde{V}(z). \tag{2.173}$$

Since $\tilde{y}(z)$ is a uniquely defined smooth function of $\tilde{B}(z)$, both $\tilde{y}(z)$ and $\tilde{V}(z)$ are smooth functions on the whole of $\mathbb{C}^n$.

After these preparations, we are ready to state the main result of this chapter.

Theorem 2.16. *By using the above notations, consider the set*

$$\tilde{S} = \{ (\tilde{y}(z), \mathrm{i} h(\lambda(z)) \Lambda(\lambda(z)) h(\lambda(z))^{-1}, \upsilon^\ell_{\mu,\nu}(\tilde{V}(z)), \upsilon^r) \mid z \in \mathbb{C}^n \}. \tag{2.174}$$

This set defines a global cross-section for the $G_+ \times G_+$-action on the constraint surface P_0. The parametrization of the elements of $\tilde{S}$ by $z \in \mathbb{C}^n$ gives rise to a symplectic diffeomorphism between $(P_{\rm red}, \Omega_{\rm red})$ and $\mathbb{C}^n$ equipped with the Darboux form $\mathrm{i} \sum_{k=1}^n dz_k \wedge d\bar{z}_k$. The spectral invariants of the 'global RSvD Lax matrix'

$$\tilde{L}(z) \equiv h(\lambda(z)) \tilde{A}(z) h(\lambda(z)) \tag{2.175}$$

yield commuting Hamiltonians on $\mathbb{C}^n$ that represent the reductions of the Hamiltonians spanning the Abelian Poisson algebra $\mathfrak{Q}^1$ (2.32).

Proof. Let us denote by

$$z \mapsto \tilde{\sigma}(z) \tag{2.176}$$

the assignment of the element of $\tilde{S}$ to $z \in \mathbb{C}^n$ as given in (2.174). The map $\tilde{\sigma}\colon \mathbb{C}^n \to P$ (2.26) is smooth (even real-analytic) and we have to verify that it possesses the following properties. First, $\tilde{\sigma}$ takes values in the constraint surface P_0. Second, with Ω in (2.26),

$$\tilde{\sigma}^*(\Omega) = \mathrm{i} \sum_{k=1}^n dz_k \wedge d\bar{z}_k. \tag{2.177}$$

Third, $\tilde{\sigma}$ is injective. Fourth, the image $\tilde{S}$ of $\tilde{\sigma}$ intersects every orbit of $G_+ \times G_+$ in P_0

in precisely one point.

Let us start by recalling from Theorem 2.3 the map $(\lambda, \theta) \mapsto \tilde{\sigma}_0(\lambda, \theta)$ that denotes the assignment of the general element of $\tilde{S}^0$ (2.81) to $(\lambda, \theta) \in C_2 \times \mathbb{T}^n$, where now we defined

$$\theta = e^{\mathrm{i}\vartheta}. \tag{2.178}$$

Then the first and second properties of $\tilde{\sigma}$ follow since we have

$$\tilde{\sigma}(z(\lambda, \theta)) = \Phi_{(m(\theta), m(\theta))}\left(\tilde{\sigma}_0(\lambda, \theta)\right), \quad \text{for all} \quad (\lambda, \theta) \in C_2 \times \mathbb{T}^n. \tag{2.179}$$

We know that $\tilde{\sigma}_0(\lambda, \theta) \in P_0$ for all $(\lambda, \theta) \in C_2 \times \mathbb{T}^n$, which implies the first property since $\tilde{\sigma}$ is continuous and P_0 is a closed subset of P. The restriction of the pull-back (2.177) to $\mathbb{C}^n_{\neq}$ is easily calculated using the parametrization $(\lambda, \theta) \mapsto z(\lambda, \theta)$ and using that by Theorem 2.3 $\tilde{\sigma}_0^*(\Omega) = \sum_{k=1}^n d\lambda_k \wedge d\vartheta_k$. Indeed, this translates into (2.177) restricted to $\mathbb{C}^n_{\neq}$, which implies the claimed equality because $\tilde{\sigma}^*(\Omega)$ is smooth on $\mathbb{C}^n$.

Before continuing, we remark that the map $(\lambda, \theta) \mapsto z(\lambda, \theta)$ naturally extends to a continuous map on the closed domain $\overline{C}_2 \times \mathbb{T}^n$ and its 'partial inverse' $z \mapsto \lambda(z)$ extends to a smooth map $\mathbb{C}^n \to \overline{C}_2$. We will use these extended maps without further notice in what follows. (The extended map $(\lambda, \theta) \mapsto z(\lambda, \theta)$ is not differentiable at the points for which $\lambda \in \partial C_2$.)

In order to show that $\tilde{\sigma}$ is injective, consider the equality

$$\tilde{\sigma}(z) = \tilde{\sigma}(\zeta) \quad \text{for some} \quad z, \zeta \in \mathbb{C}^n. \tag{2.180}$$

Looking at the 'second component' of this equality according to (2.174) we see that $\lambda(z) = \lambda(\zeta)$. Then the first component of the equality implies $\tilde{A}(z) = \tilde{A}(\zeta)$. The special case $\tilde{A}_{a,1}(z) = \tilde{A}_{a,1}(\zeta)$ of this equality gives

$$\frac{\bar{z}_a \eta_a(\lambda(z))\eta_{n+1}(\lambda(z))}{\lambda_a(z) - \lambda_1(z) - 2\mu} = \frac{\bar{\zeta}_a \eta_a(\lambda(\zeta))\eta_{n+1}(\lambda(\zeta))}{\lambda_a(\zeta) - \lambda_1(\zeta) - 2\mu}, \qquad 1 \leq a \leq n. \tag{2.181}$$

We know that the factors multiplying $\bar{z}_a$ and $\bar{\zeta}_a$ are equal and non-zero (actually negative). Thus $z = \zeta$ follows, establishing the claimed injectivity.

Next we prove that no two different element of $\tilde{S}$ are gauge equivalent to each other, i.e. $\tilde{S}$ can intersect any orbit of $G_+ \times G_+$ at most in one point. Suppose that

$$\Phi_{(g_L, g_R)}(\tilde{\sigma}(z)) = \tilde{\sigma}(\zeta) \tag{2.182}$$

for some $(g_L, g_R) \in G_+ \times G_+$ and $z, \zeta \in \mathbb{C}^n$. We conclude from the second component of this equality that $\lambda(z) = \lambda(\zeta)$. Because $\lambda(z) \in \overline{C}_2$ holds, $\lambda(z)$ is regular in the sense that it satisfies (2.87). Thus we can also conclude from the second component of the equality (2.182) that g_R belongs to the Abelian subgroup Z of G_+ given in (2.37). Then we infer from the first component

$$g_L \tilde{y}(z) g_R^{-1} = \tilde{y}(\zeta) \tag{2.183}$$

of the equality (2.182) that $g_L = g_R$. We here used that $\tilde{A}(\zeta)$ can be represented in the form (2.41) with strict inequalities in (2.39), which holds since S (2.51) is a global cross-section. Now denote $g_L = g_R = e^{\mathrm{i}\xi} \in Z$ referring to (2.37). Then we have $e^{\mathrm{i}\xi}\tilde{A}(z)e^{-\mathrm{i}\xi} = \tilde{A}(\zeta)$, and in particular

$$e^{\mathrm{i}x_a}\tilde{A}_{a,a+1}(z)e^{-\mathrm{i}x_{a+1}} = \tilde{A}_{a,a+1}(\zeta), \quad \forall a = 1, \ldots, n-1. \tag{2.184}$$

By using (2.162) and (2.171)

$$\tilde{A}_{a,a+1}(z) = -2\mu\eta_a(\lambda(z))\eta_{n+a+1}(\lambda(z)) \neq 0, \tag{2.185}$$

and thus we obtain from $\lambda(z) = \lambda(\zeta)$ that $e^{\mathrm{i}\xi}$ must be equal to a multiple of the identity element of G_+. Hence we have established that $\tilde{\sigma}(z) = \tilde{\sigma}(\zeta)$ is implied by (2.182).

It remains to demonstrate that $\tilde{S}$ intersects every gauge orbit in P_0. We have seen previously that $\mathfrak{L}^{-1}(C_2)$ is dense in P_0 and $\tilde{S}^0$ (2.81) is a cross-section for the gauge action in $\mathfrak{L}^{-1}(C_2)$. These facts imply that for any element $x \in P_0$ there exists a series $x(k) \in \mathfrak{L}^{-1}(C_2)$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty}(x(k)) = x, \tag{2.186}$$

and there also exist series $(g_L(k), g_R(k)) \in G_+ \times G_+$ and $(\lambda(k), \theta(k)) \in C_2 \times \mathbb{T}^n$ such that

$$x(k) = \Phi_{(g_L(k),g_R(k))}\left(\tilde{\sigma}_0(\lambda(k), \theta(k))\right). \tag{2.187}$$

Since $\mathfrak{L}\colon P_0 \to \mathbb{R}^n$ is continuous, we have

$$\mathfrak{L}(x) = \lim_{k\to\infty}\mathfrak{L}(x(k)) = \lim_{k\to\infty}\lambda(k). \tag{2.188}$$

This limit belongs to $\overline{C}_2$ and we denote it by λ^∞. The non-trivial case to consider is when λ^∞ belongs to the boundary ∂C_2. Now, since $G_+ \times G_+ \times \mathbb{T}^n$ is compact, there exists a convergent subseries

$$(g_L(k_i), g_R(k_i), \theta(k_i)), \quad i \in \mathbb{N}, \tag{2.189}$$

of the series $(g_L(k), g_R(k), \theta(k))$. We pick such a convergent subseries and denote its limit as

$$(g_L^\infty, g_R^\infty, \theta^\infty) = \lim_{i\to\infty}(g_L(k_i), g_R(k_i), \theta(k_i)). \tag{2.190}$$

Then we define $z^\infty \in \mathbb{C}^n$ by

$$z^\infty = \lim_{i\to\infty} z(\lambda(k_i), \theta(k_i)) = z(\lambda^\infty, \theta^\infty). \tag{2.191}$$

Since $z \mapsto \tilde{\sigma}(z)$ is continuous, we can write

$$\tilde{\sigma}(z^\infty) = \lim_{i\to\infty} \tilde{\sigma}(z(\lambda(k_i), \theta(k_i))) = \lim_{i\to\infty} \Phi_{(m(\theta(k_i)), m(\theta(k_i)))}\left(\tilde{\sigma}_0(\lambda(k_i), \theta(k_i))\right), \tag{2.192}$$

where $m(\theta)$ is defined by (2.164), with $\theta = e^{\mathrm{i}\vartheta}$. By combining these formulae, we finally obtain

$$\begin{aligned} x &= \lim_{i\to\infty} \Phi_{(g_L(k_i), g_R(k_i))}\left(\tilde{\sigma}_0(\lambda(k_i), \theta(k_i))\right) \\ &= \lim_{i\to\infty} \Phi_{(g_L(k_i) m(\theta(k_i))^{-1}, g_R(k_i) m(\theta(k_i))^{-1})}\left(\tilde{\sigma}(z(\lambda(k_i), \theta(k_i)))\right) \\ &= \Phi_{(g_L^\infty m(\theta^\infty)^{-1}, g_R^\infty m(\theta^\infty)^{-1})}(\tilde{\sigma}(z^\infty)). \end{aligned} \tag{2.193}$$

Therefore $\tilde{S}$ is a global cross-section in P_0.

The final statement of Theorem 2.16 about the global RSvD Lax matrix (2.175) follows since $\tilde{L}$ is just the restriction of the 'unreduced Lax matrix' L of (2.66) to the global cross-section $\tilde{S}$, which represents a model of the full reduced phase space P_{red}. □

2.4 Applications

2.4.1 On the equilibrium position of the Sutherland system

Since the Sutherland Lax matrix is diagonalizable, that is

$$Y(q,p) \quad \text{and} \quad \mathrm{i}\Lambda(\lambda) = \mathrm{i}\,\mathrm{diag}(\lambda, -\lambda) \tag{2.194}$$

are similar matrices, we have the following for the Sutherland Hamiltonians $H_1, \dots, H_n$

$$H_k(q,p) = \frac{1}{4k}\mathrm{tr}((-\mathrm{i}Y(q,p))^{2k}) = \frac{1}{4k}\mathrm{tr}(\Lambda(\lambda)^{2k}) = \frac{1}{2k}\sum_{j=1}^n \lambda_j^{2k} = h_k(\lambda), \quad k = 1, \dots, n. \tag{2.195}$$

In particular, for the main Hamiltonian $H_1(q,p)$ the above formula reads as

$$H(q,p) = H_1(q,p) = h_1(\lambda) = \frac{1}{2}(\lambda_1^2 + \dots + \lambda_n^2). \tag{2.196}$$

It is obvious that h_1 has a global minimum on $\overline{C_2}$ and

$$\min_{(q,p)\in C_1\times\mathbb{R}^n} H(q,p) = \min_{\lambda\in\overline{C_2}} h_1(\lambda) = h_1(\lambda^0), \tag{2.197}$$

where λ^0 is the point in $\overline{C_2}$ with the smallest (Euclidean) norm. Clearly, λ^0 is the boundary point of C_2 at which

$$\lambda_a^0 - \lambda_{a+1}^0 - 2\mu = 0, \quad a = 1,\dots,n-1 \quad \lambda_n^0 - \nu = 0, \tag{2.198}$$

hold, i.e.

$$\lambda_a^0 = (n-a)2\mu + \nu, \quad a = 1,\dots,n. \tag{2.199}$$

In terms of the “oscillator variables” $z_1,\dots,z_n \in \mathbb{C}$ this means that the equilibrium point $(q,p) = (q^e,0)$ of the Sutherland system corresponds to $z_1 = \cdots = z_n = 0$. In fact, each function H_k $(k = 1,\dots,n)$ possesses a global minimum at $z = 0$.

Now, remember that the matrices

$$-\left(h(\lambda)\check{A}(\lambda,e^{\mathrm{i}\vartheta})h(\lambda)\right)^\dagger \quad \text{and} \quad e^{\mathrm{i}Q(q)} \tag{2.200}$$

are similar. By using this and the fact that $h(\lambda)$ and m commute for any $m \in Z_{G_+}(\mathcal{A})$, one concludes that (with the special choice $m = m(e^{\mathrm{i}\vartheta})$)

$$\sigma(-(h\hat{A}h)^\dagger(z)) = \sigma(e^{2\mathrm{i}Q(q)}) = \{e^{2\mathrm{i}q_1},\dots,e^{2\mathrm{i}q_n},e^{-2\mathrm{i}q_1},\dots,e^{-2\mathrm{i}q_n}\}, \tag{2.201}$$

where $\sigma(M)$ denotes the spectrum of M. In particular, for $z = 0$ the above spectral identity provides a useful method to determine the equilibrium coordinates q^e. An interesting characterization of this equilibrium point in terms of the (q,p) variables can be found in [29].

2.4.2 Maximal superintegrability of the dual system

We have seen that the ‘unreduced RSvD Hamiltonians’

$$\tilde{\mathcal{H}}(y,Y,V) = \frac{(-1)^k}{2k}\operatorname{tr}\left(y^{-1}CyC\right)^k, \quad k = 1,\dots,n \tag{2.202}$$

take the following form in the ‘Sutherland gauge’

$$\tilde{h}_k(q) = \tilde{\mathcal{H}}_k|_S = \tilde{H}_k \circ \mathcal{R} = \frac{(-1)^k}{k}\sum_{j=1}^n \cos(2kq_j), \quad k = 1,\dots,n. \tag{2.203}$$

Note that the RSvD-type Hamiltonians depend *only* on the ‘action variables’ $q_1,\dots,q_n$. Our objective is to show that the RSvD-type dual model is maximally superintegrable.

In particular, the construction presented in [P9] will be taken out. Let us consider the $n \times n$ matrix

$$X_{a,b} = \frac{\partial \tilde{h}_a(q)}{\partial q_b}, \quad a, b = 1, \dots, n. \tag{2.204}$$

As a first step we verify that $X(q)$ is invertible at every $q \in C_1$.

Proposition 2.17. *For any $q \in C_1 = \{q \in \mathbb{R}^n \mid \pi/2 > q_1 > \cdots > q_n > 0\}$ we have $\det X(q) \neq 0$.*

Proof. Let us first compute the matrix entries $X_{a,b} = X_{a,b}(q)$. Simple differentiation shows that

$$X_{a,b} = (-1)^{a+1} 2 \sin(2aq_b), \quad a, b = 1, \dots, n. \tag{2.205}$$

Since in each row contains the constants $(-1)^{a+1}2$ $(a = 1, \dots, n)$ we have

$$\det(X_{a,b}) = (-1)^{(1+1)+\cdots+(n+1)} 2^n \det(\sin 2aq_b) = (-1)^{\frac{n(n+3)}{2}} 2^n \det(\sin 2aq_b). \tag{2.206}$$

By introducing the notation

$$\alpha_b = 2q_b, \quad b = 1, \dots, n, \tag{2.207}$$

the above determinant takes a somewhat simpler form

$$\det(X_{a,b}) = (-1)^{\frac{n(n+3)}{2}} 2^n \det(\sin a\alpha_b). \tag{2.208}$$

The trigonometric identity

$$2^r \cos^r \alpha \, \sin \alpha = \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} \left[\binom{r}{s} - \binom{r}{s-1} \right] \sin(r - 2s + 1)\alpha \tag{2.209}$$

(which can be easily proven using de Moivre's formula) implies that applying suitable column-operations on the determinant (2.208), it can be written as

$$\det(X_{a,b}) = (-1)^{\frac{n(n+3)}{2}} 2^n \det(2^{a-1} \cos^{a-1} \alpha_b \, \sin \alpha_b). \tag{2.210}$$

Hence the problem can be reduced to the computation of a Vandermonde-determinant

$$\begin{aligned} \det(X_{a,b}) &= (-1)^{\frac{n(n+3)}{2}} 2^{\frac{n(n+1)}{2}} \prod_{b=1}^{n} \sin \alpha_b \det(\cos^{a-1} \alpha_b) \\ &= (-1)^{\frac{n(n+3)}{2}} 2^{\frac{n(n+1)}{2}} \prod_{b=1}^{n} \sin \alpha_b \prod_{1 \leq b < c \leq n} (\cos \alpha_c - \cos \alpha_b). \end{aligned} \tag{2.211}$$

Now, by substituting back the q's according to (2.207) we get

$$\det(X_{a,b}(q)) = (-1)^{\frac{n(n+3)}{2}} 2^{\frac{n(n+1)}{2}} \prod_{b=1}^{n} \sin 2q_b \prod_{1\leq b<c\leq n} (\cos 2q_c - \cos 2q_b), \tag{2.212}$$

which is an obviously non-vanishing function on C_1 due to monotonicity. □

Now, by referring to [P9] and using the previous proposition for any $\tilde{H}_k$ one can construct the functions mentioned in the Discussion of [P1]

$$f_i(q,p) = \sum_{j=1}^{n} p_j (X^{-1}(q))_{j,i}, \quad i \in \{1,\dots,n\} \setminus \{k\}. \tag{2.213}$$

2.4.3 Equivalence of two sets of Hamiltonians

In this subsection, we prove the equivalence of two complete sets of Poisson commuting Hamiltonians of the (super)integrable rational BC$_n$ Ruijsenaars-Schneider system. Specifically, the commuting Hamiltonians constructed by van Diejen are shown to be linear combinations of the Hamiltonians generated by the characteristic polynomial of the Lax matrix obtained recently by Pusztai, and the explicit formula of this invertible linear transformation is found.

Hamiltonians due to van Diejen

In [140, 143] the following complete set of Poisson commuting Hamiltonians was given:

$$H_l^{\mathrm{vD}}(\lambda,\theta) = \sum_{\substack{J\subset\{1,\dots,n\},\ |J|\leq l \\ \varepsilon_j=\pm1,\ j\in J}} \cos(\theta_{\varepsilon J}) V^{1/2}_{\varepsilon J;J^c} V^{1/2}_{-\varepsilon J;J^c} U_{J^c,l-|J|}, \quad l=1,\dots,n, \tag{2.214}$$

with

$$\begin{aligned}
\theta_{\varepsilon J} &= \sum_{j\in J} \varepsilon_j \theta_j, \\
V_{\varepsilon J;K} &= \prod_{j\in J} w(\varepsilon_j\lambda_j) \prod_{\substack{j,j'\in J\\ j<j'}} v^2(\varepsilon_j\lambda_j + \varepsilon_{j'}\lambda_{j'}) \prod_{\substack{j\in J\\ k\in K}} v(\varepsilon_j\lambda_j+\lambda_k) v(\varepsilon_j\lambda_j-\lambda_k), \\
U_{K,p} &= (-1)^p \sum_{\substack{I\subset K,\ |I|=p\\ \varepsilon_i=\pm1,\ i\in I}} \Bigg(\prod_{i\in I} w(\varepsilon_i\lambda_i) \prod_{\substack{i,i'\in I\\ i<i'}} v(\varepsilon_i\lambda_i+\varepsilon_{i'}\lambda_{i'}) v(-\varepsilon_i\lambda_i-\varepsilon_{i'}\lambda_{i'}) \\
&\qquad \times \prod_{\substack{i\in I\\ k\in K\setminus I}} v(\varepsilon_i\lambda_i+\lambda_k) v(\varepsilon_i\lambda_i-\lambda_k) \Bigg).
\end{aligned} \tag{2.215}$$

We note that J^c in (2.214) denotes the complementary set of J, and the contribution to H_l^{vD} coming from $J = \emptyset$ is $U_{\emptyset^c,l}$. The relatively simple form of $U_{K,p}$ above was found in [143]. Equation (2.214) makes sense for $l = 0$, as well, giving $H_0^{\mathrm{vD}} \equiv 1$. In the rational case the functions v and w take the following form[4]

$$v(x) = \frac{x + \mathrm{i}\mu}{x}, \quad w(x) = \left[\frac{x + \mathrm{i}\nu}{x}\right]\left[\frac{x + \mathrm{i}\kappa}{x}\right]. \tag{2.216}$$

Up to irrelevant constants, H_1^{vD} reproduces the rational BC_n Ruijsenaars-Schneider Hamiltonian

$$\begin{aligned} H^{\mathrm{P}}(\lambda,\theta) = &\sum_{j=1}^{n} \cosh(\theta_j)\left[1 + \frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1 + \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} \prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1 + \frac{\mu^2}{(\lambda_j - \lambda_k)^2}\right]^{\frac{1}{2}}\left[1 + \frac{\mu^2}{(\lambda_j + \lambda_k)^2}\right]^{\frac{1}{2}} \\ &+ \frac{\nu\kappa}{\mu^2}\prod_{j=1}^{n}\left[1 + \frac{\mu^2}{\lambda_j^2}\right] - \frac{\nu\kappa}{\mu^2}. \end{aligned} \tag{2.217}$$

Indeed, one can check that $H_1^{\mathrm{vD}} = 2(H^{\mathrm{P}} - n)$. Here μ, ν, κ are real parameters for which we impose the conditions $\mu \neq 0$, $\nu \neq 0$ and $\nu\kappa \geq 0$. The generalised momenta $\theta = (\theta_1, \ldots, \theta_n)$ run over $\mathbb{R}^n$ and the 'particle positions' $\lambda = (\lambda_1, \ldots, \lambda_n)$ vary in the Weyl chamber

$$\mathfrak{c} = \{x \in \mathbb{R}^n \mid x_1 > \cdots > x_n > 0\}. \tag{2.218}$$

Now, take any point $(\lambda, \theta) \in \mathfrak{c} \times \mathbb{R}^n$ in the phase space and set $(q, p) = \mathcal{S}^{-1}(\lambda, \theta)$ to be the corresponding action-angle coordinates[5]. Consider the H^{P}-trajectory $(\lambda(t), \theta(t))$ with initial condition (λ, θ). Notice that the Hamiltonian H_l^{vD} (2.214) is constant along the H^{P}-trajectory. By utilizing the asymptotics (proved in [108])

$$\lambda_k(t) \sim t\sinh(q_k) - p_k \quad \text{and} \quad \theta_k(t) \sim q_k, \qquad k = 1, \ldots, n, \tag{2.219}$$

one can readily check that

$$(\mathcal{S}^* H_l^{\mathrm{vD}})(q,p) = \lim_{t\to\infty} H_l^{\mathrm{vD}}(\lambda(t), \theta(t)) = \sum_{\substack{J\subset\{1,\ldots,n\},\ |J|\leq l\\ \varepsilon_j = \pm 1,\ j\in J}} (-2)^{l-|J|}\binom{n - |J|}{l - |J|}\cosh(q_{\varepsilon J}). \tag{2.220}$$

From now on we let $\mathcal{H}_l^{\mathrm{vD}}$ stand for the pull-back $\mathcal{S}^* H_l^{\mathrm{vD}}$ just computed, and stress that it depends only on the variables q.

[4] The parameter β appearing in [140, 143] can be introduced via replacing $\lambda, \theta, \mu, \nu, \kappa$ by $\beta^{-1}\lambda, \beta\theta, \beta\mu, \beta\nu, \beta\kappa$, respectively. In the convention of [108], our μ, θ and q correspond to 2μ, 2θ and $2q$.

[5] Here $\mathcal{S}\colon \mathfrak{c} \times \mathbb{R}^n \to \mathfrak{c} \times \mathbb{R}^n$ is the action-angle map, that was constructed by Pusztai [107].

Hamiltonians obtained from the Lax matrix

We recall some relevant objects of [107]. First, prepare the $2n \times 2n$ Hermitian, unitary matrix

$$C = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix} \tag{2.221}$$

and the $2n \times 2n$ Hermitian matrix

$$h(\lambda) = \begin{bmatrix} a(\mathrm{diag}(\lambda)) & b(\mathrm{diag}(\lambda)) \\ -b(\mathrm{diag}(\lambda)) & a(\mathrm{diag}(\lambda)) \end{bmatrix} \tag{2.222}$$

containing the smooth functions $a(x), b(x)$ given on the interval $(0, \infty) \subset \mathbb{R}$ by

$$a(x) = \frac{\sqrt{x + \sqrt{x^2 + \kappa^2}}}{\sqrt{2x}}, \quad b(x) = \mathrm{i}\kappa \frac{1}{\sqrt{2x}} \frac{1}{\sqrt{x + \sqrt{x^2 + \kappa^2}}}. \tag{2.223}$$

Then introduce the vectors $z(\lambda) \in \mathbb{C}^n$, $F(\lambda, \theta) \in \mathbb{C}^{2n}$ by the formulae

$$z_l(\lambda) = -\left[1 + \frac{\mathrm{i}\nu}{\lambda_l}\right] \prod_{\substack{m=1 \\ (m \neq l)}}^{n} \left[1 + \frac{\mathrm{i}\mu}{\lambda_l - \lambda_m}\right]\left[1 + \frac{\mathrm{i}\mu}{\lambda_l + \lambda_m}\right], \tag{2.224}$$

and

$$F_l(\lambda, \theta) = e^{-\frac{\theta_l}{2}} |z_l(\lambda)|^{\frac{1}{2}}, \quad F_{n+l}(\lambda, \theta) = \overline{z_l(\lambda)} F_l(\lambda, \theta)^{-1}, \tag{2.225}$$

$l = 1, \dots, n$. With these notations at hand, the $2n \times 2n$ matrix

$$A_{j,k}(\lambda, \theta) = \frac{\mathrm{i}\mu F_j \overline{F_k} + \mathrm{i}(\mu - 2\nu) C_{j,k}}{\mathrm{i}\mu + \Lambda_j - \Lambda_k}, \quad j, k \in \{1, \dots, 2n\}, \tag{2.226}$$

with $\Lambda = \mathrm{diag}(\lambda, -\lambda)$ is used to define the 'RSvD Lax matrix' [107]:

$$L(\lambda, \theta) = h(\lambda)^{-1} A(\lambda, \theta) h(\lambda)^{-1}. \tag{2.227}$$

The matrices h, A, and L are invertible and satisfy the relations

$$ChC = h^{-1}, \quad CAC = A^{-1}, \quad CLC = L^{-1}. \tag{2.228}$$

Their determinants are

$$\det(h) = \det(A) = \det(L) = 1. \tag{2.229}$$

Let K_m denote the coefficients of the characteristic polynomial of L (2.227),

$$\det(L(\lambda,\theta)-x\mathbf{1}_{2n})=K_0(\lambda,\theta)x^{2n}+K_1(\lambda,\theta)x^{2n-1}+\cdots+K_{2n-1}(\lambda,\theta)x+K_{2n}(\lambda,\theta). \tag{2.230}$$

An immediate consequence of (2.228),(2.229) is that

$$K_{2n-m}\equiv K_m,\quad m=0,1,\dots,n, \tag{2.231}$$

thus the functions $K_0\equiv 1,K_1,\dots,K_n$ fully determine the characteristic polynomial (2.230). The first non-constant member of this family is proportional to H^{P} (2.217), that is $K_1=-2H^{\mathrm{P}}$. The asymptotic form of the Lax matrix L (2.229) is the diagonal matrix

$$\mathrm{diag}(e^{-q},e^{q}), \tag{2.232}$$

hence the action-angle transforms of the functions K_m $(m=0,1,\dots,n)$ can be easily computed to be

$$(\mathcal{S}^*K_m)(q,p)=(-1)^m\sum_{a=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{\substack{J\subset\{1,\dots,n\},\ |J|=m-2a\\ \varepsilon_j=\pm1,\ j\in J}}\binom{n-|J|}{a}\cosh(q_{\varepsilon J}). \tag{2.233}$$

Of course, we used the asymptotics (2.219), and that K_m is constant along the flow of H^{P}. Now we introduce the shorthand $\mathcal{K}_m=\mathcal{S}^*K_m$, and observe that it only depends on q.

It is worth emphasizing that finding a formula relating the families $\{H_l^{\mathrm{vD}}\}_{l=0}^n$ and $\{K_m\}_{m=0}^n$ is equivalent to finding a relation between their action-angle transforms $\{\mathcal{H}_l^{\mathrm{vD}}\}_{l=0}^n$ and $\{\mathcal{K}_m\}_{m=0}^n$.

Proposition 2.18. *There exists an invertible linear relation between the two families* $\{\mathcal{H}_l^{\mathrm{vD}}\}_{l=0}^n$ *and* $\{\mathcal{K}_m\}_{m=0}^n$.

Proof. Let us introduce the auxiliary functions

$$\mathcal{M}_k(q)=\sum_{\substack{J\subset\{1,\dots,n\},\ |J|=k\\ \varepsilon_j=\pm1,\ j\in J}}\cosh(q_{\varepsilon J}),\quad q\in\mathbb{R}^n,\quad k=0,1,\dots,n. \tag{2.234}$$

For any $l\in\{0,1,\dots,n\}$ the Hamiltonian $\mathcal{H}_l^{\mathrm{vD}}$ (2.220) is a linear combination of $\mathcal{M}_0,\mathcal{M}_1,\dots,\mathcal{M}_l$,

$$\mathcal{H}_l^{\mathrm{vD}}(q)=\sum_{k=0}^{l}(-2)^{l-k}\binom{n-k}{l-k}\mathcal{M}_k(q). \tag{2.235}$$

This shows that the matrix of the linear map transforming $\{\mathcal{M}_k\}_{k=0}^n$ into $\{\mathcal{H}_l^{\mathrm{vD}}\}_{l=0}^n$ is lower triangular with ones on the diagonal, hence the above relation is invertible.

Similarly, any function $\mathcal{K}_m$ (2.233), $m \in \{0, 1, \dots, n\}$ can be expressed as a linear combination of $\mathcal{M}_m, \mathcal{M}_{m-2}, \dots, \mathcal{M}_3, \mathcal{M}_1$ or $\mathcal{M}_m, \mathcal{M}_{m-2}, \dots, \mathcal{M}_2, \mathcal{M}_0$ depending on the parity of m, that is

$$\mathcal{K}_m(q) = (-1)^m \sum_{a=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-(m-2a)}{a} \mathcal{M}_{m-2a}(q). \tag{2.236}$$

Hence the linear transformation relating $\{\mathcal{M}_k\}_{k=0}^n$ to $\{\mathcal{K}_m\}_{m=0}^n$ has a lower triangular matrix with diagonal components ± 1, implying that it is invertible. This proves the existence of an invertible linear relation between the two families $\{\mathcal{H}_l^{\mathrm{vD}}\}_{l=0}^n$ and $\{\mathcal{K}_m\}_{m=0}^n$. $\square$

Now, we prove an explicit formula expressing $\mathcal{H}_l^{\mathrm{vD}}$ as linear combination of $\{\mathcal{K}_m\}_{m=0}^l$.

Proposition 2.19. *For any fixed $n \in \mathbb{N}$, $l \in \{1, \dots, n\}$ and $q \in \mathbb{R}^n$ we have*

$$(-1)^l \mathcal{H}_l^{\mathrm{vD}}(q) = \mathcal{K}_l(q) + \sum_{m=0}^{l-1} \frac{2(n-m)}{2(n-m)-(l-m)} \binom{(n-l)+(n-m)}{l-m} \mathcal{K}_m(q). \tag{2.237}$$

Proof. Substitute $\mathcal{K}_m$ (2.233) into the right-hand side of the expression above to obtain

$$\begin{aligned} &\sum_{k=0}^{l-1} \sum_{a=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{\substack{J \subset \{1,\dots,n\},\ |J|=k-2a \\ \varepsilon_j = \pm 1,\ j \in J}} (-1)^k \frac{2(n-k)}{2(n-k)-(l-k)} \binom{(n-l)+(n-k)}{l-k} \times \\ &\times \binom{n-(k-2a)}{a} \cosh(q_{\varepsilon J}) + \sum_{a=0}^{\lfloor \frac{l}{2} \rfloor} \sum_{\substack{J \subset \{1,\dots,n\},\ |J|=l-2a \\ \varepsilon_j = \pm 1,\ j \in J}} (-1)^l \binom{n-(l-2a)}{a} \cosh(q_{\varepsilon J}). \end{aligned} \tag{2.238}$$

Since $k = |J| + 2a$ it is obvious that $(-1)^k = (-1)^{-|J|}$. Multiply (2.238) by $(-1)^l$ and change the order of summations over a and J to get

$$\begin{aligned} &\sum_{\substack{J \subset \{1,\dots,n\},\ |J|<l \\ \varepsilon_j = \pm 1,\ j \in J}} (-1)^{l-|J|} \sum_{a=0}^{\lfloor \frac{l-|J|}{2} \rfloor} \frac{2[n-(|J|+2a)]}{2[n-(|J|+2a)]-[l-(|J|+2a)]} \times \\ &\times \binom{(n-l)+(n-(|J|+2a))}{l-(|J|+2a)} \binom{n-|J|}{a} \cosh(q_{\varepsilon J}) + \sum_{\substack{J \subset \{1,\dots,n\},\ |J|=l \\ \varepsilon_j = \pm 1,\ j \in J}} \cosh(q_{\varepsilon J}). \end{aligned} \tag{2.239}$$

Now, comparison of (2.235) with (2.239) leads to a relation equivalent to (2.237),

$$\sum_{a=0}^{\lfloor\frac{l-|J|}{2}\rfloor} \frac{2[n-(|J|+2a)]}{2[n-(|J|+2a)]-[l-(|J|+2a)]}\times \\ \times \binom{2n-(l+|J|+2a)}{l-(|J|+2a)}\binom{n-|J|}{a}\Big/\binom{n-|J|}{l-|J|} = 2^{l-|J|}. \quad (2.240)$$

For $n = l$ in (2.240) one obtains

$$\begin{cases} 2\sum_{a=0}^{\lfloor\frac{l-|J|}{2}\rfloor}\binom{l-|J|}{a} = 2^{l-|J|}, & \text{if } l-|J| \text{ is odd,} \\ 2\sum_{a=0}^{\frac{l-|J|}{2}-1}\binom{l-|J|}{a} + \binom{l-|J|}{\frac{l-|J|}{2}} = 2^{l-|J|}, & \text{if } l-|J| \text{ is even,} \end{cases} \quad (2.241)$$

which are well-known identities for the binomial coefficients. This means that (2.237) holds for $l = n$ for all $n \in \mathbb{N}$, which implies that if we consider $n+1$ variables it is sufficient to check the cases $l < n+1$. With that in mind let us progress by induction on n and suppose that (2.237) is verified for all $1 \leq l \leq n$ for some $n \in \mathbb{N}$.

First, notice that the Hamiltonians $\mathcal{H}_l^{\mathrm{vD}}$ (2.220) satisfy the following recursion

$$\mathcal{H}_l^{\mathrm{vD}}(q_1,\dots,q_n,q_{n+1}) = \mathcal{H}_l^{\mathrm{vD}}(q_1,\dots,q_n) + 4\sinh^2(\frac{q_{n+1}}{2})\mathcal{H}_{l-1}^{\mathrm{vD}}(q_1,\dots,q_n). \quad (2.242)$$

This can be checked either directly or by utilizing that $\mathcal{H}_l^{\mathrm{vD}}$ is the l-th elementary symmetric function with variables $\sinh^2(\frac{q_i}{2})$ (see Appendix B.2). Similarly, the functions $\mathcal{K}_k$ (2.233) satisfy

$$\mathcal{K}_k(q_1,\dots,q_n,q_{n+1}) = \mathcal{K}_k(q_1,\dots,q_n) - 2\cosh(q_{n+1})\mathcal{K}_{k-1}(q_1,\dots,q_n) + \mathcal{K}_{k-2}(q_1,\dots,q_n), \quad (2.243)$$

with $\mathcal{K}_{-1} \equiv 0$. Let us introduce some shorthand notation, such as the $\mathbb{R}^{l+1}$ vectors

$$\vec{\mathcal{H}}^{\mathrm{vD}}(n) = (\mathcal{H}_0^{\mathrm{vD}}, -\mathcal{H}_1^{\mathrm{vD}},\dots,(-1)^l\mathcal{H}_l^{\mathrm{vD}})^\top \quad \text{and} \quad \vec{\mathcal{K}}(n) = (\mathcal{K}_0,\mathcal{K}_1,\dots,\mathcal{K}_l)^\top \quad (2.244)$$

and the $\mathbb{R}^{(l+1)\times(l+1)}$ matrices

$$\mathcal{A}(n)_{j+1,k+1} = \begin{cases} \frac{2(n-k)}{2(n-k)-(j-k)}\binom{(n-j)+(n-k)}{j-k}, & \text{if } j \geq k, \\ 0, & \text{if } j < k, \end{cases} \quad (2.245)$$

where $j,k\in\{0,\dots,l\}$ and

$$\mathcal{H}^{\rm vD}(n,n+1)=\mathbf{1}_{l+1}-4\sinh^2(\frac{q_{n+1}}{2})\mathcal{I}_{-1},\quad \mathcal{K}(n,n+1)=\mathbf{1}_{l+1}-2\cosh(q_{n+1})\mathcal{I}_{-1}+\mathcal{I}_{-2} \tag{2.246}$$

with $(\mathcal{I}_{-m})_{j+1,k+1}=\delta_{j,k+m}$, $m>0$. The relations (2.242) and (2.243) can be written in the concise form

$$\vec{\mathcal{H}}^{\rm vD}(n+1)=\mathcal{H}^{\rm vD}(n,n+1)\vec{\mathcal{H}}^{\rm vD}(n),\quad \vec{\mathcal{K}}(n+1)=\mathcal{K}(n,n+1)\vec{\mathcal{K}}(n) \tag{2.247}$$

and our assumption is condensed into

$$\vec{\mathcal{H}}^{\rm vD}(n)=\mathcal{A}(n)\vec{\mathcal{K}}(n). \tag{2.248}$$

Using this notation it is clear that the desired induction step is equivalent to the matrix equation

$$\mathcal{H}^{\rm vD}(n,n+1)\mathcal{A}(n)=\mathcal{A}(n+1)\mathcal{K}(n,n+1). \tag{2.249}$$

Spelling this out at some arbitrary (j,k)-th entry gives us

$$\begin{aligned}&\frac{A+B}{A}\binom{A}{B}-4\sinh^2\left(\frac{\alpha}{2}\right)\frac{A+B}{A+1}\binom{A+1}{B-1}=\\&=\frac{A+B+2}{A+2}\binom{A+2}{B}-2\cosh(\alpha)\frac{A+B}{A+1}\binom{A+1}{B-1}+\frac{A+B-2}{A}\binom{A}{B-2},\end{aligned} \tag{2.250}$$

where

$$A=2n-j-k,\quad B=j-k,\quad \alpha=q_{n+1}. \tag{2.251}$$

A simple direct calculation shows that (2.250) indeed holds implying that (2.237) is also true for $n+1$ for any $l\leq n$. The case $l=n+1$ is given by the argument preceding induction. This completes the proof. □

Remark 2.20. We showed in Proposition 2.18 that the relation (2.237) is invertible. Without spending space on the proof, we note that the inverse relation can be written explicitly as

$$(-1)^m\mathcal{K}_m(q)=\sum_{l=0}^{m}\binom{2(n-l)}{m-l}\mathcal{H}_l^{\rm vD}(q). \tag{2.252}$$

2.5 Discussion

In this chapter, we characterized a symplectic reduction of the phase space (P,Ω) (2.26) by exhibiting two models of the reduced phase space $P_{\rm red}$ (2.29). These are provided by the global cross-sections S and $\tilde{S}$ described in Theorem 2.1 and in Theorem 2.16.

The two cross-sections naturally give rise to symplectomorphisms

$$(M,\omega)\simeq(P_{\rm red},\Omega_{\rm red})\simeq(\tilde{M},\tilde{\omega}), \tag{2.253}$$

where $M=T^*C_1$ (2.2) with the canonical symplectic form $\omega=\sum_{k=1}^n dq_k\wedge dp_k$ and $\tilde{M}=\mathbb{C}^n$ with $\tilde{\omega}=\mathrm{i}\sum_{k=1}^n dz_k\wedge d\bar{z}_k$. The Abelian Poisson algebras $\mathfrak{Q}^1$ and $\mathfrak{Q}^2$ on P (2.32) descend to reduced Abelian Poisson algebras $\mathfrak{Q}^1_{\rm red}$ and $\mathfrak{Q}^2_{\rm red}$ on $P_{\rm red}$. The construction guarantees that any element of the reduced Abelian Poisson algebras possesses complete Hamiltonian flow. These flows can be analysed by means of the standard projection algorithm as well as by utilization of the symplectomorphism (2.253).

To further discuss the interpretation of our results, consider the gauge invariant functions

$$\mathcal{H}_k(y,Y,V)=\frac{1}{4k}(-\mathrm{i}Y)^{2k}\quad\text{and}\quad\tilde{\mathcal{H}}_k(y,Y,V)=\frac{(-1)^k}{2k}\mathrm{tr}(y^{-1}CyC)^k,\quad k=1,\dots,n. \tag{2.254}$$

The restrictions of the functions $\mathcal{H}_k$ to the global cross-sections S and $\tilde{S}$ take the form

$$\mathcal{H}_k|_S=\frac{1}{4k}(-\mathrm{i}Y(q,p))^{2k}=H_k(q,p)\quad\text{and}\quad\mathcal{H}_k|_{\tilde{S}}=\frac{1}{2k}\sum_{j=1}^n\lambda_j(z)^{2k}. \tag{2.255}$$

According to (2.64), the H_k yield the commuting Hamiltonians of the Sutherland system, while the λ_j as functions on $\tilde{S}\simeq\mathbb{C}^n$ are given by (2.160). Since any smooth function on a global cross-section encodes a smooth function on $P_{\rm red}$, we conclude that the Sutherland Hamiltonians H_k and the 'eigenvalue-functions' λ_j define two alternative sets of generators for $\mathfrak{Q}^2_{\rm red}$.

The restrictions of the functions $\tilde{\mathcal{H}}_k$ read

$$\tilde{\mathcal{H}}_k|_S=\frac{(-1)^k}{k}\sum_{j=1}^n\cos(2kq_j)\quad\text{and}\quad\tilde{\mathcal{H}}_k|_{\tilde{S}}=\frac{1}{2k}\mathrm{tr}(\tilde{L}(z)^k) \tag{2.256}$$

with $\tilde{L}(z)$ is defined in (2.175). On the semi-global cross-section $\tilde{S}^0$ of Theorem 2.3, which parametrizes the dense open submanifold $\mathfrak{L}^{-1}_{\rm red}(C_2)\subset P_{\rm red}$, we have

$$\tilde{\mathcal{H}}_1|_{\tilde{S}^0}=\tilde{H}^0, \tag{2.257}$$

where $\tilde{H}^0$ is the RSvD Hamiltonian displayed in (2.4). We see from (2.256) that the functions $q_j\in C^\infty(S)$ and the commuting Hamiltonians $\tilde{\mathcal{H}}_k|_{\tilde{S}}$ engender two alternative generating sets for $\mathfrak{Q}^1_{\rm red}$. On account of the relations

$$\tilde{M}^0\simeq\tilde{S}^0\simeq C_2\times\mathbb{T}^n\simeq\mathbb{C}^n_{\neq}\subset\mathbb{C}^n\simeq\tilde{S}\simeq\tilde{M}, \tag{2.258}$$

$\tilde{\mathcal{H}}_1|_{\tilde{S}}$ yields a globally smooth extension of the many-body Hamiltonian $\tilde{H}^0$.

It is immediate from our results that both $\mathfrak{Q}^1_{\text{red}}$ and $\mathfrak{Q}^2_{\text{red}}$ define Liouville integrable systems on P_{red}, since both have n functionally independent generators. The interpretations of these Abelian Poisson algebras that stem from the models S and $\tilde{S}$ underlie the action-angle duality between the Sutherland and RSvD systems as follows. First, the generators q_k of $\mathfrak{Q}^1_{\text{red}}$ can be viewed alternatively as particle positions for the Sutherland system or as action variables for the RSvD system. Their canonical conjugates p_k are of non-compact type. Second, the generators λ_k of $\mathfrak{Q}^2_{\text{red}}$ can be viewed alternatively as action variables for the Sutherland systems or as globally well-defined 'particle positions' for the completed RSvD system. In conclusion, the symplectomorphism $\mathcal{R}\colon M \to \tilde{M}$ naturally induced by (2.253) satisfies all properties required by the notion of action-angle duality outlined in the Introduction.

We end this chapter by pointing out some further consequences. First of all, we note that the dimension of the Liouville tori of the Sutherland system drops on the locus where the action variables encoded by λ belong to the boundary of the polyhedron $\overline{C}_2$. This is a consequence of the next statement, which can be proved by direct calculation.

Proposition 2.21. *Consider the Sutherland Hamiltonians*[6] $H_k(z) = \frac{1}{2k}\sum_{j=1}^n \lambda_j(z)^{2k}$ *and for any* $z \in \mathbb{C}^n$ *define* $\mathcal{D}(z) = \#\{z_k \neq 0 \mid k = 1, \dots, n\}$. *Then one has the equality*

$$\dim\left(\operatorname{span}\{d\lambda_k(z) \mid k = 1, \dots, n\}\right) = \dim\left(\operatorname{span}\{dH_k(z) \mid k = 1, \dots, n\}\right) = \mathcal{D}(z). \tag{2.259}$$

Being in control of the action-angle variables for our dual pair of integrable systems, the following result is readily obtained.

Proposition 2.22. *Any 'Sutherland Hamiltonian'* $H_k \in C^\infty(M)$ $(k = 1, \dots, n)$ *given by* (2.63) *defines a non-degenerate Liouville integrable system, i.e. the commutant of* H_k *in the Poisson algebra* $C^\infty(M)$ *is the Abelian algebra generated by the action variables* $\lambda_1, \dots, \lambda_n$. *Any 'RSvD Hamiltonian'* $\tilde{H}_k \in C^\infty(\tilde{M})$, $k = 1, \dots, n$, *which by definition coincides with* $\tilde{\mathcal{H}}_k|_{\tilde{S}}$ *in* (2.253) *upon the identification* $\tilde{M} \simeq \tilde{S}$, *is maximally degenerate ('superintegrable') since its commutant in the Poisson algebra* $C^\infty(\tilde{M})$ *is generated by* $(2n-1)$ *elements.*

Proof. The subsequent argument relies on the 'action-angle symplectomorphisms' between (M, ω) and $(\tilde{M}, \tilde{\omega})$ corresponding to (2.253).

Let us first restrict the Sutherland Hamiltonian H_k to the submanifold parametrized by the action-angle variables varying in $C_2 \times \mathbb{T}^n$. For generic λ, we see from (2.255) that the flow of H_k is dense on the torus $\mathbb{T}^n$. Therefore any smooth function f that

[6] Here $H_k(z)$ denotes the reduction of the Hamiltonian $\mathcal{H}_k$ expressed in terms of the model $\tilde{M}$, cf. (2.255).

Poisson commutes with H_k must be constant on the non-degenerate Liouville tori of the Sutherland system. By smoothness, this implies that f Poisson commutes with all the action variables λ_j on the full phase space. Consequently, it can be expressed as a function of those variables.

Maximal superintegrability for the dual model was proved in Subsection 2.4.2. $\square$

In the end, we remark that the matrix functions $-\mathrm{i}Y(q,p)$ and $\tilde{L}(z)$, which naturally arose from the Hamiltonian reduction, serve as Lax matrices for the pertinent dual pair of integrable systems. We also notice that the z_j can be viewed as 'oscillator variables' for the Sutherland system since the actions λ_k are linear combinations in $|z_j|^2$ $(j = 1, \ldots, n)$ and the form $\tilde{\omega}$ coincides with the symplectic form of n independent harmonic oscillators. It could be worthwhile to inspect the quantization of the Sutherland system based on these oscillator variables and to compare the result to the standard quantization [58, 57, 100]. We plan to return to this issue in the future.

We demonstrated that the commuting Hamiltonians of the rational BC_n Ruijsenaars-Schneider system, constructed originally by van Diejen, are linear combinations of the coefficients of the characteristic polynomial of the Lax matrix found recently by Pusztai, and vice versa. The derivation utilized the action-angle map and the scattering theory results of [107, 108]. Our Proposition 2.18 gives rise to a determinantal representation of the somewhat complicated expressions H_l^{vD} in (2.214). It could be of some interest to provide a purely algebraic proof of the resulting formula of the characteristic polynomial of the Lax matrix.

The configuration space $\mathfrak{c}$ (2.218) is an open Weyl chamber associated with the Weyl group $W(\mathrm{BC}_n)$, and after extending this domain all Hamiltonians that we dealt with enjoy $W(\mathrm{BC}_n)$ invariance. In particular, the sets $\{\mathcal{H}_l^{\mathrm{vD}}\}_{l=0}^n$, $\{\mathcal{K}_l\}_{l=0}^n$ and $\{\mathcal{M}_l\}_{l=0}^n$ represent different free generating sets of the invariant polynomials in the functions $e^{\pm q_k}$ $(k = 1, \ldots, n)$ of the action variables q_k acted upon by the sign changes and permutations that form $W(\mathrm{BC}_n)$. In order to verify this, it is useful to point out that the $W(\mathrm{BC}_n)$ invariant polynomials in the variables $e^{\pm q_k}$ are the same as the ordinary symmetric polynomials in the variables $\cosh(q_k)$. The statement that $\{\mathcal{H}_l^{\mathrm{vD}}\}_{l=0}^n$ is a free generating set for these polynomials then follows, for example, from the identity presented in Appendix B.2.

Analogous statements hold obviously also for our trigonometric version.

An interesting open problem for future work is to extend the considerations reported here to the hyperbolic RSvD system having five independent coupling parameters.

3 A Poisson-Lie deformation

In this chapter, which based on our results reported in [P2, P3], a deformation of the classical trigonometric BC_n Sutherland system is derived via Hamiltonian reduction of the Heisenberg double of $\mathrm{SU}(2n)$. We apply a natural Poisson-Lie analogue of the Kazhdan-Kostant-Sternberg type reduction of the free particle on $\mathrm{SU}(2n)$ that led to the BC_n Sutherland system in the previous chapter. We prove that this yields a Liouville integrable Hamiltonian system and construct a globally valid model of the smooth reduced phase space wherein the commuting flows are complete. We point out that the reduced system, which contains 3 independent coupling constants besides the deformation parameter, can be recovered (at least on a dense submanifold) as a singular limit of the standard 5-coupling deformation due to van Diejen. Our findings complement and further develop those obtained recently by Marshall [86] on the hyperbolic case by reduction of the Heisenberg double of $\mathrm{SU}(n,n)$.

Here, we shall deal with a reduction of the Heisenberg double of $\mathrm{SU}(2n)$ and derive a Liouville integrable Hamiltonian system related to Marshall's one in a way similar to the connection between the original trigonometric Sutherland system and its hyperbolic variant. Although this is essentially analytic continuation, it should be noted that the resulting systems are qualitatively different in their dynamical characteristics and global features. In addition, what we hope makes our work worthwhile is that our treatment is different from the one in [86] in several respects and we go considerably further regarding the global characterization of the reduced phase space and the completeness of the relevant Hamiltonian flows.

The main Hamiltonian of the system that we obtain can be written as

$$H = \frac{e^{a+b}+e^{a-b}}{2}\sum_{j=1}^{n} e^{-2\hat{p}_j} - \sum_{j=1}^{n}\cos(\hat{q}_j)w(\hat{p}_j;a)^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{\sinh^2(x)}{\sinh^2(\hat{p}_j-\hat{p}_k)}\right]^{\frac{1}{2}} \tag{3.1}$$

with the (external) Morse potential

$$w(\hat{p}_j;a) = 1-(1+e^{2a})e^{-2\hat{p}_j}+e^{2a}e^{-4\hat{p}_j}, \tag{3.2}$$

and the real coupling constants a, b, x satisfying

$$a > 0, \quad b \neq 0, \quad \text{and} \quad x \neq 0. \tag{3.3}$$

The components of $\hat{q}$ parametrize the torus $\mathbb{T}_n$ by $e^{\mathrm{i}\hat{q}}$, and $\hat{p}$ belongs to the domain

$$\mathcal{C}_x := \{\hat{p} \in \mathbb{R}^n \mid 0 > \hat{p}_1,\ \hat{p}_k - \hat{p}_{k+1} > |x|/2\ (k = 1, \dots, n-1)\}. \tag{3.4}$$

The dynamics is then defined via the symplectic form

$$\hat{\omega} = \sum_{j=1}^{n} d\hat{q}_j \wedge d\hat{p}_j. \tag{3.5}$$

It will be shown that this system results by restricting a reduced free system on a dense open submanifold of the pertinent reduced phase space. The Hamiltonian flow is complete on the full reduced phase space, but it can leave the submanifold parametrized by $\mathcal{C}_x \times \mathbb{T}_n$. By glancing at the form of the Hamiltonian, one may say that it represents an RS type system coupled to external fields. Since differences of the 'position variables' $\hat{p}_k$ appear, one feels that this Hamiltonian somehow corresponds to an A-type root system.

To better understand the nature of this model, let us now introduce new Darboux variables q_k, p_k following essentially [86] as

$$\exp(\hat{p}_k) = \sin(q_k) \quad \text{and} \quad \hat{q}_k = p_k \tan(q_k). \tag{3.6}$$

In terms of these variables $H(\hat{p}, \hat{q}; x, a, b) = \mathcal{H}_1(q, p; x, a, b)$ with the 'new Hamiltonian'

$$\begin{aligned}\mathcal{H}_1 = \frac{e^{a+b} + e^{a-b}}{2} & \sum_{j=1}^{n} \frac{1}{\sin^2(q_j)} \\ & - \sum_{j=1}^{n} \cos(p_j \tan(q_j)) \left[1 - \frac{1 + e^{2a}}{\sin^2(q_j)} + \frac{4e^{2a}}{4\sin^2(q_j) - \sin^2(2q_j)}\right]^{\frac{1}{2}} \\ & \qquad \times \prod_{\substack{k=1 \\ (k\neq j)}}^{n} \left[1 - \frac{2\sinh^2\left(\frac{x}{2}\right)\sin^2(q_j)\sin^2(q_k)}{\sin^2(q_j - q_k)\sin^2(q_j + q_k)}\right]^{\frac{1}{2}}. \end{aligned} \tag{3.7}$$

Remarkably, only such combinations of the new 'position variables' q_k appear that are naturally associated with the BC_n root system and the Hamiltonian $\mathcal{H}_1$ enjoys symmetry under the corresponding Weyl group. Thus now one may wish to attach the Hamiltonian $\mathcal{H}_1$ to the BC_n root system. Indeed, this interpretation is preferable for the following reason. Introduce the scale parameter (corresponding to the inverse of the velocity of light in the original Ruijsenaars-Schneider system) $\beta > 0$ and make the substitutions

$$a \to \beta a, \quad b \to \beta b, \quad x \to \beta x, \quad p \to \beta p, \quad \hat{\omega} \to \beta \hat{\omega}. \tag{3.8}$$

Then consider the deformed Hamiltonian $\mathcal{H}_\beta$ defined by

$$\mathcal{H}_\beta(q,p;x,a,b) = \mathcal{H}_1(q,\beta p;\beta x,\beta a,\beta b). \tag{3.9}$$

The point is that one can then verify the following relation:

$$\lim_{\beta\to 0} \frac{\mathcal{H}_\beta(q,p;x,a,b)-n}{\beta^2} = H^{\mathrm{Suth}}_{\mathrm{BC}_n}(q,p;\gamma,\gamma_1,\gamma_2), \tag{3.10}$$

where $H^{\mathrm{Suth}}_{\mathrm{BC}_n}$ stands for the standard trigonometric BC_n Sutherland Hamiltonian (2.1) with coupling constants

$$\gamma = \frac{x^2}{4}, \quad \gamma_1 = (b^2-a^2)/2, \quad \gamma_2 = 2a^2. \tag{3.11}$$

Note that the domain of the variables $\hat{q}, \hat{p}$, and correspondingly that of q, p also depends on β, and in the $\beta \to 0$ limit it is easily seen that we recover the usual BC_n configuration space (2.2). In conclusion, we see that H (3.1) in its equivalent form $\mathcal{H}_\beta$ (3.9) is a 1-parameter deformation of the trigonometric BC_n Sutherland Hamiltonian. We remark in passing that the conditions (3.3) imply that $\gamma_2 > 0$ and $4\gamma_1+\gamma_2 > 0$, which guarantee that the flows of $H^{\mathrm{Suth}}_{\mathrm{BC}_n}$ are complete on the domain (2.2).

Marshall [86] obtained similar results for an analogous deformation of the hyperbolic BC_n Sutherland Hamiltonian. His deformed Hamiltonian differs from (3.1) above in some important signs and in the relevant domain of the 'position variables' $\hat{p}$. Although in our impression the completeness of the reduced Hamiltonian flows was not treated in a satisfactory way in [86], the completeness proof that we shall present can be adapted to Marshall's case as well, as demonstrated in Section 3.4.

It is natural to ask how the system studied in this chapter (and its cousin in [86]) is related to van Diejen's [140] 5-coupling trigonometric BC_n system? It was shown already in [140] that the 5-coupling trigonometric system is a deformation of the BC_n Sutherland system, and later [143] several other integrable systems were also derived as its ('Inozemtsev type' [64]) limits[1]. Motivated by this, we can show that the Hamiltonian (3.1) is a singular limit of van Diejen's general Hamiltonian. Incidentally, a Hamiltonian of Schneider [125] can be viewed as a subsequent singular limit of the Hamiltonian (3.1). Schneider's system was mentioned in [86], too, but the relation to van Diejen's system was not described.

The original idea behind the present work and [86] was that a natural Poisson-Lie

[1]It should be mentioned that the so-called 'Inozemtsev limit' was discovered by Ruijsenaars [115].

analogue of the Hamiltonian reduction treatment [43] of the BC_n Sutherland system should lead to a deformation of this system. It was expected that a special case of van Diejen's standard 5-coupling deformation will arise. The expectation has now been confirmed, although it came as a surprise that a singular limit is involved in the connection.

The outline of the chapter is as follows. We start in Section 3.1 by defining the reduction of interest. In Section 3.2 we observe that several technical results of [37] can be applied for analyzing the reduction at hand, and solve the momentum map constraints by taking advantage of this observation. The heart of the chapter is Section 3.3, where we characterize the reduced system. In Subsection 3.3.1 we prove that the reduced phase space is smooth, as formulated in Theorem 3.9. Then in Subsection 3.3.2 we focus on a dense open submanifold on which the Hamiltonian (3.1) lives. The demonstration of the Liouville integrability of the reduced free flows is given in Subsection 3.3.3. In particular, we prove the integrability of the completion of the system (3.1) carried by the full reduced phase space. Our main result is Theorem 3.14 (proved in Subsection 3.3.4), which establishes a globally valid model of the reduced phase space. We stress that the global structure of the phase space on which the flow of (3.1) is complete was not considered previously at all, and will be clarified as a result of our group theoretic interpretation. Section 3.5 contains our conclusions, further comments on the related paper by Marshall [86] and a discussion of open problems. This chapter is complemented by four appendices. Appendix C.1 deals with the connection to van Diejen's system; the other 3 appendices contain important details relegated from the main text.

3.1 Definition of the Hamiltonian reduction

We below introduce the 'free' Hamiltonians and define their reduction. We restrict the presentation of this background material to a minimum necessary for understanding our work. The conventions follow [37], which also contains more details. As a general reference, we recommend [27].

3.1.1 The unreduced free Hamiltonians

We fix a natural number[2] $n \geq 2$ and consider the Lie group $\mathrm{SU}(2n)$ equipped with its standard quadratic Poisson bracket defined by the compact form of the Drinfeld-Jimbo classical r-matrix,

$$r_{\mathrm{DJ}} = \mathrm{i} \sum_{1\leq\alpha<\beta\leq 2n} E_{\alpha\beta} \wedge E_{\beta\alpha}, \tag{3.12}$$

[2] The $n = 1$ case would need special treatment and is excluded in order to simplify the presentation.

where $E_{\alpha\beta}$ is the elementary matrix of size $2n$ having a single non-zero entry 1 at the $\alpha\beta$ position. In particular, the Poisson brackets of the matrix elements of $g \in \mathrm{SU}(2n)$ obey Sklyanin's formula

$$\{g \overset{\otimes}{,} g\}_{\mathrm{SU}(2n)} = [g \otimes g, r_{\mathrm{DJ}}]. \tag{3.13}$$

Thus $\mathrm{SU}(2n)$ becomes a Poisson-Lie group, i.e. the multiplication $\mathrm{SU}(2n) \times \mathrm{SU}(2n) \to \mathrm{SU}(2n)$ is a Poisson map. The cotangent bundle $T^*\mathrm{SU}(2n)$ possesses a natural Poisson-Lie analogue, the so-called Heisenberg double [126], which is provided by the real Lie group $\mathrm{SL}(2n, \mathbb{C})$ endowed with a certain symplectic form [3], ω. To describe ω, we use the Iwasawa decomposition and factorize every element $K \in \mathrm{SL}(2n, \mathbb{C})$ in two alternative ways

$$K = g_L b_R^{-1} = b_L g_R^{-1} \tag{3.14}$$

with uniquely determined

$$g_L, g_R \in \mathrm{SU}(2n), \quad b_L, b_R \in \mathrm{SB}(2n). \tag{3.15}$$

Here $\mathrm{SB}(2n)$ stands for the subgroup of $\mathrm{SL}(2n, \mathbb{C})$ consisting of upper triangular matrices with positive diagonal entries. The symplectic form ω reads

$$\omega = \frac{1}{2}\mathrm{Im}\mathrm{tr}(db_L b_L^{-1} \wedge dg_L g_L^{-1}) + \frac{1}{2}\mathrm{Im}\mathrm{tr}(db_R b_R^{-1} \wedge dg_R g_R^{-1}). \tag{3.16}$$

Before specifying free Hamiltonians on the phase space $\mathrm{SL}(2n, \mathbb{C})$, note that any smooth function h on $\mathrm{SB}(2n)$ corresponds to a function $\tilde{h}$ on the space of positive definite Hermitian matrices of determinant 1 by the relation

$$\tilde{h}(bb^\dagger) = h(b), \quad \forall b \in \mathrm{SB}(2n). \tag{3.17}$$

Then introduce the invariant functions

$$C^\infty(\mathrm{SB}(2n))^{\mathrm{SU}(2n)} \equiv \{h \in C^\infty(\mathrm{SB}(2n)) \mid \tilde{h}(bb^\dagger) = \tilde{h}(gbb^\dagger g^{-1}),\ \forall g \in \mathrm{SU}(2n), b \in \mathrm{SB}(2n)\}. \tag{3.18}$$

These in turn give rise to the following ring of functions on $\mathrm{SL}(2n, \mathbb{C})$:

$$\mathfrak{H} \equiv \{\mathcal{H} \in C^\infty(\mathrm{SL}(2n, \mathbb{C})) \mid \mathcal{H}(g_L b_R^{-1}) = h(b_R),\ h \in C^\infty(\mathrm{SB}(2n))^{\mathrm{SU}(2n)}\}, \tag{3.19}$$

where we utilized the decomposition (3.14). An important point is that $\mathfrak{H}$ forms an Abelian algebra with respect to the Poisson bracket associated with ω (3.16).

The flows of the 'free' Hamiltonians contained in $\mathfrak{H}$ can be obtained effortlessly. To describe the result, define the derivative $d^R f \in C^\infty(\mathrm{SB}(2n), \mathfrak{su}(2n))$ of any real function $f \in C^\infty(\mathrm{SB}(2n))$ by requiring

$$\frac{d}{ds}\bigg|_{s=0} f(be^{sX}) = \mathrm{Imtr}\big(Xd^R f(b)\big), \quad \forall b \in \mathrm{SB}(2n),\ \forall X \in \mathrm{Lie}(\mathrm{SB}(2n)). \tag{3.20}$$

The Hamiltonian flow generated by $\mathcal{H} \in \mathfrak{H}$ through the initial value $K(0) = g_L(0)b_R(0)^{-1}$ is in fact given by

$$K(t) = g_L(0)\exp\big[-td^R h(b_R(0))\big]b_R^{-1}(0), \tag{3.21}$$

where $\mathcal{H}$ and h are related according to (3.19). This means that $g_L(t)$ follows the orbit of a one-parameter subgroup, while $b_R(t)$ remains constant. Actually, $g_R(t)$ also varies along a similar orbit, and $b_L(t)$ is constant.

The constants of motion b_L and b_R generate a Poisson-Lie symmetry, which allows one to define Marsden-Weinstein type [85] reductions.

3.1.2 Generalized Marsden-Weinstein reduction

The free Hamiltonians in $\mathfrak{H}$ are invariant with respect to the action of $\mathrm{SU}(2n)\times\mathrm{SU}(2n)$ on $\mathrm{SL}(2n,\mathbb{C})$ given by left- and right-multiplications. This is a Poisson-Lie symmetry, which means that the corresponding action map

$$\mathrm{SU}(2n)\times\mathrm{SU}(2n)\times\mathrm{SL}(2n,\mathbb{C}) \to \mathrm{SL}(2n,C), \tag{3.22}$$

operating as

$$(\eta_L,\eta_R,K) \mapsto \eta_L K \eta_R^{-1}, \tag{3.23}$$

is a Poisson map. In (3.22) the product Poisson structure is taken using the Sklyanin bracket on $\mathrm{SU}(2n)$ and the Poisson structure on $\mathrm{SL}(2n,\mathbb{C})$ associated with the symplectic form ω (3.16). This Poisson-Lie symmetry admits a momentum map in the sense of Lu [80], given explicitly by

$$\Phi\colon \mathrm{SL}(2n,\mathbb{C}) \to \mathrm{SB}(2n)\times\mathrm{SB}(2n), \quad \Phi(K) = (b_L,b_R). \tag{3.24}$$

The key property of the momentum map is represented by the identity

$$\frac{d}{ds}\bigg|_{s=0} f(e^{sX}Ke^{-sY}) = \mathrm{Imtr}\big(X\{f,b_L\}b_L^{-1} + Y\{f,b_R\}b_R^{-1}\big), \quad \forall X,Y \in \mathfrak{su}(2n), \tag{3.25}$$

where $f \in C^\infty(\mathrm{SL}(2n,\mathbb{C}))$ is an arbitrary real function and the Poisson bracket is the one corresponding to ω (3.16). The map Φ enjoys an equivariance property and one can [80] perform Marsden-Weinstein type reduction in the same way as for usual Hamiltonian actions (for which the symmetry group has vanishing Poisson structure). To put it in a nutshell, any $\mathcal{H} \in \mathfrak{H}$ gives rise to a reduced Hamiltonian system by fixing the value of Φ and subsequently taking quotient with respect to the corresponding isotropy group. The reduced flows can be obtained by the standard restriction-projection algorithm, and under favorable circumstances the reduced phase space is a smooth symplectic manifold.

Now, consider the block-diagonal subgroup

$$G_+ := \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)) < \mathrm{SU}(2n). \tag{3.26}$$

Since G_+ is also a Poisson submanifold of $\mathrm{SU}(2n)$, the restriction of (3.23) yields a Poisson-Lie action

$$G_+ \times G_+ \times \mathrm{SL}(2n, \mathbb{C}) \to \mathrm{SL}(2n, \mathbb{C}) \tag{3.27}$$

of $G_+ \times G_+$. The momentum map for this action is provided by projecting the original momentum map Φ as follows. Let us write every element $b \in \mathrm{SB}(2n)$ in the block-form

$$b = \begin{bmatrix} b(1) & b(12) \\ \mathbf{0}_n & b(2) \end{bmatrix} \tag{3.28}$$

and define $G_+^* < \mathrm{SB}(2n)$ to be the subgroup for which $b(12) = \mathbf{0}_n$. If $\pi\colon \mathrm{SB}(2n) \to G_+^*$ denotes the projection

$$\pi\colon \begin{bmatrix} b(1) & b(12) \\ \mathbf{0}_n & b(2) \end{bmatrix} \mapsto \begin{bmatrix} b(1) & \mathbf{0}_n \\ \mathbf{0}_n & b(2) \end{bmatrix}, \tag{3.29}$$

then the momentum map $\Phi_+\colon \mathrm{SL}(2n, \mathbb{C}) \to G_+^* \times G_+^*$ is furnished by

$$\Phi_+(K) = (\pi(b_L), \pi(b_R)). \tag{3.30}$$

Indeed, it is readily checked that the analogue of (3.25) holds with X, Y taken from the block-diagonal subalgebra of $\mathfrak{su}(2n)$ and b_L, b_R replaced by their projections. The equivariance property of this momentum map means that in correspondence to

$$K \mapsto \eta_L K \eta_R^{-1} \quad \text{with} \quad (\eta_L, \eta_R) \in G_+ \times G_+, \tag{3.31}$$

one has

$$\big(\pi(b_L)\pi(b_L)^\dagger, \pi(b_R)\pi(b_R)^\dagger\big) \mapsto \big(\eta_L \pi(b_L)\pi(b_L)^\dagger \eta_L^{-1}, \eta_R \pi(b_R)\pi(b_R)^\dagger \eta_R^{-1}\big). \tag{3.32}$$

We briefly mention here that, as the notation suggests, G_+^* is itself a Poisson-Lie group that can serve as a Poisson dual of G_+. The relevant Poisson structure can be obtained by identifying the block-diagonal subgroup of $\mathrm{SB}(2n)$ with the factor group $\mathrm{SB}(2n)/L$, where L is the block-upper-triangular normal subgroup. This factor group inherits a Poisson structure from $\mathrm{SB}(2n)$, since L is a so-called coisotropic (or 'admissible') subgroup of $\mathrm{SB}(2n)$ equipped with its standard Poisson structure. The projected momentum map Φ_+ is a Poisson map with respect to this Poisson structure on the two factors G_+^* in (3.30). The details are not indispensable for us. The interested reader may find them e.g. in [18].

Inspired by the papers [43, 37, 86], we wish to study the particular Marsden-Weinstein reduction defined by imposing the following momentum map constraint:

$$\Phi_+(K) = \mu \equiv (\mu_L, \mu_R), \text{ where } \mu_L = \begin{bmatrix} e^u \nu(x) & \mathbf{0}_n \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix}, \ \mu_R = \begin{bmatrix} e^v \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & e^{-v}\mathbf{1}_n \end{bmatrix} \tag{3.33}$$

with some real constants u, v, and x. Here, $\nu(x) \in \mathrm{SB}(n)$ is the $n \times n$ upper triangular matrix defined by

$$\nu(x)_{jj} = 1, \quad \nu(x)_{jk} = (1 - e^{-x})e^{\frac{(k-j)x}{2}}, \quad j < k, \tag{3.34}$$

whose main property is that $\nu(x)\nu(x)^\dagger$ has the largest possible non-trivial isotropy group under conjugation by the elements of $\mathrm{SU}(n)$.

Our principal task is to characterize the reduced phase space

$$M \equiv \Phi_+^{-1}(\mu)/G_\mu, \tag{3.35}$$

where $\Phi_+^{-1}(\mu) = \{K \in \mathrm{SL}(2n, \mathbb{C}) \mid \Phi_+(K) = \mu\}$ and

$$G_\mu = G_+(\mu_L) \times G_+ \tag{3.36}$$

is the isotropy group of μ inside $G_+ \times G_+$. Concretely, $G_+(\mu_L)$ is the subgroup of G_+ consisting of the special unitary matrices of the form

$$\eta_L = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix}, \tag{3.37}$$

where $\eta_L(2)$ is arbitrary and

$$\eta_L(1)\nu(x)\nu(x)^\dagger\eta_L(1)^{-1} = \nu(x)\nu(x)^\dagger. \tag{3.38}$$

In words, $\eta_L(1)$ belongs to the little group of $\nu(x)\nu(x)^\dagger$ in $\mathrm{U}(n)$. We shall see that $\Phi_+^{-1}(\mu)$ and M are smooth manifolds for which the canonical projection

$$\pi_\mu \colon \Phi_+^{-1}(\mu) \to M \tag{3.39}$$

is a smooth submersion. Then M (3.35) inherits a symplectic form ω_M from ω (3.16), which satisfies

$$\iota_\mu^*(\omega) = \pi_\mu^*(\omega_M), \tag{3.40}$$

where $\iota_\mu \colon \Phi_+^{-1}(\mu) \to \mathrm{SL}(2n, \mathbb{C})$ denotes the tautological embedding.

3.2 Solution of the momentum equation

The description of the reduced phase space requires us to solve the momentum map constraints, i.e. we have to find all elements $K \in \Phi_+^{-1}(\mu)$. Of course, it is enough to do this up to the gauge transformations provided by the isotropy group G_μ (3.36). The solution of this problem will rely on the auxiliary equation (3.51) below, which is essentially equivalent to the momentum map constraint, $\Phi_+(K) = \mu$, and coincides with an equation studied previously in great detail in [37]. Thus we start in the next subsection by deriving this equation.

3.2.1 A crucial equation implied by the constraints

We begin by recalling (e.g. [87]) that any $g \in \mathrm{SU}(2n)$ can be decomposed as

$$g = g_+ \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} h_+, \tag{3.41}$$

where $g_+, h_+ \in G_+$ and $q = \mathrm{diag}(q_1, \dots, q_n) \in \mathbb{R}^n$ satisfies

$$\frac{\pi}{2} \geq q_1 \geq \cdots \geq q_n \geq 0. \tag{3.42}$$

The vector q is uniquely determined by g, while g_+ and h_+ suffer from controlled ambiguities.

First, apply the above decomposition to g_L in $K = g_L b_R^{-1} \in \Phi_+^{-1}(\mu)$ and use the right-handed momentum constraint $\pi(b_R) = \mu_R$. It is then easily seen that up to gauge transformations every element of $\Phi_+^{-1}(\mu)$ can be represented in the following form:

$$K = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{3.43}$$

Here $\rho \in \mathrm{SU}(n)$ and α is an $n \times n$ complex matrix. By using obvious block-matrix notation, we introduce $\Omega := K_{22}$ and record from (3.43) that

$$\Omega = \mathrm{i}(\sin q)\alpha + e^v \cos q. \tag{3.44}$$

For later purpose we introduce also the polar decomposition of the matrix Ω,

$$\Omega = \Lambda T, \tag{3.45}$$

where $T \in \mathrm{U}(n)$ and the Hermitian, positive semi-definite factor Λ is uniquely determined by the relation $\Omega\Omega^\dagger = \Lambda^2$.

Second, by writing $K = b_L g_R^{-1}$ the left-handed momentum constraint $\pi(b_L) = \mu_L$ tells us that b_L has the block-form

$$b_L = \begin{bmatrix} e^u \nu(x) & \chi \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix} \tag{3.46}$$

with an $n \times n$ matrix χ. Now we inspect the components of the 2×2 block-matrix identity

$$KK^\dagger = b_L b_L^\dagger, \tag{3.47}$$

which results by substituting K from (3.43). We find that the (22) component of this identity is equivalent to

$$\Omega\Omega^\dagger = \Lambda^2 = e^{-2u}\mathbf{1}_n - e^{-2v}(\sin q)^2. \tag{3.48}$$

On account of the condition (3.3), this uniquely determines Λ in terms of q, and shows also that Λ is invertible. A further important consequence is that we must have

$$q_n > 0, \tag{3.49}$$

and therefore $\sin q$ is an invertible diagonal matrix. Indeed, if $q_n = 0$, then from (3.44) and (3.48) we would get $(\Omega\Omega^\dagger)_{nn} = e^{2v} = e^{-2u}$, which is excluded by (3.3).

Next, one can check that in the presence of the relations already established, the (12) and the (21) components of the identity (3.47) are equivalent to the equation

$$\chi = \rho(\mathrm{i}\sin q)^{-1}[e^{-u}\cos q - e^{u+v}\Omega^\dagger]. \tag{3.50}$$

Observe that K uniquely determines q, T and ρ, and conversely K is uniquely defined by the above relations once q, T and ρ are found.

Now one can straightforwardly check by using the above relations that the (11) component of the identity (3.47) translates into the following equation:

$$\rho(\sin q)^{-1}T^\dagger(\sin q)^2 T(\sin q)^{-1}\rho^\dagger = \nu(x)\nu(x)^\dagger. \tag{3.51}$$

This is to be satisfied by q subject to (3.42), (3.49) and $T \in \mathrm{U}(n)$, $\rho \in \mathrm{SU}(n)$. What makes our job relatively easy is that this is the same as equation (5.7) in the paper

[37] by Fehér and Klimčík. In fact, this equation was analyzed in detail in [37], since it played a crucial role in that work, too. The correspondence with the symbols used in [37] is

$$(\rho, T, \sin q) \Longleftrightarrow (k_L, k_R^\dagger, e^{\hat{p}}). \tag{3.52}$$

This motivates to introduce the variable $\hat{p} \in \mathbb{R}^n$ in our case, by setting

$$\sin q_k = e^{\hat{p}_k}, \quad k = 1, \dots, n. \tag{3.53}$$

Notice from (3.42) and (3.49) that we have

$$0 \geq \hat{p}_1 \geq \cdots \geq \hat{p}_n > -\infty. \tag{3.54}$$

If the components of $\hat{p}$ are all different, then we can directly rely on [37] to establish both the allowed range of $\hat{p}$ and the explicit form of ρ and T. The statement that $\hat{p}_j \neq \hat{p}_k$ holds for $j \neq k$ can be proved by adopting arguments given in [37, 38]. This proof requires combining techniques of [37] and [38], whose extraction from [37, 38] is rather involved. We present it in Appendix C.2, otherwise in the next subsection we proceed by simply stating relevant applications of results from [37].

Remark 3.1. In the context of [37] the components of $\hat{p}$ are not restricted to the half-line and both k_L and k_R vary in $\mathrm{U}(n)$. These slight differences do not pose any obstacle to using the results and techniques of [37, 38]. We note that essentially the same equation (3.51) surfaced in [86] as well, but the author of that paper refrained from taking advantage of the previous analyses of this equation. In fact, some statements of [86] are not fully correct. This will be specified (and corrected) in Section 3.4.

3.2.2 Consequences of equation (3.51)

We start by pointing out the foundation of the whole analysis. For this, we first display the identity

$$\nu(x)\nu(x)^\dagger = e^{-x}\mathbf{1}_n + \operatorname{sgn}(x)\hat{v}\hat{v}^\dagger, \tag{3.55}$$

which holds with a certain n-component vector $\hat{v} = \hat{v}(x)$. By introducing

$$w = \rho^\dagger \hat{v} \tag{3.56}$$

and setting $\hat{p} \equiv \operatorname{diag}(\hat{p}_1, \dots, \hat{p}_n)$, we rewrite equation (3.51) as

$$e^{2\hat{p} - x\mathbf{1}_n} + \operatorname{sgn}(x) e^{\hat{p}} w w^\dagger e^{\hat{p}} = T^{-1} e^{2\hat{p}} T. \tag{3.57}$$

The equality of the characteristic polynomials of the matrices on the two sides of (3.57) gives a polynomial equation that contains $\hat{p}$, the absolute values $|w_j|^2$ and a complex indeterminate. Utilizing the requirement that $|w_j|^2 \geq 0$ must hold, one obtains the following result.

Proposition 3.2. *If K given by (3.43) belongs to the constraint surface $\Phi_+^{-1}(\mu)$, then the vector $\hat{p}$ (3.53) is contained in the closed polyhedron*

$$\bar{\mathcal{C}}_x := \{\hat{p} \in \mathbb{R}^n \mid 0 \geq \hat{p}_1,\ \hat{p}_k - \hat{p}_{k+1} \geq |x|/2\ (k = 1, \dots, n-1)\}. \tag{3.58}$$

Proposition 3.2 can be proved by merging the proofs of [37, Lemma 5.2] and [38, Theorem 2]. This is presented in Appendix C.2.

The above-mentioned polynomial equality permits to find the possible vectors w (3.56) as well. If $\hat{p}$ and w are given, then T is determined by equation (3.57) up to left-multiplication by a diagonal matrix and ρ is determined by (3.56) up to left-multiplication by elements from the little group of $\hat{v}(x)$. Following this line of reasoning and controlling the ambiguities in the same way as in [37], one can find the explicit form of the most general ρ and T at *any* fixed $\hat{p} \in \bar{\mathcal{C}}_x$. In particular, it turns out that the range of the vector $\hat{p}$ equals $\bar{\mathcal{C}}_x$.

Before presenting the result, we need to prepare some notations. First of all, we pick an arbitrary $\hat{p} \in \bar{\mathcal{C}}_x$ and define the $n \times n$ matrix $\theta(x, \hat{p})$ as follows:

$$\theta(x,\hat{p})_{jk} := \frac{\sinh\left(\frac{x}{2}\right)}{\sinh(\hat{p}_k - \hat{p}_j)} \prod_{\substack{m=1\\(m\neq j,k)}}^{n} \left[\frac{\sinh(\hat{p}_j - \hat{p}_m - \frac{x}{2})\sinh(\hat{p}_k - \hat{p}_m + \frac{x}{2})}{\sinh(\hat{p}_j - \hat{p}_m)\sinh(\hat{p}_k - \hat{p}_m)}\right]^{\frac{1}{2}}, \quad j \neq k, \tag{3.59}$$

and

$$\theta(x,\hat{p})_{jj} := \prod_{\substack{m=1\\(m\neq j)}}^{n} \left[\frac{\sinh(\hat{p}_j - \hat{p}_m - \frac{x}{2})\sinh(\hat{p}_j - \hat{p}_m + \frac{x}{2})}{\sinh^2(\hat{p}_j - \hat{p}_m)}\right]^{\frac{1}{2}}. \tag{3.60}$$

All expressions under square root are non-negative and non-negative square roots are taken. Note that $\theta(x, \hat{p})$ is a real orthogonal matrix of determinant 1 for which $\theta(x, \hat{p})^{-1} = \theta(-x, \hat{p})$ holds, too.

Next, define the real vector $r(x, \hat{p}) \in \mathbb{R}^n$ with non-negative components

$$r(x,\hat{p})_j = \sqrt{\frac{1 - e^{-x}}{1 - e^{-nx}}} \prod_{\substack{k=1\\(k\neq j)}}^{n} \sqrt{\frac{1 - e^{2\hat{p}_j - 2\hat{p}_k - x}}{1 - e^{2\hat{p}_j - 2\hat{p}_k}}}, \quad j = 1, \dots, n, \tag{3.61}$$

and the real $n \times n$ matrix $\zeta(x, \hat{p})$,

$$\begin{aligned} &\zeta(x,\hat{p})_{aa} = r(x,\hat{p})_a, \quad \zeta(x,\hat{p})_{ij} = \delta_{ij} - \frac{r(x,\hat{p})_i r(x,\hat{p})_j}{1 + r(x,\hat{p})_a}, \\ &\zeta(x,\hat{p})_{ia} = -\zeta(x,\hat{p})_{ai} = r(x,\hat{p})_i, \quad i,j \neq a, \end{aligned} \tag{3.62}$$

where $a = n$ if $x > 0$ and $a = 1$ if $x < 0$. Introduce also the vector $v = v(x)$:

$$v(x)_j = \sqrt{\frac{n(e^x - 1)}{1 - e^{-nx}}} e^{-\frac{jx}{2}}, \quad j = 1, \dots, n, \tag{3.63}$$

which is related to $\hat{v}$ in (3.55) by

$$\hat{v}(x) = \sqrt{\mathrm{sgn}(x) e^{-x} \frac{e^{nx} - 1}{n}} v(x). \tag{3.64}$$

Finally, define the $n \times n$ matrix $\kappa(x)$ as

$$\begin{aligned} \kappa(x)_{aa} = \frac{v(x)_a}{\sqrt{n}}, \quad \kappa(x)_{ij} = \delta_{ij} - \frac{v(x)_i v(x)_j}{n + \sqrt{n} v(x)_a}, \\ \kappa(x)_{ia} = -\kappa(x)_{ai} = \frac{v(x)_i}{\sqrt{n}}, \quad i, j \neq a, \end{aligned} \tag{3.65}$$

where, again, $a = n$ if $x > 0$ and $a = 1$ if $x < 0$. It can be shown that both $\kappa(x)$ and $\zeta(x, \hat{p})$ are orthogonal matrices of determinant 1 for any $\hat{p} \in \bar{\mathcal{C}}_x$.

Now we can state the main result of this section, whose proof is omitted since it is a direct application of the analysis of the solutions of (3.51) presented in [37, Section 5].

Proposition 3.3. *Take any $\hat{p} \in \bar{\mathcal{C}}_x$ and any diagonal unitary matrix $e^{\mathrm{i}\hat{q}} \in \mathbb{T}_n$. By using the preceding notations define $K \in \mathrm{SL}(2n, \mathbb{C})$ (3.43) by setting*

$$T = e^{\mathrm{i}\hat{q}} \theta(-x, \hat{p}), \quad \rho = \kappa(x) \zeta(x, \hat{p})^{-1}, \tag{3.66}$$

and also applying the equations (3.44), (3.45), (3.48), and (3.53). Then the element K belongs to the constraint surface $\Phi_+^{-1}(\mu)$, and every orbit of the gauge group G_μ (3.36) in $\Phi_+^{-1}(\mu)$ intersects the set of elements K just constructed.

Remark 3.4. It is worth spelling out the expression of the element K given by Proposition 3.3. Indeed, we have

$$K(\hat{p}, e^{\mathrm{i}\hat{q}}) = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{1}_n - e^{2\hat{p}}} & \mathrm{i}e^{\hat{p}} \\ \mathrm{i}e^{\hat{p}} & \sqrt{\mathbf{1}_n - e^{2\hat{p}}} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix} \tag{3.67}$$

using the above definitions and

$$\alpha = -\mathrm{i}\left[e^{\mathrm{i}\hat{q}} \sqrt{e^{-2u} e^{-2\hat{p}} - e^{-2v}\mathbf{1}_n}\, \theta(-x, \hat{p}) - e^{v} \sqrt{e^{-2\hat{p}} - \mathbf{1}_n} \right]. \tag{3.68}$$

Remark 3.5. Let us call S the set of the elements $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ constructed above, and observe that this set is homeomorphic to

$$\bar{\mathcal{C}}_x \times \mathbb{T}_n = \{(\hat{p}, e^{\mathrm{i}\hat{q}})\} \tag{3.69}$$

by its very definition. This is not a smooth manifold, because of the presence of the boundary of $\bar{\mathcal{C}}_x$. However, this does not indicate any 'trouble' since it is not true (at the boundary of $\bar{\mathcal{C}}_x$) that S intersects every gauge orbit in $\Phi_+^{-1}(\mu)$ in a *single* point. Indeed, it is instructive to verify that if $\hat{p}$ is the special vertex of $\bar{\mathcal{C}}_x$ for which $\hat{p}_k = (1-k)|x|/2$ for $k = 1, \ldots, n$, then all points $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ lie on a single gauge orbit. This, and further inspection, can lead to the idea that the variables $\hat{q}_j$ should be identified with arguments of complex numbers, which lose their meaning at the origin that should correspond to the boundary of $\bar{\mathcal{C}}_x$. Our Theorem 3.14 will show that this idea is correct. It is proper to stress that we arrived at such idea under the supporting influence of previous works [117, 37].

3.3 Characterization of the reduced system

The smoothness of the reduced phase space and the completeness of the reduced free flows follows immediately if we can show that the gauge group G_μ acts in such a way on $\Phi_+^{-1}(\mu)$ that the isotropy group of every point is just the finite center of the symmetry group. In Subsection 3.3.1, we prove that the factor of G_μ by the center acts freely on $\Phi_+^{-1}(\mu)$. Then in Subsection 3.3.2 we explain that $\mathcal{C}_x \times \mathbb{T}_n$ provides a model of a dense open subset of the reduced phase space by means of the corresponding subset of $\Phi_+^{-1}(\mu)$ defined by Proposition 3.3. Adopting a key calculation from [86], it turns out that $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n$ are Darboux coordinates on this dense open subset. In Subsection 3.3.3, we demonstrate that the reduction of the Abelian Poisson algebra of free Hamiltonians (3.19) yields an integrable system. Finally, in Subsection 3.3.4, we present a model of the full reduced phase space, which is our main result in this chapter.

3.3.1 Smoothness of the reduced phase space

It is clear that the normal subgroup of the full symmetry group $G_+ \times G_+$ consisting of matrices of the form

$$(\eta, \eta) \quad \text{with} \quad \eta = \mathrm{diag}(z\mathbf{1}_n, z\mathbf{1}_n), \quad z^{2n} = 1 \tag{3.70}$$

acts trivially on the phase space. This subgroup is contained in G_μ (3.36). The corresponding factor group of G_μ is called 'effective gauge group' and is denoted by $\bar{G}_\mu$. We wish to show that $\bar{G}_\mu$ acts freely on the constraint surface $\Phi_+^{-1}(\mu)$.

We need the following elementary lemmas.

Lemma 3.6. *Suppose that*

$$g_+ \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} h_+ = g_+' \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} h_+' \tag{3.71}$$

with $g_+, h_+, g_+', h_+' \in G_+$ *and* $q = \mathrm{diag}(q_1, \dots, q_n)$ *subject to*

$$\frac{\pi}{2} \geq q_1 > \cdots > q_n > 0. \tag{3.72}$$

Then there exist diagonal matrices $m_1, m_2 \in \mathbb{T}_n$ *having the form*

$$m_1 = \mathrm{diag}(a, \xi), \quad m_2 = \mathrm{diag}(b, \xi), \quad \xi \in \mathbb{T}_{n-1},\ a, b \in \mathbb{T}_1, \quad \det(m_1 m_2) = 1, \tag{3.73}$$

for which

$$(g_+', h_+') = (g_+ \mathrm{diag}(m_1, m_2), \mathrm{diag}(m_2^{-1}, m_1^{-1}) h_+). \tag{3.74}$$

If (3.72) *holds with strict inequality* $\frac{\pi}{2} > q_1$*, then* $m_1 = m_2$*, i.e.* $a = b$*.*

Lemma 3.7. *Pick any* $\hat{p} \in \bar{\mathcal{C}}_x$ *and consider the matrix* $\theta(x, \hat{p})$ *given by* (3.59) *and* (3.60)*. Then the entries* $\theta_{n,1}(x, \hat{p})$ *and* $\theta_{j,j+1}(x, \hat{p})$ *are all non-zero if* $x > 0$ *and the entries* $\theta_{1,n}(x, \hat{p})$ *and* $\theta_{j+1,j}(x, \hat{p})$ *are all non-zero if* $x < 0$*.*

For convenience, we present the proof of Lemma 3.6 in Appendix C.3. The property recorded in Lemma 3.7 is known [117, 37], and is easily checked by inspection.

Proposition 3.8. *The effective gauge group* $\bar{G}_\mu$ *acts freely on* $\Phi_+^{-1}(\mu)$*.*

Proof. Since every gauge orbit intersects the set S specified by Proposition 3.3, it is enough to show that if $(\eta_L, \eta_R) \in G_\mu$ maps $K \in S$ (3.67) to itself, then (η_L, η_R) equals some element (η, η) given in (3.70). For K of the form (3.43), we can spell out $K' \equiv \eta_L K \eta_R^{-1}$ as

$$K' = \begin{bmatrix} \eta_L(1)\rho & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \begin{bmatrix} \eta_R(1)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2)^{-1} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \eta_R(1)\alpha\eta_R(2)^{-1} \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{3.75}$$

The equality $K' = K$ implies by the uniqueness of the Iwasawa decomposition and Lemma 3.6 that we must have

$$\eta_L(2) = \eta_R(1) = m_2, \quad \eta_R(2) = m_1, \quad \eta_L(1)\rho = \rho m_1, \tag{3.76}$$

with some diagonal unitary matrices having the form (3.73). By using that $\eta_R(1) = m_2$ and $\eta_R(2) = m_1$, the Iwasawa decomposition of $K' = K$ in (3.67) also entails the relation

$$\alpha = m_2 \alpha m_1^{-1}. \tag{3.77}$$

Because of (3.68), the off-diagonal components of the matrix equation (3.77) yield

$$\theta(-x,\hat{p})_{jk} = \left(m_2 \theta(-x,\hat{p}) m_1^{-1}\right)_{jk}, \quad \forall j \neq k. \tag{3.78}$$

This implies by means of Lemma 3.7 and equation (3.73) that $m_1 = m_2 = z\mathbf{1}_n$ is a scalar matrix. But then $\eta_L(1) = m_1$ follows from $\eta_L(1)\rho = \rho m_1$, and the proof is complete. □

Proposition 3.8 and the general results gathered in Appendix C.4 imply the following theorem, which is one of our main results.

Theorem 3.9. *The constraint surface $\Phi_+^{-1}(\mu)$ is an embedded submanifold of* $\mathrm{SL}(2n,\mathbb{C})$ *and the reduced phase space M (3.35) is a smooth manifold for which the natural projection $\pi_\mu \colon \Phi_+^{-1}(\mu) \to M$ is a smooth submersion.*

3.3.2 Model of a dense open subset of the reduced phase space

Let us denote by $S^o \subset S$ the subset of the elements K given by Proposition 3.3 with $\hat{p}$ in the interior $\mathcal{C}_x$ of the polyhedron $\bar{\mathcal{C}}_x$ (3.58). Explicitly, we have

$$S^o = \{K(\hat{p}, e^{\mathrm{i}\hat{q}}) \mid (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n\}, \tag{3.79}$$

where $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ stands for the expression (3.67). Note that S^o is in bijection with $\mathcal{C}_x \times \mathbb{T}_n$. The next lemma says that no two different point of S^o are gauge equivalent.

Lemma 3.10. *The intersection of any gauge orbit with S^o consists of at most one point.*

Proof. Suppose that

$$K' := K(\hat{p}', e^{\mathrm{i}\hat{q}'}) = \eta_L K(\hat{p}, e^{\mathrm{i}\hat{q}}) \eta_R^{-1} \tag{3.80}$$

with some $(\eta_L, \eta_R) \in G_\mu$. By spelling out the gauge transformation as in (3.75), using the shorthand $\sin q = e^{\hat{p}}$, we observe that $\hat{p}' = \hat{p}$ since q in (3.41) does not change under the action of $G_+ \times G_+$. Since now we have $\frac{\pi}{2} > q_1$ (which is equivalent to $0 > \hat{p}_1$), the arguments applied in the proof of Proposition 3.8 permit to translate the equality (3.80) into the relations

$$\eta_L(2) = \eta_R(1) = \eta_R(2) = m, \quad \eta_L(1)\rho = \rho m, \tag{3.81}$$

complemented with the condition

$$\alpha(\hat{p}, e^{\mathrm{i}\hat{q}'}) = m\alpha(\hat{p}, e^{\mathrm{i}\hat{q}}) m^{-1}, \tag{3.82}$$

which is equivalent to

$$e^{\mathrm{i}\hat{q}'}\theta(-x,\hat{p}) = me^{\mathrm{i}\hat{q}}\theta(-x,\hat{p})m^{-1}. \tag{3.83}$$

We stress that $m \in \mathbb{T}_n$ and notice from (3.60) that for $\hat{p} \in \mathcal{C}_x$ all the diagonal entries $\theta(-x,\hat{p})_{jj}$ are non-zero. Therefore we conclude from (3.83) that $e^{\mathrm{i}\hat{q}'} = e^{\mathrm{i}q}$. This finishes the proof, but of course we can also confirm that $m = z\mathbf{1}_n$, consistently with Proposition 3.8. □

Now we introduce the map $\mathcal{P}\colon \mathrm{SL}(2n,\mathbb{C}) \to \mathbb{R}^n$ by

$$\mathcal{P}\colon K = g_L b_R^{-1} \mapsto \hat{p}, \tag{3.84}$$

defined by writing g_L in the form (3.41) with $\sin q = e^{\hat{p}}$. The map $\mathcal{P}$ gives rise to a map $\bar{\mathcal{P}}\colon M \to \mathbb{R}^n$ verifying

$$\bar{\mathcal{P}}(\pi_\mu(K)) = \mathcal{P}(K), \quad \forall K \in \Phi_+^{-1}(\mu), \tag{3.85}$$

where π_μ is the canonical projection (3.39). We notice that, since the 'eigenvalue parameters' $\hat{p}_j$ $(j = 1,\dots,n)$ are pairwise different for any $K \in \Phi_+^{-1}(\mu)$, $\bar{\mathcal{P}}$ is a smooth map. The continuity of $\bar{\mathcal{P}}$ implies that

$$M^o := \bar{\mathcal{P}}^{-1}(\mathcal{C}_x) = \pi_\mu(S^o) \subset M \tag{3.86}$$

is an open subset. The second equality is a direct consequence of our foregoing results about S and S^o. Note that $\bar{\mathcal{P}}^{-1}(\bar{\mathcal{C}}_x) = \pi_\mu(S) = M$. Since π_μ is continuous (actually smooth) and any point of S is the limit of a sequence in S^o, M^o is *dense* in the reduced phase space M. The dense open subset M^o can be parametrized by $\mathcal{C}_x \times \mathbb{T}_n$ according to

$$(\hat{p}, e^{\mathrm{i}\hat{q}}) \mapsto \pi_\mu(K(\hat{p}, e^{\mathrm{i}\hat{q}})), \tag{3.87}$$

which also allows us to view $S^o \simeq \mathcal{C}_x \times \mathbb{T}_n$ as a model of $M^o \subset M$. In principle, the restriction of the reduced symplectic form to M^o can now be computed by inserting the explicit formula $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ (3.67) into the Alekseev-Malkin form (3.16). In the analogous reduction of the Heisenberg double of $\mathrm{SU}(n,n)$, Marshall [86] found a nice way to circumvent such a tedious calculation. By taking the same route, we have verified that $\hat{p}$ and $\hat{q}$ are Darboux coordinates on M^o.

The outcome of the above considerations is summarized by the next theorem.

Theorem 3.11. *M^o defined by equation (3.86) is a dense open subset of the reduced phase space M. Parametrizing M^o by $\mathcal{C}_x \times \mathbb{T}_n$ according to (3.87), the restriction of reduced symplectic form ω_M (3.40) to M^o is equal to $\hat{\omega} = \sum_{j=1}^n d\hat{q}_j \wedge d\hat{p}_j$ (3.5).*

3.3.3 Liouville integrability of the reduced free Hamiltonians

The Abelian Poisson algebra $\mathfrak{H}$ (3.19) consists of $(G_+ \times G_+)$-invariant functions[3] generating complete flows, given explicitly by (3.21), on the unreduced phase space. Thus each element of $\mathfrak{H}$ descends to a smooth reduced Hamiltonian on M (3.35), and generates a complete flow via the reduced symplectic form ω_M. This flow is the projection of the corresponding unreduced flow, which preserves the constraint surface $\Phi_+^{-1}(\mu)$. It also follows from the construction that $\mathfrak{H}$ gives rise to an Abelian Poisson algebra, $\mathfrak{H}_M$, on (M, ω_M). Now the question is whether the Hamiltonian vector fields of $\mathfrak{H}_M$ span an n-dimensional subspace of the tangent space at the points of a dense open submanifold of M. If yes, then $\mathfrak{H}_M$ yields a Liouville integrable system, since $\dim(M) = 2n$.

Before settling the above question, let us focus on the Hamiltonian $\mathcal{H} \in \mathfrak{H}$ defined by

$$\mathcal{H}(K) := \frac{1}{2}\mathrm{tr}\big((K^\dagger K)^{-1}\big) = \frac{1}{2}\mathrm{tr}(b_R^\dagger b_R). \tag{3.88}$$

Using the formula of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ in Remark 3.4, it is readily verified that

$$\mathcal{H}(K(\hat{p}, e^{\mathrm{i}\hat{q}})) = H(\hat{p}, \hat{q}; x, v-u, v+u), \quad \forall (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n, \tag{3.89}$$

with the Hamiltonian H displayed in equation (3.1). *Consequently, H in (3.1) is identified as the restriction of the reduction of $\mathcal{H}$ (3.88) to the dense open submanifold M^o (3.86) of the reduced phase space, wherein the flow of every element of $\mathfrak{H}_M$ is complete.*

Turning to the demonstration of Liouville integrability, consider the n functions

$$\mathcal{H}_k(K) := \frac{1}{2k}\mathrm{tr}\big((K^\dagger K)^{-1}\big)^k = \frac{1}{2k}\mathrm{tr}(b_R^\dagger b_R)^k, \quad k = 1, \dots, n. \tag{3.90}$$

The restriction of the corresponding elements of $\mathfrak{H}_M$ on $M^o \simeq \mathcal{C}_x \times \mathbb{T}_n$ gives the functions

$$H_k(\hat{p}, \hat{q}) = \frac{1}{2k}\mathrm{tr}\begin{bmatrix} e^{2v}\mathbf{1}_n & -e^v\alpha \\ -e^v\alpha^\dagger & (e^{-2v}\mathbf{1}_n + \alpha^\dagger\alpha) \end{bmatrix}^k, \tag{3.91}$$

where α has the form (3.68). These are real-analytic functions on $\mathcal{C}_x \times \mathbb{T}_n$. It is enough to show that their exterior derivatives are linearly independent on a dense open subset of $\mathcal{C}_x \times \mathbb{T}_n$. This follows if we show that the function

$$f(\hat{p}, \hat{q}) = \det\left[d_{\hat{q}}H_1, d_{\hat{q}}H_2, \dots, d_{\hat{q}}H_n\right] \tag{3.92}$$

is not identically zero on $\mathcal{C}_x \times \mathbb{T}_n$. Indeed, since f is an analytic function and $\mathcal{C}_x \times \mathbb{T}_n$ is connected, if f is not identically zero then its zero set cannot contain any accumulation

[3]More precisely, $\mathfrak{H} = C^\infty(\mathrm{SL}(2n, \mathbb{C}))^{\mathrm{SU}(2n)\times\mathrm{SU}(2n)}$.

point. This, in turn, implies that f is non-zero on a dense open subset of $\mathcal{C}_x \times \mathbb{T}_n \simeq M^o$, which is also dense and open in the full reduced phase space M. In other words, the reductions of $\mathcal{H}_k$ $(k = 1, \dots, n)$ possess the property of Liouville integrability. It is rather obvious that the function f is not identically zero, since H_k involves dependence on $\hat{q}$ through $e^{\pm \mathrm{i}k\hat{q}}$ and lower powers of $e^{\pm \mathrm{i}\hat{q}}$. It is not difficult to inspect the function $f(\hat{p}, \hat{q})$ in the 'asymptotic domain' where all differences $|\hat{p}_j - \hat{p}_m|$ $(m \neq j)$ tend to infinity, since in this domain α becomes close to a diagonal matrix. We omit the details of this inspection, whereby we checked that f is indeed not identically zero.

The above arguments prove the Liouville integrability of the reduced free Hamiltonians, i.e. the elements of $\mathfrak{H}_M$. Presumably, there exists a dual set of integrable many-body Hamiltonians that live on the space of action-angle variables of the Hamiltonians in $\mathfrak{H}_M$. The construction of such dual Hamiltonians is an interesting task for the future, which will be further commented upon in Section 3.5.

3.3.4 The global structure of the reduced phase space

We here construct a global cross-section of the gauge orbits in the constraint surface $\Phi_+^{-1}(\mu)$. This engenders a symplectic diffeomorphism between the reduced phase space (M, ω_M) and the manifold $(\hat{M}_c, \hat{\omega}_c)$ below. It is worth noting that $(\hat{M}_c, \hat{\omega}_c)$ is symplectomorphic to $\mathbb{R}^{2n}$ carrying the standard Darboux 2-form, and one can easily find an explicit symplectomorphism if desired. Our construction was inspired by the previous papers [117, 37], but detailed inspection of the specific example was also required for finding the final result given by Theorem 3.14. After a cursory glance, the reader is advised to go directly to this theorem and follow the definitions backwards as becomes necessary. See also Remark 3.15 for the rationale behind the subsequent definitions.

To begin, consider the product manifold

$$\hat{M}_c := \mathbb{C}^{n-1} \times \mathbb{D}, \tag{3.93}$$

where $\mathbb{D}$ stands for the open unit disk, i.e. $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, and equip it with the symplectic form

$$\hat{\omega}_c = \mathrm{i} \sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j + \frac{\mathrm{i} dz_n \wedge d\bar{z}_n}{1 - z_n \bar{z}_n}. \tag{3.94}$$

The subscript c refers to 'complex variables'. Define the surjective map

$$\hat{\mathcal{Z}}_x \colon \bar{\mathcal{C}}_x \times \mathbb{T}_n \to \hat{M}_c, \quad (\hat{p}, e^{\mathrm{i}\hat{q}}) \mapsto z(\hat{p}, e^{\mathrm{i}\hat{q}}) \tag{3.95}$$

by the formulae

$$
\begin{aligned}
z_j(\hat{p}, e^{\mathrm{i}\hat{q}}) &= (\hat{p}_j - \hat{p}_{j+1} - |x|/2)^{\frac{1}{2}} \prod_{k=j+1}^{n} e^{\mathrm{i}\hat{q}_k}, \quad j = 1, \dots, n-1, \\
z_n(\hat{p}, e^{\mathrm{i}\hat{q}}) &= (1 - e^{\hat{p}_1})^{\frac{1}{2}} \prod_{k=1}^{n} e^{\mathrm{i}\hat{q}_k}.
\end{aligned}
\tag{3.96}
$$

Notice that the restriction $\mathcal{Z}_x$ of $\hat{\mathcal{Z}}_x$ to $\mathcal{C}_x \times \mathbb{T}_n$ is a diffeomorphism onto the dense open submanifold

$$
\hat{M}_c^o = \{z \in \hat{M}_c \mid \prod_{j=1}^{n} z_j \neq 0\}. \tag{3.97}
$$

It verifies

$$
\mathcal{Z}_x^*(\hat{\omega}_c) = \hat{\omega} = \sum_{j=1}^{n} d\hat{q}_j \wedge d\hat{p}_j, \tag{3.98}
$$

which means that $\mathcal{Z}_x$ is a symplectic embedding of $(\mathcal{C}_x \times \mathbb{T}_n, \hat{\omega})$ into $(\hat{M}_c, \hat{\omega}_c)$. The inverse $\mathcal{Z}_x^{-1} \colon \hat{M}_c^o \to \mathcal{C}_x \times \mathbb{T}_n$ operates according to

$$
\begin{gathered}
\hat{p}_1(z) = \log(1 - |z_n|^2), \quad \hat{p}_j(z) = \log(1 - |z_n|^2) - \sum_{k=1}^{j-1} (|z_k|^2 + |x|/2) \quad (j = 2, \dots, n) \\
e^{\mathrm{i}\hat{q}_1}(z) = \frac{z_n \bar{z}_1}{|z_n \bar{z}_1|}, \quad e^{\mathrm{i}\hat{q}_m}(z) = \frac{z_{m-1} \bar{z}_m}{|z_{m-1} \bar{z}_m|} \quad (m = 2, \dots, n-1), \quad e^{\mathrm{i}\hat{q}_n}(z) = \frac{z_{n-1}}{|z_{n-1}|}.
\end{gathered}
\tag{3.99}
$$

It is important to remark that the $\hat{p}_k(z)$ $(k = 1, \dots, n)$ given above yield smooth functions on the whole of $\hat{M}_c$, while the angles $\hat{q}_k$ are of course not well-defined on the complementary locus of $\hat{M}_c^o$. Our construction of the global cross-section will rely on the building blocks collected in the following long definition.

Definition 3.12. For any $(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ consider the smooth functions

$$
Q_{jk}(x, z) = \left[\frac{\sinh(\sum_{\ell=j}^{k-1} z_\ell \bar{z}_\ell + (k-j)|x|/2 - x/2)}{\sinh(\sum_{\ell=j}^{k-1} z_\ell \bar{z}_\ell + (k-j)|x|/2)} \right]^{\frac{1}{2}}, \quad 1 \leq j < k \leq n, \tag{3.100}
$$

and set $Q_{jk}(x, z) := Q_{kj}(-x, z)$ for $j > k$. Applying these as well as the real analytic function

$$
J(y) := \sqrt{\frac{\sinh(y)}{y}}, \quad y \neq 0, \quad J(0) := 1, \tag{3.101}
$$

and recalling (3.61), introduce the $n \times n$ matrix $\hat{\zeta}(x,z)$ by the formulae

$$\begin{aligned}
\hat{\zeta}(x,z)_{aa} &= r(x,\hat{p}(z))_a, \quad \hat{\zeta}(x,z)_{aj} = -\overline{\hat{\zeta}(x,z)_{ja}}, \quad j \neq a,\\
\hat{\zeta}(x,z)_{jn} &= \sqrt{\frac{\sinh(\frac{x}{2})}{\sinh(\frac{nx}{2})}\frac{z_j J(z_j\bar{z}_j)}{\sinh(z_j\bar{z}_j + \frac{x}{2})}} \prod_{\substack{\ell=1\\(\ell\neq j,j+1)}}^{n} Q_{j\ell}(x,z), \quad x>0, \quad j\neq n,\\
\hat{\zeta}(x,z)_{j1} &= \sqrt{\frac{\sinh(\frac{x}{2})}{\sinh(\frac{nx}{2})}\frac{\bar{z}_{j-1} J(z_{j-1}\bar{z}_{j-1})}{\sinh(z_{j-1}\bar{z}_{j-1} - \frac{x}{2})}} \prod_{\substack{\ell=1\\(\ell\neq j-1,j)}}^{n} Q_{j\ell}(x,z), \quad x<0, \quad j\neq 1,\\
\hat{\zeta}(x,z)_{jk} &= \delta_{j,k} + \frac{\hat{\zeta}(x,z)_{ja}\hat{\zeta}(x,z)_{ak}}{1+\hat{\zeta}(x,z)_{aa}}, \quad j,k\neq a,
\end{aligned} \tag{3.102}$$

where $a=n$ if $x>0$ and $a=1$ if $x<0$. Then introduce the matrix $\hat{\theta}(x,z)$ for $x>0$ as

$$\begin{aligned}
\hat{\theta}(x,z)_{jk} &= \frac{\sinh(\frac{nx}{2})\operatorname{sgn}(k-j-1)\hat{\zeta}(x,z)_{jn}\hat{\zeta}(-x,\bar{z})_{1k}}{\sinh(\sum_{\ell=\min(j,k)}^{\max(j,k)-1} z_\ell\bar{z}_\ell + |k-j-1|\frac{x}{2})}, \quad k\neq j+1,\\
\hat{\theta}(x,z)_{j,j+1} &= \frac{-\sinh(\frac{x}{2})}{\sinh(z_j\bar{z}_j + \frac{x}{2})} \prod_{\substack{\ell=1\\(\ell\neq j,j+1)}}^{n} Q_{j\ell}(x,z)Q_{j+1,\ell}(-x,z),
\end{aligned} \tag{3.103}$$

and for $x<0$ as

$$\hat{\theta}(x,z) = \hat{\theta}(-x,\bar{z})^\dagger. \tag{3.104}$$

Finally, for any $z \in \hat{M}_c$ define the matrix $\hat{\gamma}(x,z) = \operatorname{diag}(\hat{\gamma}_1,\dots,\hat{\gamma}_n)$ with

$$\hat{\gamma}(z)_1 = z_n\sqrt{2 - z_n\bar{z}_n}, \quad \hat{\gamma}(x,z)_j = \left[1-(1-z_n\bar{z}_n)^2 e^{-2\sum_{\ell=1}^{j-1}(z_\ell\bar{z}_\ell + |x|/2)}\right]^{\frac{1}{2}}, \quad j=2,\dots,n, \tag{3.105}$$

and the matrix

$$\hat{\alpha}(x,u,v,z) = -\mathrm{i}\big[\sqrt{e^{-2u}e^{-2\hat{p}(z)} - e^{-2v}\mathbf{1}_n}\,\hat{\theta}(-x,\bar{z}) - e^{v}e^{-\hat{p}(z)}\hat{\gamma}(x,z)^\dagger\big], \tag{3.106}$$

using the constants $x, u=(b-a)/2, v=(a+b)/2$ subject to (3.3).

Although the variable z_n appears only in $\hat{\gamma}_1$, we can regard all objects defined above as smooth functions on $\hat{M}_c$, and we shall do so below.

The key properties of the matrices $\hat{\zeta}$, $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\gamma}$ are given by the following lemma, which can be verified by straightforward inspection. The role of these identities and their origin will be enlightened by Theorem 3.14.

Lemma 3.13. *Prepare the notations*

$$\tau_{(x)} := \mathrm{diag}(\tau_2, \dots, \tau_n, 1) \quad \text{if} \quad x > 0 \quad \text{and} \quad \tau_{(x)} := \mathrm{diag}(1, \tau_2^{-1}, \dots, \tau_n^{-1}) \quad \text{if} \quad x < 0, \tag{3.107}$$

$$\tilde{\tau}_{(x)} := \mathrm{diag}(1, \tau_2, \dots, \tau_n) \quad \text{if} \quad x > 0 \quad \text{and} \quad \tilde{\tau}_{(x)} := \mathrm{diag}(\tau_2^{-1}, \dots, \tau_n^{-1}, 1) \quad \text{if} \quad x < 0 \tag{3.108}$$

with

$$\tau_j = \prod_{k=j}^{n} e^{\mathrm{i}\hat{q}_k}. \tag{3.109}$$

Then the following identities hold for all $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n$*:*

$$\hat{\zeta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \tau_{(x)}\zeta(x, \hat{p})\tau_{(x)}^{-1}, \tag{3.110}$$

$$\hat{\theta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \tau_{(x)}\theta(x, \hat{p})\tilde{\tau}_{(x)}^{-1}, \tag{3.111}$$

$$\hat{\gamma}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = e^{\mathrm{i}\hat{q}}\tau_{(x)}\tilde{\tau}_{(x)}^{-1}\sqrt{\mathbf{1}_n - e^{2\hat{p}}}, \tag{3.112}$$

$$\hat{\alpha}(x, u, v, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = e^{-\mathrm{i}\hat{q}}\tilde{\tau}_{(x)}\alpha(x, u, v, \hat{p}, e^{\mathrm{i}\hat{q}})\tau_{(x)}^{-1}. \tag{3.113}$$

Here we use Definition 3.12 and the functions on $\bar{\mathcal{C}}_x \times \mathbb{T}_n$ *introduced in Subsection 3.2.2.*

For the verification of the above identities, we remark that the vector r (3.61) can be expressed as a smooth function of the complex variables as

$$r(x, \hat{p}(z))_j = \sqrt{\frac{\sinh(\frac{x}{2})}{\sinh(\frac{nx}{2})}} \prod_{\substack{k=1\\(k\neq j)}}^{n} Q_{jk}(x, z), \quad j = 1, \dots, n. \tag{3.114}$$

With all necessary preparations now done, we state the main new result of the chapter.

Theorem 3.14. *The image of the smooth map* $\hat{K}\colon \hat{M}_c \to \mathrm{SL}(2n, \mathbb{C})$ *given by the formula*

$$\hat{K}(z) = \begin{bmatrix} \kappa(x)\hat{\zeta}(x,z)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \hat{\gamma}(x,z) & \mathrm{i}e^{\hat{p}(z)} \\ \mathrm{i}e^{\hat{p}(z)} & \hat{\gamma}(x,z)^\dagger \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \hat{\alpha}(x,u,v,z) \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix} \tag{3.115}$$

lies in $\Phi_+^{-1}(\mu)$*, intersects every gauge orbit in precisely one point, and* $\hat{K}$ *is injective. The pull-back of the Alekseev-Malkin* 2*-form* ω (3.16) *by* $\hat{K}$ *is* $\hat{\omega}_c$ (3.94)*. Consequently,* $\pi_\mu \circ \hat{K}\colon \hat{M}_c \to M$ *is a symplectomorphism, whereby* $(\hat{M}_c, \hat{\omega}_c)$ *provides a model of the reduced phase space* (M, ω_M) *defined in Subsection 3.1.2.*

Proof. The proof is based upon the identity

$$\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \begin{bmatrix} \kappa(x)\tau_{(x)}\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} \end{bmatrix} K(\hat{p}, e^{\mathrm{i}\hat{q}}) \begin{bmatrix} \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} & \mathbf{0}_n \\ \mathbf{0}_n & \tau_{(x)} \end{bmatrix}^{-1}, \quad \forall(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n, \tag{3.116}$$

which is readily seen to be equivalent to the set of identities displayed in Lemma 3.13. It means that $\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}}))$ is a gauge transform of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ in (3.67). Indeed, the above transformation of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ has the form (3.31) with

$$\eta_L = c\begin{bmatrix} \kappa(x)\tau_{(x)}\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} \end{bmatrix}, \qquad \eta_R = c\begin{bmatrix} \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} & \mathbf{0}_n \\ \mathbf{0}_n & \tau_{(x)} \end{bmatrix}, \tag{3.117}$$

where c is a harmless scalar inserted to ensure $\det(\eta_L) = \det(\eta_R) = 1$. Using (3.65) and (3.107), one can check that $\kappa(x)\tau_{(x)}\kappa(x)^{-1}\hat{v}(x) = \hat{v}(x)$ for the vector $\hat{v}(x)$ in (3.64), which implies via the relation (3.55) that (η_L, η_R) belongs to the isotropy group G_μ (3.36), the gauge group acting on $\Phi_+^{-1}(\mu)$.

It follows from Proposition 3.3 and the identity (3.116) that the set

$$\hat{S} := \{\hat{K}(z) \mid z \in \hat{M}_c\} \tag{3.118}$$

lies in $\Phi_+^{-1}(\mu)$ and intersects every gauge orbit. Since the dense subset

$$\hat{S}^o := \{\hat{K}(z) \mid z \in \hat{M}_c^o\} \tag{3.119}$$

is gauge equivalent to S^o in (3.79), we obtain the equality

$$\hat{K}^*(\omega) = \hat{\omega}_c \tag{3.120}$$

by using Theorem 3.11 and equation (3.98). More precisely, we here also utilized that $\hat{K}^*(\omega)$ is (obviously) smooth and $\hat{M}_c^o$ is dense in $\hat{M}_c$.

The only statements that remain to be proved are that the intersection of $\hat{S}$ with any gauge orbit consists of a single point and that $\hat{K}$ is injective. (These are already clear for $\hat{S}^o \subset \hat{S}$ and for $\hat{K}|_{\hat{M}_c^o}$.) Now suppose that

$$\hat{K}(z') = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \hat{K}(z) \begin{bmatrix} \eta_R(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2) \end{bmatrix}^{-1} \tag{3.121}$$

for some gauge transformation and $z, z' \in \hat{M}_c$. Let us observe from the definitions that we can write

$$\begin{bmatrix} \hat{\gamma}(x,z) & \mathrm{i}e^{\hat{p}(z)} \\ \mathrm{i}e^{\hat{p}(z)} & \hat{\gamma}(x,z)^\dagger \end{bmatrix} = D(z) \begin{bmatrix} \cos q(z) & \mathrm{i}\sin q(z) \\ \mathrm{i}\sin q(z) & \cos q(z) \end{bmatrix} D(z), \tag{3.122}$$

where $\sin q(z) = e^{\hat{p}(z)}$, with $\frac{\pi}{2} \geq q_1 > \cdots > q_n > 0$, and $D(z)$ is a diagonal unitary matrix of the form $D(z) = \mathrm{diag}(d_1, \mathbf{1}_{n-1}, \bar{d}_1, \mathbf{1}_{n-1})$. Then the uniqueness properties of the Iwasawa decomposition of $\mathrm{SL}(2n,\mathbb{C})$ and the generalized Cartan decomposition (3.41) of $\mathrm{SU}(2n)$ allow to establish the following consequences of (3.121). First,

$$\hat{p}(z) = \hat{p}(z'). \tag{3.123}$$

Second, using Lemma 3.6,

$$\begin{bmatrix} \eta_R(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2) \end{bmatrix} = \begin{bmatrix} m_2 & \mathbf{0}_n \\ \mathbf{0}_n & m_1 \end{bmatrix} \tag{3.124}$$

for some diagonal unitary matrices of the form (3.73). Third, we have

$$\hat{\alpha}(z') = \eta_R(1)\hat{\alpha}(z)\eta_R(2)^{-1} = m_2\hat{\alpha}(z)m_1^{-1}. \tag{3.125}$$

For definiteness, let us focus on the case $x > 0$. Then we see from the definitions that the components $\hat{\alpha}_{k+1,k}$ and $\hat{\alpha}_{1,n}$ depend only on $\hat{p}(z)$ and are non-zero. By using this, we find from (3.125) that $m_1 = m_2 = C\mathbf{1}_n$ with a scalar C, and therefore

$$\hat{\alpha}(z') = \hat{\alpha}(z). \tag{3.126}$$

Inspection of the components $(1,2),\dots,(1,n-1)$ of this matrix equality and (3.123) permit to conclude that $z_2' = z_2, \dots, z_{n-1}' = z_{n-1}$, respectively. Then, the equality of the $(2,n)$ entries in (3.126) gives $z_1' = z_1$ which used in the $(1,1)$ position implies $z_n' = z_n$. Thus we see that $z' = z$ and the proof is complete. (Everything written below (3.125) is quite similar for $x < 0$.) □

Remark 3.15. Let us hint at the way the global structure was found. The first point to notice was that all or some of the phases $e^{\mathrm{i}\hat{q}_j}$ cannot encode gauge invariant quantities if $\hat{p}$ belongs to the boundary of $\bar{\mathcal{C}}_x$, as was already mentioned in Remark 3.5. Motivated by [37], then we searched for complex variables by requiring that a suitable gauge transform of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ in (3.67) should be expressible as a smooth function of those variables. Given the similarities to [37], only the definition of z_n was a true open question. After trial and error, the idea came in a flash that the gauge transformation at issue should be constructed from a transformation that appears in Lemma C.2. Then it was not difficult to find the correct result.

Remark 3.16. Let us elaborate on how the trajectories $\hat{p}(t)$ corresponding to the flows of the reduced free Hamiltonians, arising from $\mathcal{H}_k$ (3.90) for $k = 1, \dots, n$, can be obtained. Recall that for $k = 1$ the reduction of $\mathcal{H}_1$ completes the main Hamiltonian H (3.1). Since $\mathcal{H}_k(K) = h_k(b_R)$ with $h_k(b) = \frac{1}{2k}\mathrm{tr}(b^\dagger b)^k$, the free flow generated by $\mathcal{H}_k$ through the initial value $K(0) = g_L(0)b_R^{-1}(0)$ is given by (3.21) with $d^R h_k(b) = \mathrm{i}(b^\dagger b)^k$. Thus the curve $g_L(t)$ (3.21) has the form

$$g_L(t) = g_L(0)\exp\big(-\mathrm{i}t\big[\mathcal{L}(0)^k - \frac{1}{2n}\mathrm{tr}(\mathcal{L}(0)^k)\mathbf{1}_{2n}\big]\big) \quad \text{with} \quad \mathcal{L}(0) = b_R(0)^\dagger b_R(0). \tag{3.127}$$

The reduced flow results by the usual projection algorithm. This starts by picking an initial value $z(0) \in \hat{M}_c$ and setting $K(0) = \hat{K}(z(0))$ by applying (3.115), which directly determines $g_L(0)$ and $b_R(0)$ as well. Then the map $\mathcal{P}$ (3.84) gives rise to $\hat{p}(t)$ via the decomposition of $g_L(t) \in \mathrm{SU}(2n)$ as displayed in (3.41), that is

$$\hat{p}(t) = \mathcal{P}(K(t)). \tag{3.128}$$

More explicitly, if $\mathcal{D}(t)$ stands for the (11) block of $g_L(t)$, then the eigenvalues of $\mathcal{D}(t)\mathcal{D}(t)^\dagger$ are

$$\sigma(\mathcal{D}(t)\mathcal{D}(t)^\dagger) = \{\cos^2 q_j(t) \mid j = 1, \dots, n\}, \tag{3.129}$$

from which $\hat{p}_j(t)$ can be obtained using (3.53). In particular, the 'particle positions' evolve according to an 'eigenvalue dynamics' similarly to other many-body systems. This involves the one-parameter group $e^{-\mathrm{i}t\mathcal{L}(0)^k}$, where $\mathcal{L}(0)$ is the initial value of the Lax matrix (cf. (3.91))

$$\mathcal{L}(z) = \begin{bmatrix} e^{2v}\mathbf{1}_n & -e^v\hat{\alpha}(z) \\ -e^v\hat{\alpha}(z)^\dagger & (e^{-2v}\mathbf{1}_n + \hat{\alpha}(z)^\dagger\hat{\alpha}(z)) \end{bmatrix}, \tag{3.130}$$

where we suppressed the dependence of $\hat{\alpha}$ (3.106) on the parameters x, u, v. A more detailed characterization of the dynamics will be provided elsewhere.

3.4 Full phase space of the hyperbolic version

In this section, we complete the earlier derivation [86] of the hyperbolic analogue of the Ruijsenaars type system that we studied in the previous sections. This hyperbolic version arises as a reduction of the natural free system on the Heisenberg double of $\mathrm{SU}(n,n)$. The previous analysis by Marshall focused on a dense open submanifold of the reduced phase space, and here we describe the full phase space wherein Liouville integrability of the system holds by construction.

3.4.1 Definitions and first steps

The starting point in this case is the group

$$\mathrm{SU}(n,n) = \{g \in \mathrm{SL}(2n,\mathbb{C}) \mid g^\dagger \boldsymbol{J} g = \boldsymbol{J}\} \tag{3.131}$$

with $\boldsymbol{J} = \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n)$. Then consider the open submanifold $\mathrm{SL}(2n,\mathbb{C})' \subset \mathrm{SL}(2n,\mathbb{C})$ consisting of those elements, K, that admit both Iwasawa-like decompositions of the form

$$K = g_L b_R^{-1} = b_L g_R^{-1}, \qquad g_L, g_R \in \mathrm{SU}(n,n), \quad b_L, b_R \in \mathrm{SB}(2n), \tag{3.132}$$

where $\mathrm{SB}(2n) < \mathrm{SL}(2n,\mathbb{C})$ is the group of upper triangular matrices having positive entries along the diagonal. Both decompositions are unique and the constituent factors depend smoothly on $K \in \mathrm{SL}(2n,\mathbb{C})'$. The manifold $\mathrm{SL}(2n,\mathbb{C})'$ inherits a symplectic form ω [3], which has the same form as the one seen in (3.16). On this symplectic manifold $(\mathrm{SL}(2n,\mathbb{C})', \omega)$, which is a symplectic leaf of the Heisenberg double of the Poisson-Lie group $\mathrm{SU}(n,n)$, one has the pairwise Poisson commuting Hamiltonians

$$\mathcal{H}_j(K) = \frac{1}{2j}\mathrm{tr}(K\boldsymbol{J}K^\dagger \boldsymbol{J})^j, \qquad j \in \mathbb{Z}^*. \tag{3.133}$$

They generate complete flows that can be written down explicitly (see Section 3.5). We are concerned with a reduction of these Hamiltonians based on the symmetry group $G_+ \times G_+$, where G_+ is the block-diagonal subgroup (3.26). Throughout, we refer to the obvious 2×2 block-matrix structure corresponding to $\boldsymbol{J}$. The action of $G_+ \times G_+$ on $\mathrm{SL}(2n,\mathbb{C})'$ is given by the map

$$G_+ \times G_+ \times \mathrm{SL}(2n,\mathbb{C})' \to \mathrm{SL}(2n,\mathbb{C})' \tag{3.134}$$

that works according to

$$(\eta_L, \eta_R, K) \mapsto \eta_L K \eta_R^{-1}. \tag{3.135}$$

One can check that this map is well-defined, i.e. $\eta_L K \eta_R^{-1}$ stays in $\mathrm{SL}(2n,\mathbb{C})'$, and has the Poisson property with respect to the product Poisson structure on the left-hand side [126, 3], where on G_+ the standard Sklyanin bracket is used and the Poisson structure on $\mathrm{SL}(2n,\mathbb{C})'$ is engendered by ω. Moreover, this $G_+ \times G_+$ action is associated with a momentum map in the sense of Lu [80]. The momentum map

$$\Phi_+ \colon \mathrm{SL}(2n,\mathbb{C})' \to G_+^* \times G_+^* \tag{3.136}$$

works the same way as (3.30) did. Namely, it takes an element $K \in \mathrm{SL}(2n, \mathbb{C})'$ (3.132) and maps it to a pair of matrices obtained from (b_L, b_R) by replacing the off-diagonal blocks by null matrices. The Hamiltonians $\mathcal{H}_j$ (3.133) are invariant with respect to the symmetry group $G_+ \times G_+$ and Φ_+ is constant along their flows.

The general theory [80] ensures that one can now perform Marsden-Weinstein type reduction. This amounts to imposing the constraint $\Phi_+(K) = \mu = (\mu_L, \mu_R)$ with some constant $\mu \in G_+^* \times G_+^*$ and then taking the quotient of $\Phi_+^{-1}(\mu)$ by the corresponding isotropy group, denoted below as G_μ.

We pick the value of the momentum map to be $\mu = (\mu_L, \mu_R)$ (3.33). For simplicity, we now assume that the parameter x in (3.33) is positive. The corresponding isotropy group is G_μ (3.36). It will turn out that the reduced phase space

$$M = \Phi_+^{-1}(\mu)/G_\mu \tag{3.137}$$

is a smooth manifold. Our task is to characterize this manifold, which carries the reduced symplectic form ω_M defined by the relation

$$\iota_\mu^* \omega = \pi_\mu^* \omega_M, \tag{3.138}$$

where $\iota_\mu \colon \Phi_+^{-1}(\mu) \to \mathrm{SL}(2n, \mathbb{C})'$ is the tautological injection and $\pi_\mu \colon \Phi_+^{-1}(\mu) \to M$ is the natural projection.

Consider the following central subgroup $\mathbb{Z}_{2n}$ of $G_+ \times G_+$,

$$\mathbb{Z}_{2n} = \{(w\mathbf{1}_{2n}, w\mathbf{1}_{2n}) \mid w \in \mathbb{C},\ w^{2n} = 1\}, \tag{3.139}$$

which acts trivially according to (3.135) and is contained in G_μ. Later we shall refer to the factor group

$$\bar{G}_\mu = G_\mu/\mathbb{Z}_{2n} \tag{3.140}$$

as the 'effective gauge group'. Obviously, we have $\Phi_+^{-1}(\mu)/G_\mu = \Phi_+^{-1}(\mu)/\bar{G}_\mu$.

Our aim is to obtain a model of the quotient space M by explicitly exhibiting a global cross-section of the orbits of G_μ in $\Phi_+^{-1}(\mu)$. The construction uses the generalized Cartan decomposition of $\mathrm{SU}(n, n)$, which says that every $g \in \mathrm{SU}(n, n)$ can be written as

$$g = g_+ \begin{bmatrix} \cosh q & \sinh q \\ \sinh q & \cosh q \end{bmatrix} h_+, \tag{3.141}$$

where $g_+, h_+ \in G_+$ and $q = \mathrm{diag}(q_1, \dots, q_n)$ is a real diagonal matrix verifying

$$q_1 \geq \cdots \geq q_n \geq 0. \tag{3.142}$$

The components q_j are uniquely determined by g, and yield smooth functions on the locus where they are all distinct. In what follows we shall often identify diagonal matrices like q with the corresponding elements of $\mathbb{R}^n$.

As the first step towards describing M, we apply the decomposition (3.141) to g_L in $K = g_L b_R^{-1}$ and impose the right-handed momentum constraint $\pi(b_R) = \mu_R$. It is then easily seen that up to G_μ-transformations every element of $\Phi_+^{-1}(\mu)$ can be represented in the following form:

$$K = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \cosh q & \sinh q \\ \sinh q & \cosh q \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{3.143}$$

Here $\rho \in \mathrm{SU}(n)$ and α is an $n \times n$ complex matrix. Referring to the 2×2 block-matrix notation, we introduce $\Omega = K_{22}$ and record from (3.143) that

$$\Omega = (\sinh q)\alpha + e^v \cosh q. \tag{3.144}$$

Just as in (3.45), it proves to be advantageous to seek for Ω in the polar-decomposed form,

$$\Omega = \Lambda T, \tag{3.145}$$

where $T \in \mathrm{U}(n)$ and Λ is a Hermitian, positive semi-definite matrix.

The next step is to implement the left-handed momentum constraint $\pi(b_L) = \mu_L$ by writing $K = b_L g_R^{-1}$ with

$$b_L = \begin{bmatrix} e^u \nu(x) & \chi \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix}, \tag{3.146}$$

where χ is an unknown $n \times n$ matrix. Then we inspect the components of the 2×2 block-matrix identity

$$K \boldsymbol{J} K^\dagger = b_L \boldsymbol{J} b_L^\dagger, \tag{3.147}$$

which results by substituting K from (3.143). We find that the (22) component of this identity is equivalent to

$$\Omega\Omega^\dagger = \Lambda^2 = e^{-2u}\mathbf{1}_n + e^{-2v}(\sinh q)^2. \tag{3.148}$$

This uniquely determines Λ in terms of q and also shows that Λ is invertible. As in the trigonometric case, the condition $u + v \neq 0$ ensures that

$$q_n > 0, \tag{3.149}$$

and therefore $\sinh q$ is an invertible diagonal matrix.

By using the above relations, it is simple algebra to convert the (12) and the (21) components of the identity (3.147) into the equation

$$\chi = \rho(\sinh q)^{-1}[e^{-u}\cosh q - e^{u+v}\Omega^\dagger]. \tag{3.150}$$

Finally, the (11) entry of the identity (3.147) translates into the following crucial equation (which is the hyperbolic analogue of (3.51)):

$$\rho(\sinh q)^{-1}T^\dagger(\sinh q)^2 T(\sinh q)^{-1}\rho^\dagger = \nu(x)\nu(x)^\dagger. \tag{3.151}$$

This is to be satisfied by q subject to (3.142), (3.149) and $T \in \mathrm{U}(n)$, $\rho \in \mathrm{SU}(n)$. After finding q, T and ρ, one can reconstruct K (3.143) by applying the formulas derived above.

From our viewpoint, a key observation is that (3.151) coincides completely with equation (5.7) in the paper [37], where its general solution was found. The correspondence between the notations used here and in [37] is

$$(\rho, T, \sinh q) \Longleftrightarrow (k_L, k_R^\dagger, e^{\hat{p}}). \tag{3.152}$$

For this reason, we introduce the new variable $\hat{p} \in \mathbb{R}^n$ by the definition

$$\sinh q_k = e^{\hat{p}_k}, \quad k = 1, \dots, n. \tag{3.153}$$

Because of (3.142) and (3.149), the variables $\hat{p}_k$ satisfy

$$\hat{p}_1 \geq \dots \geq \hat{p}_n. \tag{3.154}$$

We do not see an a priori reason why the very different reduction procedures led to the same equation (3.151) here and in [37]. However, we are going to take full advantage of this situation. We note that essentially every formula written in this section appears in [86] as well (with slightly different notations), but in Marshall's work the previously obtained results about the solutions of (3.151) were not used.

3.4.2 The reduced phase space

The statement of Proposition 3.19 characterizes a submanifold of M (3.137), which was erroneously claimed in [86] to be equal to M. After describing this 'local picture', we shall present a globally valid model of M.

The local picture

By applying results of [37, 38] in the same way as we did in the trigonometric case, one can prove the following lemma.

Lemma 3.17. *The constraint surface $\Phi_+^{-1}(\mu)$ contains an element of the form (3.143) if and only if $\hat{p}$ defined by (3.153) lies in the closed polyhedron*

$$\bar{\mathcal{C}}'_x = \{\hat{p} \in \mathbb{R}^n \mid \hat{p}_k - \hat{p}_{k+1} \geq x/2 \ (k = 1, \dots, n-1)\}. \tag{3.155}$$

The polyhedron $\bar{\mathcal{C}}'_x$ is the closure of its interior, $\mathcal{C}'_x$, defined by strict inequalities. We note that in [86] the elements of the boundary $\bar{\mathcal{C}}'_x \setminus \mathcal{C}'_x$ were omitted.

For any fixed $\hat{p} \in \bar{\mathcal{C}}'_x$, one can write down the solutions of (3.151) for T and ρ explicitly [37]. By inserting those into the formula (3.143), using the relations (3.144), (3.145), (3.148) to determine the matrix α, one arrives at the next lemma. It refers to the $n \times n$ real matrices $\theta(x, \hat{p})$, $\zeta(x, \hat{p})$, $\kappa(x)$ displayed in (3.61)-(3.65), which belong to the group $\mathrm{SO}(n)$.

Proposition 3.18. *For any parameters u, v, x subject to $u+v \neq 0$, $x > 0$, and variables $\hat{p} \in \bar{\mathcal{C}}'_x$ and $e^{\mathrm{i}\hat{q}}$ from the n-torus $\mathbb{T}_n$, define the matrix*

$$K(\hat{p}, e^{\mathrm{i}\hat{q}}) = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{1}_n + e^{2\hat{p}}} & e^{\hat{p}} \\ e^{\hat{p}} & \sqrt{\mathbf{1}_n + e^{2\hat{p}}} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix} \tag{3.156}$$

by employing

$$\rho = \rho(x, \hat{p}) = \kappa(x)\zeta(x, \hat{p})^{-1} \tag{3.157}$$

and

$$\alpha = \alpha(x, u, v, \hat{p}, e^{\mathrm{i}\hat{q}}) = e^{\mathrm{i}\hat{q}}\sqrt{e^{-2u}e^{-2\hat{p}} + e^{-2v}\mathbf{1}_n}\,\theta(x, \hat{p})^{-1} - e^{v}\sqrt{e^{-2\hat{p}} + \mathbf{1}_n}. \tag{3.158}$$

Then $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ resides in the constraint surface $\Phi_+^{-1}(\mu)$ and the set

$$S = \{K(\hat{p}, e^{\mathrm{i}\hat{q}}) \mid (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}'_x \times \mathbb{T}_n\} \tag{3.159}$$

intersects every orbit of G_μ in $\Phi_+^{-1}(\mu)$.

By arguing verbatim along the lines of the previous section, and referring to [86] for the calculation of the reduced symplectic form, one can establish the validity of the subsequent proposition.

Proposition 3.19. *The effective gauge group $\bar{G}_\mu$ (3.140) acts freely on $\Phi_+^{-1}(\mu)$ and thus the quotient space M (3.137) is a smooth manifold. The restriction of the natural projection $\pi_\mu \colon \Phi_+^{-1}(\mu) \to M$ to*

$$S^o = \{K(\hat{p}, e^{\mathrm{i}\hat{q}}) \mid (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}'_x \times \mathbb{T}_n\} \tag{3.160}$$

gives rise to a diffeomorphism between $\mathcal{C}'_x \times \mathbb{T}_n$ and the open, dense submanifold of M provided by $\pi_\mu(S^o)$. Taking S^o as model of $\pi_\mu(S^o)$, the corresponding restriction of the *reduced symplectic form ω_M becomes the Darboux form*

$$\omega_{S^o} = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k. \tag{3.161}$$

Remark 3.20. In the formula (3.156) $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ appears in the decomposed form $K = g_L b_R^{-1}$ and it is not immediately obvious that it belongs to $\mathrm{SL}(2n,\mathbb{C})'$, i.e. that it can be decomposed alternatively as $b_L g_R^{-1}$. However, by defining $b_L(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathrm{SB}(2n)$ by the formula (3.146) using χ in (3.150) with the change of variables $\sinh q = e^{\hat{p}}$, the matrix ρ as given above, and $T = e^{\mathrm{i}\hat{q}}\theta(x,\hat{p})^{-1}$ that enters (3.156), we can verify that for these elements $g_R^{-1} = b_L^{-1}K$ satisfies the defining relation of $\mathrm{SU}(n,n)$ (3.131), as required. The reader may perform this verification, which relies only on the constraint equations displayed in Subsection 3.4.1.

The global picture

The train of thought leading to the construction below can be outlined as follows. Proposition 3.19 tells us, in particular, that any G_μ-orbit passing through S^o intersects S^o in a single point. Direct inspection shows that the analogous statement is false for $S \setminus S^o$, which corresponds to $(\bar{\mathcal{C}}_x' \setminus \mathcal{C}_x') \times \mathbb{T}_n$ in a one-to-one manner. Thus a global model of M should result by identifying those points of $S \setminus S^o$ that lie on the same G_μ-orbit. By using the bijective map from $\bar{\mathcal{C}}_x' \times \mathbb{T}_n$ onto S given by the formula (3.156), the desired identification will be achieved by constructing such complex variables out of $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x' \times \mathbb{T}_n$ that coincide precisely for gauge equivalent elements of S.

Turning to the implementation of the above plan, we introduce the space of complex variables

$$\hat{M}_c = \mathbb{C}^{n-1} \times \mathbb{C}^\times, \qquad (\mathbb{C}^\times = \mathbb{C} \setminus \{0\}), \tag{3.162}$$

carrying the symplectic form

$$\hat{\omega}_c = \mathrm{i}\sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j + \frac{\mathrm{i} dz_n \wedge d\bar{z}_n}{2 z_n \bar{z}_n}. \tag{3.163}$$

We also define the surjective map

$$\hat{\mathcal{Z}}_x \colon \bar{\mathcal{C}}_x' \times \mathbb{T}_n \to \hat{M}_c, \quad (\hat{p}, e^{\mathrm{i}\hat{q}}) \mapsto z(\hat{p}, e^{\mathrm{i}\hat{q}}) \tag{3.164}$$

by setting

$$
\begin{aligned}
z_j(\hat{p}, e^{\mathrm{i}\hat{q}}) &= (\hat{p}_j - \hat{p}_{j+1} - x/2)^{\frac{1}{2}} \prod_{k=j+1}^{n} e^{\mathrm{i}\hat{q}_k}, \quad j = 1, \dots, n-1, \\
z_n(\hat{p}, e^{\mathrm{i}\hat{q}}) &= e^{-\hat{p}_1} \prod_{k=1}^{n} e^{\mathrm{i}\hat{q}_k}.
\end{aligned}
\tag{3.165}
$$

The restriction $\mathcal{Z}_x$ of $\hat{\mathcal{Z}}_x$ to $\mathcal{C}'_x \times \mathbb{T}_n$ is a diffeomorphism onto the open subset

$$
\hat{M}_c^o = \left\{ z \in \hat{M}_c \,\middle|\, \prod_{j=1}^{n-1} z_j \neq 0 \right\}, \tag{3.166}
$$

and it verifies the relation

$$
\mathcal{Z}_x^* \hat{\omega}_c = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k. \tag{3.167}
$$

Thus we manufactured a change of variables $\mathcal{C}'_x \times \mathbb{T}_n \longleftrightarrow \hat{M}_c^o$. The inverse $\mathcal{Z}_x^{-1} \colon \hat{M}_c^o \to \mathcal{C}'_x \times \mathbb{T}_n$ involves the functions

$$
\hat{p}_1(z) = -\log|z_n|, \quad \hat{p}_j(z) = -\log|z_n| - \sum_{k=1}^{j-1} (|z_k|^2 + x/2) \quad (j = 2, \dots, n). \tag{3.168}
$$

These extend smoothly to $\hat{M}_c$ wherein $\hat{M}_c^o$ sits as a dense submanifold.

Now we state a lemma, which is a simple adaptation from [37, 117].

Lemma 3.21. *By using the shorthand $\sigma_j = \prod_{k=j+1}^{n} e^{\mathrm{i}\hat{q}_k}$ for $j = 1, \dots, n-1$ (cf. (3.165)), let us define*

$$
\sigma_+(e^{\mathrm{i}\hat{q}}) = \operatorname{diag}(\sigma_1, \dots, \sigma_{n-1}, 1) \quad \text{and} \quad \sigma_-(e^{\mathrm{i}\hat{q}}) = \operatorname{diag}(1, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}). \tag{3.169}
$$

Then there exist unique smooth functions $\hat{\zeta}(x, z)$, $\hat{\theta}(x, z)$ and $\hat{\alpha}(x, u, v, z)$ of $z \in \hat{M}_c$ that satisfy the following identities for any $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}'_x \times \mathbb{T}_n$:

$$
\hat{\zeta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \sigma_+(e^{\mathrm{i}\hat{q}}) \zeta(x, \hat{p}) \sigma_+(e^{\mathrm{i}\hat{q}})^{-1}, \tag{3.170}
$$

$$
\hat{\theta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \sigma_+(e^{\mathrm{i}\hat{q}}) \theta(x, \hat{p}) \sigma_-(e^{\mathrm{i}\hat{q}}), \tag{3.171}
$$

$$
\hat{\alpha}(x, u, v, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \sigma_+(e^{\mathrm{i}\hat{q}}) \alpha(x, u, v, \hat{p}, e^{\mathrm{i}\hat{q}}) \sigma_+(e^{\mathrm{i}\hat{q}})^{-1}. \tag{3.172}
$$

Here we refer to the functions on $\bar{\mathcal{C}}'_x \times \mathbb{T}_n$ displayed in equations (3.61)-(3.65), and (3.158).

The explicit formulas of the functions on $\hat{M}_c$ that appear in the above identities are easily found by first determining them on $\hat{M}_c^o$ using the change of variables $\mathcal{Z}_x$, and then noticing that they automatically extend to $\hat{M}_c$. The expressions of the functions

$\hat{\zeta}$ and $\hat{\theta}$, which depend only on $z_1, \dots, z_{n-1}$, are the same as given in [37, Definition 3.3]. (For most purposes the above definitions and the formulas (3.61)-(3.65) suffice.) As for $\hat{\alpha}$, by defining

$$\Delta(z) = \operatorname{diag}(z_n, e^{-\hat{p}_2(z)}, \dots, e^{-\hat{p}_n(z)}) \tag{3.173}$$

we have

$$\hat{\alpha}(x,u,v,z) = \sqrt{e^{-2v}e^{2\hat{p}(z)} + e^{-2u}\mathbf{1}_n}\,\Delta(z)\hat{\theta}(x,z)^{-1} - e^{v}\sqrt{e^{-2\hat{p}(z)} + \mathbf{1}_n} \tag{3.174}$$

that satisfies relation (3.172) due to the identity

$$\Delta(z(\hat{p}, e^{\mathrm{i}\hat{q}})) = e^{-\hat{p}}e^{\mathrm{i}\hat{q}}\sigma_+(e^{\mathrm{i}\hat{q}})\sigma_-(e^{\mathrm{i}\hat{q}}), \qquad \forall (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x' \times \mathbb{T}_n. \tag{3.175}$$

With these preparations at hand, we can formulate the main result of this section.

Theorem 3.22. *Define the smooth map $\hat{K}\colon \hat{M}_c \to \mathrm{SL}(2n,\mathbb{C})'$ by the formula*

$$\hat{K}(z) = \begin{bmatrix} \kappa(x)\hat{\zeta}(x,z)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{1}_n + e^{2\hat{p}(z)}} & e^{\hat{p}(z)} \\ e^{\hat{p}(z)} & \sqrt{\mathbf{1}_n + e^{2\hat{p}(z)}} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \hat{\alpha}(x,u,v,z) \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{3.176}$$

The image of $\hat{K}$ belongs to the submanifold $\Phi_+^{-1}(\mu)$ and the induced mapping $\pi_\mu \circ \hat{K}$, obtained by using the natural projection $\pi_\mu\colon \Phi_+^{-1}(\mu) \to M = \Phi^{-1}(\mu)/G_\mu$, is a symplectomorphism between $(\hat{M}_c, \hat{\omega}_c)$, defined by (3.162), (3.163), and the reduced phase space (M, ω_M).

Proof. We start by pointing out that for any $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x' \times \mathbb{T}_n$ the identity

$$\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \begin{bmatrix} \kappa(x)\sigma_+(e^{\mathrm{i}\hat{q}})\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix} K(\hat{p}, e^{\mathrm{i}\hat{q}}) \begin{bmatrix} \sigma_+(e^{\mathrm{i}\hat{q}}) & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix}^{-1} \tag{3.177}$$

is equivalent to the identities listed in Lemma 3.21. We see from this that $\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}}))$ is a G_μ-transform of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ (3.156), and thus $\hat{K}(z)$ belongs to $\Phi_+^{-1}(\mu)$. Indeed, the right-hand side of (3.177) can be written as $\eta_L K(\hat{p}, e^{\mathrm{i}\hat{q}})\eta_R^{-1}$ with

$$\eta_L = c\begin{bmatrix} \kappa(x)\sigma_+(e^{\mathrm{i}\hat{q}})\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix}, \qquad \eta_R = c\begin{bmatrix} \sigma_+(e^{\mathrm{i}\hat{q}}) & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix}, \tag{3.178}$$

where c is a scalar ensuring $\det(\eta_L) = \det(\eta_R) = 1$, and one can check that this (η_L, η_R) lies in the isotropy group G_μ. Indeed, both $\kappa(x)$ and $\zeta(x, \hat{p})$ are orthogonal matrices of determinant 1. The main feature of $\kappa(x)$ is that the matrix $\kappa(x)^{-1}\nu(x)\nu(x)^\dagger\kappa(x)$ (with $\nu(x)$ in (3.34)) is diagonal. This implies that $\eta_L(1) = \kappa(x)\tau\kappa(x)^{-1} \in \mathrm{U}(n)$ satisfies (3.38) for any $\tau \in \mathbb{T}_n$.

To proceed further, we let $\hat{K}_o$ denote the restriction of $\hat{K}$ to the dense open subset

$\hat{M}_c^o$ and also let $K_o\colon \mathcal{C}'_x \times \mathbb{T}_n \to \mathrm{SL}(2n,\mathbb{C})'$ denote the map defined by the corresponding restriction of the formula (3.156). Notice that, in addition to (3.138), we have the relations

$$\pi_\mu \circ \hat{K}_0 = \pi_\mu \circ K_o \circ \mathcal{Z}_x^{-1} \quad \text{and} \quad (\pi_\mu \circ K_o)^* \omega_M = \sum_{k=1}^n d\hat{q}_k \wedge d\hat{p}_k, \tag{3.179}$$

which follow from (3.177) and the last sentence of Proposition 3.19. By using (3.167) (together with $\hat{K}_o = \iota_\mu \circ \hat{K}_o$ and $K_o = \iota_\mu \circ K_o$) the above relations imply the restriction of the equality

$$(\pi_\mu \circ \hat{K})^* \omega_M = \hat{\omega}_c \tag{3.180}$$

on $\hat{M}_c^o$. This equality is then valid on the full $\hat{M}_c$ since the 2-forms concerned are smooth.

It is a direct consequence of (3.177) and Proposition 3.18 that $\pi_\mu \circ \hat{K}$ is surjective. Since, on account of (3.180), it is a local diffeomorphism, it only remains to demonstrate that the map $\pi_\mu \circ \hat{K}$ is injective. The relation $\pi_\mu(\hat{K}(z)) = \pi_\mu(\hat{K}(z'))$ for $z, z' \in \hat{M}_c$ requires that

$$\hat{K}(z') = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \hat{K}(z) \begin{bmatrix} \eta_R(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2) \end{bmatrix}^{-1} \tag{3.181}$$

for some $(\eta_L, \eta_R) \in G_\mu$. Supposing that (3.181) holds, application of the decomposition $\hat{K}(z) = g_L(z) b_R(z)^{-1}$ to the formula (3.176) implies that

$$\hat{\alpha}(z') = \eta_R(1) \hat{\alpha}(z) \eta_R(2)^{-1} \tag{3.182}$$

and

$$g_L(z') = \eta_L g_L(z) \eta_R^{-1}. \tag{3.183}$$

The matrices on the two sides of (3.183) appear in the form (3.141), and standard uniqueness properties of the constituents in this generalized Cartan decomposition now imply that

$$\hat{p}(z') = \hat{p}(z) \tag{3.184}$$

and

$$\eta_R(1) = \eta_R(2) = m \in \mathbb{T}_n. \tag{3.185}$$

We continue by looking at the $(k+1,k)$ components of the equality (3.182) for $k = 1, \dots, n-1$ using that $\hat{\alpha}_{k+1,k}$ depends on z only through $\hat{p}(z)$ and it never vanishes. (This follows from (3.173)-(3.174) by utilizing that $\hat{\theta}(x,z)_{k,k+1} = \theta(x, \hat{p}(z))_{k,k+1}$ by (3.171), which is nonzero for each $\hat{p}(z) \in \bar{\mathcal{C}}'_x$ as seen from (3.59).) Putting (3.185) into

(3.182), we obtain that $m = C\mathbf{1}_n$ with a scalar C, and therefore

$$\hat{\alpha}(z') = \hat{\alpha}(z). \tag{3.186}$$

The rest is an inspection of this matrix equality. In view of (3.184) and the forms of $\Delta(z)$ (3.173) and $\hat{\alpha}(z)$ (3.174), the last column of the equality (3.186) entails that

$$\hat{\theta}(x,z)_{nk} = \hat{\theta}(x,z')_{nk}, \quad k = 2,\dots,n, \tag{3.187}$$

where we re-instated the dependence on x that was suppressed above. One can check directly from the formulas (3.165), (3.171) and (3.59), (3.60) that

$$\hat{\theta}(x,z)_{nk} = \bar{z}_{k-1}F_k(x,\hat{p}(z)), \quad k = 2,\dots,n, \tag{3.188}$$

where $F_k(x,\hat{p}(z))$ is a smooth, strictly positive function. Hence we obtain that $z_j = z'_j$ for $j = 1,\dots,n-1$. With this in hand, since the variable z_n appears only in $\Delta(z)$, we conclude from (3.186) that $\Delta(z) = \Delta(z')$. This plainly implies that $z_n = z'_n$, whereby the proof is complete.

We note in passing that by continuing the above line of arguments the free action of G_μ is easily confirmed. Indeed, for $z' = z$ (3.183) also implies, besides (3.185), the equalities $\eta_L(2) = m$ and $\eta_L(1)\kappa(x)\hat{\zeta}(x,z)^{-1} = \kappa(x)\hat{\zeta}(x,z)^{-1}m$. Since $m = C\mathbf{1}_n$, as was already established, we must have $(\eta_L,\eta_R) = C(\mathbf{1}_{2n},\mathbf{1}_{2n}) \in \mathbb{Z}_{2n}$ (3.139). By using that the image of $\hat{K}$ intersects every G_μ-orbit, we can conclude that $\bar{G}_\mu$ (3.140) acts freely on $\Phi_+^{-1}(\mu)$. □

Remark 3.23. Observe from Theorem 3.22 that $\hat{S} = \{\hat{K}(z) \mid z \in \hat{M}_c\}$ is a global cross-section for the action of G_μ on $\Phi_+^{-1}(\mu)$. Hence $\hat{S}$ carrying the pull-back of ω as well as $(\hat{M}_c,\hat{\omega}_c)$ yield globally valid models of the reduced phase space (M,ω_M). The submanifold of $\hat{S}$ corresponding to $\hat{M}_c^o$ (3.166) is gauge equivalent to S^o (3.160) that features in Proposition 3.19.

3.5 Discussion

In this chapter we derived a deformation of the trigonometric BC_n Sutherland system by means of Hamiltonian reduction of a free system on the Heisenberg double of $\mathrm{SU}(2n)$. Our main result is the global characterization of the reduced phase space given by Theorem 3.14. The Liouville integrability of our system holds on this phase space, wherein the reduced free flows are complete. These flows can be obtained by the usual projection method applied to the original free flows described in Section 3.1.

The local form of our reduced 'main Hamiltonian' (3.1) is similar to the Hamiltonian derived in [86], which deforms the hyperbolic BC_n Sutherland system. However, besides a sign difference corresponding to the difference of the undeformed Hamiltonians, the local domain of our system, $\mathcal{C}_x \times \mathbb{T}_n$ in (3.4), is different from the local domain appearing in [86], which in effect has the form $\mathcal{C}'_x \times \mathbb{T}_n$ with the open polyhedron[4]

$$\mathcal{C}'_x := \{\hat{p} \in \mathbb{R}^n \mid \hat{p}_k - \hat{p}_{k+1} > |x|/2 \ (k = 1, \dots, n-1)\}. \tag{3.189}$$

We here wish to point out once more that $\mathcal{C}'_x \times \mathbb{T}_n$ is *not* the full reduced phase space that arises from the reduction considered in [86]. In fact, similarly to our case, the constraint surface contains a submanifold of the form $\bar{\mathcal{C}}'_x \times \mathbb{T}_n$ in the case of [86], where $\bar{\mathcal{C}}'_x$ is the closure of $\mathcal{C}'_x$. Then a global model of the reduced phase space can be constructed by introducing complex variables suitably accommodating the procedure that we utilized in Subsection 3.3.4. In Section 3.4 we clarified the global structure of the reduced phase space M (3.137), and thus completed the previous analysis [86] that dealt with the submanifold parametrized by $\mathcal{C}'_x \times \mathbb{T}_n$. In terms of the model $\hat{M}_c$ (3.162) of M, the complement of the submanifold in question is simply the zero set of the product of the complex variables. The phase space $\hat{M}_c$ and the embedding of $\mathcal{C}'_x \times \mathbb{T}_n$ into it coincides with what occurs for the so-called $\widetilde{\mathrm{III}}$-system of Ruijsenaars [117, 37], which is the action-angle dual of the standard trigonometric Ruijsenaars-Schneider system. This circumstance is not surprising in light of the fact [86] that the reduced 'main Hamiltonian' arising from $\mathcal{H}_1$ (3.133) is a $\widetilde{\mathrm{III}}$-type Hamiltonian coupled to external fields. We display this Hamiltonian below after exhibiting the corresponding Lax matrices.

The unreduced free Hamiltonians $\mathcal{H}_j$, $j \in \mathbb{Z}^*$ (3.133), mentioned in Section 3.4, can be written alternatively as

$$\mathcal{H}_j(K) = \frac{1}{2j}\mathrm{tr}(K\boldsymbol{J}K^\dagger\boldsymbol{J})^j = \frac{1}{2j}\mathrm{tr}(K^\dagger\boldsymbol{J}K\boldsymbol{J})^j. \tag{3.190}$$

One can verify (for example by using the standard r-matrix formula of the Poisson bracket on the Heisenberg double [126]) that the Hamiltonian flow generated by $\mathcal{H}_j$ reads

$$\begin{aligned} K(t_j) &= \exp\left[\mathrm{i}t_j\left((K(0)\boldsymbol{J}K(0)^\dagger\boldsymbol{J})^j - \frac{1}{2n}\mathrm{tr}(K(0)\boldsymbol{J}K(0)^\dagger\boldsymbol{J})^j\mathbf{1}_{2n}\right)\right]K(0)\\ &= K(0)\exp\left[\mathrm{i}t_j\left((\boldsymbol{J}K(0)^\dagger\boldsymbol{J}K(0))^j - \frac{1}{2n}\mathrm{tr}(\boldsymbol{J}K(0)^\dagger\boldsymbol{J}K(0))^j\mathbf{1}_{2n}\right)\right]. \end{aligned} \tag{3.191}$$

Since the exponentiated elements reside in the Lie algebra $\mathfrak{su}(n,n)$, these alternative formulas show that the flow stays in $\mathrm{SL}(2n,\mathbb{C})'$, as it must, and imply that the building blocks g_L and g_R of $K = b_L g_R^{-1} = g_L b_R^{-1}$ follow geodesics on $\mathrm{SU}(n,n)$, while b_L and b_R

[4] The notational correspondence with [86] is: $(q, p, \alpha, x, y) \leftrightarrow (\hat{p}, \hat{q}, e^{-\frac{x}{2}}, e^{-v}, e^{-u})$.

provide constants of motion. Equivalently, the last statement means that

$$K\boldsymbol{J}K^\dagger\boldsymbol{J} = b_L\boldsymbol{J}b_L^\dagger\boldsymbol{J} \quad \text{and} \quad K^\dagger\boldsymbol{J}K\boldsymbol{J} = (b_R^{-1})^\dagger\boldsymbol{J}b_R^{-1}\boldsymbol{J} \tag{3.192}$$

stay constant along the unreduced free flows.

To elaborate the reduced Hamiltonians, note that for an element K of the form (3.143) we have

$$(b_R^{-1})^\dagger\boldsymbol{J}b_R^{-1}\boldsymbol{J} = \begin{bmatrix} e^{-2v}\mathbf{1}_n & -e^{-v}\alpha \\ e^{-v}\alpha^\dagger & e^{2v}\mathbf{1}_n - \alpha^\dagger\alpha \end{bmatrix}. \tag{3.193}$$

By using this, as explained in Appendix C.5, one can prove that on $\Phi_+^{-1}(\mu)$ the Hamiltonians $\mathcal{H}_j$ can be written (for all j), up to additive constants, as linear combinations of the expressions

$$h_k = \operatorname{tr}(\alpha^\dagger\alpha)^k, \qquad k = 1, \dots, n. \tag{3.194}$$

Since in this way the Hermitian matrix $L = \alpha\alpha^\dagger$ generates the commuting reduced Hamiltonians, it provides a Lax matrix for the reduced system. By inserting α from (3.158), we obtain the explicit formula

$$\begin{aligned} L(\hat{p}, e^{\mathrm{i}\hat{q}}) =& (e^{2v} + e^{-2u})e^{-2\hat{p}} + (e^{2v} + e^{-2v})\mathbf{1}_n \\ &- \sqrt{e^{-2u}e^{-2\hat{p}} + e^{-2v}\mathbf{1}_n}\, e^{\mathrm{i}\hat{q}}\theta(x, \hat{p})^{-1} e^v \sqrt{e^{-2\hat{p}} + \mathbf{1}_n} \\ &- e^v\sqrt{e^{-2\hat{p}} + \mathbf{1}_n}\,\theta(x, \hat{p})e^{-\mathrm{i}\hat{q}}\sqrt{e^{-2u}e^{-2\hat{p}} + e^{-2v}\mathbf{1}_n}. \end{aligned} \tag{3.195}$$

On the other hand, the Lax matrix of Ruijsenaars's $\widetilde{\mathrm{III}}$-system can be taken to be [117, 37]

$$\tilde{L}(\hat{p}, e^{\mathrm{i}\hat{q}}) = e^{\mathrm{i}\hat{q}}\theta(x, \hat{p})^{-1} + \theta(x, \hat{p})e^{-\mathrm{i}\hat{q}}. \tag{3.196}$$

The similarity of the structures of these Lax matrices as well as the presence of the external field couplings in (3.195) is clear upon comparison. The extension of the Lax matrix $\alpha\alpha^\dagger$ (3.195) to the full phase space $M \simeq \hat{M}_c$ is of course given by $\hat{\alpha}\hat{\alpha}^\dagger$ by means of (3.174).

The main reduced Hamiltonian found in [86] reads as follows:

$$\begin{aligned} \mathcal{H}_1(K(\hat{p}, e^{\mathrm{i}\hat{q}})) =& -\frac{e^{-2u} + e^{2v}}{2}\sum_{j=1}^n e^{-2\hat{p}_j} + \\ &+ \sum_{j=1}^n \cos(\hat{q}_j)\big[1 + (1 + e^{2(v-u)})e^{-2\hat{p}_j} + e^{2(v-u)}e^{-4\hat{p}_j}\big]^{\frac{1}{2}} \prod_{\substack{k=1\\(k\neq j)}}^n \left[1 - \frac{\sinh^2\left(\frac{x}{2}\right)}{\sinh^2(\hat{p}_j - \hat{p}_k)}\right]^{\frac{1}{2}}. \end{aligned} \tag{3.197}$$

Liouville integrability holds since the functional independence of the involutive family obtained by reducing $\mathcal{H}_1, \dots, \mathcal{H}_n$ (3.190) is readily established and the projections of the free flows (3.191) to M are automatically complete. Similarly to its trigonometric analogue the Hamiltonian (3.197) can be identified as an Inozemtsev type limit of a specialization of van Diejen's 5-coupling deformation of the hyperbolic BC_n Sutherland Hamiltonian [140]. This fact suggests that it should be possible to extract the local form of dual Hamiltonians from [145] and references therein, which contain interesting results about closely related quantum mechanical systems and their bispectral properties. Indeed, in several examples, classical Hamiltonians enjoying action-angle duality correspond to bispectral pairs of Hamiltonian operators after quantization.

Throughout the text we assumed that $n > 1$, but we now note that the reduced system can be specialized to $n = 1$ and the reduction procedure works in this case as well. The assumption was made merely to save words. The formalism actually simplifies for $n = 1$ since the Poisson structure on $G_+ = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) < \mathrm{SU}(2)$ is trivial.

As explained in Appendix C.1, the Hamiltonian (3.1) is a singular limit of a specialization of the trigonometric van Diejen Hamiltonian [140], which (in addition to the deformation parameter) contains 5 coupling constants. As a result, at least classically, van Diejen's system can be degenerated into the trigonometric BC_n Sutherland system either directly, as described in [140], or in two stages, going through our system. Of course, a similar statement holds in relation to hyperbolic BC_n Sutherland and the system of [86].

Until recently, no Lax matrix was known that would generate van Diejen's commuting Hamiltonians, except in the rational limit [107]. In the reduction approach a Lax matrix arises automatically, in our case it features in equations (3.91) and (3.130). This might be helpful in further investigations for a Lax matrix behind van Diejen's 5-coupling Hamiltonian. The search would be easy if one could derive van Diejen's system by Hamiltonian reduction. It is a long standing open problem to find such derivation. Perhaps one should consider some 'classical analogue' of the quantum group interpretation of the Koornwinder (BC_n Macdonald) polynomials found in [93], since those polynomials diagonalize van Diejen's quantized Hamiltonians [142].

Another problem is to construct action-angle duals of the deformed BC_n Sutherland systems. Duality relations are not only intriguing on their own right, but are also very useful for extracting information about the dynamics [113, 117, 118, 108]. The duality was used in [P9, P1] to show that the hyperbolic BC_n Sutherland system is maximally superintegrable, the trigonometric BC_n Sutherland system has precisely n constants of motion, and the relevant dual systems are maximally superintegrable in both cases.

We mention that based on the results of this chapter Fehér and Marshall [41] recently explored the action-angle dual of the Hamiltonian (3.197) in the reduction framework. The question of duality for the system derived in [86] is still open.

Finally, we wish to mention the recent paper [145] dealing with the quantum mechanics of a lattice version of a 4-parameter Inozemtsev type limit of van Diejen's trigonometric/hyperbolic system. The systems studied in [86] and in this chapter correspond to further limits of specializations of this one. The statements about quantum mechanical dualities contained in [145] and its references should be related to classical dualities. We hope to return to this question in the future.

Part II

Developments in the Ruijsenaars-Schneider family

4 Lax representation of the hyperbolic BC_n van Diejen system

In this chapter, which follows [P7], we construct a Lax pair for the classical hyperbolic van Diejen system with two independent coupling parameters. Built upon this construction, we show that the dynamics can be solved by a projection method, which in turn allows us to initiate the study of the scattering properties. As a consequence, we prove the equivalence between the first integrals provided by the eigenvalues of the Lax matrix and the family of van Diejen's commuting Hamiltonians. Also, at the end of the chapter, we propose a candidate for the Lax matrix of the hyperbolic van Diejen system with three independent coupling constants.

The Ruijsenaars-Schneider-van Diejen (RSvD) systems, or simply van Diejen systems [142, 140, 143], are multi-parametric generalisations of the translation invariant Ruijsenaars-Schneider (RS) models [111, 112]. Moreover, in the so-called 'non-relativistic' limit, they reproduce the Calogero-Moser-Sutherland (CMS) models [21, 132, 88, 95] associated with the BC-type root systems. However, compared to the translation invariant A-type models, the geometrical picture underlying the most general classical van Diejen models is far less developed. The most probable explanation of this fact is the lack of Lax representation for the van Diejen dynamics. For this reason, working mainly in a symplectic reduction framework, in the last couple of years Pusztai undertook the study of the BC-type rational van Diejen models [106, 107, 108, 109] [P1, P5]. By going one stage up, in this chapter we wish to report on our first results about the hyperbolic variants of the van Diejen family.

In order to describe the Hamiltonian systems of our interest, let us recall that the configuration space of the hyperbolic n-particle van Diejen model is the open subset

$$Q = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n | \lambda_1 > \dots > \lambda_n > 0\} \subseteq \mathbb{R}^n, \tag{4.1}$$

that can be seen as an open Weyl chamber of type BC_n. The cotangent bundle of Q is trivial, and it can be naturally identified with the open subset

$$P = Q \times \mathbb{R}^n = \{(\lambda, \theta) = (\lambda_1, \dots, \lambda_n, \theta_1, \dots, \theta_n) \in \mathbb{R}^{2n} | \lambda_1 > \dots > \lambda_n > 0\} \subseteq \mathbb{R}^{2n}. \tag{4.2}$$

Following the widespread custom, throughout the chapter we shall occasionally think of the letters λ_a and θ_a $(1 \leq a \leq n)$ as globally defined coordinate functions on P. For example, using this latter interpretation, the canonical symplectic form on the phase space $P \cong T^*Q$ can be written as

$$\omega = \sum_{c=1}^{n} \mathrm{d}\lambda_c \wedge \mathrm{d}\theta_c, \tag{4.3}$$

whereas the fundamental Poisson brackets take the form

$$\{\lambda_a, \lambda_b\} = 0, \quad \{\theta_a, \theta_b\} = 0, \quad \{\lambda_a, \theta_b\} = \delta_{a,b} \qquad (1 \leq a, b \leq n). \tag{4.4}$$

The principal goal of this chapter is to study the dynamics generated by the smooth Hamiltonian function

$$H = \sum_{a=1}^{n} \cosh(\theta_a) \left[1 + \frac{\sin(\nu)^2}{\sinh(2\lambda_a)^2}\right]^{\frac{1}{2}} \prod_{\substack{c=1 \\ (c \neq a)}}^{n} \left[1 + \frac{\sin(\mu)^2}{\sinh(\lambda_a - \lambda_c)^2}\right]^{\frac{1}{2}} \left[1 + \frac{\sin(\mu)^2}{\sinh(\lambda_a + \lambda_c)^2}\right]^{\frac{1}{2}}, \tag{4.5}$$

where $\mu, \nu \in \mathbb{R}$ are arbitrary coupling constants satisfying the conditions

$$\sin(\mu) \neq 0 \neq \sin(\nu). \tag{4.6}$$

Note that H (4.5) does belong to the family of the hyperbolic n-particle van Diejen Hamiltonians with *two independent parameters* μ and ν (cf. (4.215)). Of course, the values of the parameters μ and ν really matter only modulo π.

Now, we briefly outline the content of the chapter. In the subsequent section, we start with a short overview on some relevant facts and notations from Lie theory. Having equipped with the necessary background material, in Section 4.2 we define our Lax matrix (4.34) for the van Diejen system (4.5), and also investigate its main algebraic properties. In Section 4.3 we turn to the study of the Hamiltonian flow generated by (4.5). As the first step, in Theorem 4.5 we formulate the completeness of the corresponding Hamiltonian vector field. Most importantly, in Theorem 4.8 we provide a Lax representation of the dynamics, whereas in Theorem 4.12 we establish a solution algorithm of purely algebraic nature. Making use of the projection method formulated in Theorem 4.12, we also initiate the study of the scattering properties of the system (4.5). Our rigorous results on the temporal asymptotics of the maximally defined trajectories are summarized in Lemma 4.13. Section 4.4 serves essentially two purposes. In Subsection 4.4.1 we elaborate the link between our special 2-parameter family of Hamiltonians (4.5) and the most general 5-parameter family of hyperbolic van Diejen systems (4.209). At the level of the coupling parameters the relationship can be read off from the equation (4.212). Furthermore, in Lemma 4.14 we affirm the equivalence between van Diejen's commuting family of Hamiltonians and the coefficients of the characteristic polynomial of the Lax matrix (4.34). Based on this technical result, in Theorem 4.15 we can infer that the eigenvalues of the proposed Lax matrix (4.34) provide a commuting family of first integrals for the Hamiltonian system (4.5). We conclude the chapter with Section 4.5, where we discuss the potential applications, and also offer some open problems and conjectures. In particular, in (D.5) we propose a Lax matrix for the 3-parameter family of hyperbolic van Diejen systems defined in (D.7).

4.1 Preliminaries from group theory

This section has two main objectives. Besides fixing the notations used throughout the chapter, we also provide a brief account on some relevant facts from Lie theory underlying our study of the 2-parameter family of hyperbolic van Diejen systems (4.5). For convenience, our conventions closely follow Knapp's book [72].

As before, by $n \in \mathbb{N} = \{1, 2, \dots\}$ we denote the number of particles. Let $N = 2n$, and also introduce the shorthand notations

$$\mathbb{N}_n = \{1, \dots, n\} \quad \text{and} \quad \mathbb{N}_N = \{1, \dots, N\}. \tag{4.7}$$

With the aid of the $N \times N$ matrix

$$C = \begin{bmatrix} 0_n & \mathbf{1}_n \\ \mathbf{1}_n & 0_n \end{bmatrix} \tag{4.8}$$

we define the non-compact real reductive matrix Lie group

$$G = \mathrm{U}(n, n) = \{y \in GL(N, \mathbb{C}) \,|\, y^* C y = C\}, \tag{4.9}$$

in which the set of unitary elements

$$K = \{y \in G \,|\, y^* y = \mathbf{1}_N\} \cong \mathrm{U}(n) \times \mathrm{U}(n) \tag{4.10}$$

forms a maximal compact subgroup. The Lie algebra of G (4.9) takes the form

$$\mathfrak{g} = \mathfrak{u}(u, n) = \{Y \in \mathfrak{gl}(N, \mathbb{C}) \,|\, Y^* C + C Y = 0\}, \tag{4.11}$$

whereas for the Lie subalgebra corresponding to K (4.10) we have the identification

$$\mathfrak{k} = \{Y \in \mathfrak{g} \,|\, Y^* + Y = 0\} \cong \mathfrak{u}(n) \oplus \mathfrak{u}(n). \tag{4.12}$$

Upon introducing the subspace

$$\mathfrak{p} = \{Y \in \mathfrak{g} \,|\, Y^* = Y\}, \tag{4.13}$$

we can write the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which is orthogonal with respect to the usual trace pairing defined on the matrix Lie algebra $\mathfrak{g}$. Let us note that the restriction of the exponential map onto the complementary subspace $\mathfrak{p}$ (4.13) is injective. Moreover, the image of $\mathfrak{p}$ under the exponential map can be identified with the set of the positive definite elements of the group $\mathrm{U}(n,n)$; that is,

$$\exp(\mathfrak{p}) = \{y \in \mathrm{U}(n,n) \mid y > 0\}. \tag{4.14}$$

Notice that, due to the Cartan decomposition $G = \exp(\mathfrak{p})K$, the above set can be also naturally identified with the non-compact symmetric space associated with the pair (G, K), i.e.,

$$\exp(\mathfrak{p}) \cong \mathrm{U}(n,n)/(\mathrm{U}(n) \times \mathrm{U}(n)) \cong \mathrm{SU}(n,n)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)). \tag{4.15}$$

To get a more detailed picture about the structure of the reductive Lie group $\mathrm{U}(n,n)$, in $\mathfrak{p}$ (4.13) we introduce the maximal Abelian subspace

$$\mathfrak{a} = \{X = \operatorname{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}. \tag{4.16}$$

Let us recall that we can attain every element of $\mathfrak{p}$ by conjugating the elements of $\mathfrak{a}$ with the elements of the compact subgroup K (4.10). More precisely, the map

$$\mathfrak{a} \times K \ni (X, k) \mapsto kXk^{-1} \in \mathfrak{p} \tag{4.17}$$

is well-defined and onto. As for the centralizer of $\mathfrak{a}$ inside K (4.10), it turns out to be the Abelian Lie group

$$M = Z_K(\mathfrak{a}) = \{\operatorname{diag}(e^{\mathrm{i}\chi_1}, \dots, e^{\mathrm{i}\chi_n}, e^{\mathrm{i}\chi_1}, \dots, e^{\mathrm{i}\chi_n}) \mid \chi_1, \dots, \chi_n \in \mathbb{R}\} \tag{4.18}$$

with Lie algebra

$$\mathfrak{m} = \{\operatorname{diag}(\mathrm{i}\chi_1, \dots, \mathrm{i}\chi_n, \mathrm{i}\chi_1, \dots, \mathrm{i}\chi_n) \mid \chi_1, \dots, \chi_n \in \mathbb{R}\}. \tag{4.19}$$

Let $\mathfrak{m}^\perp$ and $\mathfrak{a}^\perp$ denote the sets of the off-diagonal elements in the subspaces $\mathfrak{k}$ and $\mathfrak{p}$, respectively; then clearly we can write the refined orthogonal decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp \oplus \mathfrak{a} \oplus \mathfrak{a}^\perp. \tag{4.20}$$

To put it simple, each Lie algebra element $Y \in \mathfrak{g}$ can be decomposed as

$$Y = Y_{\mathfrak{m}} + Y_{\mathfrak{m}^\perp} + Y_{\mathfrak{a}} + Y_{\mathfrak{a}^\perp} \tag{4.21}$$

with unique components belonging to the subspaces indicated by the subscripts.

Throughout our work the commuting family of linear operators

$$\mathrm{ad}_X \colon \mathfrak{gl}(N,\mathbb{C}) \to \mathfrak{gl}(N,\mathbb{C}), \quad Y \mapsto [X,Y] \tag{4.22}$$

defined for the diagonal matrices $X \in \mathfrak{a}$ plays a distinguished role. Let us note that the (real) subspace $\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp \subseteq \mathfrak{gl}(N,\mathbb{C})$ is invariant under ad_X, whence the restriction

$$\widetilde{\mathrm{ad}}_X = \mathrm{ad}_X|_{\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp} \in \mathfrak{gl}(\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp) \tag{4.23}$$

is a well-defined operator for each $X = \mathrm{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) \in \mathfrak{a}$ with spectrum

$$\mathrm{Spec}(\widetilde{\mathrm{ad}}_X) = \{x_a - x_b, \pm(x_a + x_b), \pm 2x_c \mid a, b, c \in \mathbb{N}_n,\ a \neq b\}. \tag{4.24}$$

Now, recall that the regular part of the Abelian subalgebra $\mathfrak{a}$ (4.16) is defined by the subset

$$\mathfrak{a}_{\mathrm{reg}} = \{X \in \mathfrak{a} \mid \widetilde{\mathrm{ad}}_X \text{ is invertible}\}, \tag{4.25}$$

in which the standard open Weyl chamber

$$\mathfrak{c} = \{X = \mathrm{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) \in \mathfrak{a} \,|\, x_1 > \dots > x_n > 0\} \tag{4.26}$$

is a connected component. Let us observe that it can be naturally identified with the configuration space Q (4.1); that is, $Q \cong \mathfrak{c}$. Finally, let us recall that the regular part of $\mathfrak{p}$ (4.13) is defined as

$$\mathfrak{p}_{\mathrm{reg}} = \{kXk^{-1} \in \mathfrak{p} \,|\, X \in \mathfrak{a}_{\mathrm{reg}} \text{ and } k \in K\}. \tag{4.27}$$

As a matter of fact, from the map (4.17) we can derive a particularly useful characterization for the open subset $\mathfrak{p}_{\mathrm{reg}} \subseteq \mathfrak{p}$. Indeed, the map

$$\mathfrak{c} \times (K/M) \ni (X, kM) \mapsto kXk^{-1} \in \mathfrak{p}_{\mathrm{reg}} \tag{4.28}$$

turns out to be a diffeomorphism, providing the identification $\mathfrak{p}_{\mathrm{reg}} \cong \mathfrak{c} \times (K/M)$.

4.2 Algebraic properties of the Lax matrix

Having reviewed the necessary notions and notations from Lie theory, in this section we propose a Lax matrix for the hyperbolic van Diejen system of our interest (4.5). To make the presentation simpler, with any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ we associate the real N-tuples

$$\Lambda = (\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n) \quad \text{and} \quad \Theta = (\theta_1, \dots, \theta_n, -\theta_1, \dots, -\theta_n), \tag{4.29}$$

respectively, and also define the $N \times N$ diagonal matrix

$$\boldsymbol{\Lambda} = \operatorname{diag}(\Lambda_1, \dots, \Lambda_N) = \operatorname{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n) \in \mathfrak{a}. \tag{4.30}$$

Notice that if $\lambda \in \mathbb{R}^n$ is a regular element in the sense that the corresponding diagonal matrix $\boldsymbol{\Lambda}$ (4.30) belongs to $\mathfrak{a}_{\text{reg}}$ (4.25), then for each $j \in \mathbb{N}_N$ the complex number

$$z_j = -\frac{\sinh(\mathrm{i}\nu + 2\Lambda_j)}{\sinh(2\Lambda_j)} \prod_{\substack{c=1 \\ (c \neq j, j-n)}}^{n} \frac{\sinh(\mathrm{i}\mu + \Lambda_j - \lambda_c)}{\sinh(\Lambda_j - \lambda_c)} \frac{\sinh(\mathrm{i}\mu + \Lambda_j + \lambda_c)}{\sinh(\Lambda_j + \lambda_c)} \tag{4.31}$$

is well-defined. Thinking of z_j as a function of λ, let us observe that its modulus $u_j = |z_j|$ takes the form

$$u_j = \left(1 + \frac{\sin(\nu)^2}{\sinh(2\Lambda_j)^2}\right)^{\frac{1}{2}} \prod_{\substack{c=1 \\ (c \neq j, j-n)}}^{n} \left(1 + \frac{\sin(\mu)^2}{\sinh(\Lambda_j - \lambda_c)^2}\right)^{\frac{1}{2}} \left(1 + \frac{\sin(\mu)^2}{\sinh(\Lambda_j + \lambda_c)^2}\right)^{\frac{1}{2}}, \tag{4.32}$$

and the property $z_{n+a} = \bar{z}_a$ $(a \in \mathbb{N}_n)$ is also clear. Next, built upon the functions z_j and u_j, we introduce the column vector $F \in \mathbb{C}^N$ with components

$$F_a = e^{\frac{\theta_a}{2}} u_a^{\frac{1}{2}} \quad \text{and} \quad F_{n+a} = e^{-\frac{\theta_a}{2}} \bar{z}_a u_a^{-\frac{1}{2}} \qquad (a \in \mathbb{N}_n). \tag{4.33}$$

At this point we are in a position to define our Lax matrix $L \in \mathfrak{gl}(N, \mathbb{C})$ with the entries

$$L_{k,l} = \frac{\mathrm{i}\sin(\mu) F_k \bar{F}_l + \mathrm{i}\sin(\mu - \nu) C_{k,l}}{\sinh(\mathrm{i}\mu + \Lambda_k - \Lambda_l)} \qquad (k, l \in \mathbb{N}_N). \tag{4.34}$$

Note that the matrix valued function L is well-defined at each point $(\lambda, \theta) \in \mathbb{R}^N$ satisfying the regularity condition $\boldsymbol{\Lambda} \in \mathfrak{a}_{\text{reg}}$. Since $\mathfrak{c} \subseteq \mathfrak{a}_{\text{reg}}$ (4.26), L makes sense at each point of the phase space P (4.2) as well. To give a motivation for the definition of $L = L(\lambda, \theta; \mu, \nu)$ (4.34), let us observe that in its 'rational limit' we get back the Lax matrix of the rational van Diejen system with two parameters. Indeed, up to some irrelevant numerical factors caused by a slightly different convention, in the $\alpha \to 0^+$

limit the matrix $L(\alpha\lambda, \theta; \alpha\mu, \alpha\nu)$ tends to the rational Lax matrix $\mathcal{A} = \mathcal{A}(\lambda, \theta; \mu, \nu)$ as defined in [106, eqs. (4.2)-(4.5)]. In [106] the matrix $\mathcal{A}$ has many peculiar algebraic properties, that we wish to generalize for the proposed hyperbolic Lax matrix L in the rest of this section.

4.2.1 Lax matrix: explicit form, inverse, and positivity

By inspecting the matrix entries (4.34), it is obvious that L is Hermitian. However, it is a less trivial fact that L is closely tied with the non-compact Lie group $\mathrm{U}(n,n)$ (4.9). The purpose of this subsection is to explore this surprising relationship.

Proposition 4.1. *The matrix L (4.34) obeys the quadratic equation $LCL = C$. In other words, the matrix valued function L takes values in the Lie group $\mathrm{U}(n,n)$.*

Proof. Take an arbitrary element $(\lambda, \theta) \in \mathbb{R}^N$ satisfying the regularity condition $\mathbf{\Lambda} \in \mathfrak{a}_{\mathrm{reg}}$. We start by observing that for each $a \in \mathbb{N}_n$ the complex conjugates of z_a (4.31) and F_{n+a} (4.33) can be obtained by changing the sign of the single component λ_a of λ. Therefore, if $a, b \in \mathbb{N}_n$ are arbitrary indices, then by interchanging the components λ_a and λ_b of the n-tuple λ, the expression $(LCL)_{a,b} F_a^{-1} \bar{F}_b^{-1}$ readily transforms into $(LCL)_{n+a,n+b} F_{n+a}^{-1} \bar{F}_{n+b}^{-1}$. We capture this fact by writing

$$\frac{(LCL)_{a,b}}{F_a \bar{F}_b} \underset{\lambda_a \leftrightarrow \lambda_b}{\rightsquigarrow} \frac{(LCL)_{n+a,n+b}}{F_{n+a} \bar{F}_{n+b}} \qquad (a, b \in \mathbb{N}_n). \tag{4.35}$$

Similarly, if $a \neq b$, then from $(LCL)_{a,b} F_a^{-1} \bar{F}_b^{-1}$ we can recover $(LCL)_{n+a,b} F_{n+a}^{-1} \bar{F}_b^{-1}$ by exchanging λ_a for $-\lambda_a$. Schematically, we have

$$\frac{(LCL)_{a,b}}{F_a \bar{F}_b} \underset{\lambda_a \leftrightarrow -\lambda_a}{\rightsquigarrow} \frac{(LCL)_{n+a,b}}{F_{n+a} \bar{F}_b} \qquad (a, b \in \mathbb{N}_n,\ a \neq b). \tag{4.36}$$

Furthermore, the expression $(LCL)_{a,b} F_a^{-1} \bar{F}_b^{-1}$ reproduces $(LCL)_{a,n+b} F_a^{-1} \bar{F}_{n+b}^{-1}$ upon swapping λ_b for $-\lambda_b$, i.e.,

$$\frac{(LCL)_{a,b}}{F_a \bar{F}_b} \underset{\lambda_b \leftrightarrow -\lambda_b}{\rightsquigarrow} \frac{(LCL)_{a,n+b}}{F_a \bar{F}_{n+b}} \qquad (a, b \in \mathbb{N}_n, a \neq b). \tag{4.37}$$

Finally, the relationship between the remaining entries is given by the exchange

$$(LCL)_{a,n+a} \underset{\lambda_a \leftrightarrow -\lambda_a}{\rightsquigarrow} (LCL)_{n+a,a} \qquad (a \in \mathbb{N}_n). \tag{4.38}$$

The message of the above equations (4.35)-(4.38) is quite evident. Indeed, in order to prove the desired matrix equation $LCL = C$, it does suffice to show that $(LCL)_{a,b} = 0$ for all $a, b \in \mathbb{N}_n$, and also that $(LCL)_{a,n+a} = 1$ for all $a \in \mathbb{N}_n$.

Recalling the formulae (4.33) and (4.34), it is clear that for all $a \in \mathbb{N}_n$ we can write

$$\frac{(LCL)_{a,a}}{F_a\bar{F}_a} = 2\mathrm{Re}\bigg(\frac{\mathrm{i}\sin(\mu)z_a + \mathrm{i}\sin(\mu-\nu)}{\sinh(\mathrm{i}\mu + 2\lambda_a)} - \sum_{\substack{c=1\\(c\neq a)}}^{n} \frac{\sin(\mu)^2 z_c}{\sinh(\mathrm{i}\mu + \lambda_a + \lambda_c)\sinh(\mathrm{i}\mu - \lambda_a + \lambda_c)}\bigg). \tag{4.39}$$

To proceed further, we introduce a complex valued function f_a depending on a single complex variable w obtained simply by replacing λ_a with $\lambda_a + w$ in the right-hand side of the above equation (4.39). Remembering (4.31), it is obvious that the resulting function is meromorphic with at most first order poles at the points

$$w \equiv -\lambda_a,\ w \equiv \pm\mathrm{i}\mu/2 - \lambda_a,\ w \equiv \Lambda_j - \lambda_a\,(j \in \mathbb{N}_N) \pmod{\mathrm{i}\pi}. \tag{4.40}$$

However, by inspecting the terms appearing in the explicit expression of f_a, a straightforward computation reveals immediately that the residue of f_a at each of these points is zero, i.e., the singularities are in fact removable. As a consequence, f_a can be uniquely extended onto the whole complex plane as a periodic entire function with period $2\pi\mathrm{i}$. Moreover, since $f_a(w)$ vanishes as $\mathrm{Re}(w) \to \infty$, the function f_a is clearly bounded. By invoking Liouville's theorem, we conclude that $f_a(w) = 0$ for all $w \in \mathbb{C}$, and so

$$\frac{(LCL)_{a,a}}{F_a\bar{F}_a} = f_a(0) = 0. \tag{4.41}$$

Next, let $a, b \in \mathbb{N}_n$ be arbitrary indices satisfying $a \neq b$. Keeping in mind the definitions (4.33) and (4.34), we find at once that

$$\begin{aligned}\frac{(LCL)_{a,b}}{F_a\bar{F}_b} =& \frac{\mathrm{i}\sin(\mu)\big(\mathrm{i}\sin(\mu)z_a + \mathrm{i}\sin(\mu-\nu)\big)}{\sinh(\mathrm{i}\mu + \lambda_a - \lambda_b)\sinh(\mathrm{i}\mu + 2\lambda_a)} + \frac{\mathrm{i}\sin(\mu)\big(\mathrm{i}\sin(\mu)\bar{z}_b + \mathrm{i}\sin(\mu-\nu)\big)}{\sinh(\mathrm{i}\mu + \lambda_a - \lambda_b)\sinh(\mathrm{i}\mu - 2\lambda_b)}\\ &+ \frac{\mathrm{i}\sin(\mu)\bar{z}_a}{\sinh(\mathrm{i}\mu - \lambda_a - \lambda_b)} + \frac{\mathrm{i}\sin(\mu)z_b}{\sinh(\mathrm{i}\mu + \lambda_a + \lambda_b)}\\ &- \sum_{\substack{j=1\\(j\neq a,b,n+a,n+b)}}^{N} \frac{\sin(\mu)^2 z_j}{\sinh(\mathrm{i}\mu + \lambda_a + \Lambda_j)\sinh(\mathrm{i}\mu - \lambda_b + \Lambda_j)}.\end{aligned} \tag{4.42}$$

Although this equation looks considerably more complicated than (4.39), it can be analyzed by the same techniques. Indeed, by replacing λ_a with $\lambda_a + w$ in the right-hand side of (4.42), we may obtain a meromorphic function $f_{a,b}$ of $w \in \mathbb{C}$ that has at most first order poles at the points

$$w \equiv -\lambda_a,\ w \equiv -\mathrm{i}\mu/2 - \lambda_a,\ w \equiv -\mathrm{i}\mu - \lambda_a + \lambda_b,\ w \equiv \Lambda_j - \lambda_a\,(j \in \mathbb{N}_N) \pmod{\mathrm{i}\pi}. \tag{4.43}$$

However, the residue of $f_{a,b}$ at each of these points turns out to be zero, and $f_{a,b}(w)$

also vanishes as $\mathrm{Re}(w) \to \infty$. Due to Liouville's theorem we get $f_{a,b}(w) = 0$ for all $w \in \mathbb{C}$, thus

$$\frac{(LCL)_{a,b}}{F_a \bar{F}_b} = f_{a,b}(0) = 0. \tag{4.44}$$

Finally, by taking an arbitrary $a \in \mathbb{N}_n$, from (4.33) and (4.34) we see that

$$(LCL)_{a,n+a} = u_a^2 + \frac{(\mathrm{i}\sin(\mu) z_a + \mathrm{i}\sin(\mu - \nu))^2}{\sinh(\mathrm{i}\mu + 2\lambda_a)^2} - \sum_{\substack{j=1 \\ (j \neq a, n+a)}}^{N} \frac{\sin(\mu)^2 z_a z_j}{\sinh(\mathrm{i}\mu + \lambda_a + \Lambda_j)^2}. \tag{4.45}$$

By replacing λ_a with $\lambda_a + w$ in the right-hand side of (4.45), we end up with a meromorphic function f_{n+a} of the complex variable w that has at most second order poles at the points

$$w \equiv -\lambda_a,\ w \equiv -\mathrm{i}\mu/2 - \lambda_a,\ w \equiv \Lambda_j - \lambda_a\, (j \in \mathbb{N}_N) \pmod{\mathrm{i}\pi}. \tag{4.46}$$

Though the calculations are a bit more involved as in the previous cases, one can show that the singularities of f_{n+a} are actually removable. Moreover, it is evident that $f_{n+a}(w) \to 1$ as $\mathrm{Re}(w) \to \infty$. Liouville's theorem applies again, implying that $f_{n+a}(w) = 1$ for all $w \in \mathbb{C}$. Thus the relationship

$$(LCL)_{a,n+a} = f_{n+a}(0) = 1 \tag{4.47}$$

also follows, whence the proof is complete. □

In the earlier paper [106] we saw that the rational analogue of L (4.34) takes values in the symmetric space $\exp(\mathfrak{p})$ (4.15). We find it reassuring that the proof of Lemma 7 of paper [106] allows a straightforward generalization into the present hyperbolic context, too.

Lemma 4.2. *At each point of the phase space we have $L \in \exp(\mathfrak{p})$.*

Proof. Recalling the identification (4.14) and Proposition 4.1, it is enough to prove that the Hermitian matrix L (4.34) is positive definite. For this reason, take an arbitrary point $(\lambda, \theta) \in P$ and keep it fixed. To prove the Lemma, below we offer a standard continuity argument by analyzing the dependence of L solely on the coupling parameters.

In the very special case when the pair (μ, ν) formed by the coupling parameters obey the relationship $\sin(\mu - \nu) = 0$, the Lax matrix L (4.34) becomes a hyperbolic Cauchy-like matrix and the generalized Cauchy determinant formula (see e.g. [113, eq. (1.2)]) readily implies the positivity of all its leading principal minors. Thus, recalling Sylvester's criterion, we conclude that L is positive definite.

Turning to the general case, suppose that the pair (μ, ν) is restricted only by the conditions displayed in (4.6). It is clear that in the 2-dimensional space of the admissible coupling parameters characterized by (4.6) one can find a continuous curve with endpoints (μ, ν) and (μ_0, ν_0), where μ_0 and ν_0 satisfy the additional requirement $\sin(\mu_0 - \nu_0) = 0$. Since the dependence of the Hermitian matrix L on the coupling parameters is smooth, along this curve the smallest eigenvalue of L moves continuously. However, it cannot cross zero, since by Proposition 4.1 the matrix L remains invertible during this deformation. Therefore, since the eigenvalues of L are strictly positive at the endpoint (μ_0, ν_0), they must be strictly positive at the other endpoint (μ, ν) as well. □

4.2.2 Commutation relation and regularity

As Ruijsenaars has observed in his seminal paper on the translation invariant CMS and RS type pure soliton systems, one of the key ingredients in their analysis is the fact that their Lax matrices obey certain non-trivial commutation relations with some diagonal matrices (for details, see equation (2.4) and the surrounding ideas in [113]). As a momentum map constraint, an analogous commutation relation has also played a key role in the geometric study of the rational C_n and BC_n RSvD systems (see [106, 107] [P1]). Due to its importance, our first goal in this subsection is to set up a Ruijsenaars type commutation relation for the proposed Lax matrix L (4.34), too. As a technical remark, we mention in passing that from now on we shall apply frequently the standard functional calculus on the linear operators $\mathrm{ad}_{\boldsymbol{\Lambda}}$ (4.22) and $\widetilde{\mathrm{ad}}_{\boldsymbol{\Lambda}}$ (4.23) associated with the diagonal matrix $\boldsymbol{\Lambda} \in \mathfrak{c}$ (4.30).

Lemma 4.3. *The matrix L (4.34) and the diagonal matrix $e^{\boldsymbol{\Lambda}}$ obey the Ruijsenaars type commutation relation*

$$e^{\mathrm{i}\mu} e^{\mathrm{ad}_{\boldsymbol{\Lambda}}} L - e^{-\mathrm{i}\mu} e^{-\mathrm{ad}_{\boldsymbol{\Lambda}}} L = 2\mathrm{i}\sin(\mu) F F^* + 2\mathrm{i}\sin(\mu - \nu) C. \tag{4.48}$$

Proof. Recalling the matrix entries of L, for all $k, l \in \mathbb{N}_N$ we can write that

$$\begin{aligned}
&\left(e^{\mathrm{i}\mu} e^{\mathrm{ad}_{\boldsymbol{\Lambda}}} L - e^{-\mathrm{i}\mu} e^{-\mathrm{ad}_{\boldsymbol{\Lambda}}} L\right)_{k,l} = \left(e^{\mathrm{i}\mu} e^{\boldsymbol{\Lambda}} L e^{-\boldsymbol{\Lambda}} - e^{-\mathrm{i}\mu} e^{-\boldsymbol{\Lambda}} L e^{\boldsymbol{\Lambda}}\right)_{k,l} \\
&= e^{\mathrm{i}\mu} e^{\Lambda_k} L_{k,l} e^{-\Lambda_l} - e^{-\mathrm{i}\mu} e^{-\Lambda_k} L_{k,l} e^{\Lambda_l} = 2\sinh(\mathrm{i}\mu + \Lambda_k - \Lambda_l) L_{k,l} \\
&= 2\mathrm{i}\sin(\mu) F_k \bar{F}_l + 2\mathrm{i}\sin(\mu - \nu) C_{k,l} = (2\mathrm{i}\sin(\mu) F F^* + 2\mathrm{i}\sin(\mu - \nu) C)_{k,l},
\end{aligned} \tag{4.49}$$

thus (4.48) follows at once. □

Though the proof of Lemma 4.3 is almost trivial, it proves to be quite handy in the forthcoming calculations. In particular, based on the commutation relation (4.48), we shall now prove that the spectrum of L is *simple*. Heading toward our present goal, first let us recall that Lemma 4.2 tells us that $L \in \exp(\mathfrak{p})$. Therefore, as we can infer easily from (4.17), one can find some $y \in K$ and a real n-tuple $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_n) \in \mathbb{R}^n$ satisfying

$$\hat{\theta}_1 \geq \ldots \geq \hat{\theta}_n \geq 0, \tag{4.50}$$

such that with the diagonal matrix

$$\hat{\boldsymbol{\Theta}} = \mathrm{diag}(\hat{\Theta}_1, \dots, \hat{\Theta}_N) = \mathrm{diag}(\hat{\theta}_1, \dots, \hat{\theta}_n, -\hat{\theta}_1, \dots, -\hat{\theta}_n) \in \mathfrak{a} \tag{4.51}$$

we can write

$$L = y e^{2\hat{\boldsymbol{\Theta}}} y^{-1}. \tag{4.52}$$

Now, upon defining

$$\hat{L} = y^{-1} e^{2\boldsymbol{\Lambda}} y \in \exp(\mathfrak{p}) \quad \text{and} \quad \hat{F} = e^{-\hat{\boldsymbol{\Theta}}} y^{-1} e^{\boldsymbol{\Lambda}} F \in \mathbb{C}^N, \tag{4.53}$$

for these new objects we can also set up a commutation relation analogous to (4.48). Indeed, from (4.48) one can derive that

$$e^{\mathrm{i}\mu} e^{-\hat{\boldsymbol{\Theta}}} \hat{L} e^{\hat{\boldsymbol{\Theta}}} - e^{-\mathrm{i}\mu} e^{\hat{\boldsymbol{\Theta}}} \hat{L} e^{-\hat{\boldsymbol{\Theta}}} = 2\mathrm{i}\sin(\mu) \hat{F} \hat{F}^* + 2\mathrm{i}\sin(\mu - \nu) C. \tag{4.54}$$

Componentwise, from (4.54) we conclude that

$$\hat{L}_{k,l} = \frac{\mathrm{i}\sin(\mu) \hat{F}_k \bar{\hat{F}}_l + \mathrm{i}\sin(\mu - \nu) C_{k,l}}{\sinh(\mathrm{i}\mu - \hat{\Theta}_k + \hat{\Theta}_l)} \qquad (k, l \in \mathbb{N}_N). \tag{4.55}$$

Since $\hat{L}$ (4.53) is a positive definite matrix, its diagonal entries are strictly positive. Therefore, by exploiting (4.55), we can write

$$0 < \hat{L}_{k,k} = |\hat{F}_k|^2. \tag{4.56}$$

The upshot of this trivial observation is that $\hat{F}_k \neq 0$ for all $k \in \mathbb{N}_N$.

To proceed further, notice that for the inverse matrix $\hat{L}^{-1} = C\hat{L}C$ we can also cook up an equation analogous to (4.54). Indeed, by simply multiplying both sides of (4.54) with the matrix C (4.8), we obtain

$$e^{\mathrm{i}\mu} e^{\hat{\boldsymbol{\Theta}}} \hat{L}^{-1} e^{-\hat{\boldsymbol{\Theta}}} - e^{-\mathrm{i}\mu} e^{-\hat{\boldsymbol{\Theta}}} \hat{L}^{-1} e^{\hat{\boldsymbol{\Theta}}} = 2\mathrm{i}\sin(\mu) (C\hat{F})(C\hat{F})^* + 2\mathrm{i}\sin(\mu - \nu) C, \tag{4.57}$$

that leads immediately to the matrix entries

$$(\hat{L}^{-1})_{k,l} = \frac{\mathrm{i}\sin(\mu) (C\hat{F})_k \overline{(C\hat{F})_l} + \mathrm{i}\sin(\mu - \nu) C_{k,l}}{\sinh(\mathrm{i}\mu + \hat{\Theta}_k - \hat{\Theta}_l)} \qquad (k, l \in \mathbb{N}_N). \tag{4.58}$$

For further reference, we now spell out the trivial equation

$$\delta_{k,l} = \sum_{j=1}^{N} \hat{L}_{k,j}(\hat{L}^{-1})_{j,l} \tag{4.59}$$

for certain values of $k, l \in \mathbb{N}_N$. First, by plugging the explicit formulae (4.55) and (4.58) into the relationship (4.59), with the special choice of indices $k = l = a \in \mathbb{N}_n$ one finds that

$$\begin{aligned} 0 =& 1 + \frac{\sin(\mu-\nu)^2}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)^2} + \frac{2\sin(\mu)\sin(\mu-\nu)\hat{F}_a\bar{\hat{F}}_{n+a}}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)^2} \\ &+ \sin(\mu)^2\hat{F}_a\bar{\hat{F}}_{n+a}\sum_{c=1}^{n}\left(\frac{\bar{\hat{F}}_c\hat{F}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a + \hat{\theta}_c)^2} + \frac{\hat{F}_c\bar{\hat{F}}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a - \hat{\theta}_c)^2}\right). \end{aligned} \tag{4.60}$$

Second, if $k = a$ and $l = n + a$ with some $a \in \mathbb{N}_n$, then from (4.59) we obtain

$$\begin{aligned} &\sin(\mu)^2\sum_{c=1}^{n}\left(\frac{\bar{\hat{F}}_c\hat{F}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a + \hat{\theta}_c)\sinh(\mathrm{i}\mu + \hat{\theta}_c + \hat{\theta}_a)} + \frac{\hat{F}_c\bar{\hat{F}}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a - \hat{\theta}_c)\sinh(\mathrm{i}\mu - \hat{\theta}_c + \hat{\theta}_a)}\right) \\ &= \mathrm{i}\sin(\mu-\nu)\left(\frac{1}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)} + \frac{1}{\sinh(\mathrm{i}\mu + 2\hat{\theta}_a)}\right). \end{aligned} \tag{4.61}$$

Third, if $k = a$ and $l = b$ with some $a, b \in \mathbb{N}_n$ satisfying $a \neq b$, then the relationship (4.59) immediately leads to the equation

$$\begin{aligned} &\sin(\mu)^2\sum_{c=1}^{n}\left(\frac{\bar{\hat{F}}_c\hat{F}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a + \hat{\theta}_c)\sinh(\mathrm{i}\mu + \hat{\theta}_c - \hat{\theta}_b)} + \frac{\hat{F}_c\bar{\hat{F}}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a - \hat{\theta}_c)\sinh(\mathrm{i}\mu - \hat{\theta}_c - \hat{\theta}_b)}\right) \\ &= -\frac{\sin(\mu)\sin(\mu-\nu)}{\sinh(\mathrm{i}\mu - \hat{\theta}_a - \hat{\theta}_b)}\left(\frac{1}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)} + \frac{1}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_b)}\right). \end{aligned} \tag{4.62}$$

At this point we wish to emphasize that during the derivation of the last two equations (4.61) and (4.62) it proves to be essential that each component of the column vector $\hat{F}$ (4.53) is nonzero, as we have seen in (4.56).

Lemma 4.4. *Under the additional assumption on the coupling parameters*

$$\sin(2\mu - \nu) \neq 0, \tag{4.63}$$

the spectrum of the matrix L (4.34) is simple of the form

$$\operatorname{Spec}(L) = \{e^{\pm 2\hat{\theta}_a} \mid a \in \mathbb{N}_n\}, \tag{4.64}$$

where $\hat{\theta}_1 > \ldots > \hat{\theta}_n > 0$. In other words, L is regular in the sense that $L \in \exp(\mathfrak{p}_{\mathrm{reg}})$.

Proof. First, let us *suppose* that $\hat{\theta}_a = 0$ for some $a \in \mathbb{N}_n$. With this particular index a, from equation (4.60) we infer that

$$\begin{aligned} 0 =& 1 - \frac{\sin(\mu-\nu)^2}{\sin(\mu)^2} - \frac{2\sin(\mu-\nu)\hat{F}_a\bar{\hat{F}}_{n+a}}{\sin(\mu)} \\ &+ \sin(\mu)^2 \hat{F}_a \bar{\hat{F}}_{n+a} \sum_{c=1}^{n} \left(\frac{\bar{\hat{F}}_c \hat{F}_{n+c}}{\sinh(\mathrm{i}\mu + \hat{\theta}_c)^2} + \frac{\hat{F}_c \bar{\hat{F}}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_c)^2} \right), \end{aligned} \tag{4.65}$$

while (4.61) leads to the relationship

$$\sin(\mu)^2 \sum_{c=1}^{n} \left(\frac{\bar{\hat{F}}_c \hat{F}_{n+c}}{\sinh(\mathrm{i}\mu + \hat{\theta}_c)^2} + \frac{\hat{F}_c \bar{\hat{F}}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_c)^2} \right) = \frac{2\sin(\mu-\nu)}{\sin(\mu)}. \tag{4.66}$$

Now, by plugging (4.66) into (4.65), we obtain

$$0 = 1 - \frac{\sin(\mu-\nu)^2}{\sin(\mu)^2} = \frac{\sin(\mu)^2 - \sin(\mu-\nu)^2}{\sin(\mu)^2} = \frac{\sin(\nu)\sin(2\mu-\nu)}{\sin(\mu)^2}, \tag{4.67}$$

which clearly contradicts the assumptions imposed in the equations (4.6) and (4.63). Thus, we are forced to conclude that for all $a \in \mathbb{N}_n$ we have $\hat{\theta}_a \neq 0$.

Second, let us *suppose* that $\hat{\theta}_a = \hat{\theta}_b$ for some $a, b \in \mathbb{N}_n$ satisfying $a \neq b$. With these particular indices a and b, equation (4.62) takes the form

$$\sin(\mu)^2 \sum_{c=1}^{n} \left(\frac{\bar{\hat{F}}_c \hat{F}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a + \hat{\theta}_c)^2} + \frac{\hat{F}_c \bar{\hat{F}}_{n+c}}{\sinh(\mathrm{i}\mu - \hat{\theta}_a - \hat{\theta}_c)^2} \right) = -\frac{2\sin(\mu)\sin(\mu-\nu)}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)^2}. \tag{4.68}$$

Now, by plugging this formula into (4.60), we obtain immediately that

$$0 = 1 + \frac{\sin(\mu-\nu)^2}{\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)^2}, \tag{4.69}$$

which in turn implies that

$$\begin{aligned} \sin(\mu-\nu)^2 &= -\sinh(\mathrm{i}\mu - 2\hat{\theta}_a)^2 \\ &= \sin(\mu)^2\cosh(2\hat{\theta}_a)^2 - \cos(\mu)^2\sinh(2\hat{\theta}_a)^2 + \mathrm{i}\sin(\mu)\cos(\mu)\sinh(4\hat{\theta}_a). \end{aligned} \tag{4.70}$$

Since $\hat{\theta}_a \neq 0$ and since $\sin(\mu) \neq 0$, the imaginary part of the above equation leads to the relation $\cos(\mu) = 0$, whence $\sin(\mu)^2 = 1$ also follows. Now, by plugging these observations into the real part of (4.70), we end up with the contradiction

$$1 \geq \sin(\mu-\nu)^2 = \cosh(2\hat{\theta}_a)^2 > 1. \tag{4.71}$$

Thus, if $a, b \in \mathbb{N}_n$ and $a \neq b$, then necessarily we have $\hat{\theta}_a \neq \hat{\theta}_b$. $\square$

Since the spectrum of L (4.34) is simple, it follows that the dependence of the eigenvalues on the matrix entries is smooth. Therefore, recalling (4.64), it is clear that each $\hat{\theta}_c$ ($c \in \mathbb{N}_n$) can be seen as a smooth function on P (4.2), i.e.,

$$\hat{\theta}_c \in C^\infty(P). \tag{4.72}$$

To conclude this subsection, we also offer a few remarks on the additional constraint appearing in (4.63), that we keep in effect in the rest of the chapter. Naively, this assumption excludes a 1-dimensional subset from the 2-dimensional space of the parameters (μ, ν). However, looking back to the Hamiltonian H (4.5), it is clear that the effective coupling constants of our van Diejen systems are rather the positive numbers $\sin(\mu)^2$ and $\sin(\nu)^2$. Therefore, keeping in mind (4.6), on the parameters μ and ν we could have imposed the requirement, say,

$$(\mu, \nu) \in ((0, \pi/4) \times [-\pi/2, 0)) \cup ([\pi/4, \pi/2] \times (0, \pi/2]), \tag{4.73}$$

at the outset. The point is that, under the requirement (4.73), the equation $\sin(2\mu - \nu) = 0$ is equivalent to the pair of equations $\sin(\mu)^2 = 1/2$ and $\sin(\nu)^2 = 1$. To put it differently, our observation is that under the assumptions (4.6) and (4.63) the pair $(\sin(\mu)^2, \sin(\nu)^2)$ formed by the relevant coupling constants can take on any values from the 'square' $(0, 1] \times (0, 1]$, except the *single point* $(1/2, 1)$. From the proof of Lemma 4.4, especially from equation (4.66), one may get the impression that even this very slight technical assumption can be relaxed by further analyzing the properties of column vector $\hat{F}$ (4.53). However, we do not wish to pursue this direction in this chapter.

4.3 Analyzing the dynamics

In this section we wish to study the dynamics generated by the Hamiltonian H (4.5). Recalling the formulae (4.32) and (4.34), by the obvious relationship

$$H = \sum_{c=1}^{n} \cosh(\theta_c) u_c = \frac{1}{2}\operatorname{tr}(L) \tag{4.74}$$

we can make the first contact of our van Diejen system with the proposed Lax matrix L (4.34). As an important ingredient of the forthcoming analysis, let us introduce the Hamiltonian vector field $\boldsymbol{X}_H \in \mathfrak{X}(P)$ with the usual definition

$$\boldsymbol{X}_H[f] = \{f, H\} \qquad (f \in C^\infty(P)). \tag{4.75}$$

Working with the convention (4.4), for the time evolution of the global coordinate functions λ_a and θ_a ($a \in \mathbb{N}_n$) we can clearly write

$$\dot{\lambda}_a = \boldsymbol{X}_H[\lambda_a] = \frac{\partial H}{\partial \theta_a} = \sinh(\theta_a) u_a, \tag{4.76}$$

$$\dot{\theta}_a = \boldsymbol{X}_H[\theta_a] = -\frac{\partial H}{\partial \lambda_a} = -\sum_{c=1}^{n} \cosh(\theta_c) u_c \frac{\partial \ln(u_c)}{\partial \lambda_a}. \tag{4.77}$$

To make the right-hand side of (4.77) more explicit, let us display the logarithmic derivatives of the constituent functions u_c. Notice that for all $a \in \mathbb{N}_n$ we can write

$$\frac{\partial \ln(u_a)}{\partial \lambda_a} = -\mathrm{Re}\bigg(\frac{2\mathrm{i}\sin(\nu)}{\sinh(2\lambda_a)\sinh(\mathrm{i}\nu + 2\lambda_a)} + \sum_{\substack{j=1 \\ (j \neq a, n+a)}}^{N} \frac{\mathrm{i}\sin(\mu)}{\sinh(\lambda_a - \Lambda_j)\sinh(\mathrm{i}\mu + \lambda_a - \Lambda_j)} \bigg), \tag{4.78}$$

while if $c \in \mathbb{N}_n$ and $c \neq a$, then we have

$$\frac{\partial \ln(u_c)}{\partial \lambda_a} = \mathrm{Re}\bigg(\frac{\mathrm{i}\sin(\mu)}{\sinh(\lambda_a - \lambda_c)\sinh(\mathrm{i}\mu + \lambda_a - \lambda_c)} - \frac{\mathrm{i}\sin(\mu)}{\sinh(\lambda_a + \lambda_c)\sinh(\mathrm{i}\mu + \lambda_a + \lambda_c)} \bigg). \tag{4.79}$$

The rest of this section is devoted to the study of the Hamiltonian dynamical system (4.76)-(4.77).

4.3.1 Completeness of the Hamiltonian vector field

Undoubtedly, the Hamiltonian (4.5) does not take the usual form one finds in the standard textbooks on classical mechanics. It is thus inevitable that we have even less intuition about the generated dynamics than in the case of the 'natural systems' characterized by a kinetic term plus a potential. To get a finer picture about the solutions of the Hamiltonian dynamics (4.76)-(4.77), we start our study with a brief analysis on the completeness of the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75).

As the first step, we introduce the strictly positive constant

$$\mathcal{S} = \min\{|\sin(\mu)|, |\sin(\nu)|\} \in (0, 1]. \tag{4.80}$$

Giving a glance at (4.32), it is evident that

$$u_n > \left(1 + \frac{\sin(\nu)^2}{\sinh(2\lambda_n)^2}\right)^{\frac{1}{2}} > \frac{|\sin(\nu)|}{\sinh(2\lambda_n)} \geq \frac{\mathcal{S}}{\sinh(2\lambda_n)}, \tag{4.81}$$

while for all $c \in \mathbb{N}_{n-1}$ we can write

$$u_c > \left(1 + \frac{\sin(\mu)^2}{\sinh(\lambda_c - \lambda_{c+1})^2}\right)^{\frac{1}{2}} > \frac{|\sin(\mu)|}{\sinh(\lambda_c - \lambda_{c+1})} \geq \frac{\mathcal{S}}{\sinh(\lambda_c - \lambda_{c+1})}. \tag{4.82}$$

Keeping in mind the above trivial inequalities, we are ready to prove the following result.

Theorem 4.5. *The Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) generated by the van Diejen type Hamiltonian function H (4.5) is complete. That is, the maximum interval of existence of each integral curve of $\boldsymbol{X}_H$ is the whole real axis $\mathbb{R}$.*

Proof. Take an arbitrary point

$$\gamma_0 = (\lambda^{(0)}, \theta^{(0)}) \in P, \tag{4.83}$$

and let

$$\gamma\colon (\alpha, \beta) \to P, \qquad t \mapsto \gamma(t) = (\lambda(t), \theta(t)) \tag{4.84}$$

be the unique maximally defined integral curve of $\boldsymbol{X}_H$ with $-\infty \leq \alpha < 0 < \beta \leq \infty$ satisfying the initial condition $\gamma(0) = \gamma_0$. Since the Hamiltonian H is smooth, the existence, the uniqueness, and also the smoothness of such a maximal solution are obvious. Our goal is to show that for the domain of the maximally defined trajectory γ (4.84) we have $(\alpha, \beta) = \mathbb{R}$; that is, $\alpha = -\infty$ and $\beta = \infty$.

Arguing by contradiction, first let us *suppose* that $\beta < \infty$. Since the Hamiltonian H is a first integral, for all $t \in (\alpha, \beta)$ and for all $a \in \mathbb{N}_n$ we can write

$$H(\gamma_0) = H(\gamma(t)) = \sum_{c=1}^{n} \cosh(\theta_c(t)) u_c(\lambda(t)) > \cosh(\theta_a(t)) u_a(\lambda(t)), \tag{4.85}$$

whence the estimation

$$H(\gamma_0) > \cosh(|\theta_a(t)|) \geq \frac{1}{2} e^{|\theta_a(t)|} \tag{4.86}$$

is also immediate. Thus, upon introducing the cube

$$\mathcal{C} = [-\ln(2H(\gamma_0)), \ln(2H(\gamma_0))]^n \subseteq \mathbb{R}^n, \tag{4.87}$$

from (4.86) we infer at once that

$$\theta(t) \in \mathcal{C} \qquad (t \in (\alpha, \beta)). \tag{4.88}$$

Turning to the equations (4.76) and (4.85), we can cook up an estimation on the growing of the vector $\lambda(t)$, too. Indeed, we see that

$$|\dot{\lambda}_1(t)| = \sinh(|\theta_1(t)|) u_1(\lambda(t)) \leq \cosh(|\theta_1(t)|) u_1(\lambda(t)) < H(\gamma_0) \qquad (t \in (\alpha, \beta)), \tag{4.89}$$

that implies immediately that for all $t \in [0, \beta)$ we have

$$|\lambda_1(t) - \lambda_1^{(0)}| = |\lambda_1(t) - \lambda_1(0)| = \left| \int_0^t \dot{\lambda}_1(s)\, \mathrm{d}s \right| \leq \int_0^t |\dot{\lambda}_1(s)|\, \mathrm{d}s \leq tH(\gamma_0) < \beta H(\gamma_0). \tag{4.90}$$

Therefore, with the aid of the strictly positive constant

$$\rho = \lambda_1^{(0)} + \beta H(\gamma_0) \in (0, \infty), \tag{4.91}$$

we end up with the estimation

$$\lambda_1(t) = |\lambda_1(t)| = |\lambda_1^{(0)} + \lambda_1(t) - \lambda_1^{(0)}| \leq |\lambda_1^{(0)}| + |\lambda_1(t) - \lambda_1^{(0)}| < \rho \qquad (t \in [0, \beta)). \tag{4.92}$$

Since $\lambda(t)$ moves in the configuration space Q (4.1), the above observation entails that

$$\rho > \lambda_1(t) > \ldots > \lambda_n(t) > 0 \qquad (t \in [0, \beta)). \tag{4.93}$$

To proceed further, now for all $\varepsilon > 0$ we define the subset $Q_\varepsilon \subseteq \mathbb{R}^n$ consisting of those real n-tuples $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfy the inequalities

$$\rho \geq x_1 \text{ and } 2x_n \geq \varepsilon \text{ and } x_c \geq x_{c+1} + \varepsilon \text{ for all } c \in \mathbb{N}_{n-1}, \tag{4.94}$$

simultaneously. In other words,

$$Q_\varepsilon = \{x \in \mathbb{R}^n \,|\, \rho \geq x_1\} \cap \{x \in \mathbb{R}^n \,|\, 2x_n \geq \varepsilon\} \cap \bigcap_{c=1}^{n-1} \{x \in \mathbb{R}^n \,|\, x_c - x_{c+1} \geq \varepsilon\}. \tag{4.95}$$

Notice that Q_ε is a bounded and closed subset of $\mathbb{R}^n$. Moreover, by comparing the definitions (4.1) and (4.94), it is evident that $Q_\varepsilon \subseteq Q$. Since the cube $\mathcal{C}$ (4.87) is also a compact subset of $\mathbb{R}^n$, we conclude that the Cartesian product $Q_\varepsilon \times \mathcal{C}$ is a *compact* subset of the phase space P (4.2). Therefore, due to the assumption $\beta < \infty$, after some time the maximally defined trajectory γ (4.84) escapes from $Q_\varepsilon \times \mathcal{C}$, as can be read off from any standard reference on dynamical systems (see e.g. [2, Theorem 2.1.18]). More precisely, there is some $\tau_\varepsilon \in [0, \beta)$ such that

$$(\lambda(t), \theta(t)) \in P \setminus (Q_\varepsilon \times \mathcal{C}) = ((Q \setminus Q_\varepsilon) \times \mathcal{C}) \cup (Q \times (\mathbb{R}^n \setminus \mathcal{C})) \qquad (t \in (\tau_\varepsilon, \beta)), \tag{4.96}$$

where the union above is actually a disjoint union. For instance, due to the relationship (4.88), at the mid-point

$$t_\varepsilon = \frac{\tau_\varepsilon + \beta}{2} \in (\tau_\varepsilon, \beta) \tag{4.97}$$

we can write that

$$\lambda(t_\varepsilon) \in Q \setminus Q_\varepsilon \subseteq \mathbb{R}^n \setminus Q_\varepsilon. \tag{4.98}$$

Therefore, simply by taking the complement of Q_ε (4.95), and also keeping in mind (4.92), it is evident that

$$\min\{\lambda_1(t_\varepsilon) - \lambda_2(t_\varepsilon), \dots, \lambda_{n-1}(t_\varepsilon) - \lambda_n(t_\varepsilon), 2\lambda_n(t_\varepsilon)\} < \varepsilon, \tag{4.99}$$

which in turn implies that

$$\max\left\{\frac{1}{\sinh(\lambda_1(t_\varepsilon) - \lambda_2(t_\varepsilon))}, \dots, \frac{1}{\sinh(\lambda_{n-1}(t_\varepsilon) - \lambda_n(t_\varepsilon))}, \frac{1}{\sinh(2\lambda_n(t_\varepsilon))}\right\} > \frac{1}{\sinh(\varepsilon)}. \tag{4.100}$$

Now, since $\varepsilon > 0$ was arbitrary, the estimations (4.81) and (4.82) immediately lead to the contradiction

$$\begin{aligned} H(\gamma_0) = H(\gamma(t_\varepsilon)) &= \sum_{c=1}^{n} \cosh(\theta_c(t_\varepsilon)) u_c(\lambda(t_\varepsilon)) \\ &\geq \sum_{c=1}^{n} u_c(\lambda(t_\varepsilon)) > \frac{\mathcal{S}}{\sinh(2\lambda_n(t_\varepsilon))} + \sum_{c=1}^{n-1} \frac{\mathcal{S}}{\sinh(\lambda_c(t_\varepsilon) - \lambda_{c+1}(t_\varepsilon))} > \frac{\mathcal{S}}{\sinh(\varepsilon)}. \end{aligned} \tag{4.101}$$

Therefore, necessarily, $\beta = \infty$.

Either by repeating the above ideas, or by invoking a time-reversal argument, one can also show that $\alpha = -\infty$, whence the proof is complete. □

4.3.2 Dynamics of the vector F

Looking back to the definition (4.34), we see that the column vector F (4.33) is important building block of the matrix L. Therefore, the study of the derivative of L along the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) does require close control over the derivative of the components of F, too. Upon introducing the auxiliary functions

$$\varphi_k = \frac{1}{F_k} \boldsymbol{X}_H[F_k] \qquad (k \in \mathbb{N}_N), \tag{4.102}$$

for all $a \in \mathbb{N}_n$ we can write

$$\begin{aligned} 2\varphi_a = \boldsymbol{X}_H[\ln(F_a^2)] &= \boldsymbol{X}_H[\theta_a + \ln(u_a)] = \{\theta_a + \ln(u_a), H\} \\ &= \sum_{c=1}^{n} \left(\sinh(\theta_c) u_c \frac{\partial \ln(u_a)}{\partial \lambda_c} - \cosh(\theta_c) u_c \frac{\partial \ln(u_c)}{\partial \lambda_a} \right). \end{aligned} \tag{4.103}$$

Therefore, due to the explicit formulae (4.78) and (4.79), we have complete control over the first n components of (4.102). Turning to the remaining components, from the definition (4.33) it is evident that $F_{n+a} = F_a^{-1}\bar{z}_a$, whence the relationship

$$\varphi_{n+a} = -\varphi_a + \frac{1}{\bar{z}_a}\boldsymbol{X}_H[\bar{z}_a] = -\varphi_a + \sum_{c=1}^{n} \sinh(\theta_c) u_c \frac{1}{\bar{z}_a}\frac{\partial \bar{z}_a}{\partial \lambda_c} \tag{4.104}$$

follows immediately. Notice that for all $a \in \mathbb{N}_n$ we can write that

$$\frac{1}{z_a}\frac{\partial z_a}{\partial \lambda_a} = -\frac{2\mathrm{i}\sin(\nu)}{\sinh(2\lambda_a)\sinh(\mathrm{i}\nu + 2\lambda_a)} - \sum_{\substack{j=1 \\ (j\neq a, n+a)}}^{N} \frac{\mathrm{i}\sin(\mu)}{\sinh(\lambda_a - \Lambda_j)\sinh(\mathrm{i}\mu + \lambda_a - \Lambda_j)}, \tag{4.105}$$

whereas if $c \in \mathbb{N}_n$ and $c \neq a$, then we find immediately that

$$\frac{1}{z_a}\frac{\partial z_a}{\partial \lambda_c} = \frac{\mathrm{i}\sin(\mu)}{\sinh(\lambda_a - \lambda_c)\sinh(\mathrm{i}\mu + \lambda_a - \lambda_c)} - \frac{\mathrm{i}\sin(\mu)}{\sinh(\lambda_a + \lambda_c)\sinh(\mathrm{i}\mu + \lambda_a + \lambda_c)}. \tag{4.106}$$

The above observations can be summarized as follows.

Proposition 4.6. *For the derivative of the components of the function F (4.33) along the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) we have*

$$\boldsymbol{X}_H[F_k] = \varphi_k F_k \qquad (k \in \mathbb{N}_N), \tag{4.107}$$

where for each $a \in \mathbb{N}_n$ we can write

$$\varphi_a = \mathrm{Re}\bigg(\frac{\mathrm{i}\sin(\nu)e^{-\theta_a}u_a}{\sinh(2\lambda_a)\sinh(\mathrm{i}\nu + 2\lambda_a)} + \frac{1}{2}\sum_{\substack{j=1 \\ (j\neq a, n+a)}}^{N} \frac{\mathrm{i}\sin(\mu)(e^{-\theta_a}u_a + e^{\Theta_j}u_j)}{\sinh(\lambda_a - \Lambda_j)\sinh(\mathrm{i}\mu + \lambda_a - \Lambda_j)}\bigg), \tag{4.108}$$

whereas

$$\varphi_{n+a} = -\varphi_a - \frac{2\mathrm{i}\sin(\nu)\sinh(\theta_a)u_a}{\sinh(2\lambda_a)\sinh(\mathrm{i}\nu - 2\lambda_a)} - \sum_{\substack{j=1 \\ (j\neq a, n+a)}}^{N} \frac{\mathrm{i}\sin(\mu)(\sinh(\theta_a)u_a - \sinh(\Theta_j)u_j)}{\sinh(\lambda_a - \Lambda_j)\sinh(\mathrm{i}\mu - \lambda_a + \Lambda_j)}. \tag{4.109}$$

By invoking Proposition 4.1, let us observe that for the inverse of the matrix L (4.34) we can write that $L^{-1} = CLC$, whence for the Hermitian matrix $L - L^{-1}$ we have

$$(L - L^{-1})C + C(L - L^{-1}) = LC - CLC^2 + CL - C^2LC = 0. \tag{4.110}$$

Thus, the matrix valued smooth function $(L - L^{-1})/2$ defined on the phase space P

(4.2) takes values in the subspace $\mathfrak{p}$ (4.13). Therefore, by taking its projection onto the Abelian subspace $\mathfrak{a}$ (4.16), we obtain the diagonal matrix

$$D = (L - L^{-1})_{\mathfrak{a}}/2 \in \mathfrak{a} \tag{4.111}$$

with diagonal entries

$$D_{j,j} = \sinh(\Theta_j) u_j \qquad (j \in \mathbb{N}_N). \tag{4.112}$$

Next, by projecting the function $(L - L^{-1})/2$ onto the complementary subspace $\mathfrak{a}^\perp$, we obtain the off-diagonal matrix

$$Y = (L - L^{-1})_{\mathfrak{a}^\perp}/2 \in \mathfrak{a}^\perp, \tag{4.113}$$

which in turn allows us to introduce the matrix valued smooth function

$$Z = \sinh(\widetilde{\mathrm{ad}}_{\boldsymbol{\Lambda}})^{-1} Y \in \mathfrak{m}^\perp, \tag{4.114}$$

too. Since $\lambda \in Q$ (4.1), the corresponding diagonal matrix $\boldsymbol{\Lambda}$ (4.30) is regular in the sense that it takes values in the open Weyl chamber $\mathfrak{c} \subseteq \mathfrak{a}_{\mathrm{reg}}$ (4.26). Therefore, Z is indeed a well-defined off-diagonal $N \times N$ matrix, and its non-trivial entries take the form

$$Z_{k,l} = \frac{Y_{k,l}}{\sinh(\Lambda_k - \Lambda_l)} = \frac{L_{k,l} - (L^{-1})_{k,l}}{2\sinh(\Lambda_k - \Lambda_l)} \qquad (k, l \in \mathbb{N}_N,\ k \neq l). \tag{4.115}$$

Utilizing Z, for each $a \in \mathbb{N}_n$ we also define the function

$$\mathcal{M}_a = \frac{\mathrm{i}}{F_a} \mathrm{Im}((ZF)_a) = \frac{\mathrm{i}}{F_a} \mathrm{Im}\left(\sum_{j=1}^{N} Z_{a,j} F_j\right) \in C^\infty(P). \tag{4.116}$$

Recalling the subspace $\mathfrak{m}$ (4.19), it is clear that

$$B_{\mathfrak{m}} = \mathrm{diag}(\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{M}_1, \dots, \mathcal{M}_n) \in \mathfrak{m} \tag{4.117}$$

is a well-defined function. Having the above objects at our disposal, the content of Proposition 4.6 can be recast into a more convenient matrix form as follows.

Lemma 4.7. *With the aid of the smooth functions Z (4.114) and $B_{\mathfrak{m}}$ (4.117), for the derivative of the column vector F (4.33) along the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) we can write*

$$\boldsymbol{X}_H[F] = (Z - B_{\mathfrak{m}})F. \tag{4.118}$$

Proof. Upon introducing the column vector

$$J = \boldsymbol{X}_H[F] + B_{\mathfrak{m}}F - ZF \in \mathbb{C}^N, \tag{4.119}$$

it is enough to prove that $J_k = 0$ for all $k \in \mathbb{N}_N$, at each point (λ, θ) of the phase space P (4.2). Starting with the upper n components of J, notice that by Proposition 4.6 and the formulae (4.115)-(4.117) we can write that

$$J_a = \frac{1}{2} e^{-\frac{\theta_a}{2}} u_a^{\frac{3}{2}} G_a \qquad (a \in \mathbb{N}_n), \tag{4.120}$$

where G_a is an appropriate function depending only on λ. More precisely, it has the form

$$G_a = \mathrm{Re}\bigg(\frac{2\mathrm{i}\sin(\nu)}{\sinh(2\lambda_a)\sinh(\mathrm{i}\nu + 2\lambda_a)} + \sum_{\substack{j=1 \\ (j \neq a, n+a)}}^{N} \frac{\mathrm{i}\sin(\mu)(1 + \bar{z}_j \bar{z}_a^{-1})}{\sinh(\lambda_a - \Lambda_j)\sinh(\mathrm{i}\mu + \lambda_a - \Lambda_j)} \\ + \frac{\mathrm{i}\sin(\mu)(z_a \bar{z}_a^{-1} - 1) + \mathrm{i}\sin(\mu - \nu)(\bar{z}_a^{-1} - z_a^{-1})}{\sinh(2\lambda_a)\sinh(\mathrm{i}\mu + 2\lambda_a)} \bigg), \tag{4.121}$$

that can be made quite explicit by exploiting the definition of the constituent functions z_j (4.31). Now, following the same strategy we applied in the proof of Proposition 4.1, let us introduce a complex valued function g_a depending only on a single complex variable w, obtained simply by replacing λ_a with $\lambda_a + w$ in the explicit expression of right-hand side of the above equation (4.121). In mod $\mathrm{i}\pi$ sense this meromorphic function has at most first order poles at the points

$$w \equiv -\lambda_a,\ w \equiv \pm\mathrm{i}\mu/2 - \lambda_a,\ w \equiv \pm\mathrm{i}\nu/2 - \lambda_a,\ w \equiv \Lambda_j - \lambda_a,\ w \equiv \pm(\mathrm{i}\mu + \Lambda_j) - \lambda_a\ (j \in \mathbb{N}_N). \tag{4.122}$$

However, at each of these points the residue of g_a turns out to be zero. Moreover, it is obvious that $g_a(w)$ vanishes as $\mathrm{Re}(w) \to \infty$, therefore Liouville's theorem implies that $g_a(w) = 0$ for all $w \in \mathbb{C}$. In particular $G_a = g_a(0) = 0$, and so by (4.120) we conclude that $J_a = 0$.

Turning to the lower n components of the column vector J (4.119), let us note that our previous result $J_a = 0$ allows us to write that

$$J_{n+a} = -\sinh(\theta_a) e^{-\frac{\theta_a}{2}} u_a^{\frac{1}{2}} \bar{z}_a G_{n+a} \qquad (a \in \mathbb{N}_n), \tag{4.123}$$

where G_{n+a} is again an appropriate smooth function depending only on λ, as can be seen from the formula

$$\begin{aligned}G_{n+a} =&\frac{2\mathrm{i}\sin(\nu)}{\sinh(2\lambda_a)\sinh(\mathrm{i}\nu-2\lambda_a)} - \frac{\mathrm{i}\sin(\mu)+\mathrm{i}\sin(\mu-\nu)\bar{z}_a^{-1}}{\sinh(2\lambda_a)\sinh(\mathrm{i}\mu-2\lambda_a)} + \bar{z}_a^{-1}\frac{\mathrm{i}\sin(\mu)z_a+\mathrm{i}\sin(\mu-\nu)}{\sinh(2\lambda_a)\sinh(\mathrm{i}\mu+2\lambda_a)}\\ &+ \sum_{\substack{j=1\\(j\neq a,n+a)}}^{N}\frac{1}{\sinh(\lambda_a-\Lambda_j)}\left(\frac{\mathrm{i}\sin(\mu)}{\sinh(\mathrm{i}\mu-\lambda_a+\Lambda_j)} + \frac{\mathrm{i}\sin(\mu)\bar{z}_a^{-1}\bar{z}_j}{\sinh(\mathrm{i}\mu+\lambda_a-\Lambda_j)}\right).\end{aligned} \tag{4.124}$$

Next, let us plug the definition of z_j (4.31) into the above expression (4.124) and introduce the complex valued function g_{n+a} of $w \in \mathbb{C}$ by replacing λ_a with $\lambda_a + w$ in the resulting formula. Note that g_{n+a} has at most first order poles at the points

$$w \equiv -\lambda_a,\ w \equiv \pm\mathrm{i}\mu/2-\lambda_a,\ w \equiv \Lambda_j-\lambda_a\ (j \in \mathbb{N}_N) \pmod{\mathrm{i}\pi}, \tag{4.125}$$

but all these singularities are removable. Since $g_{n+a}(w) \to 0$ as $\mathrm{Re}(w) \to \infty$, the boundedness of the periodic function g_{n+a} is also obvious. Thus, Liouville's theorem entails that $g_{n+a} = 0$ on the whole complex plane, whence the relationship $G_{n+a} = g_{n+a}(0) = 0$ also follows. Now, looking back to the equation (4.123), we end up with the desired equation $J_{n+a} = 0$. □

4.3.3 Lax representation of the dynamics

Based on our proposed Lax matrix (4.34), in this subsection we wish to construct a Lax representation for the dynamics of the van Diejen system (4.5). As it turns out, Lemmas 4.3 and 4.7 prove to be instrumental in our approach. As the first step, by applying the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) on the Ruijsenaars type commutation relation (4.48), let us observe that the Leibniz rule yields

$$\begin{aligned}&e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\boldsymbol{\Lambda}}}\left(\boldsymbol{X}_H[L]-\left[L,e^{-\boldsymbol{\Lambda}}\boldsymbol{X}_H[e^{\boldsymbol{\Lambda}}]\right]\right)-e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\boldsymbol{\Lambda}}}\left(\boldsymbol{X}_H[L]+\left[L,\boldsymbol{X}_H[e^{\boldsymbol{\Lambda}}]e^{-\boldsymbol{\Lambda}}\right]\right)\\ &\quad= 2\mathrm{i}\sin(\mu)\left(\boldsymbol{X}_H[F]F^*+F(\boldsymbol{X}_H[F])^*\right).\end{aligned} \tag{4.126}$$

By comparing the formula appearing in (4.76) with the matrix entries (4.112) of the diagonal matrix D, it is clear that

$$\boldsymbol{X}_H[\boldsymbol{\Lambda}] = D, \tag{4.127}$$

which in turn implies that

$$e^{-\boldsymbol{\Lambda}}\boldsymbol{X}_H[e^{\boldsymbol{\Lambda}}] = \boldsymbol{X}_H[e^{\boldsymbol{\Lambda}}]e^{-\boldsymbol{\Lambda}} = D. \tag{4.128}$$

Thus, the above equation (4.126) can be cast into the fairly explicit form

$$\begin{aligned} &e^{\mathrm{i}\mu} e^{\mathrm{ad}_{\boldsymbol{\Lambda}}} (\boldsymbol{X}_H[L] - [L, D]) - e^{-\mathrm{i}\mu} e^{-\mathrm{ad}_{\boldsymbol{\Lambda}}} (\boldsymbol{X}_H[L] + [L, D]) \\ &\quad = 2\mathrm{i} \sin(\mu) \left(\boldsymbol{X}_H[F] F^* + F(\boldsymbol{X}_H[F])^* \right), \end{aligned} \tag{4.129}$$

which serves as the starting point in our analysis on the derivative $\boldsymbol{X}_H[L]$. Before formulating the main result of this subsection, over the phase space P (4.2) we define the matrix valued function

$$B_{\mathfrak{m}^\perp} = -\coth(\widetilde{\mathrm{ad}_{\boldsymbol{\Lambda}}}) Y \in \mathfrak{m}^\perp. \tag{4.130}$$

Recalling the definition (4.113), we see that $B_{\mathfrak{m}^\perp}$ is actually an off-diagonal matrix. Furthermore, for its non-trivial entries we have the explicit expressions

$$(B_{\mathfrak{m}^\perp})_{k,l} = -\coth(\Lambda_k - \Lambda_l) \frac{L_{k,l} - (L^{-1})_{k,l}}{2} \qquad (k, l \in \mathbb{N}_N,\ k \neq l). \tag{4.131}$$

Finally, with the aid of the diagonal matrix $B_{\mathfrak{m}}$ (4.117), over the phase space P (4.2) we also define the $\mathfrak{k}$-valued smooth function

$$B = B_{\mathfrak{m}} + B_{\mathfrak{m}^\perp} \in \mathfrak{k}. \tag{4.132}$$

Theorem 4.8. *The derivative of the matrix valued function L (4.34) along the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) takes the Lax form*

$$\boldsymbol{X}_H[L] = [L, B]. \tag{4.133}$$

In other words, the matrices L (4.34) and B (4.132) provide a Lax pair for the dynamics generated by the Hamiltonian (4.5).

Proof. For simplicity, let us introduce the matrix valued smooth functions

$$\Psi = \boldsymbol{X}_H[L] - [L, B] \quad \text{and} \quad R = \sinh(\mathrm{i}\mu \mathrm{Id}_{\mathfrak{gl}(N,\mathbb{C})} + \mathrm{ad}_{\boldsymbol{\Lambda}}) \Psi \tag{4.134}$$

defined on the phase space P (4.2). Our goal is to prove that $\Psi = 0$. However, since $\sin(\mu) \neq 0$, the linear operator

$$\sinh(\mathrm{i}\mu \mathrm{Id}_{\mathfrak{gl}(N,\mathbb{C})} + \mathrm{ad}_{\boldsymbol{\Lambda}}) \in \mathrm{End}(\mathfrak{gl}(N, \mathbb{C})) \tag{4.135}$$

is invertible at each point of P, whence it is enough to show that $R = 0$. For this reason, notice that from the relationship (4.129) we can infer that

$$\begin{aligned}2R =&e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}\Psi - e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}\Psi\\ =&2\mathrm{i}\sin(\mu)\left(\boldsymbol{X}_H[F]F^* + F(\boldsymbol{X}_H[F])^*\right) - \left(e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[L, B_{\mathfrak{m}^\perp}] - e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[L, B_{\mathfrak{m}^\perp}]\right)\\ &- \left(e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[L, B_{\mathfrak{m}}] - e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[L, B_{\mathfrak{m}}]\right) - \left(e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[D, L] + e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[D, L]\right).\end{aligned}\tag{4.136}$$

Our strategy is to inspect the right-hand side of the above equation term-by-term.

As a preparatory step, from the definitions of D (4.111) and Y (4.113) we see that

$$(L - L^{-1})/2 = D + Y,\tag{4.137}$$

thus the commutation relation

$$[L, Y] = [L, -D + (L - L^{-1})/2] = [D, L]\tag{4.138}$$

readily follows. Keeping in mind the relationship (4.138) and the standard hyperbolic functional equations

$$\coth(w) \pm 1 = \frac{e^{\pm w}}{\sinh(w)} \qquad (w \in \mathbb{C}),\tag{4.139}$$

from the definitions of Z (4.114) and $B_{\mathfrak{m}^\perp}$ (4.130) we infer that

$$\begin{aligned}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[L, B_{\mathfrak{m}^\perp}] =& - e^{\mathrm{ad}_{\mathbf{\Lambda}}}[L, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y]\\ =& - e^{\mathrm{ad}_{\mathbf{\Lambda}}}\left([L, (\coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}}) - \mathrm{Id}_{\mathfrak{m}^\perp\oplus\mathfrak{a}^\perp})Y] + [L, Y]\right)\\ =& - e^{\mathrm{ad}_{\mathbf{\Lambda}}}\left([L, e^{-\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}}}\sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] + [D, L]\right)\\ =& - [e^{\mathrm{ad}_{\mathbf{\Lambda}}}L, Z] - e^{\mathrm{ad}_{\mathbf{\Lambda}}}[D, L].\end{aligned}\tag{4.140}$$

Along the same lines, one finds immediately that

$$e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[L, B_{\mathfrak{m}^\perp}] = -[e^{-\mathrm{ad}_{\mathbf{\Lambda}}}L, Z] + e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[D, L].\tag{4.141}$$

At this point let us recall that Z (4.114) takes values in the subspace $\mathfrak{m}^\perp \subseteq \mathfrak{k}$, thus it is anti-Hermitian and commutes with the matrix C (4.8). Therefore, by utilizing equations (4.140) and (4.141), the application of the commutation relation (4.48) leads to the relationship

$$\begin{aligned}
&e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[L,B_{\mathfrak{m}^\perp}]-e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[L,B_{\mathfrak{m}^\perp}]\\
&\quad=-[e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}L-e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}L,Z]-\big(e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[D,L]+e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[D,L]\big)\\
&\quad=-[2\mathrm{i}\sin(\mu)FF^*+2\mathrm{i}\sin(\mu-\nu)C,Z]-\big(e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[D,L]+e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[D,L]\big)\\
&\quad=2\mathrm{i}\sin(\mu)\left((ZF)F^*+F(ZF)^*\right)-\big(e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[D,L]+e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[D,L]\big)\,.
\end{aligned}\tag{4.142}$$

To proceed further, let us recall that $B_{\mathfrak{m}}$ (4.117) takes values in $\mathfrak{m}\subseteq\mathfrak{k}$, whence it is also anti-Hermitian and also commutes with the matrix C (4.8). Thus, by applying commutation relation (4.48) again, we obtain at once that

$$\begin{aligned}
&e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[L,B_{\mathfrak{m}}]-e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[L,B_{\mathfrak{m}}]\\
&\quad=e^{\mathrm{i}\mu}[e^{\mathrm{ad}_{\mathbf{\Lambda}}}L,e^{\mathrm{ad}_{\mathbf{\Lambda}}}B_{\mathfrak{m}}]-e^{-\mathrm{i}\mu}[e^{-\mathrm{ad}_{\mathbf{\Lambda}}}L,e^{-\mathrm{ad}_{\mathbf{\Lambda}}}B_{\mathfrak{m}}]=[e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}L-e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}L,B_{\mathfrak{m}}]\\
&\quad=[2\mathrm{i}\sin(\mu)FF^*+2\mathrm{i}\sin(\mu-\nu)C,B_{\mathfrak{m}}]=-2\mathrm{i}\sin(\mu)\left((B_{\mathfrak{m}}F)F^*+F(B_{\mathfrak{m}}F)^*\right).
\end{aligned}\tag{4.143}$$

Now, by plugging the expressions (4.142) and (4.143) into (4.136), we obtain that

$$R=\mathrm{i}\sin(\mu)\left((\boldsymbol{X}_H[F]-ZF+B_{\mathfrak{m}}F)F^*+F(\boldsymbol{X}_H[F]-ZF+B_{\mathfrak{m}}F)^*\right).\tag{4.144}$$

Giving a glance at Lemma 4.7, we conclude that $R=0$, thus the Theorem follows. □

At this point we wish to make a short comment on matrix $B=B(\lambda,\theta;\mu,\nu)$ (4.132) appearing in the Lax representation (4.133) of the dynamics (4.5). It is an important fact that by taking its 'rational limit' we can recover the second member of the Lax pair of the rational C_n van Diejen system with two parameters μ and ν. More precisely, up to some irrelevant numerical factors, in the $\alpha\to 0^+$ limit the matrix $\alpha B(\alpha\lambda,\theta;\alpha\mu,\alpha\nu)$ tends to the second member $\hat{\mathcal{B}}(\lambda,\theta;\mu,\nu,\kappa=0)$ of the rational Lax pair, that first appeared in equation (4.60) of the recent paper [109]. In other words, matrix B (4.132) is an appropriate hyperbolic generalization of the 'rational' matrix $\hat{\mathcal{B}}$ with two coupling parameters. We can safely state that the results presented in [109] has played a decisive role in our present work. As a matter of fact, most probably we could not have guessed the form of the non-trivial building blocks (4.117) and (4.130) without the knowledge of rational analogue of B.

In order to harvest some consequences of the Lax representation (4.133), we continue with a simple corollary of Theorem 4.8, that proves to be quite handy in the developments of the next subsection.

Proposition 4.9. *For the derivatives of the matrix valued smooth functions D (4.111)* *and* Y (4.113) *along the Hamiltonian vector field* $\boldsymbol{X}_H$ (4.75) *we have*

$$\boldsymbol{X}_H[D] = [Y, B_{\mathfrak{m}^\perp}]_{\mathfrak{a}} \quad \text{and} \quad \boldsymbol{X}_H[Y] = [Y, B_{\mathfrak{m}^\perp}]_{\mathfrak{a}^\perp} + [D, B_{\mathfrak{m}^\perp}] + [Y, B_{\mathfrak{m}}]. \tag{4.145}$$

Proof. As a consequence of Proposition 4.1, for the inverse of L we can write that $L^{-1} = CLC$. Since the matrix valued function B (4.132) takes values in $\mathfrak{k}$ (4.12), from Theorem 4.8 we infer that

$$\boldsymbol{X}[L^{-1}] = C\boldsymbol{X}_H[L]C = C[L, B]C = [CLC, CBC] = [L^{-1}, B], \tag{4.146}$$

thus the equation

$$\boldsymbol{X}_H[(L - L^{-1})/2] = [(L - L^{-1})/2, B] \tag{4.147}$$

is immediate. Due to the relationship (4.137), by simply projecting of the above equation onto the subspaces $\mathfrak{a}$ and $\mathfrak{a}^\perp$, respectively, the derivatives displayed in (4.145) follow at once. □

4.3.4 Geodesic interpretation

The geometric study of the CMS type integrable systems goes back to the fundamental works of Olshanetsky and Perelomov (see e.g. [95, 98]). Since their landmark papers the so-called projection method has been vastly generalized to cover many variants of the CMS type particle systems. By now some result are available in the context of the RSvD models, too. For details, see e.g. [68, 42, 43, 36, 106, 107]. The primary goal of this subsection is to show that the Hamiltonian flow generated by the Hamiltonian (4.5) can be also obtained by an appropriate 'projection method' from the geodesic flow of the Lie group $\mathrm{U}(n, n)$. In order to make this statement more precise, take the maximal integral curve

$$\mathbb{R} \ni t \mapsto (\lambda(t), \theta(t)) = (\lambda_1(t), \dots, \lambda_n(t), \theta_1(t), \dots, \theta_n(t)) \in P \tag{4.148}$$

of the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) satisfying the initial condition

$$\gamma(0) = \gamma_0, \tag{4.149}$$

where $\gamma_0 \in P$ is an arbitrary point. By exploiting Proposition 4.9, we start our analysis with the following observation.

Proposition 4.10. *Along the maximally defined trajectory* (4.148)*, the time evolution of the diagonal matrix* $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(t) \in \mathfrak{c}$ (4.30) *obeys the second order differential equation*

$$\ddot{\boldsymbol{\Lambda}} + [Y, \coth(\widetilde{\mathrm{ad}}_{\boldsymbol{\Lambda}})Y]_{\mathfrak{a}} = 0, \tag{4.150}$$

whilst for the evolution of $Y = Y(t)$ (4.113) we have the first order equation

$$\dot{Y} + [Y, \coth(\widetilde{\mathrm{ad}}_{\boldsymbol{\Lambda}})Y]_{\mathfrak{a}^\perp} - [Y, B_{\mathfrak{m}}] + [\dot{\boldsymbol{\Lambda}}, \coth(\widetilde{\mathrm{ad}}_{\boldsymbol{\Lambda}})Y] = 0. \tag{4.151}$$

Proof. Due to equation (4.127), along the solution curve (4.148) we can write

$$\dot{\boldsymbol{\Lambda}} = D, \tag{4.152}$$

whereas from the relationships displayed in (4.145) we get

$$\dot{D} = [Y, B_{\mathfrak{m}^\perp}]_{\mathfrak{a}} \quad \text{and} \quad \dot{Y} = [Y, B_{\mathfrak{m}^\perp}]_{\mathfrak{a}^\perp} + [D, B_{\mathfrak{m}^\perp}] + [Y, B_{\mathfrak{m}}]. \tag{4.153}$$

Recalling the definition (4.130), equations (4.150) and (4.151) clearly follow. □

Next, by evaluating the matrices Z (4.114) and $B_{\mathfrak{m}}$ (4.117) along the fixed trajectory (4.148), for all $t \in \mathbb{R}$ we define

$$\mathcal{K}(t) = B_{\mathfrak{m}}(t) - Z(t) \in \mathfrak{k}. \tag{4.154}$$

Since the dependence of $\mathcal{K}$ on t is smooth, there is a unique maximal smooth solution

$$\mathbb{R} \ni t \mapsto k(t) \in GL(N, \mathbb{C}) \tag{4.155}$$

of the first order differential equation

$$\dot{k}(t) = k(t)\mathcal{K}(t) \qquad (t \in \mathbb{R}) \tag{4.156}$$

satisfying the initial condition

$$k(0) = \mathbf{1}_N. \tag{4.157}$$

Since (4.156) is a linear differential equation for k, the existence of such a global fundamental solution is obvious. Moreover, since $\mathcal{K}$ (4.154) takes values in the Lie algebra $\mathfrak{k}$ (4.12), the trivial observations

$$\frac{\mathrm{d}(kCk^*)}{\mathrm{d}t} = \dot{k}Ck^* + kC\dot{k}^* = k(\mathcal{K}C + C\mathcal{K}^*)k^* = 0 \quad \text{and} \quad k(0)Ck(0)^* = C \tag{4.158}$$

imply immediately that k (4.155) actually takes values in the subgroup K (4.10); that is,

$$k(t) \in K \qquad (t \in \mathbb{R}). \tag{4.159}$$

Utilizing k, we can formulate the most important technical result of this subsection.

Lemma 4.11. *The smooth function*

$$\mathbb{R} \ni t \mapsto A(t) = k(t)e^{2\mathbf{\Lambda}(t)}k(t)^{-1} \in \exp(\mathfrak{p}_{\mathrm{reg}}) \tag{4.160}$$

satisfies the second order geodesic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}A(t)}{\mathrm{d}t}A(t)^{-1}\right) = 0 \qquad (t \in \mathbb{R}). \tag{4.161}$$

Proof. First, let us observe that (4.160) is a well-defined map. Indeed, since along the trajectory (4.148) we have $\mathbf{\Lambda}(t) \in \mathfrak{c}$, from (4.28) we see that A does take values in $\exp(\mathfrak{p}_{\mathrm{reg}})$. Continuing with the proof proper, notice that for all $t \in \mathbb{R}$ we have $A^{-1} = ke^{-2\mathbf{\Lambda}}k^{-1}$ and

$$\dot{A} = \dot{k}e^{2\mathbf{\Lambda}}k^{-1} + ke^{2\mathbf{\Lambda}}2\dot{\mathbf{\Lambda}}k^{-1} - ke^{2\mathbf{\Lambda}}k^{-1}\dot{k}k^{-1}, \tag{4.162}$$

thus the formulae

$$\dot{A}A^{-1} = k\left(2\dot{\mathbf{\Lambda}} - e^{2\mathrm{ad}_{\mathbf{\Lambda}}}\mathcal{K} + \mathcal{K}\right)k^{-1} \quad \text{and} \quad A^{-1}\dot{A} = k\left(2\dot{\mathbf{\Lambda}} + e^{-2\mathrm{ad}_{\mathbf{\Lambda}}}\mathcal{K} - \mathcal{K}\right)k^{-1} \tag{4.163}$$

are immediate. Upon introducing the shorthand notations

$$\mathcal{L}(t) = \dot{\mathbf{\Lambda}}(t) + \cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}(t)})Y(t) \in \mathfrak{p}, \tag{4.164}$$

$$\mathcal{N}(t) = \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}(t)})Y(t) \in \mathfrak{k}, \tag{4.165}$$

from (4.163) we conclude that

$$\begin{aligned}\frac{\dot{A}A^{-1} + A^{-1}\dot{A}}{4} &= k\left(\dot{\mathbf{\Lambda}} - \frac{1}{2}\sinh(2\mathrm{ad}_{\mathbf{\Lambda}})\mathcal{K}\right)k^{-1} = k\left(\dot{\mathbf{\Lambda}} - \frac{1}{2}\sinh(2\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})\mathcal{K}_{\mathfrak{m}^\perp}\right)k^{-1}\\ &= k\left(\dot{\mathbf{\Lambda}} + \cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})\sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Z\right)k^{-1} = k\mathcal{L}k^{-1},\end{aligned} \tag{4.166}$$

and the relationship

$$\begin{aligned}\frac{\dot{A}A^{-1} - A^{-1}\dot{A}}{4} &= k\frac{\mathcal{K} - \cosh(2\mathrm{ad}_{\mathbf{\Lambda}})\mathcal{K}}{2}k^{-1} = -k\left(\sinh(\mathrm{ad}_{\mathbf{\Lambda}})^2\mathcal{K}\right)k^{-1}\\ &= k\left(\sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^2 Z\right)k^{-1} = k\mathcal{N}k^{-1}\end{aligned} \tag{4.167}$$

also follows.

Now, by differentiating (4.166) with respect to time t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{A}A^{-1} + A^{-1}\dot{A}}{4} = k\left(\dot{\mathcal{L}} - [\mathcal{L}, \mathcal{K}]\right)k^{-1}. \tag{4.168}$$

Recalling the definition (4.164), Leibniz rule yields

$$\dot{\mathcal{L}} = \ddot{\mathbf{\Lambda}} + [\dot{\mathbf{\Lambda}}, \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y] + \cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})\dot{Y}, \tag{4.169}$$

and the commutator

$$[\mathcal{L}, \mathcal{K}] = -[\dot{\mathbf{\Lambda}}, \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] + [\cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y, B_{\mathfrak{m}}] - [\cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y, \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] \tag{4.170}$$

is also immediate. By inspecting the right-hand side of the above equation, for the second term one can easily derive that

$$\begin{aligned}[\cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y, B_{\mathfrak{m}}] &= \frac{1}{2}[e^{\mathrm{ad}_{\mathbf{\Lambda}}}Y, B_{\mathfrak{m}}] + \frac{1}{2}[e^{-\mathrm{ad}_{\mathbf{\Lambda}}}Y, B_{\mathfrak{m}}] = \frac{1}{2}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[Y, B_{\mathfrak{m}}] + \frac{1}{2}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[Y, B_{\mathfrak{m}}] \\ &= \cosh(\mathrm{ad}_{\mathbf{\Lambda}})[Y, B_{\mathfrak{m}}] = \cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})[Y, B_{\mathfrak{m}}].\end{aligned} \tag{4.171}$$

Furthermore, bearing in mind the identities appearing in (4.139), a slightly longer calculation also reveals that the third term in (4.170) can be cast into the form

$$\begin{aligned}&[\cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y, \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] = \frac{1}{2}[e^{\mathrm{ad}_{\mathbf{\Lambda}}}Y, \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] + \frac{1}{2}[e^{-\mathrm{ad}_{\mathbf{\Lambda}}}Y, \sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] \\ &= \frac{1}{2}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[Y, e^{-\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}}}\sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] + \frac{1}{2}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[Y, e^{\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}}}\sinh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})^{-1}Y] \\ &= \cosh(\mathrm{ad}_{\mathbf{\Lambda}})[Y, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y] = [Y, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y]_{\mathfrak{a}} + \cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})[Y, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y]_{\mathfrak{a}^\perp}.\end{aligned} \tag{4.172}$$

Now, by plugging the expressions (4.171) and (4.172) into (4.170), and by applying the hyperbolic identity

$$\sinh(w) + \frac{1}{\sinh(w)} = \cosh(w)\coth(w) \qquad (w \in \mathbb{C}), \tag{4.173}$$

one finds immediately that

$$\begin{aligned}\dot{\mathcal{L}} - [\mathcal{L}, \mathcal{K}] =& \ddot{\mathbf{\Lambda}} + [Y, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y]_{\mathfrak{a}} \\ &+ \cosh(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})\left(\dot{Y} + [Y, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y]_{\mathfrak{a}^\perp} - [Y, B_{\mathfrak{m}}] + [\dot{\mathbf{\Lambda}}, \coth(\widetilde{\mathrm{ad}}_{\mathbf{\Lambda}})Y]\right).\end{aligned} \tag{4.174}$$

Looking back to Proposition 4.10, we see that $\dot{\mathcal{L}} - [\mathcal{L}, \mathcal{K}] = 0$, thus by (4.168) we end up with the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{A}A^{-1} + A^{-1}\dot{A}}{4} = 0. \tag{4.175}$$

Next, upon differentiating (4.167) with respect to t, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{A}A^{-1}-A^{-1}\dot{A}}{4}=k\left(\dot{\mathcal{N}}-[\mathcal{N},\mathcal{K}]\right)k^{-1}. \tag{4.176}$$

Remembering the form of $\mathcal{N}$ (4.165), Leibniz rule yields

$$\dot{\mathcal{N}}=\cosh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})[\dot{\mathbf{\Lambda}},Y]+\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})\dot{Y}=\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})\left(\coth(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})[\dot{\mathbf{\Lambda}},Y]+\dot{Y}\right), \tag{4.177}$$

and the formula

$$[\mathcal{N},\mathcal{K}]=[\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y,B_{\mathfrak{m}}]-[\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y,\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})^{-1}Y] \tag{4.178}$$

is also immediate. Now, let us observe that the first term on the right-hand side of the above equation can be transformed into the equivalent form

$$\begin{aligned}[\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y,B_{\mathfrak{m}}]&=\frac{1}{2}[e^{\mathrm{ad}_{\mathbf{\Lambda}}}Y,B_{\mathfrak{m}}]-\frac{1}{2}[e^{-\mathrm{ad}_{\mathbf{\Lambda}}}Y,B_{\mathfrak{m}}]=\frac{1}{2}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[Y,B_{\mathfrak{m}}]-\frac{1}{2}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[Y,B_{\mathfrak{m}}]\\&=\sinh(\mathrm{ad}_{\mathbf{\Lambda}})[Y,B_{\mathfrak{m}}]=\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})[Y,B_{\mathfrak{m}}],\end{aligned} \tag{4.179}$$

while for the second term we get

$$\begin{aligned}&[\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y,\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})^{-1}Y]=\frac{1}{2}[e^{\mathrm{ad}_{\mathbf{\Lambda}}}Y,\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})^{-1}Y]-\frac{1}{2}[e^{-\mathrm{ad}_{\mathbf{\Lambda}}}Y,\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})^{-1}Y]\\&\quad=\frac{1}{2}e^{\mathrm{ad}_{\mathbf{\Lambda}}}[Y,e^{-\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}}}\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})^{-1}Y]-\frac{1}{2}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}[Y,e^{\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}}}\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})^{-1}Y]\\&\quad=\sinh(\mathrm{ad}_{\mathbf{\Lambda}})[Y,\coth(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y]=\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})[Y,\coth(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y]_{\mathfrak{a}^\perp}.\end{aligned} \tag{4.180}$$

Taking into account the above expressions, we obtain that

$$\dot{\mathcal{N}}-[\mathcal{N},\mathcal{K}]=\sinh(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})\left(\dot{Y}+[Y,\coth(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y]_{\mathfrak{a}^\perp}-[Y,B_{\mathfrak{m}}]+[\dot{\mathbf{\Lambda}},\coth(\widetilde{\mathrm{ad}_{\mathbf{\Lambda}}})Y]\right), \tag{4.181}$$

whence by Proposition 4.10 we are entitled to write that $\dot{\mathcal{N}}-[\mathcal{N},\mathcal{K}]=0$. Giving a glance at the relationship (4.176), it readily follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{A}A^{-1}-A^{-1}\dot{A}}{4}=0. \tag{4.182}$$

To complete the proof, observe that the desired geodesic equation (4.161) is a trivial consequence of the equations (4.175) and (4.182). □

To proceed further, let us observe that by integrating the differential equation

(4.161), we obtain immediately that

$$\dot{A}(t)A(t)^{-1} = \dot{A}(0)A(0)^{-1} \qquad (t \in \mathbb{R}). \tag{4.183}$$

However, recalling the definitions (4.164) and (4.165), and also the relationships (4.152) and (4.137), from the equations (4.166), (4.167) and (4.157) we infer that

$$\begin{aligned}\dot{A}(0)A(0)^{-1} &= 2k(0)(\mathcal{L}(0) + \mathcal{N}(0))k(0)^{-1} = 2(\dot{\mathbf{\Lambda}}(0) + e^{\mathrm{ad}_{\mathbf{\Lambda}(0)}}Y(0)) \\ &= 2e^{\mathrm{ad}_{\mathbf{\Lambda}(0)}}(D(0) + Y(0)) = e^{\mathbf{\Lambda}(0)}(L(0) - L(0)^{-1})e^{-\mathbf{\Lambda}(0)}.\end{aligned} \tag{4.184}$$

Moreover, remembering (4.157) and the definition (4.160), at $t = 0$ we can also write that

$$A(0) = k(0)e^{2\mathbf{\Lambda}(0)}k(0)^{-1} = e^{2\mathbf{\Lambda}(0)}. \tag{4.185}$$

Putting the above observations together, it is now evident that the unique maximal solution of the first order differential equation (4.183) with the initial condition (4.185) is the smooth curve

$$A(t) = e^{te^{\mathbf{\Lambda}(0)}(L(0)-L(0)^{-1})e^{-\mathbf{\Lambda}(0)}}e^{2\mathbf{\Lambda}(0)} = e^{\mathbf{\Lambda}(0)}e^{t(L(0)-L(0)^{-1})}e^{\mathbf{\Lambda}(0)} \qquad (t \in \mathbb{R}). \tag{4.186}$$

Comparing this formula with (4.160), the following result is immediate.

Theorem 4.12. *Take an arbitrary maximal solution (4.148) of the van Diejen system (4.5), then at each $t \in \mathbb{R}$ it can be recovered uniquely from the spectral identification*

$$\{e^{\pm 2\lambda_a(t)} \mid a \in \mathbb{N}_n\} = \mathrm{Spec}(e^{\mathbf{\Lambda}(0)}e^{t(L(0)-L(0)^{-1})}e^{\mathbf{\Lambda}(0)}). \tag{4.187}$$

The essence of the above theorem is that any solution (4.148) of the van Diejen system (4.5) can be obtained by a purely algebraic process based on the diagonalization of a matrix flow. Indeed, once one finds the evolution of $\lambda(t)$ from (4.187), the evolution of $\theta(t)$ also becomes accessible by the formula

$$\theta_a(t) = \operatorname{arcsinh}\left(\frac{\dot{\lambda}_a(t)}{u_a(\lambda(t))}\right) \qquad (a \in \mathbb{N}_n), \tag{4.188}$$

as dictated by the equation of motion (4.76).

4.3.5 Temporal asymptotics

One of the immediate consequences of the projection method formulated in the previous subsection is that the Hamiltonian (4.5) describes a 'repelling' particle system, thus it is fully justified to inquire about its scattering properties. Although rigorous scattering theory is in general a hard subject, a careful study of the algebraic solution algorithm described in Theorem 4.12 allows us to investigate the asymptotic properties of any maximally defined trajectory (4.148) as $t \to \pm\infty$. In this respect our main tool is Ruijsenaars' theorem on the spectral asymptotics of exponential type matrix flows (see [113, Theorem A2]). To make it work, let us look at the relationship (4.52) and Lemma 4.4, from where we see that there is a group element $y \in K$ and a unique real n-tuple $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_n) \in \mathbb{R}^n$ satisfying

$$\hat{\theta}_1 > \ldots > \hat{\theta}_n > 0, \tag{4.189}$$

such that with the (regular) diagonal matrix $\hat{\boldsymbol{\Theta}} \in \mathfrak{c}$ defined in (4.51) we can write that

$$L(0) = y e^{2\hat{\boldsymbol{\Theta}}} y^{-1}. \tag{4.190}$$

Following the notations of the previous subsection, here $L(0)$ still stands for the Lax matrix (4.34) evaluated along the trajectory (4.148) at $t = 0$. Since

$$L(0) - L(0)^{-1} = 2y \sinh(2\hat{\boldsymbol{\Theta}}) y^{-1}, \tag{4.191}$$

with the aid of the positive definite matrix

$$\hat{L} = y^{-1} e^{2\boldsymbol{\Lambda}(0)} y \in \exp(\mathfrak{p}) \tag{4.192}$$

for the spectrum of the matrix flow appearing in (4.186) we obtain at once that

$$\operatorname{Spec}(e^{\boldsymbol{\Lambda}(0)} e^{t(L(0)-L(0)^{-1})} e^{\boldsymbol{\Lambda}(0)}) = \operatorname{Spec}(\hat{L} e^{2t\sinh(2\hat{\boldsymbol{\Theta}})}). \tag{4.193}$$

In order to make a closer contact with Ruijsenaars' theorem, let us also introduce the Hermitian $n \times n$ matrix $\mathcal{R}$ with entries

$$\mathcal{R}_{a,b} = \delta_{a+b,n+1}. \tag{4.194}$$

Since $\mathcal{R}^2 = \mathbf{1}_n$, we have $\mathcal{R}^{-1} = \mathcal{R}$, whence the block-diagonal matrix

$$\mathcal{W} = \begin{bmatrix} \mathbf{1}_n & 0_n \\ 0_n & \mathcal{R}_n \end{bmatrix} \in GL(N, \mathbb{C}), \tag{4.195}$$

also satisfies the relations $\mathcal{W}^{-1} = \mathcal{W} = \mathcal{W}^*$. As the most important ingredients of our present analysis, now we introduce the matrices

$$\boldsymbol{\Theta}^+ = 2\mathcal{W}\hat{\boldsymbol{\Theta}}\mathcal{W}^{-1} \quad \text{and} \quad \tilde{L} = \mathcal{W}\hat{L}\mathcal{W}^{-1}. \tag{4.196}$$

Recalling the relationships (4.187) and (4.193), it is clear that for all $t \in \mathbb{R}$ we can write that

$$\{e^{\pm 2\lambda_a(t)} \mid a \in \mathbb{N}_n\} = \operatorname{Spec}(\tilde{L} e^{2t \sinh(\boldsymbol{\Theta}^+)}). \tag{4.197}$$

However, upon performing the conjugations with the unitary matrix $\mathcal{W}$ (4.195) in the defining equations displayed in (4.196), we find immediately that

$$\boldsymbol{\Theta}^+ = \operatorname{diag}(\theta_1^+, \dots, \theta_n^+, -\theta_n^+, \dots, -\theta_1^+), \tag{4.198}$$

where

$$\theta_a^+ = 2\hat{\theta}_a \qquad (a \in \mathbb{N}_n). \tag{4.199}$$

The point is that, due to our regularity result formulated in Lemma 4.4, the diagonal matrix (4.198) has a simple spectrum, and its eigenvalues are in strictly decreasing order along the diagonal (see (4.189)). Moreover, since $\hat{L}$ (4.192) is positive definite, so is $\tilde{L}$. In particular, the leading principal minors of matrix $\tilde{L}$ are all strictly positive. So, the exponential type matrix flow

$$\mathbb{R} \ni t \mapsto \tilde{L} e^{2t \sinh(\boldsymbol{\Theta}^+)} \in GL(N, \mathbb{C}) \tag{4.200}$$

does meet all the requirements of Ruijsenaars' aforementioned theorem. Therefore, essentially by taking the logarithm of the quotients of the consecutive leading principal minors of the $n \times n$ submatrix taken from the upper-left-hand corner of $\tilde{L}$, one finds a unique real n-tuple

$$\lambda^+ = (\lambda_1^+, \dots, \lambda_n^+) \in \mathbb{R}^n \tag{4.201}$$

such that for all $a \in \mathbb{N}_n$ we can write

$$\lambda_a(t) \sim t \sinh(\theta_a^+) + \lambda_a^+ \quad \text{and} \quad \theta_a(t) \sim \theta_a^+, \tag{4.202}$$

up to exponentially vanishing small terms as $t \to \infty$. It is obvious that the same ideas work for the case $t \to -\infty$, too, with complete control over the asymptotic momenta θ_a^- and the asymptotic phases λ_a^- as well. The above observations can be summarized as follows.

Lemma 4.13. *For an arbitrary maximal solution (4.148) of the hyperbolic n-particle van Diejen system (4.5) the particles move asymptotically freely as $|t| \to \infty$. More precisely, for all $a \in \mathbb{N}_n$ we have the asymptotics*

$$\lambda_a(t) \sim t \sinh(\theta_a^\pm) + \lambda_a^\pm \quad \textit{and} \quad \theta_a(t) \sim \theta_a^\pm \qquad (t \to \pm\infty), \tag{4.203}$$

where the asymptotic momenta obey

$$\theta_a^- = -\theta_a^+ \quad \textit{and} \quad \theta_1^+ > \ldots > \theta_n^+ > 0. \tag{4.204}$$

We find it quite remarkable that, up to an overall sign, the asymptotic momenta are preserved (4.204). Following Ruijsenaars' terminology [113, 114], we may say that the 2-parameter family of van Diejen systems (4.5) are *finite dimensional pure soliton systems*. Now, let us remember that for each pure soliton system analysed in the earlier literature, the scattering map has a factorized form. That is, the n-particle scattering can be completely reconstructed from the 2-particle processes, and also by the 1-particle scattering on the external potential (see e.g. [77, 90, 113, 114, 108]). Albeit the results we shall present in rest of this chapter do not rely on this peculiar feature of the scattering process, still, it would be of considerable interest to prove this property for the hyperbolic van Diejen systems (4.5), too. However, because of its subtleties, we wish to work out the details of the scattering theory in a later publication.

4.4 Spectral invariants of the Lax matrix

The ultimate goal of this section is to prove that the eigenvalues of the Lax matrix L (4.34) are in involution. Superficially, one could say that it follows easily from the scattering theoretical results presented in the previous section. A convincing argument would go as follows. Recalling the notations (4.148) and (4.149), let us consider the flow

$$\Phi \colon \mathbb{R} \times P \to P, \qquad (t, \gamma_0) \mapsto \Phi_t(\gamma_0) = \gamma(t) \tag{4.205}$$

generated by the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75). Since for all $t \in \mathbb{R}$ the map $\Phi_t \colon P \to P$ is a symplectomorphism, for all $a, b \in \mathbb{N}_n$ we can write that

$$\{\theta_a \circ \Phi_t, \theta_b \circ \Phi_t\} = \{\theta_a, \theta_b\} \circ \Phi_t = 0. \tag{4.206}$$

On the other hand, from (4.203) it is also clear that at each point of the phase space P, for all $c \in \mathbb{N}_n$ we have

$$\theta_c \circ \Phi_t \to \theta_c^+ \qquad (t \to \infty). \tag{4.207}$$

Recalling (4.72) and (4.199), it is evident that $\theta_c^+ \in C^\infty(P)$. Therefore, by a 'simple interchange of limits', from (4.206) and (4.207) one could infer that the asymptotic momenta θ_c^+ ($c \in \mathbb{N}_n$) Poisson commute. Bearing in mind the relationships (4.199) and (4.64), it would also follow that the eigenvalues of L (4.34) generate a maximal Abelian Poisson subalgebra. However, to justify the interchange of limits, one does need a deeper knowledge about the scattering properties than the pointwise limit formulated in (4.207). Since we wish to work out the full scattering theory elsewhere, in this chapter we choose an alternative approach by merging the temporal asymptotics of the trajectories with van Diejen's earlier results [142, 140, 143].

4.4.1 Link to the 5-parameter family of van Diejen systems

As is known from the seminal papers [140, 143], the definition of the classical hyperbolic van Diejen system is based on the smooth functions $v, w\colon \mathbb{R} \setminus \{0\} \to \mathbb{C}$ defined by the formulae

$$v(x) = \frac{\sinh(\mathrm{i}g + x)}{\sinh(x)}, \quad w(x) = \frac{\sinh(\mathrm{i}g_0 + x)}{\sinh(x)} \frac{\cosh(\mathrm{i}g_1 + x)}{\cosh(x)} \frac{\sinh(\mathrm{i}g_0' + x)}{\sinh(x)} \frac{\cosh(\mathrm{i}g_1' + x)}{\cosh(x)}, \tag{4.208}$$

where the five independent real numbers g, g_0, g_1, g_0', g_1' are the coupling constants. Parameter g in the 'potential' function v controls the strength of inter-particle interaction, whereas the remaining four constants appearing in the 'external potential' w are responsible for the influence of the ambient field. Conforming to the notations introduced in the aforementioned papers, let us recall that the set of Poisson commuting functions found by van Diejen can be succinctly written as

$$H_l = \sum_{\substack{J \subseteq \mathbb{N}_n,\ |J| \leq l \\ \varepsilon_j = \pm 1,\ j \in J}} \cosh(\theta_{\varepsilon J}) |V_{\varepsilon J; J^c}| U_{J^c, l-|J|} \qquad (l \in \mathbb{N}_n), \tag{4.209}$$

where the various constituents are defined by the formulae

$$\theta_{\varepsilon J} = \sum_{j \in J} \varepsilon_j \theta_j, \quad V_{\varepsilon J; J^c} = \prod_{j \in J} w(\varepsilon_j \lambda_j) \prod_{\substack{j, j' \in J \\ (j < j')}} v(\varepsilon_j \lambda_j + \varepsilon_{j'} \lambda_{j'})^2 \prod_{\substack{j \in J \\ k \in J^c}} v(\varepsilon_j \lambda_j + \lambda_k) v(\varepsilon_j \lambda_j - \lambda_k), \tag{4.210}$$

together with the expression

$$U_{J^c, l-|J|} = (-1)^{l-|J|} \sum_{\substack{I \subseteq J^c,\ |I| = l-|J| \\ \varepsilon_i = \pm 1,\ i \in I}} \prod_{i \in I} w(\varepsilon_i \lambda_i) \prod_{\substack{i, i' \in I \\ (i < i')}} |v(\varepsilon_i \lambda_i + \varepsilon_{i'} \lambda_{i'})|^2 \prod_{\substack{i \in I \\ k \in J^c \setminus I}} v(\varepsilon_i \lambda_i + \lambda_k) v(\varepsilon_i \lambda_i - \lambda_k). \tag{4.211}$$

At this point two short technical remarks are in order. First, we extend the family of the first integrals (4.209) with the constant function $H_0 = 1$. Analogously, in the last equation (4.211) it is understood that $U_{J^c, 0} = 1$.

To make contact with the 2-parameter family of van Diejen systems of our interest (4.5), for the coupling parameters of the potential functions (4.208) we make the special choice

$$g = \mu, \quad g_0 = g_1 = \frac{\nu}{2}, \quad g_0' = g_1' = 0. \tag{4.212}$$

Under this assumption, from the definitions (4.31) and (4.210) it is evident that with the singleton $J = \{a\}$ we can write that

$$V_{\{a\};\{a\}^c} = -z_a \qquad (a \in \mathbb{N}_n). \tag{4.213}$$

Giving a glance at (4.211), it is also clear that the term corresponding to $J = \emptyset$ in the defining sum of H_1 (4.209) is a constant function of the form

$$U_{\mathbb{N}_n,1} = 2\sum_{a=1}^{n} \mathrm{Re}(z_a) = -2\cos\left(\nu + (n-1)\mu\right)\frac{\sin(n\mu)}{\sin(\mu)}. \tag{4.214}$$

Plugging the above formulae into van Diejen's main Hamiltonian H_1 (4.209), one finds immediately that

$$H_1 + 2\cos\big(\nu + (n-1)\mu\big)\frac{\sin(n\mu)}{\sin(\mu)} = 2H = \mathrm{tr}(L). \tag{4.215}$$

That is, up to some irrelevant constants, our Hamiltonian H (4.5) can be identified with H_1 (4.209), provided the coupling parameters are related by the equations displayed in (4.212). At this point one may suspect that the quantities $\mathrm{tr}(L^l)$ are also expressible with the aid of the Poisson commuting family of functions H_l (4.209). Clearly, it would imply immediately that the eigenvalues of the Lax matrix L (4.34) are in involution. However, due to the complexity of the underlying objects (4.210)-(4.211), this naive approach would lead to a formidable combinatorial task, that we do not wish to pursue in this chapter. To circumvent the difficulties, below we rather resort to a clean analytical approach by exploiting the scattering theoretical results formulated in the previous section.

4.4.2 Poisson brackets of the eigenvalues of L

Take an arbitrary point $\gamma_0 \in P$ and consider the unique maximal integral curve

$$\mathbb{R} \ni t \mapsto \gamma(t) = (\lambda(t), \theta(t)) \in P \tag{4.216}$$

of the Hamiltonian vector field $\boldsymbol{X}_H$ (4.75) satisfying the initial condition

$$\gamma(0) = \gamma_0. \tag{4.217}$$

Since the functions H_l (4.209) are first integrals of the dynamics, their values at the point γ_0 can be recovered by inspecting the limit of $H_l(\gamma(t))$ as $t \to \infty$. Now, recalling the potentials (4.209) and the specialization of the coupling parameters (4.212), it is evident that

$$\lim_{x\to\pm\infty} v(x) = e^{\pm \mathrm{i}\mu} \quad \text{and} \quad \lim_{x\to\pm\infty} w(x) = e^{\pm \mathrm{i}\nu}. \tag{4.218}$$

Therefore, taking into account the regularity properties (4.204) of the asymptotic momenta θ_c^+ (4.203), from Lemma 4.13 and the definitions (4.209)-(4.211) one finds immediately that

$$H_l(\gamma_0) = \lim_{t\to\infty} H_l(\gamma(t)) = \sum_{\substack{J\subseteq\mathbb{N}_n,\ |J|\leq l\\ \varepsilon_j=\pm 1,\ j\in J}} \cosh(\theta_{\varepsilon J}^+)\, \mathcal{U}_{J^c, l-|J|} \qquad (l\in\mathbb{N}_n), \tag{4.219}$$

where

$$\mathcal{U}_{J^c, l-|J|} = (-1)^{l-|J|} \sum_{\substack{I\subseteq J^c,\ |I|=l-|J|\\ \varepsilon_j=\pm 1,\ j\in I}} \prod_{j\in I} e^{\varepsilon_j \mathrm{i}\nu} \prod_{\substack{j\in I,\ k\in J^c\setminus I\\ (j<k)}} e^{\varepsilon_j 2\mathrm{i}\mu}. \tag{4.220}$$

By inspecting the above expression, let us observe that the value of $\mathcal{U}_{J^c,l-|J|}$ does *not* depend on the specific choice of the subset J, but only on its cardinality $|J|$. More precisely, if $J\subseteq\mathbb{N}_n$ is an arbitrary subset of cardinality $|J|=k$ $(0\leq k\leq l-1)$, then we can write that

$$\mathcal{U}_{J^c, l-|J|} = (-1)^{l-k} \sum_{\substack{1\leq j_1<\dots<j_{l-k}\leq n-k\\ \varepsilon_1=\pm 1,\dots,\varepsilon_{l-k}=\pm 1}} \exp\left(\mathrm{i}\sum_{m=1}^{l-k} \varepsilon_m\left(\nu + 2(n-l+m-j_m)\mu\right)\right). \tag{4.221}$$

To proceed further, let us now turn to the study of the Lax matrix L (4.34). Due to the Lax representation of the dynamics that we established in Theorem 4.8, the eigenvalues of L are conserved quantities. Consequently, the coefficients $K_0, K_1, \dots, K_N \in C^\infty(P)$ of the characteristic polynomial

$$\det(L - y\mathbf{1}_N) = \sum_{m=0}^{N} K_{N-m} y^m \qquad (y\in\mathbb{C}) \tag{4.222}$$

are also first integrals. As expected, the special algebraic properties of L formulated in Proposition 4.1 and Lemma 4.2 have a profound impact on these coefficients as well, as can be seen from the relations

$$K_{N-m} = K_m \qquad (m=0,1,\dots,N). \tag{4.223}$$

So, it is enough to analyze the properties of the members $K_0 = 1, K_1, \dots, K_n$. In this respect the most important ingredient is the relationship

$$\lim_{t\to\infty} L(\gamma(t)) = \exp(\mathbf{\Theta}^+), \tag{4.224}$$

where $\mathbf{\Theta}^+$ is the $N\times N$ diagonal matrix (4.198) containing the asymptotic momenta. Therefore, looking back to the definition (4.222), for any $m=0,1,\dots,n$ we obtain at once that

$$K_m(\gamma_0) = \lim_{t\to\infty} K_m(\gamma(t)) = (-1)^m \sum_{a=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\substack{J\subseteq \mathbb{N}_n,\ |J|=m-2a \\ \varepsilon_j=\pm 1,\ j\in J}} \binom{n-|J|}{a} \cosh(\theta^+_{\varepsilon J}). \tag{4.225}$$

Based on the formulae (4.219) and (4.225), we can prove the following important technical result.

Lemma 4.14. *The two distinguished families of first integrals $\{H_l\}_{l=0}^n$ and $\{K_m\}_{m=0}^n$ are connected by an invertible linear relation with purely numerical coefficients depending only on the coupling parameters μ and ν.*

Proof. For brevity, let us introduce the notation

$$\mathcal{A}_k = \sum_{\substack{J\subseteq \mathbb{N}_n,\ |J|=k \\ \varepsilon_j=\pm 1,\ j\in J}} \cosh(\theta^+_{\varepsilon J}) \qquad (k=0,1,\dots,n). \tag{4.226}$$

As we have seen in (4.221), the coefficients $\mathcal{U}_{J^c,l-|J|}$ appearing in the formula (4.219) depend only on the cardinality of J, whence for any $l \in \{0,1,\dots,n\}$ we can write that

$$H_l(\gamma_0) = \sum_{k=0}^{l} \mathcal{U}_{\mathbb{N}_{n-k},l-k} \mathcal{A}_k. \tag{4.227}$$

Since $\mathcal{U}_{\mathbb{N}_{n-l},0} = 1$, the matrix transforming $\{\mathcal{A}_k\}_{k=0}^n$ into $\{H_l(\gamma_0)\}_{l=0}^n$ is lower triangular with plus ones on the diagonal, whence the above linear relation (4.227) is invertible. Comparing the formulae (4.225) and (4.226), it is also clear that

$$K_m(\gamma_0) = (-1)^m \sum_{a=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-(m-2a)}{a} \mathcal{A}_{m-2a}, \tag{4.228}$$

which in turn implies that the matrix relating $\{\mathcal{A}_k\}_{k=0}^n$ to $\{K_m(\gamma_0)\}_{m=0}^n$ is lower triangular with diagonal entries ± 1. Hence the linear relationship (4.228) is also invertible. Putting together the above observations, it is clear that there is an invertible $(n+1)\times(n+1)$ matrix $\mathcal{C}$ with purely numerical entries $\mathcal{C}_{m,l}$ depending only on μ and ν such that

$$K_m(\gamma_0) = \sum_{l=0}^{n} \mathcal{C}_{m,l} H_l(\gamma_0). \tag{4.229}$$

Since γ_0 is an arbitrary point of the phase space P (4.2), the Lemma follows. ☐

The scattering theoretical idea in the proof the above Lemma goes back to the fundamental works of Moser (see e.g. [88]). However, in the recent paper [P5] it has been revitalized in the context of the rational BC_n van Diejen model, too. Compared to the rational case, it is a significant difference that our coefficients $\mathcal{U}_{J^c,l-|J|}$ (4.220) do depend on the parameters μ and ν in a non-trivial manner, whence the observations surrounding the derivations of formula (4.221) turns out to be crucial in our presentation.

Since the family of functions $\{H_l\}_{l=0}^n$ Poisson commute, Lemma 4.14 readily implies that the first integrals $\{K_m\}_{m=0}^n$ are also in involution. Now, let us recall that the spectrum of the Lax matrix L is simple, as we have seen in Lemma 4.4. As a consequence, the eigenvalues of L can be realized as smooth functions of the coefficients of the characteristic polynomial (4.222), thus the following result is immediate.

Theorem 4.15. *The eigenvalues of the Lax matrix L (4.34) are in involution.*

To conclude this section, let us note that the proof of Theorem 4.15 is quite indirect in the sense that it hinges on the commutativity of the family of functions (4.209). However, the only available proof of this highly non-trivial fact is based on the observation that the Hamiltonians (4.209) can be realized as classical limits of van Diejen's commuting analytic difference operators [142]. As a more elementary approach, let us note that Theorem 4.15 would also follow from the existence of an r-matrix encoding the tensorial Poisson bracket of the Lax matrix L (4.34). Due to Lemma 4.14, it would imply the commutativity of the family (4.209), too, at least under the specialization (4.212). To find such an r-matrix, one may wish to generalize the analogous results on the rational system [109].

4.5 Discussion

One of the most important objects in the study of integrable systems is the Lax representation of the dynamics. By generalizing the earlier results on the rational BC_n RSvD models [106, 109], in this chapter we succeeded in constructing a Lax pair for the 2-parameter family of hyperbolic van Diejen systems (4.5). Making use of this construction, we showed that the dynamics can be solved by a projection method, which in turn allowed us to initiate the study of the scattering properties of (4.5). Moreover, by combining our scattering theoretical results with the ideas of the recent paper [P5], we proved that the first integrals provided by the eigenvalues of the proposed Lax matrix (4.34) are in fact in involution. To sum up, it is fully justified to say that the matrices L (4.34) and B (4.132) form a Lax pair for the hyperbolic van Diejen system (4.5).

Apart from taking a non-trivial step toward the construction of Lax matrices for the most general hyperbolic van Diejen many-particle systems (4.209), let us not forget about the potential applications of our results. In analogy with the translation invariant RS systems, we expect that the van Diejen models may play a crucial role in clarifying the particle-soliton picture in the context of integrable boundary field theories. While the relationship between the A-type RS models and the soliton equations defined on the whole line is under control (see e.g. [111, 113, 10, 116, 117]), the link between the van Diejen models and the soliton systems defined on the half-line is less understood (see e.g. [131, 67]). As in the translation invariant case, the Lax matrices of the van Diejen systems could turn out to be instrumental for elaborating this correspondence.

Turning to the more recent activities surrounding the CMS and the RS many-particle models, let us recall the so-called classical/quantum duality (see e.g. [91, 5, 55, 139, 13]), which relates the spectra of certain quantum spin chains with the Lax matrices of the classical CMS and RS systems. An equally remarkable development is the emergence of new integrable tops based on the Lax matrices of the CMS and the RS systems [6, 79]. Relatedly, it would be interesting to see whether the Lax matrix (4.34) of the hyperbolic van Diejen system (4.5) can be fit into these frameworks.

One of the most interesting aspects of the CMS and the RSvD systems we have not addressed in this chapter is the so-called Ruijsenaars duality, or action-angle duality. Based on hard analytical techniques, this remarkable property was first exhibited by Ruijsenaars [113] in the context of the translation invariant non-elliptic models. Let us note that in the recent papers [36, 35, 37, 38] almost all of these duality relationships have been successfully reinterpreted in a nice geometrical framework provided by powerful symplectic reduction methods. Moreover, by now some duality results are available also for the CMS and the RSvD models associated with the BC-type root systems [106, 107] [P1].

As for the key player of this chapter, we have no doubt that the 2-parameter family of hyperbolic van Diejen systems (4.5) is self-dual. Indeed, upon diagonalizing the Lax matrix L (4.34), we see that the transformed objects defined in (4.52)-(4.53) obey the relationship (4.54), that has the same form as the Ruijsenaars type commutation relation (4.48) we set up in Lemma 4.3. Based on the method presented in [113], we expect that the transformed matrix $\hat{L}$ (4.53) shall provide a Lax matrix for the dual system. Therefore, comparing the matrix entries displayed in (4.34) and (4.55), the self-duality of the system (4.5) seems to be more than plausible. Admittedly, many subtle details are still missing for a complete proof. As for filling these gaps, the immediate idea is that either one could mimic Ruijsenaars' scattering theoretical approach, or invent an appropriate symplectic reduction framework. However, notice that the non-standard form of the Hamiltonian (4.5) poses severe analytical difficulties on the study of the scattering theory, whereas the weakness of the geometrical approach lies in the fact that up to now even the translation invariant hyperbolic RS model has not been derived from symplectic reduction. Nevertheless, by taking the analytical continuation of the Lax matrix L (4.34), it is conceivable that the self-duality of the compactified trigonometric version of (4.5) can be proved by adapting the quasi-Hamiltonian reduction approach advocated by Fehér and Klimčík [38]. For further motivation, let us recall that the duality properties are indispensable in the study of the recently introduced integrable random matrix ensembles [15, 16, 49], too.

5 Trigonometric and elliptic Ruijsenaars-Schneider models on $\mathbb{CP}^{n-1}$

Following [P4], we present a direct construction of compact real forms of the trigonometric and elliptic n-particle Ruijsenaars-Schneider systems whose completed center-of-mass phase space is the complex projective space $\mathbb{CP}^{n-1}$ with the Fubini-Study symplectic structure. These systems are labelled by an integer $p \in \{1, \dots, n-1\}$ relative prime to n and a coupling parameter y, which can vary in a certain punctured interval around $p\pi/n$. Our work extends Ruijsenaars's pioneering study of compactifications that imposed the restriction $0 < y < \pi/n$, and also builds on an earlier derivation of more general compact trigonometric systems by Hamiltonian reduction.

The phase spaces of the particle systems we encountered so far are usually the cotangent bundles of the configuration spaces, hence they are never compact due to the infinite range of the canonical momenta. For example, the standard Ruijsenaars-Schneider Hamiltonian depends on the momenta ϕ_k through the function $\cosh(\phi_k)$, but by analytic continuation this may be replaced by $\cos(\phi_k)$, which effectively compactifies the momenta on a circle. If the dependence on the position variables x_k is also through a periodic function, then the phase space of the system can be taken to be bounded. This possibility was examined in [117], where the Hamiltonian

$$H = \sum_{k=1}^{n} \cos(\phi_k) \sqrt{\prod_{\substack{j=1 \\ (j \neq k)}}^{n} \left[1 - \frac{\sin^2 y}{\sin^2(x_j - x_k)}\right]} \tag{5.1}$$

containing a real coupling parameter $0 < y < \pi/2$ was considered. Ruijsenaars called this the $\mathrm{III_b}$ system, with III referring to the trigonometric character of the interaction, as in [98], and the suffix standing for 'bounded'. (One may also introduce the deformation parameter β into the $\mathrm{III_b}$ system, by replacing ϕ_k by $\beta\phi_k$.) The domain of the 'angular position variables' $\{(x_1, \dots, x_n)\} \subset [0, \pi]^n$ must be restricted in such a way that the Hamiltonian (5.1) is real and smooth. This may be ensured by prescribing

$$x_{i+1} - x_i > y \quad (i = 1, \dots, n-1), \quad x_n - x_1 < \pi - y, \tag{5.2}$$

which obviously implies Ruijsenaars's condition

$$0 < y < \frac{\pi}{n}. \tag{5.3}$$

Although the Hamiltonian is then real, its flow is not complete on the naive phase space, because it may reach the boundary $x_{k+1} - x_k = y$ (with $x_{k+n} \equiv x_k + \pi$) at finite time [117]. Completeness of the commuting flows is a crucial property of any bona fide integrable system, but one cannot directly add the boundary to the phase space because that would not yield a smooth manifold. One of the seminal results of [117] is the solution of this conundrum. In fact, Ruijsenaars constructed a symplectic embedding of the center-of-mass phase space of the system into the complex projective space $\mathbb{CP}^{n-1}$, such that the image of the embedding is a dense open submanifold and the Hamiltonian (5.1) as well as its commuting family extend to smooth functions on the full $\mathbb{CP}^{n-1}$. As $\mathbb{CP}^{n-1}$ is compact, the corresponding Hamiltonian flows are complete. The resulting 'compactified trigonometric RS system' has been studied at the classical level in detail [117], and after an initial exploration of the rank 1 case [114], its quantum mechanical version was also solved [146]. These classical systems are self-dual in the sense that their position and action variables can be exchanged by a canonical transformation of order 4, somewhat akin to the mapping $(x, \phi) \mapsto (-\phi, x)$ for a free particle, and their quantum mechanical versions enjoy the bispectral property [114, 146].

The possibility of an analogous compactification of the elliptic RS system having the Hamiltonian

$$H = \sum_{k=1}^{n} \cos(\phi_k) \sqrt{\prod_{\substack{j=1 \\ (j \neq k)}}^{n} \left[\mathrm{s}(y)^2 \big(\wp(y) - \wp(x_j - x_k)\big) \right]} \tag{5.4}$$

with functions $\wp$ (5.71) and s (5.72) was pointed out in [114, 118], but it was not described in detail.

Even though it was only proved [117] that the restrictions (5.2), (5.3) are sufficient to allow compactification, equation (5.3) was customarily mentioned in the literature [38, 54, 115, 118, 146] as a necessary condition for the systems to make sense. However, in a recent work [40] a completion of the $\mathrm{III_b}$ system on a compact phase space was obtained for any generic parameter

$$0 < y < \pi. \tag{5.5}$$

The paper [40] relied on deriving compactified RS systems in the center-of-mass frame via reduction of a 'free system' on the quasi-Hamiltonian [4] double $\mathrm{SU}(n) \times \mathrm{SU}(n)$.

This was achieved by setting the relevant group-valued moment map equal to the constant matrix $\mu_0(y) = \mathrm{diag}(e^{2\mathrm{i}y}, \dots, e^{2\mathrm{i}y}, e^{-2(n-1)\mathrm{i}y})$, and it makes perfect sense for any (generic) y. The corresponding domain of the position variables depends on y and differs from the one posited in (5.2). The possibility to relax the condition (5.3) on y also appeared in [15].

The principal motivation for our present work comes from the classification of the coupling parameter y found in [40]. Namely, it turned out that the reduction is applicable except for a finite set of y-values, and the rest of the set $(0, \pi)$ decomposes into two subsets, containing so-called type (i) and type (ii) y-values. The 'main reduced Hamiltonian' always takes the $\mathrm{III_b}$ form (5.1) on a dense open subset of the reduced phase space. In the type (i) cases the particles cannot collide and the action variables of the reduced system naturally engender an isomorphism with the Hamiltonian toric manifold $\mathbb{CP}^{n-1}$. In type (ii) cases, that exist for any $n > 3$, the reduction constraints admit solutions $(a, b) \in \mathrm{SU}(n) \times \mathrm{SU}(n)$ for which the eigenvalues of a or b are not all distinct, entailing that the particles of the reduced system can collide. For a detailed exposition of these succinct statements, the reader may consult [40]. We here only add the remark that the connected domain of the positions always contains the equal-distance configuration $x_{k+1} - x_k = \pi/n$ $(\forall k)$ for which the number of negative factors in each product under the square root in (5.1) is $2\lfloor ny/\pi \rfloor$ if $0 < y < \pi/2$ and $2\lfloor n(\pi - y)/\pi \rfloor$ if $\pi/2 < y < \pi$.

5.1 Embedding of the local phase space into $\mathbb{CP}^{n-1}$

In this section we first recall the local phase space of the $\mathrm{III_b}$ model from [40], and then present its symplectic embedding into $\mathbb{CP}^{n-1}$ in every type (i) case.

The $\mathrm{III_b}$ model can be thought of as n interacting particles on the unit circle with positions $\delta_k = e^{2\mathrm{i}x_k}$. We impose the condition $\prod_{k=1}^n \delta_k = 1$, which means that we work in the 'center-of-mass frame', and parametrize the positions as

$$\delta_1(\xi) = e^{\frac{2\mathrm{i}}{n}\sum_{j=1}^n j\xi_j}, \qquad \delta_k(\xi) = e^{2\mathrm{i}\xi_{k-1}}\delta_{k-1}(\xi), \quad k = 2, \dots, n, \tag{5.6}$$

where ξ belongs to a certain open subset $\mathcal{A}_y^+$ inside the 'Weyl alcove'

$$\mathcal{A} = \{\xi \in \mathbb{R}^n \mid \xi_k \geq 0 \ (k = 1, \dots, n),\ \xi_1 + \dots + \xi_n = \pi\}. \tag{5.7}$$

Note that $\mathcal{A}$ is a simplex in the $(n-1)$-dimensional affine space

$$E = \{\xi \in \mathbb{R}^n \mid \xi_1 + \dots + \xi_n = \pi\}. \tag{5.8}$$

The local phase space can be described as the product manifold

$$P_y^{\mathrm{loc}} = \{(\xi, e^{\mathrm{i}\theta}) \mid \xi \in \mathcal{A}_y^+,\ e^{\mathrm{i}\theta} \in \mathbb{T}^{n-1}\}, \tag{5.9}$$

where $\mathbb{T}^{n-1}$ is the $(n-1)$-torus, equipped with the standard symplectic form

$$\omega^{\mathrm{loc}} = \sum_{k=1}^{n-1} d\theta_k \wedge d\xi_k. \tag{5.10}$$

The dynamics is governed by the Hamiltonian

$$H_y^{\mathrm{loc}}(\xi, \theta) = \sum_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \sqrt{\prod_{m=j+1}^{j+n-1} \left[1 - \frac{\sin^2 y}{\sin^2(\sum_{k=j}^{m-1} \xi_k)}\right]}. \tag{5.11}$$

Here, $\theta_0 = \theta_n = 0$ have been introduced and the indices are understood modulo n, i.e.

$$\xi_{m+n} = \xi_m, \quad \forall m. \tag{5.12}$$

The product under the square root is positive for every $\xi \in \mathcal{A}_y^+$, and thus $H_y^{\mathrm{loc}} \in C^\infty(P_y^{\mathrm{loc}})$. This model was considered in [40] for any y chosen from the interval $(0, \pi)$ except the excluded values that satisfy $e^{2\mathrm{i}my} = 1$ for some $m = 1, \dots, n$.

According to [40], there are two different kinds of intervals for y to be in, named type (i) and (ii). The type (i) couplings can be described as follows. For a fixed positive integer $n \geq 2$, choose $p \in \{1, \dots, n-1\}$ to be a coprime to n, i.e. $\gcd(n, p) = 1$, and let q denote the multiplicative inverse of p in the ring $\mathbb{Z}_n$, that is $pq \equiv 1 \pmod{n}$. Then the parameter y can take its values according to either

$$\left(\frac{p}{n} - \frac{1}{nq}\right)\pi < y < \frac{p\pi}{n} \qquad \text{or} \qquad \frac{p\pi}{n} < y < \left(\frac{p}{n} + \frac{1}{(n-q)n}\right)\pi. \tag{5.13}$$

For such a type (i) parameter y, the local configuration space $\mathcal{A}_y^+$ is the interior of a simplex $\mathcal{A}_y$ in E (5.8) bounded by the hyperplanes

$$\xi_j + \dots + \xi_{j+p-1} = y, \quad j = 1, \dots, n, \tag{5.14}$$

where (5.12) is understood. To give a more detailed description of $\mathcal{A}_y$, we introduce

$$M = p\pi - ny, \tag{5.15}$$

and note that (5.13) gives $M > 0$ and $M < 0$, respectively. Then any $\xi \in \mathcal{A}_y$ must satisfy

$$\mathrm{sgn}(M)(\xi_j + \dots + \xi_{j+p-1} - y) \geq 0, \quad j = 1, \dots, n. \tag{5.16}$$

In terms of the particle coordinates x_k, which are ordered as $x_{k+1} \geq x_k$ and extended by the convention $x_{k+n} = x_k + \pi$, the above condition says that

$$x_{j+p} - x_j \geq y \quad \text{if } M > 0 \quad \text{and} \quad x_{j+p} - x_j \leq y \quad \text{if } M < 0 \tag{5.17}$$

for every j. Therefore the distances of the p-th neighbouring particles on the circle are constrained. The n vertices of the simplex $\mathcal{A}_y$ are explicitly given in [40, Proposition 11 and Lemma 8 *op. cit.*]. Every vertex and thus $\mathcal{A}_y$ itself lies inside the larger simplex $\mathcal{A}$ (5.7), entailing that $x_{j+1} - x_j$ possesses a positive lower bound in each type (i) case.

The type (ii) cases correspond to those admissible y-values that do not satisfy (5.13) for any p relative prime to n. In such cases $\mathcal{A}_y^+$ has a different structure [40]. Type (ii) cases exist for every $n \geq 4$. See Figure 7 for an illustration.

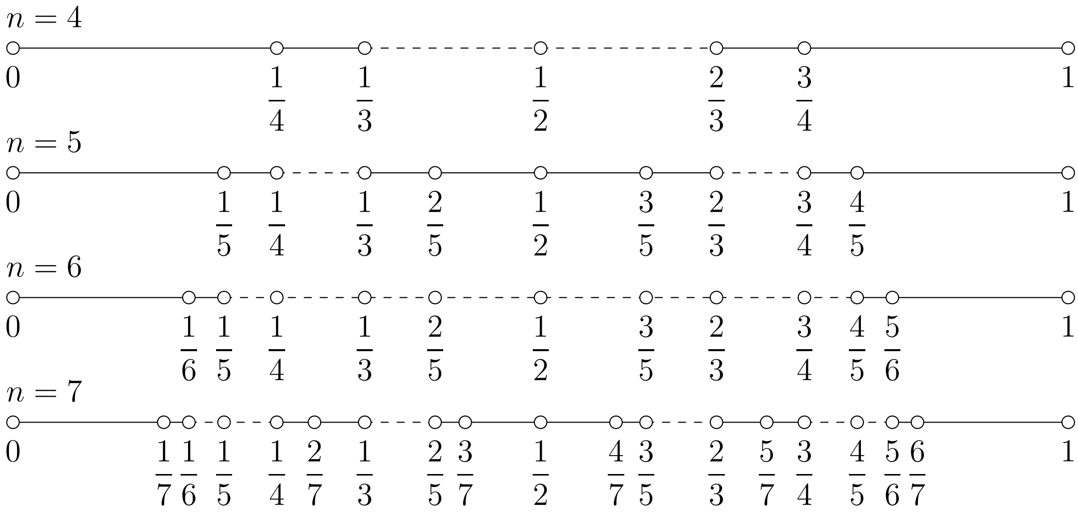

Figure 7: The range of y/π for $n = 4, 5, 6, 7$. The displayed numbers are excluded values. Admissible values of y form intervals of type (i) (solid) and type (ii) (dashed) couplings.

We further continue with the assumption that y satisfies (5.13). Motivated by [117, 38], we now introduce the map

$$\mathcal{E} \colon \mathcal{A}_y^+ \times \mathbb{T}^{n-1} \to \mathbb{C}^n, \quad (\xi, e^{\mathrm{i}\theta}) \mapsto (u_1, \dots, u_n) \tag{5.18}$$

with the complex coordinates having the squared absolute values

$$|u_j|^2 = \operatorname{sgn}(M)(\xi_j + \cdots + \xi_{j+p-1} - y), \quad j = 1, \dots, n, \tag{5.19}$$

and the arguments

$$\arg(u_j) = \operatorname{sgn}(M) \sum_{k=1}^{n-1} \Omega_{j,k} \theta_k, \quad j = 1, \dots, n-1, \qquad \arg(u_n) = 0, \tag{5.20}$$

where the $\Omega_{j,k}$ ($j,k=1,\dots,n-1$) are integers chosen in such a way that

$$\mathcal{E}^*\bigg(\mathrm{i}\sum_{j=1}^{n} d\bar{u}_j\wedge du_j\bigg)=\sum_{k=1}^{n-1} d\theta_k\wedge d\xi_k. \tag{5.21}$$

In order for (5.21) to be achieved Ω has to be the inverse transpose of the $(n-1)\times(n-1)$ coefficient matrix of $\xi_1,\dots,\xi_{n-1}$ extracted from eqs. (5.19) by applying $\xi_1+\dots+\xi_n=\pi$. In other words, the squared absolute values $|u_j|^2$ are written as

$$|u_j|^2=\begin{cases}\mathrm{sgn}(M)\big(\sum_{k=1}^{n-1}A_{j,k}\xi_k-y\big), & \text{if } 1\leq j\leq n-p,\\ \mathrm{sgn}(M)\big(\sum_{k=1}^{n-1}A_{j,k}\xi_k-y+\pi\big), & \text{if } n-p<j\leq n-1,\end{cases} \tag{5.22}$$

where A stands for the above-mentioned coefficient matrix, which has the components

$$A_{j,k}=\begin{cases}+1, & \text{if } 1\leq j\leq n-p \text{ and } j\leq k<j+p,\\ -1, & \text{if } n-p<j\leq n-1 \text{ and } j+p-n\leq k<j,\\ 0, & \text{otherwise.}\end{cases} \tag{5.23}$$

A close inspection of the structure of A reveals that

$$\det(A)=(-1)^{(n-p)(p-1)}\prod_{j=1}^{n-p}A_{j,j+p-1}\prod_{k=1}^{p-1}A_{n-p+k,k}=(-1)^{(n-p+1)(p-1)}=+1, \tag{5.24}$$

therefore $\Omega=(A^{-1})^\top$ exists and consists of integers, as required in (5.20). Next, we give Ω explicitly.

Proposition 5.1. *The transpose of the inverse of the matrix A (5.23) can be written as*

$$\Omega=B-C, \tag{5.25}$$

where B is a $(0,1)$-matrix of size $(n-1)$ with zeros along certain diagonals given by

$$B_{m,k}=\begin{cases}0, & \textit{if } k-m\equiv\ell p \pmod n \textit{ for some } \ell\in\{1,\dots,n-q\},\\ 1, & \textit{otherwise,}\end{cases} \tag{5.26}$$

and C is also a binary matrix of size $(n-1)$ with zeros along columns given by

$$C_{m,k}=\begin{cases}0, & \textit{if } k\equiv\ell p \pmod n \textit{ for some } \ell\in\{1,\dots,n-q\},\\ 1, & \textit{otherwise.}\end{cases} \tag{5.27}$$

Proof. We start by presenting a useful auxiliary statement. Let us introduce the subsets

S and S_i of the ring $\mathbb{Z}_n$ as

$$S = \{\ell p \,(\bmod\ n) \mid \ell = 1, \dots, n-q\}, \quad S_i = \{i+\ell\,(\bmod\ n) \mid \ell = 0, \dots, p-1\}, \tag{5.28}$$

for any $i \in \mathbb{Z}_n$. Then define $I_i \in \mathbb{N}$ to be the number of elements in the intersection $S_i \cap S$. Notice that $i \in S$ if and only if $(i+p) \in S$ except for $i \equiv (n-1) \equiv (n-q)p$ (mod n), for which $(n-1)+p \equiv (n-q+1)p \pmod n$ does not belong to S. It follows that

$$I_1 = \cdots = I_{n-1} = I_n + 1. \tag{5.29}$$

Our aim is to show that $(A\Omega^\top)_{j,m} = \delta_{j,m}$ $(\forall j, m)$ with Ω defined by (5.25)-(5.27). First, by the formula of A (5.23) for any $1 \leq j \leq n-p$ and $1 \leq m \leq n-1$ we have

$$(A\Omega^\top)_{j,m} = \sum_{k=1}^{n-1} A_{j,k}\Omega_{m,k} = \sum_{k=j}^{j+p-1} \Omega_{m,k} = \sum_{k=j}^{j+p-1} (B_{m,k} - C_{m,k}). \tag{5.30}$$

The definition of the matrices B (5.26) and C (5.27) gives directly that

$$\sum_{k=j}^{j+p-1} B_{m,k} = p - I_{j-m}, \qquad \sum_{k=j}^{j+p-1} C_{m,k} = p - I_j. \tag{5.31}$$

By using (5.29), this readily implies that $(A\Omega^\top)_{j,m} = \delta_{j,m}$ holds for the case at hand.

Second, for any $n-p < j \leq n-1$ and $1 \leq m \leq n-1$ we have

$$(A\Omega^\top)_{j,m} = \sum_{k=1}^{n-1} A_{j,k}\Omega_{m,k} = \sum_{k=j+p-n}^{j-1} (-1)\Omega_{m,k} = \sum_{k=j+p-n}^{j-1} (C_{m,k} - B_{m,k}). \tag{5.32}$$

From this point on the reasoning is quite similar to the previous case, and we obtain that $(A\Omega^\top)_{j,m} = \delta_{j,m}$ always holds. $\square$

To enlighten the geometric meaning of the map $\mathcal{E}$ (5.18), notice from (5.19) that

$$\sum_{j=1}^{n} |u_j|^2 = \operatorname{sgn}(M)\big(p(\xi_1 + \cdots + \xi_n) - ny\big) = \operatorname{sgn}(M)\big(p\pi - ny\big) = |M|. \tag{5.33}$$

Then represent the complex projective space $\mathbb{CP}^{n-1}$ as

$$\mathbb{CP}^{n-1} = S^{2n-1}_{|M|}/\mathrm{U}(1) \tag{5.34}$$

with

$$S^{2n-1}_{|M|} = \{(u_1, \dots, u_n) \in \mathbb{C}^n \mid |u_1|^2 + \cdots + |u_n|^2 = |M|\}. \tag{5.35}$$

Correspondingly, let

$$\pi_{|M|}\colon S^{2n-1}_{|M|} \to \mathbb{CP}^{n-1} \tag{5.36}$$

denote the natural projection and equip $\mathbb{CP}^{n-1}$ with the rescaled Fubini-Study symplectic form $|M|\omega_{\rm FS}$ characterized by the relation

$$\pi^*_{|M|}(|M|\omega_{\rm FS}) = \mathrm{i}\sum_{j=1}^{n} d\bar{u}_j \wedge du_j, \tag{5.37}$$

where the u_j's are regarded as functions on $S^{2n-1}_{|M|}$. It is readily seen from the definitions that the map

$$\pi_{|M|} \circ \mathcal{E}\colon \mathcal{A}_y^+ \times \mathbb{T}^{n-1} \to \mathbb{CP}^{n-1} \tag{5.38}$$

is smooth, injective and its image is the open submanifold for which $\prod_{j=1}^n |u_j|^2 \neq 0$. Equations (5.10), (5.21) and (5.37) together imply the symplectic property

$$(\pi_{|M|} \circ \mathcal{E})^*(|M|\omega_{\rm FS}) = \omega^{\rm loc}, \tag{5.39}$$

from which it follows that this map is an *embedding*.

To summarize, in this section we have constructed the symplectic diffeomorphism $\pi_{|M|} \circ \mathcal{E}$ between the local phase space $P_y^{\rm loc}$ (5.9) and the dense open submanifold of $\mathbb{CP}^{n-1}$ on which the product of the homogeneous coordinates is nowhere zero. If desired, the explicit formula of the smooth inverse mapping can be easily found as well.

5.2 Global extension of the trigonometric Lax matrix

It was proved in [40] with the aid of quasi-Hamiltonian reduction that the global phase space of the $\mathrm{III_b}$ model is $\mathbb{CP}^{n-1}$ for the type (i) couplings, which we continue to consider. Here, we utilize the symplectic embedding (5.38) to construct a global Lax matrix on $\mathbb{CP}^{n-1}$ explicitly, starting from the local RS Lax matrix defined on $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$. This issue was not investigated previously except for the $p = 1$ case of (5.13), see [117, 38, 40].

The local Lax matrix $L_y^{\rm loc}(\xi, e^{\mathrm{i}\theta}) \in \mathrm{SU}(n)$ used in [40] contains the trigonometric Cauchy matrix C_y given with the help of (5.6) by

$$C_y(\xi)_{j,\ell} = \frac{e^{\mathrm{i}y} - e^{-\mathrm{i}y}}{e^{\mathrm{i}y}\delta_j(\xi)^{1/2}\delta_\ell(\xi)^{-1/2} - e^{-\mathrm{i}y}\delta_j(\xi)^{-1/2}\delta_\ell(\xi)^{1/2}}. \tag{5.40}$$

Thanks to the relation $\delta_k(\xi) = e^{2\mathrm{i}x_k}$, this is equivalent to

$$C_y(\xi)_{j,\ell} = \frac{\sin(y)}{\sin(x_j - x_\ell + y)}. \tag{5.41}$$

Then we have

$$L_y^{\rm loc}(\xi, e^{\mathrm{i}\theta})_{j,\ell} = C_y(\xi)_{j,\ell} v_j(\xi, y) v_\ell(\xi, -y)\rho(\theta)_\ell, \qquad \forall (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1}, \tag{5.42}$$

where $\rho(\theta)_\ell = e^{\mathrm{i}(\theta_{\ell-1}-\theta_\ell)}$ (applying $\theta_0 = \theta_n = 0$) and

$$v_\ell(\xi, \pm y) = \sqrt{z_\ell(\xi, \pm y)} \quad \text{with} \quad z_\ell(\xi, \pm y) = \operatorname{sgn}(\sin(ny)) \prod_{m=\ell+1}^{\ell+n-1} \frac{\sin(\sum_{k=\ell}^{m-1} \xi_k \mp y)}{\sin(\sum_{k=\ell}^{m-1} \xi_k)}. \tag{5.43}$$

A key point [40] (which is detailed below) is that $z_\ell(\xi, \pm y)$ is positive for any $\xi \in \mathcal{A}_y^+$. We note for clarity that z_ℓ and v_ℓ above differ from those in [40] by a harmless multiplicative constant, and also mention that $L_y^{\rm loc}$ is a specialization of (a similarity transform of) the standard RS Lax matrix [118].

The spectral invariants of $L_y^{\rm loc}$ (5.42) yield a Poisson commuting family of functional dimension $(n-1)$ [118, 40], containing the Hamiltonian $H_y^{\rm loc}$ (5.11) due to the equation

$$\mathrm{Re}\big(\mathrm{tr} L_y^{\rm loc}(\xi, e^{\mathrm{i}\theta})\big) = H_y^{\rm loc}(\xi, \theta). \tag{5.44}$$

There are two important observations to be made here. First, for each $1 \leq \ell \leq n$, there is only one factor in $z_\ell(\xi, \pm y)$ (5.43) that (up to sign) contains the sine of the squared absolute value (5.19) of one of the complex variables in its numerator:

- For $z_\ell(\xi, y)$, it is the factor corresponding to $m = \ell + p$, whose numerator is

$$\operatorname{sgn}(M) \sin(|u_\ell|^2). \tag{5.45}$$

- For $z_\ell(\xi, -y)$, it is the factor with $m = \ell + n - p$, whose the numerator is either

$$\sin(\pi - \operatorname{sgn}(M)|u_{\ell+n-p}|^2) = \operatorname{sgn}(M)\sin(|u_{\ell+n-p}|^2), \quad \text{if } 1 \leq \ell \leq p, \tag{5.46}$$

  or

$$\sin(\pi - \operatorname{sgn}(M)|u_{\ell-p}|^2) = \operatorname{sgn}(M)\sin(|u_{\ell-p}|^2), \quad \text{if } p < \ell \leq n. \tag{5.47}$$

  Here we made use of $\xi_1 + \cdots + \xi_n = \pi$, $\sin(\pi - \alpha) = \sin(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$.

Second, the $(p-1)$ factors in $z_\ell(\xi, \pm y)$ with $m < \ell + p$ and $m > \ell + n - p$, respectively, are strictly negative and the factors corresponding to $m > \ell + p$ and $m < \ell + n - p$, respectively, are strictly positive for all ξ in the *closed* simplex $\mathcal{A}_y$. In particular, for any $\xi \in \mathcal{A}_y^+$ the sign of the ξ-dependent product in (5.43) equals $(-1)^{p-1}\operatorname{sgn}(M) = \operatorname{sgn}(\sin(ny))$, and therefore

$$z_\ell(\xi, \pm y) \geq 0, \quad \forall \xi \in \mathcal{A}_y, \quad \ell = 1, \ldots, n. \tag{5.48}$$

We saw that z_ℓ can only vanish due to the numerators (5.45) and (5.46), (5.47), respectively. Consequently, in (5.43) the positive square root of $z_\ell(\xi, \pm y)$ can be taken for any $\xi \in \mathcal{A}_y^+$.

Now notice that, for all $\xi \in \mathcal{A}_y^+$, we have

$$v_j(\xi, y) = |u_j| w_j(\xi, y), \quad 1 \leq j \leq n, \tag{5.49}$$

where the $w_j(\xi, y)$ are positive and smooth functions of the form

$$w_j(\xi, y) = \left[\frac{\sin(|u_j|^2)}{|u_j|^2} \frac{(-1)^{p-1}}{\sin(\sum_{k=j}^{j+p-1} \xi_k)} \prod_{\substack{m=j+1 \\ (m \neq j+p)}}^{j+n-1} \frac{\sin(\sum_{k=j}^{m-1} \xi_k - y)}{\sin(\sum_{k=j}^{m-1} \xi_k)}\right]^{\frac{1}{2}}. \tag{5.50}$$

Similarly, we have

$$v_\ell(\xi, -y) = \begin{cases} |u_{\ell+n-p}| w_\ell(\xi, -y), & \text{if } 1 \leq \ell \leq p, \\ |u_{\ell-p}| w_\ell(\xi, -y), & \text{if } p < \ell \leq n \end{cases} \tag{5.51}$$

with the positive and smooth functions

$$w_\ell(\xi, -y) = \left[\frac{\sin(|u_{\ell+n-p}|^2)}{|u_{\ell+n-p}|^2} \frac{(-1)^{p-1}}{\sin(\sum_{k=\ell}^{\ell+n-p-1} \xi_k)} \prod_{\substack{m=\ell+1 \\ (m \neq \ell+n-p)}}^{\ell+n-1} \frac{\sin(\sum_{k=\ell}^{m-1} \xi_k + y)}{\sin(\sum_{k=\ell}^{m-1} \xi_k)}\right]^{\frac{1}{2}} \tag{5.52}$$

for $1 \leq \ell \leq p$, and

$$w_\ell(\xi, -y) = \left[\frac{\sin(|u_{\ell-p}|^2)}{|u_{\ell-p}|^2} \frac{(-1)^{p-1}}{\sin(\sum_{k=\ell}^{\ell+n-p-1} \xi_k)} \prod_{\substack{m=\ell+1 \\ (m \neq \ell+n-p)}}^{\ell+n-1} \frac{\sin(\sum_{k=\ell}^{m-1} \xi_k + y)}{\sin(\sum_{k=\ell}^{m-1} \xi_k)}\right]^{\frac{1}{2}} \tag{5.53}$$

for $p < \ell \leq n$.

The relation (5.22) allows us to express the ξ_k in terms of the complex variables for $k = 1, \ldots, n-1$ as

$$\xi_k(u) = \sum_{j=1}^{n-1} \Omega_{j,k}\big(\operatorname{sgn}(M)|u_j|^2 + c_j\big), \quad \text{with } c_j = \begin{cases} y, & \text{if } 1 \leq j \leq n-p, \\ y - \pi, & \text{if } n-p < j \leq n-1, \end{cases} \tag{5.54}$$

and $\xi_n(u) = \pi - \xi_1(u) - \cdots - \xi_{n-1}(u)$. These formulas extend to U(1)-invariant smooth functions on $S^{2n-1}_{|M|}$, which represent smooth functions on $\mathbb{CP}^{n-1}$ on account of (5.34). By applying these, the above expressions $w_j(\xi(u), \pm y)$ $(j = 1, \ldots, n)$ *give rise to smooth functions on* $\mathbb{CP}^{n-1}$.

Definition 5.2. By setting $\theta_k = 0$ $(\forall k)$ in the local Lax matrix L_y^{loc} (5.42) with y

(5.13), we define the functions $\Lambda^y_{j,\ell}\colon \mathcal{A}^+_y \to \mathbb{R}$ $(j,\ell = 1,\dots,n)$ via the equations

$$\Lambda^y_{j,j+p}(\xi) = L^{\rm loc}_y(\xi, \mathbf{1}_{n-1})_{j,j+p}, \quad 1 \leq j \leq n-p, \tag{5.55}$$

$$\Lambda^y_{j,j+p-n}(\xi) = L^{\rm loc}_y(\xi, \mathbf{1}_{n-1})_{j,j+p-n}, \quad n-p < j \leq n, \tag{5.56}$$

$$\Lambda^y_{j,\ell}(\xi) = L^{\rm loc}_y(\xi, \mathbf{1}_{n-1})_{j,\ell}(|u_j||u_{\ell+n-p}|)^{-1}, \quad 1 \leq j \leq n,\ 1 \leq \ell \leq p \quad (\ell \neq j+p-n), \tag{5.57}$$

$$\Lambda^y_{j,\ell}(\xi) = L^{\rm loc}_y(\xi, \mathbf{1}_{n-1})_{j,\ell}(|u_j||u_{\ell-p}|)^{-1}, \quad 1 \leq j \leq n,\ p < \ell \leq n \quad (\ell \neq j+p). \tag{5.58}$$

The foregoing results lead to explicit formulas for $\Lambda^y_{j,\ell}$ (see Appendix E.1). Using the identification (5.34) and (5.54), it is readily seen that the $\Lambda^y_{j,\ell}(\xi(u))$ given by Definition 5.2 extend to smooth functions on $\mathbb{CP}^{n-1}$.

Remark 5.3. The explicit formulas of $\Lambda^y_{j,\ell}(\xi(u))$ contain products of square roots of strictly positive functions depending on $|u_k|^2 \in C^\infty(S^{2n-1}_{|M|})^{\mathrm{U}(1)}$ for $k = 1,\dots,n$. In particular, they contain the square root of the function J given by

$$J(|u_k|^2) = \frac{\sin(|u_k|^2)}{|u_k|^2}, \tag{5.59}$$

which remains smooth (even real-analytic) at $|u_k|^2 = 0$ and is positive since we have $0 \leq |u_k|^2 \leq |M| < \pi$. Indeed, $|M| < \pi/q$ and $|M| < \pi/(n-q)$, respectively, for the two intervals of the type (i) couplings in (5.13).

The above observations allow us to introduce the following functions, which will be used to construct the global Lax matrix.

Definition 5.4. For $M > 0$ (5.15), define the smooth functions $L^{y,+}_{j,\ell}\colon \mathbb{CP}^{n-1} \to \mathbb{C}$ by

$$L^{y,+}_{j,\ell} \circ \pi_{|M|}(u) = \begin{cases} \Lambda^y_{j,j+p}(\xi(u)), & \text{if } 1 \leq j \leq n-p,\ \ell = j+p, \\ \Lambda^y_{j,j+p-n}(\xi(u)), & \text{if } n-p < j \leq n,\ \ell = j+p-n, \\ \bar{u}_j u_{\ell+n-p}\Lambda^y_{j,\ell}(\xi(u)), & \text{if } 1 \leq j \leq n,\ 1 \leq \ell \leq p,\ \ell \neq j+p-n, \\ \bar{u}_j u_{\ell-p}\Lambda^y_{j,\ell}(\xi(u)), & \text{if } 1 \leq j \leq n,\ p < \ell \leq n,\ \ell \neq j+p, \end{cases} \tag{5.60}$$

where u varies in $S^{2n-1}_{|M|}$. Then, for $M < 0$, define $L^{y,-}_{j,\ell}\colon \mathbb{CP}^{n-1} \to \mathbb{C}$ by

$$L^{y,-}_{j,\ell} \circ \pi_{|M|}(u) = L^{y,+}_{j,\ell} \circ \pi_{|M|}(\bar{u}), \tag{5.61}$$

referring to the right-hand-side of (5.60) with the understanding that now $y > p\pi/n$.

Next, we prove that the matrices $L^{\rm loc}_y$ and $L^{y,\pm} \circ \pi_{|M|} \circ \mathcal{E}$, are similar and can be transformed into each other by a unitary matrix. This is one of our main results.

Theorem 5.5. *The smooth matrix function $L^{y,\pm}\colon \mathbb{CP}^{n-1} \to \mathbb{C}^{n\times n}$ with components $L^{y,\pm}_{j,\ell}$ given by (5.60),(5.61) satisfies the following identity*

$$(L^{y,\pm}\circ\pi_{|M|}\circ\mathcal{E})(\xi,e^{\mathrm{i}\theta}) = \Delta(e^{\mathrm{i}\theta})^{-1}L^{\mathrm{loc}}_y(\xi,e^{\mathrm{i}\theta})\Delta(e^{\mathrm{i}\theta}),\quad \forall(\xi,e^{\mathrm{i}\theta})\in\mathcal{A}^+_y\times\mathbb{T}^{n-1}, \tag{5.62}$$

where $\Delta(e^{\mathrm{i}\theta}) = \mathrm{diag}(\Delta_1,\dots,\Delta_n)\in \mathrm{U}(n)$ with

$$\Delta_j = \exp\left(\mathrm{i}\sum_{k=1}^{n-1}\Omega_{j,k}\theta_k\right),\quad j=1,\dots,n-1,\quad \Delta_n = 1. \tag{5.63}$$

Consequently, $L^{y,\pm}(\pi_{|M|}(u))\in\mathrm{SU}(n)$ for every $u\in S^{2n-1}_{|M|}$, and $L^{y,\pm}$ provides an extension of the local Lax matrix L^{loc}_y (5.42) to the global phase space $\mathbb{CP}^{n-1}$.

Proof. The form of the local Lax matrix L^{loc}_y (5.42) and Definitions 5.2 and 5.4 show that (5.62) is equivalent to the equations

$$\Delta_j = \begin{cases}\Delta_{j+p}\rho_{j+p}, & \text{if}\quad 1\leq j\leq n-p,\\ \Delta_{j+p-n}\rho_{j+p-n}, & \text{if}\quad n-p<j\leq n.\end{cases} \tag{5.64}$$

The two sides of (5.64) can be written as exponentials of linear combinations of the variables θ_k ($1\leq k\leq n-1$). We next spell out the relations that ensure the exact matching of the coefficients of the θ_k in these exponentials. Plugging the components of Δ and ρ into (5.64), the case $1\leq j<n-p$ gives

$$\begin{aligned}\Omega_{j,j+p-1} &= \Omega_{j+p,j+p-1}+1,\quad (\text{coefficients of }\theta_{j+p-1})\\ \Omega_{j,j+p} &= \Omega_{j+p,j+p}-1,\quad (\text{coefficients of }\theta_{j+p})\\ \Omega_{j,k} &= \Omega_{j+p,k},\quad (\text{coefficients of }\theta_k,\ k\neq j+p-1,j+p),\end{aligned} \tag{5.65}$$

while for $j=n-p$ we get

$$\begin{aligned}\Omega_{n-p,n-1} &= 1,\quad (\text{coefficients of }\theta_{n-1})\\ \Omega_{n-p,k} &= 0,\quad (\text{coefficients of }\theta_k,\ k\neq n-1).\end{aligned} \tag{5.66}$$

The case $n-p<j<n$ (and $p>1$) leads to

$$\begin{aligned}\Omega_{j,j+p-n-1} &= \Omega_{j+p-n,j+p-n-1}+1,\quad (\text{coefficients of }\theta_{j+p-n-1})\\ \Omega_{j,j+p-n} &= \Omega_{j+p-n,j+p-n}-1,\quad (\text{coefficients of }\theta_{j+p-n})\\ \Omega_{j,k} &= \Omega_{j+p-n,k},\quad (\text{coefficients of }\theta_k,\ k\neq j+p-n-1,j+p-n).\end{aligned} \tag{5.67}$$

For $j = n$ there are two possibilities. If $p = 1$ then we obtain

$$\begin{aligned} \Omega_{1,1} &= 1, \quad (\text{coefficients of } \theta_1) \\ \Omega_{1,k} &= 0, \quad (\text{coefficients of } \theta_k,\ k \neq 1), \end{aligned} \tag{5.68}$$

and if $p > 1$ then we require

$$\begin{aligned} \Omega_{p,p-1} &= -1, \quad (\text{coefficients of } \theta_{p-1}) \\ \Omega_{p,p} &= 1, \quad (\text{coefficients of } \theta_p) \\ \Omega_{p,k} &= 0, \quad (\text{coefficients of } \theta_k,\ k \neq p-1, p). \end{aligned} \tag{5.69}$$

Using the explicit formula given by Proposition 5.1, we now show that Ω satisfies (5.65). Since $\Omega_{j,k} = B_{j,k} - C_{j,k}$ for all j, k, where $B_{j,k}$ (5.26) depends on $(k-j)$ and $C_{j,k}$ (5.27) depends only on k, the equations (5.65) reduce to

$$\begin{aligned} B_{j,j+p-1} &= B_{j+p,j+p-1} + 1, \\ B_{j,j+p} &= B_{j+p,j+p} - 1, \\ B_{j,k} &= B_{j+p,k}, \quad k \neq j+p-1, j+p. \end{aligned} \tag{5.70}$$

The first equation holds, because $(j+p-1) - j = p - 1 \equiv (n-q+1)p \pmod{n}$ implies $B_{j,j+p-1} = 1$ and $(j+p-1) - (j+p) = -1 \equiv (n-q)p \pmod{n}$ implies $B_{j+p,j+p-1} = 0$. For the second equation, we plainly have $B_{j,j+p} = 0$, and $(j+p) - (j+p) = 0 \equiv np \pmod{n}$ gives $B_{j+p,j+p} = 1$. Regarding the third equation, notice that $B_{j,k} = 0$ in (5.70) when $k - j \equiv \ell p \pmod{n}$ for some $\ell \in \{2, \dots, n-q\}$, and then $B_{j+p,k} = 0$ holds, too. Conversely, $B_{j+p,k} = 0$ in (5.70) means that $(k-j) - p \equiv \ell p \pmod{n}$ for some $\ell \in \{1, \dots, n-q-1\}$, from which $(k-j) \equiv (\ell+1)p \pmod{n}$ and thus $B_{j,k} = 0$ follows. As B is a $(0,1)$-matrix, we conclude that (5.70) is valid. Proceeding in a similar manner, we have verified the rest of the relations (5.66)-(5.69) as well. Since the relations (5.65)-(5.69) imply (5.64), the proof is complete. $\square$

It is an immediate consequence of Theorem 5.5 that the spectral invariants of the global Lax matrix $L^{y,\pm} \in C^\infty(\mathbb{CP}^{n-1}, \mathrm{SU}(n))$ yield a Liouville integrable system. Because of (5.44) the corresponding Poisson commuting family contains the extension of the $\mathrm{III_b}$ Hamiltonian H_y^{loc} to $\mathbb{CP}^{n-1}$ for any type (i) coupling. The self-duality of this compactified RS system was established in [40], and it will be studied in more detail elsewhere.

5.3 New compact forms of the elliptic Ruijsenaars-Schneider system

In this section we explain that type (i) compactifications of the elliptic RS system can be constructed in exactly the same way as we saw for the trigonometric system. This is due to the fact that the local elliptic Lax matrix is built from the s-function (5.72) similarly as its trigonometric counterpart is built from the sine function, and on the real axis these two functions have the same zeros, signs, parity and antiperiodicity property.

We start by recalling some formulas of the relevant elliptic functions. First, let ω, ω' stand for the half-periods of the Weierstrass $\wp$ function defined by

$$\wp(z;\omega,\omega') = \frac{1}{z^2} + \sum_{\substack{m,m'=-\infty \\ (m,m')\neq(0,0)}}^{\infty} \left[\frac{1}{(z-\omega_{m,m'})^2} - \frac{1}{\omega_{m,m'}^2}\right], \tag{5.71}$$

with $\omega_{m,m'} = 2m\omega + 2m'\omega'$. We adopt the convention $\omega, -\mathrm{i}\omega' \in (0,\infty)$, which ensures that $\wp$ is positive on the real axis. Next, introduce the following 's-function':

$$\mathrm{s}(z;\omega,\omega') = \frac{2\omega}{\pi}\sin\!\Big(\frac{\pi z}{2\omega}\Big)\prod_{m=1}^{\infty}\left[1 + \frac{\sin^2(\pi z/(2\omega))}{\sinh^2(m\pi|\omega'|/\omega)}\right], \tag{5.72}$$

related to the Weierstrass σ and ζ functions by $\mathrm{s}(z) = \sigma(z)\exp(-\eta z^2/(2\omega))$ with the constant $\eta = \zeta(\omega)$. A useful identity connecting $\wp$ and s is

$$\frac{\mathrm{s}(z+z')\,\mathrm{s}(z-z')}{\mathrm{s}^2(z)\,\mathrm{s}^2(z')} = \wp(z') - \wp(z), \qquad z, z' \in \mathbb{C}. \tag{5.73}$$

The s-function is odd, has simple zeros at $\omega_{m,m'}$ $(m, m' \in \mathbb{Z})$ and enjoys the scaling property $\mathrm{s}(tz; t\omega, t\omega') = t\,\mathrm{s}(z;\omega,\omega')$. From now on we take

$$\omega = \frac{\pi}{2}, \tag{5.74}$$

whereby $\mathrm{s}(z+\pi) = -\,\mathrm{s}(z)$ holds as well. The trigonometric limit is obtained according to

$$\lim_{-\mathrm{i}\omega'\to\infty}\wp(z;\pi/2,\omega') = \frac{1}{\sin^2(z)} - \frac{1}{3}, \qquad \lim_{-\mathrm{i}\omega'\to\infty}\mathrm{s}(z;\pi/2,\omega') = \sin(z). \tag{5.75}$$

Let us now pick a type (i) coupling parameter y (5.13) and choose the domain of the dynamical variables to be the same $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$ as in the trigonometric case. Then consider the following $\mathrm{IV_b}$ variant of the standard [112, 118] elliptic RS Lax matrix:

$$L_y^{\mathrm{loc}}(\xi, e^{\mathrm{i}\theta}|\lambda)_{j,\ell} = \frac{\mathrm{s}(y)}{\mathrm{s}(\lambda)} \frac{\mathrm{s}(x_j - x_\ell + \lambda)}{\mathrm{s}(x_j - x_\ell + y)} v_j(\xi, y) v_\ell(\xi, -y) \rho(\theta)_\ell, \ \ \forall (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1}, \tag{5.76}$$

where $\lambda \in \mathbb{C} \setminus \{\omega_{m,m'} : m, m' \in \mathbb{Z}\}$ is a spectral parameter and $v_\ell(\xi, \pm y) = \sqrt{z_\ell(\xi, \pm y)}$ with

$$z_\ell(\xi, \pm y) = \mathrm{sgn}(\mathrm{s}(ny)) \prod_{m=\ell+1}^{\ell+n-1} \frac{\mathrm{s}(\sum_{k=\ell}^{m-1} \xi_k \mp y)}{\mathrm{s}(\sum_{k=\ell}^{m-1} \xi_k)}. \tag{5.77}$$

These formulas are to be compared with the trigonometric case. Since $\mathrm{s}(z)$ and $\sin(z)$ have matching properties on the real line, we can repeat the arguments presented in Section 5.2 to verify that $z_\ell(\xi, \pm y) > 0$ for every $\xi \in \mathcal{A}_y^+$. Taking positive square roots, and applying the relation $x_{k+1} - x_k = \xi_k$ to express $x_j - x_\ell$ in terms of ξ, we conclude that the above local Lax matrix is a smooth function on $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$ for every allowed value of the spectral parameter. The fact that it is a specialization of the standard elliptic Lax matrix ensures [112, 118] that its characteristic polynomial generates $(n-1)$ independent real Hamiltonians in involution with respect to the symplectic form (5.10). Indeed, the characteristic polynomial has the form

$$\det\left(L_y^{\mathrm{loc}}(\xi, e^{\mathrm{i}\theta}|\lambda) - \alpha \mathbf{1}_n\right) = \sum_{k=0}^{n} (-\alpha)^{n-k} c_k(\lambda, y) \mathcal{S}_k^{\mathrm{loc}}(\xi, e^{\mathrm{i}\theta}, y), \tag{5.78}$$

where the functions $\mathcal{S}_k^{\mathrm{loc}}$ as well as their real and imaginary parts Poisson commute, and $\mathrm{Re}(\mathcal{S}_k^{\mathrm{loc}})$ for $k = 1, \dots, n-1$ are functionally independent. Explicit formulas of the c_k (that do not depend on the phase space variables) and $\mathcal{S}_k^{\mathrm{loc}}$ (that do not depend on λ) can be found in [112, 118]. The function $\mathrm{Re}(\mathcal{S}_1^{\mathrm{loc}})$ is the RS Hamiltonian of $\mathrm{IV_b}$ type

$$\mathrm{Re}\big(\mathrm{tr} L_y^{\mathrm{loc}}(\xi, e^{\mathrm{i}\theta}|\lambda)\big) = \sum_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \sqrt{\prod_{m=j+1}^{j+n-1} \left[\mathrm{s}(y)^2 (\wp(y) - \wp(\textstyle\sum_{k=j}^{m-1} \xi_k))\right]}. \tag{5.79}$$

We note in passing that in Ruijsenaars's papers [112, 118] one finds the elliptic Lax matrix $V L_y^{\mathrm{loc}} V^{-1}$, where V is the diagonal matrix $V = \rho(\theta) \mathrm{diag}(v_1(\xi, -y), \dots, v_n(\xi, -y))$. This difference is irrelevant, since it has no effect on the generated spectral invariants. Another difference is that we work in the center-of-mass frame.

Now the complete train of thought applied in the previous section remains valid if we simply replace the sine function with the s-function everywhere. In particular, the direct analogues of the formulas (5.49)-(5.53) hold with smooth functions $w_k(\xi, \pm y) > 0$, for $\xi \in \mathcal{A}_y$. Due to this fact, we can introduce a smooth elliptic Lax matrix defined on the global phase space $\mathbb{CP}^{n-1}$. The subsequent definition refers to the explicit formulas of Appendix E.1, which in the elliptic case contain the function

$$\mathcal{J}(|u_k|^2) = \frac{\mathrm{s}(|u_k|^2)}{|u_k|^2}. \tag{5.80}$$

This has the same smoothness and positivity properties at and around zero as J (5.59) does. We also use $\xi(u)$ (5.54) and the functions $(x_j - x_\ell)(\xi)$ determined by $x_{k+1} - x_k = \xi_k$.

Definition 5.6. Take a type (i) y from (5.13) and represent the points of $\mathbb{CP}^{n-1}$ as $\pi_{|M|}(u)$ with $u \in S^{2n-1}_{|M|}$. For $M > 0$ (5.15), define the smooth functions $\mathcal{L}^{y,+}_{j,\ell}$ on $\mathbb{CP}^{n-1}$ by

$$\mathcal{L}^{y,+}_{j,\ell}(\pi_{|M|}(u)) = \begin{cases} \Lambda^y_{j,j+p}(\xi(u)), & \text{if } 1 \leq j \leq n-p,\ \ell = j+p, \\ \Lambda^y_{j,j+p-n}(\xi(u)), & \text{if } n-p < j \leq n,\ \ell = j+p-n, \\ \bar{u}_j u_{\ell+n-p} \Lambda^y_{j,\ell}(\xi(u)), & \text{if } 1 \leq j \leq n,\ 1 \leq \ell \leq p,\ \ell \neq j+p-n, \\ \bar{u}_j u_{\ell-p} \Lambda^y_{j,\ell}(\xi(u)), & \text{if } 1 \leq j \leq n,\ p < \ell \leq n,\ \ell \neq j+p, \end{cases} \tag{5.81}$$

with $\Lambda^y_{j,\ell}$ given in Appendix E.1. For $M < 0$, set $\mathcal{L}^{y,-}_{j,\ell}$ to be

$$\mathcal{L}^{y,-}_{j,\ell}(\pi_{|M|}(u)) = \mathcal{L}^{y,+}_{j,\ell}(\pi_{|M|}(\bar{u})) \tag{5.82}$$

with the understanding that in this case $y > p\pi/n$. Finally, define the λ-dependent elliptic Lax matrix $L^{y,\pm}$ on $\mathbb{CP}^{n-1}$ by

$$L^{y,\pm}_{j,\ell}(\pi_{|M|}(u)|\lambda) = \frac{\mathrm{s}((x_j - x_\ell)(\xi(u)) + \lambda)}{\mathrm{s}(\lambda)} \mathcal{L}^{y,\pm}_{j,\ell}(\pi_{|M|}(u)), \tag{5.83}$$

where u runs over $S^{2n-1}_{|M|}$ and the spectral parameter λ varies in $\mathbb{C} \setminus \{\omega_{m,m'} : m, m' \in \mathbb{Z}\}$.

Theorem 5.7. *The spectral parameter dependent elliptic Lax matrix* $L^{y,\pm}(\pi_{|M|}(u)|\lambda)$ (5.83) *is a smooth global extension of* $L^{\mathrm{loc}}_y(\xi, e^{\mathrm{i}\theta}|\lambda)$ (5.76) *to the complex projective space* $\mathbb{CP}^{n-1}$ *since it satisfies*

$$L^{y,\pm}((\pi_{|M|} \circ \mathcal{E})(\xi, e^{\mathrm{i}\theta})|\lambda) = \Delta(e^{\mathrm{i}\theta})^{-1} L^{\mathrm{loc}}_y(\xi, e^{\mathrm{i}\theta}|\lambda) \Delta(e^{\mathrm{i}\theta}), \quad \forall (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}^+_y \times \mathbb{T}^{n-1}, \tag{5.84}$$

where Δ *is given by* (5.63) *and* $\pi_{|M|} \circ \mathcal{E} \colon \mathcal{A}^+_y \times \mathbb{T}^{n-1} \to \mathbb{CP}^{n-1}$ *is the symplectic embedding defined in Section 5.1.*

The proof of Theorem 5.7 follows the lines of the proof of Theorem 5.5. The characteristic polynomial $\det\left(L^{y,\pm}(\pi_{|M|}(u)|\lambda) - \alpha \mathbf{1}_n\right)$ of the global Lax matrix depends smoothly on $\pi_{|M|}(u) \in \mathbb{CP}^{n-1}$ and as a consequence of (5.84) it satisfies

$$\det\left(L^{y,\pm}((\pi_{|M|} \circ \mathcal{E})(\xi, e^{\mathrm{i}\theta})|\lambda) - \alpha\mathbf{1}_n\right) = \det\left(L^{\mathrm{loc}}_y(\xi, e^{\mathrm{i}\theta}|\lambda) - \alpha\mathbf{1}_n\right). \tag{5.85}$$

Since this holds for all α and λ, we see that the local $\mathrm{IV_b}$ Hamiltonian (5.79) together with its constants of motion $\mathrm{Re}(\mathcal{S}^{\mathrm{loc}}_k)$, $k = 2, \dots, n-1$ extends to an integrable system on $\mathbb{CP}^{n-1}$. This was pointed out previously [118] for the special case $0 < y < \pi/n$ in (5.13).

In the trigonometric limit $-\mathrm{i}\omega' \to \infty$ the s-function becomes the sine function, and we obtain a spectral parameter dependent trigonometric Lax matrix from the elliptic one. Then, setting the spectral parameter to be on the imaginary axis and taking the limit $-\mathrm{i}\lambda \to \infty$ reproduces, up to conjugation by a diagonal matrix, the trigonometric global Lax matrix of Definition 5.4. Correspondingly, the global extension of the $\mathrm{IV_b}$ Hamiltonian (5.79) and its commuting family reduces to the global extension of the $\mathrm{III_b}$ Hamiltonian (5.11) and its constants of motion.

5.4 Discussion

In this chapter we have demonstrated by direct construction that the local phase space $\mathcal{A}^+_y \times \mathbb{T}^{n-1}$ of the $\mathrm{III_b}$ and $\mathrm{IV_b}$ RS models (where $\mathcal{A}^+_y$ is the interior of the simplex (5.16)) can be embedded into $\mathbb{CP}^{n-1}$ for any type (i) coupling y (5.13) in such a way that a suitable conjugate of the local Lax matrix extends to a smooth (actually real-analytic) function. Theorems 5.5 and 5.7 together with Appendix E.1 provide explicit formulas for the resulting global Lax matrices. Their characteristic polynomials give rise to Poisson commuting real Hamiltonians on $\mathbb{CP}^{n-1}$ that yield the Liouville integrable compactified trigonometric and elliptic RS systems.

Our direct construction was inspired by the earlier derivation of compactified $\mathrm{III_b}$ systems by quasi-Hamiltonian reduction [40]. The reduction identifies the $\mathrm{III_b}$ system with a topological Chern-Simons field theory for any generic coupling parameter y. It appears natural to ask if an analogous derivation and relation to some topological field theory could exist for $\mathrm{IV_b}$ systems, too. We also would like to obtain a better understanding of the type (ii) trigonometric systems and their possible elliptic analogues.

Besides further studying the systems that we described, it would be also interesting to search for compactifications of generalized RS systems. We have in mind especially the BC_n systems due to van Diejen [140] and the recently introduced supersymmetric systems [14]. Regarding the former case, and even for general root systems, the results of [144] could be relevant, as well as the construction of Lax matrices for some of the BC_n systems reported in Chapter 4.

Throughout the text, we worked in the 'center-of-mass frame' and now we end by a comment on how the center-of-mass coordinate can be introduced into our systems. One possibility is to take the full phase space to be the Cartesian product of $\mathbb{CP}^{n-1}$ with $\mathrm{U}(1)\times\mathrm{U}(1) = \{(e^{2\mathrm{i}X}, e^{\mathrm{i}\Phi})\}$ endowed with the symplectic form $|M|\,\omega_{\mathrm{FS}}+dX\wedge d\Phi$. Here, $e^{2\mathrm{i}X}$ is interpreted as a center-of-mass variable for the n particles on the circle. Then n functions in involution result by adding an arbitrary function of $e^{\mathrm{i}\Phi}$ to the $(n-1)$ commuting Hamiltonians generated by the 'total Lax matrix' $e^{-\mathrm{i}\Phi}L^{y,\pm}$. On the dense open domain the total Lax matrix is obtained by replacing $\rho(\theta)$ in (5.76) by $\rho(\theta)e^{-\mathrm{i}\Phi}$. By setting $e^{\mathrm{i}\Phi}$ to 1 and quotienting by the canonical transformations generated by the functions of $e^{\mathrm{i}\Phi}$ one recovers the phase space of the relative motion, $\mathbb{CP}^{n-1}$. There are also several other possibilities, as was discussed for analogous situations in [117, 35]. For example, one may replace $\mathrm{U}(1)\times\mathrm{U}(1)$ by its covering space $\mathbb{R}\times\mathbb{R}$.

In the near future, we wish to explore the classical dynamics and quantization of the $\mathrm{III_b}$ systems. For arbitrary type (i) couplings, geometric quantization yields the joint spectra of the quantized action variables effortlessly [39]. (It is necessary to introduce a second parameter into the systems before quantization, which can be achieved by taking an arbitrary multiple of the symplectic form.) The joint eigenfunctions of the quantized Ruijsenaars-Schneider Hamiltonian and its commuting family should be derived by generalizing the results of van Diejen and Vinet [146].

Appendices

A Appendix to Chapter 1

A.1 An alternative proof of Theorem 1.2

In this appendix, we give an alternative proof for Theorem 1.2, which is based on the scattering behaviour of particles in the rational Calogero-Moser model (see Figure 8).

Recall the Lax pair found by Moser [88]

$$L_{jk} = p_j\delta_{jk} + \mathrm{i}g\frac{1-\delta_{jk}}{q_j - q_k}, \quad B_{jk} = \mathrm{i}g\delta_{jk}\sum_{\substack{l=1\\(l\neq k)}}^{n}\frac{1}{(q_k-q_l)^2} - \mathrm{i}g\frac{1-\delta_{jk}}{(q_j-q_k)^2}, \tag{A.1}$$

and consider $Q = \mathrm{diag}(q_1, \dots, q_n)$. These matrices satisfy the commutation relation

$$[z\mathbf{1}_n - L, Q] = \mathrm{i}g(vv^\dagger - \mathbf{1}_n), \tag{A.2}$$

for any scalar $z \in \mathbb{C}$, where $v = (1\dots1)^\dagger$ and $\mathbf{1}_n$ stands for the $n\times n$ identity matrix. They also enjoy the following relations along solutions

$$\dot{L} = [L, B], \quad \dot{Q} = [Q, B] + L. \tag{A.3}$$

The asymptotic form of solutions of the Calogero-Moser system is

$$q_k(t) \sim p_k^\pm t + q_k^\pm, \quad p_k(t) \sim p_k^\pm, \quad t \to \pm\infty. \tag{A.4}$$

Theorem 1.2 connects the following functions

$$C(z) = \mathrm{tr}(Q(z\mathbf{1}_n - L)^\vee vv^\dagger), \quad D(z) = \mathrm{tr}(Q(z\mathbf{1}_n - L)^\vee), \tag{A.5}$$

where $M^\vee$ denotes the adjugate of M, i.e. the transpose of its cofactor matrix.

Theorem A.1. *For any $n\in\mathbb{N}$, $p, q\in\mathbb{R}^n$ with $q_j \neq q_k$ $(j\neq k)$, and $z\in\mathbb{C}$ we have*

$$C(z) = D(z) + \frac{\mathrm{i}g}{2}\frac{d^2}{dz^2}\det(z\mathbf{1}_n - L). \tag{A.6}$$

Proof. Pick any point in the phase space and consider the solution passing through it.

The Lax matrix L is isospectral along the solutions, thus its characteristic polynomial is constant, viz. $\frac{d}{dt}\det(z\mathbf{1}_n - L) = 0$. Consequently, its second derivative w.r.t. z is also constant, that is

$$\frac{d}{dt}\left(\frac{d^2}{dz^2}\det(z\mathbf{1}_n - L)\right) = 0. \tag{A.7}$$

The difference of the functions $C(z)$ and $D(z)$ reads

$$C(z) - D(z) = \operatorname{tr}\big(Q(z\mathbf{1}_n - L)^\vee(vv^\dagger - \mathbf{1}_n)\big). \tag{A.8}$$

By utilizing the commutation relation (A.2) of L and Q, the above casts into

$$\mathrm{i}g(C(z) - D(z)) = \det(z\mathbf{1}_n - L)\operatorname{tr}(Q^2) - \operatorname{tr}\big(Q(z\mathbf{1}_n - L)^\vee Q(z\mathbf{1}_n - L)\big), \tag{A.9}$$

where we made use of the matrix identity $MM^\vee = \det(M)$. By applying (A.3) and

$$\frac{d}{dt}(z\mathbf{1}_n - L)^\vee = [(z\mathbf{1}_n - L)^\vee, B], \tag{A.10}$$

as well as, the Leibniz rule and the cyclic property of the trace one finds that

$$\frac{d}{dt}\big(C(z) - D(z)\big) = 0. \tag{A.11}$$

Putting (A.7) and (A.11) together shows that $C(z) - D(z) + \frac{\mathrm{i}g}{2}\frac{d^2}{dz^2}\det(z\mathbf{1}_n - L)$ is constant. However, due to (A.4), in the asymptotic limit a closer inspection of (A.8) shows that

$$\lim_{t\to\infty}\left(C(z) - D(z) + \frac{\mathrm{i}g}{2}\frac{d^2}{dz^2}\det(z\mathbf{1}_n - L)\right) = 0. \tag{A.12}$$

This concludes the proof. $\square$

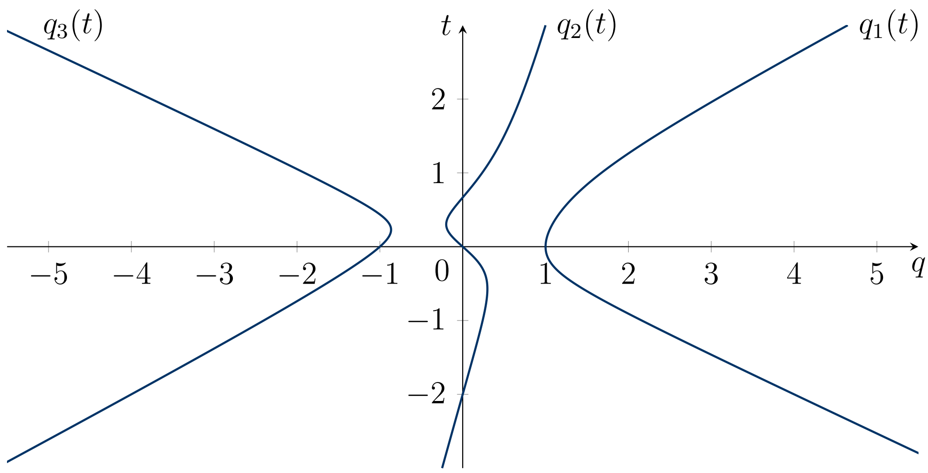

Figure 8: Space-time diagram of particle scattering in the rational Calogero-Moser model with coupling $g = 1$, and initial state $q(0) = (1, 0, -1)$, $p(0) = (1, -1, 1)$.

B Appendices to Chapter 2

B.1 Application of Jacobi's theorem on complementary minors

In this appendix, we complete the proof of Lemma 2.7 by a calculation based on Jacobi's theorem on complementary minors (e.g. [104]), which will be recalled shortly. Our reasoning below is adapted from Pusztai [105]. A significant difference is that in our case we need the strong regularity conditions (2.87) and (2.100) to avoid dividing by zero during the calculation. In fact, this appendix is presented mainly to explain the origin of the strong regularity conditions.

For an $m \times m$ matrix M let $M\left(\begin{smallmatrix} r_1 & \cdots & r_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)$ denote the determinant formed from the entries lying on the intersection of the rows $r_1, \dots, r_k$ with the columns $c_1, \dots, c_k$ of M ($k \leq m$),

$$M\begin{pmatrix} r_1 & \dots & r_k \\ c_1 & \dots & c_k \end{pmatrix} = \det(M_{r_i,c_j})_{i,j=1}^k.$$

Theorem B.1 (Jacobi)**.** *Let A be an invertible $N \times N$ matrix with $\det(A) = 1$ and $B = (A^{-1})^\top$. For a fixed permutation $\left(\begin{smallmatrix} j_1 & \cdots & j_N \\ k_1 & \cdots & k_N \end{smallmatrix}\right)$ of the pairwise distinct indices $j_1, \dots, j_N \in \{1, \dots, N\}$ and any $1 \leq p < N$ we have*

$$B\begin{pmatrix} j_1 & \dots & j_p \\ k_1 & \dots & k_p \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} j_1 & \dots & j_N \\ k_1 & \dots & k_N \end{pmatrix} A\begin{pmatrix} j_{p+1} & \dots & j_N \\ k_{p+1} & \dots & k_N \end{pmatrix}. \tag{B.1}$$

Applying Jacobi's theorem to $\check{A}$ (2.101) we now derive the two equations (2.103) and (2.104) for the pair of functions (W_a, W_{n+a}) for each $a = 1, \dots, n$, which are defined by $W_k = w_k \mathcal{F}_k$ with $\mathcal{F}_k = |F_k|^2$ (2.93) and w_k in (2.102).

Lemma B.2. *Fix any strongly regular λ, i.e. $\lambda \in \mathbb{R}^n$ for which (2.87) and (2.100) hold, and use the above notations for (W_a, W_{n+a}). If $\check{A}$ given by (2.101) is a unitary matrix, then (W_a, W_{n+a}) satisfies the two equations (2.103) and (2.104) for each $a = 1, \dots, n$.*

Proof. Let $\check{B} = (\check{A}^{-1})^\top$, i.e. $\check{B}_{j,k} = \overline{\check{A}_{j,k}}$, $j, k \in \{1, \dots, N\}$ and $a \in \{1, \dots, n\}$ be a fixed index. Since $\det(\check{A}) = 1$, by Jacobi's theorem with $j_b = b$, ($b \in \mathbb{N}_N$) and $k_c = c$,

($c \in \mathbb{N}_N \setminus \{a, n+a\}$), $k_a = n+a$, $k_{n+a} = a$ and $p = n$ we have

$$\check{B}\begin{pmatrix} 1 & \dots & a & \dots & n \\ 1 & \dots & n+a & \dots & n \end{pmatrix} = -\check{A}\begin{pmatrix} n+1 & \dots & n+a & \dots & N \\ n+1 & \dots & a & \dots & N \end{pmatrix}. \tag{B.2}$$

Denote the corresponding $n \times n$ submatrices of $\check{B}$ and $\check{A}$ by ξ and η, respectively. One can check that

$$\xi = \Psi - \frac{\mu - \nu}{\mu - \lambda_a} E_{a,a}, \quad \eta = \Xi - \frac{\mu - \nu}{\mu + \lambda_a} E_{a,a}, \tag{B.3}$$

where $E_{j,k}$ stands for the $n \times n$ elementary matrix $(E_{j,k})_{j',k'} = \delta_{j,j'}\delta_{k,k'}$ and Ψ and Ξ are the Cauchy-like matrices

$$\Psi_{j,k} = \begin{cases} \dfrac{2\mu \overline{F}_j F_{n+k}}{2\mu - \lambda_j + \lambda_k}, & \text{if } k \neq a, \\[2ex] \dfrac{2\mu \overline{F}_j F_a}{2\mu - \lambda_j - \lambda_a}, & \text{if } k = a, \end{cases} \quad \text{and} \quad \Xi_{j,k} = \begin{cases} \dfrac{2\mu F_{n+j} \overline{F}_k}{2\mu + \lambda_j - \lambda_k}, & \text{if } k \neq a, \\[2ex] \dfrac{2\mu F_{n+j} \overline{F}_{n+a}}{2\mu + \lambda_j + \lambda_a}, & \text{if } k = a, \end{cases} \tag{B.4}$$

$j, k \in \{1, \dots, n\}$. Expanding $\det(\xi)$ and $\det(\eta)$ along the a-th column we obtain the formulae

$$\det(\xi) = \det(\Psi) - \frac{\mu - \nu}{\mu - \lambda_a} \mathcal{C}_{a,a}, \quad \det(\eta) = \det(\Xi) - \frac{\mu - \nu}{\mu + \lambda_a} \mathcal{C}_{a,a}, \tag{B.5}$$

where $\mathcal{C}_{a,a}$ is the cofactor of Ψ associated with entry $\Psi_{a,a}$. Since Ψ and Ξ are both Cauchy-like matrices we have

$$\det(\Psi) = \frac{1}{\mu - \lambda_a} D_a W_a, \quad \det(\Xi) = \frac{1}{\mu + \lambda_a} D_a W_{n+a}, \tag{B.6}$$

where

$$D_a = \prod_{\substack{b=1 \\ (b \neq a)}}^{n} \overline{F}_b F_{n+b} \prod_{\substack{c,d=1 \\ (a \neq c \neq d \neq a)}}^{n} \frac{\lambda_c - \lambda_d}{2\mu + \lambda_c - \lambda_d}. \tag{B.7}$$

It can be easily seen that $\mathcal{C}_{a,a} = D_a$, therefore formulae (B.2), (B.5), (B.6) lead to the equation

$$(\mu + \lambda_a) W_a + (\mu - \lambda_a) W_{n+a} - 2(\mu - \nu) = 0. \tag{B.8}$$

It should be noticed that in the last step we divided by D_a, which is legitimate since D_a is non-vanishing due to the strong-regularity condition given by (2.87) and (2.100). To see this, assume momentarily that $F_i = 0$ for some $i = 1, \dots, n$ at some strongly regular λ. The denominator in (2.101) does not vanish, and the unitarity of $\check{A}$ implies that we must have $\check{A}_{i,i+n} = 1$ or $\check{A}_{i,i+n} = -1$. These in turn are equivalent to

$$\lambda_i = 2\mu - \nu \quad \text{or} \quad \lambda_i = \nu, \tag{B.9}$$

which are excluded by (2.100). One can similarly check that the vanishing of F_{n+i} would require

$$\lambda_i = \nu - 2\mu \quad \text{or} \quad \lambda_i = -\nu, \tag{B.10}$$

which are also excluded. These remarks pinpoint the origin of the second half of the conditions imposed in (2.100).

Next, we apply Jacobi's theorem by setting $j_b = k_b = b$, $(b \in \mathbb{N}_n)$, $j_{n+1} = k_{n+1} = n + a$, $j_{n+c} = k_{n+c} = n + c - 1$, $(c \in \mathbb{N}_{n-1})$ and $p = n + 1$. Thus

$$\breve{B}\begin{pmatrix} 1 & \dots & n & n+a \\ 1 & \dots & n & n+a \end{pmatrix} = \breve{A}\begin{pmatrix} n+1 & \dots & \widehat{n+a} & \dots & N \\ n+1 & \dots & \widehat{n+a} & \dots & N \end{pmatrix}, \tag{B.11}$$

where $\widehat{n+a}$ indicates that the $(n+a)$-th row and column are omitted. Now denote the submatrices of size $(n+1)$ and $(n-1)$ corresponding to the determinants in (B.11) by X and Y, respectively. From (B.11) and (2.101) it follows that $\det(X) = \det(Y) = D_a$ (B.7). The submatrix X can be written in the form

$$X = \Phi - \frac{\mu - \nu}{\mu - \lambda_a} E_{a,n+1} - \frac{\mu - \nu}{\mu + \lambda_a} E_{n+1,a}, \tag{B.12}$$

i.e. X is a rank two perturbation of the Cauchy-like matrix Φ having the entries

$$\begin{aligned} \Phi_{j,k} &= \frac{2\mu \overline{F}_j F_{n+k}}{2\mu - \lambda_j + \lambda_k}, \quad \Phi_{j,n+1} = \frac{2\mu \overline{F}_j F_a}{2\mu - \lambda_j - \lambda_a}, \\ \Phi_{n+1,k} &= \frac{2\mu \overline{F}_{n+a} F_{n+k}}{2\mu + \lambda_a + \lambda_k}, \quad \Phi_{n+1,n+1} = \overline{F}_{n+a} F_a, \end{aligned} \tag{B.13}$$

where $j, k \in \{1, \dots, n\}$. The determinant of Φ is

$$\det(\Phi) = -\frac{\lambda_a^2}{\mu^2 - \lambda_a^2} D_a W_a W_{n+a}, \tag{B.14}$$

which cannot vanish because λ is strongly regular. Since X is a rank two perturbation of Φ we obtain

$$\det(X) = \det(\Phi) - (\mu - \nu)\left(\frac{\mathcal{C}_{a,n+1}}{\mu - \lambda_a} + \frac{\mathcal{C}_{n+1,a}}{\mu + \lambda_a}\right) + (\mu - \nu)^2 \frac{\mathcal{C}_{a,n+1}\mathcal{C}_{n+1,a} - \mathcal{C}_{a,a}\mathcal{C}_{n+1,n+1}}{(\mu - \lambda_a)(\mu + \lambda_a)\det(\Phi)}, \tag{B.15}$$

where $\mathcal{C}$ now is used to denote the cofactors of Φ. By calculating the necessary cofactors we derive

$$\begin{gathered} \mathcal{C}_{a,a}\mathcal{C}_{n+1,n+1} = D_a^2 W_a W_{n+a}, \\ \mathcal{C}_{a,n+1} = -\frac{1}{\mu + \lambda_a} D_a W_{n+a}, \quad \mathcal{C}_{n+1,a} = -\frac{1}{\mu - \lambda_a} D_a W_a. \end{gathered} \tag{B.16}$$

Equations (B.14)-(B.16) together with $\det(X) = D_a$ imply

$$\lambda_a^2(W_a W_{n+a} - 1) - \mu(\mu - \nu)(W_a + W_{n+a} - 2) + \nu^2 = 0. \tag{B.17}$$

Equations (B.8) and (B.17) coincide with (2.103) and (2.104), respectively. $\square$

B.2 $\mathcal{H}_l^{\mathrm{vD}}$ as elementary symmetric function

Fix an arbitrary $n \in \mathbb{N}$ and $l \in \{0, 1, \dots, n\}$ and let e_l stand for the l-th elementary symmetric polynomial in n variables $x_1, \dots, x_n$, i.e. $e_0(x_1, \dots, x_n) = 1$ and for $l \geq 1$

$$e_l(x_1, \dots, x_n) = \sum_{1 \leq j_1 \dots < j_l \leq n} x_{j_1} \dots x_{j_l}. \tag{B.18}$$

At the end of Chapter 2, we referred to the following useful result due to van Diejen [142, Proposition 2.3]. For convenience, we present it together with a direct proof.

Proposition B.3. *By using* (2.220) *it can be shown that*

$$\mathcal{H}_l^{\mathrm{vD}}(q) = 4^l e_l(\sinh^2 \frac{q_1}{2}, \dots, \sinh^2 \frac{q_n}{2}). \tag{B.19}$$

Proof. First, e_l has the equivalent form

$$e_l(\sinh^2 \frac{q_1}{2}, \dots, \sinh^2 \frac{q_n}{2}) = \sum_{J \subset \{1, \dots, n\},\ |J| = l} \prod_{j \in J} \sinh^2 \frac{q_j}{2}. \tag{B.20}$$

Utilizing the identity $\sinh^2(\alpha/2) = [\cosh(\alpha) - 1]/2$ casts the right-hand side into

$$\sum_{J \subset \{1, \dots, n\},\ |J| = l} 2^{-l} \prod_{j \in J} [\cosh(q_j) - 1] = \sum_{J \subset \{1, \dots, n\},\ |J| = l} 2^{-l} \sum_{K \subset J} (-1)^{l - |K|} \prod_{k \in K} \cosh(q_k). \tag{B.21}$$

The two sums on the right-hand side can be merged into one, but the multiplicity of subsets must remain the same. This results in the appearance of a binomial coefficient

$$\sum_{J \subset \{1, \dots, n\},\ |J| \leq l} \frac{(-1)^{l - |J|}}{2^l} \binom{n - |J|}{l - |J|} \prod_{j \in J} \cosh(q_j) =$$

$$= \sum_{\substack{J \subset \{1, \dots, n\},\ |J| \leq l \\ \varepsilon_j = \pm 1,\ j \in J}} \frac{(-1)^{l - |J|}}{2^{l + |J|}} \binom{n - |J|}{l - |J|} \prod_{j \in J} \cosh(\varepsilon_j q_j), \tag{B.22}$$

where we also used that cosh is an even function and compensated the 'over-counting' of terms. Now, let us simply pull a 4^{-l} factor out of the sum to get

$$4^{-l} \sum_{\substack{J\subset\{1,\dots,n\},\ |J|\le l \\ \varepsilon_j=\pm1,\ j\in J}} (-2)^{l-|J|} \binom{n-|J|}{l-|J|} \prod_{j\in J} \cosh(\varepsilon_j q_j). \tag{B.23}$$

Recall the following identity for the hyperbolic cosine of the sum of a finite number, say N, real arguments (see [62, Art. 132] and apply $\cos(\mathrm{i}\alpha) = \cosh(\alpha)$)

$$\cosh\left(\sum_{k=1}^{N} \alpha_k\right) = \left[\prod_{k=1}^{N} \cosh(\alpha_k)\right]\left[\sum_{m=0}^{\lfloor \frac{N}{2} \rfloor} e_{2m}(\tanh(\alpha_1),\dots,\tanh(\alpha_N))\right], \tag{B.24}$$

where e_{2m} are now elementary symmetric functions with arguments $\tanh(\alpha_k)$. Note that for any $m>0$ and set of signs ε there is another one ε', such that $e_{2m}^{J,\varepsilon'} = -e_{2m}^{J,\varepsilon}$, therefore by using (B.24) we see that (B.23) equals to

$$\begin{aligned} 4^{-l} \sum_{\substack{J\subset\{1,\dots,n\},\ |J|\le l \\ \varepsilon_j=\pm1,\ j\in J}} (-2)^{l-|J|} \binom{n-|J|}{l-|J|} \prod_{j\in J} \cosh(\varepsilon_j q_j) \sum_{m=0}^{\lfloor \frac{|J|}{2} \rfloor} s_{2m}^{J,\varepsilon} = \\ = 4^{-l} \sum_{\substack{J\subset\{1,\dots,n\},\ |J|\le l \\ \varepsilon_j=\pm1,\ j\in J}} (-2)^{l-|J|} \binom{n-|J|}{l-|J|} \cosh(q_{\varepsilon J}). \end{aligned} \tag{B.25}$$

Applying (2.220) concludes the proof. □

C Appendices to Chapter 3

C.1 Links to systems of van Diejen and Schneider

Recall that the trigonometric BC_n van Diejen system [140] has the Hamiltonian

$$H_{\mathrm{vD}}(\lambda,\theta)=\sum_{j=1}^{n}\big(\cosh(\theta_j)\mathsf{V}_j(\lambda)^{1/2}\mathsf{V}_{-j}(\lambda)^{1/2}-[\mathsf{V}_j(\lambda)+\mathsf{V}_{-j}(\lambda)]/2\big),\tag{C.1}$$

with $\mathsf{V}_{\pm j}$ $(j=1,\dots,n)$ defined by

$$\mathsf{V}_{\pm j}(\lambda)=\mathsf{w}(\pm\lambda_j)\prod_{\substack{k=1\\(k\neq j)}}^{n}\mathsf{v}(\pm\lambda_j+\lambda_k)\mathsf{v}(\pm\lambda_j-\lambda_k),\tag{C.2}$$

and v,w denoting the trigonometric potentials

$$\mathsf{v}(z)=\frac{\sin(\mu+z)}{\sin(z)}\quad\text{and}\quad\mathsf{w}(z)=\frac{\sin(\mu_0+z)}{\sin(z)}\frac{\cos(\mu_1+z)}{\cos(z)}\frac{\sin(\mu_0'+z)}{\sin(z)}\frac{\cos(\mu_1'+z)}{\cos(z)},\tag{C.3}$$

where $\mu,\mu_0,\mu_1,\mu_0',\mu_1'$ are arbitrary parameters. By making the substitutions

$$\begin{matrix}\lambda_j\to\mathrm{i}(\hat{p}_j+R),\\ \theta_j\to\mathrm{i}\hat{q}_j,\end{matrix}\quad\forall j\quad\text{and}\quad\mu\to\mathrm{i}g/2,\quad\begin{matrix}\mu_0\to\mathrm{i}(g_0+R), & \mu_1\to\mathrm{i}g_1+\pi/2,\\ \mu_0'\to\mathrm{i}(g_0'-R), & \mu_1'\to\mathrm{i}g_1'+\pi/2\end{matrix}\tag{C.4}$$

the potentials become hyperbolic functions and their $R\to\infty$ limit exists, namely

$$\lim_{R\to\infty}\mathsf{v}(\pm(\lambda_j+\lambda_k))=e^{\pm g/2},\quad\lim_{R\to\infty}\mathsf{v}(\pm(\lambda_j-\lambda_k))=\frac{\sinh(\pm g/2+\hat{p}_j-\hat{p}_k)}{\sinh(\hat{p}_j-\hat{p}_k)},\quad\forall j,k\tag{C.5}$$

and

$$\lim_{R\to\infty}\mathsf{w}(\pm\lambda_j)=e^{g_0-g_0'\pm(g_1+g_1')-2\hat{p}_j}-e^{\pm(g_0+g_0'+g_1+g_1')},\quad\forall j.\tag{C.6}$$

In the 1-particle case we have $V_\pm(\lambda)=\mathsf{w}(\pm\lambda)$, thus H_{vD} takes the following form

$$H_{\mathrm{vD}}(\lambda,\theta)=\cosh(\theta)\mathsf{w}(\lambda)^{1/2}\mathsf{w}(-\lambda)^{1/2}-[\mathsf{w}(\lambda)+\mathsf{w}(-\lambda)]/2.\tag{C.7}$$

By utilizing (C.6) one obtains

$$\lim_{R\to\infty} \mathsf{w}(\lambda)^{1/2}\mathsf{w}(-\lambda)^{1/2} = \left[1 - (e^{2g_0} + e^{-2g_0'})e^{-2\hat{p}} + e^{2g_0-2g_0'-4\hat{p}}\right]^{1/2},$$
$$\lim_{R\to\infty} [\mathsf{w}(\lambda) + \mathsf{w}(-\lambda)]/2 = \frac{e^{g_0-g_0'+g_1+g_1'} + e^{g_0-g_0'-g_1-g_1'}}{2} e^{-2\hat{p}} - \cosh(g_0 + g_0' + g_1 + g_1'). \tag{C.8}$$

Equating the $R \to \infty$ limit of $H_{\mathrm{vD}}(\lambda, \theta)$ (C.7) with the Hamiltonian $H(\hat{p}, \hat{q}; x, a, b)$ (3.1) yields a system of linear equations involving g_0, g_1, g_0', g_1' as unknowns and u, v as parameters. Actually, four sets of linear equations can be constructed, each with infinitely many solutions depending on one (real) parameter, but these sets are 'equivalent' under the exchanges: $g_0 \leftrightarrow g_0'$ or $g_1 \leftrightarrow g_1'$. Therefore it is sufficient to give only one set of solutions, e.g.

$$g_0 = a, \quad g_0' = 0, \quad g_1 = b - g_1', \quad g_1' \in \mathbb{R}. \tag{C.9}$$

Setting $g = x$ and $g_1' = 0$ provides the following special choice of couplings in (C.4)

$$\mu = \mathrm{i}x/2, \quad \mu_0 = \mathrm{i}(a + R), \quad \mu_0' = -\mathrm{i}R, \quad \mu_1 = \mathrm{i}b + \pi/2, \quad \mu_1' = \pi/2, \tag{C.10}$$

and one finds the following

$$\lim_{R\to\infty} H_{\mathrm{vD}}\big(\lambda(\hat{p}, R), \theta(\hat{q})\big) = -H(\hat{p}, \hat{q}; x, a, b) + \cosh(b - a). \tag{C.11}$$

In the n-particle case, by using (C.5) and (C.6) it can be shown that with (C.10) one has

$$\lim_{R\to\infty} H_{\mathrm{vD}}\big(\lambda(\hat{p}, R), \theta(\hat{q})\big) = -H(\hat{p}, \hat{q}; x, a, b) + \sum_{j=1}^{n} \cosh\big((j-1)x + b - a\big), \tag{C.12}$$

i.e., the Hamiltonian H (3.1) is recovered as a singular limit of H_{vD} (C.1).

Consider now the function $H(\hat{p}, \hat{q}; x, a, b)$ and introduce the real parameter σ through the substitutions

$$b \to b - 2\sigma \tag{C.13}$$

and apply the canonical transformation

$$\hat{p}_j \to -Q_j + \sigma, \quad \hat{q}_j \to -P_j, \quad \forall j. \tag{C.14}$$

Then we have

$$\lim_{\sigma\to\infty} H(\hat{p}(Q, \sigma), \hat{q}(P), x, a, b(\sigma)) = H_{\mathrm{Sch}}(Q, P, x, a - b), \tag{C.15}$$

with Schneider's [125] Hamiltonian

$$H_{\mathrm{Sch}}(Q,P,x,a-b) = \frac{e^{a-b}}{2}\sum_{j=1}^{n} e^{2Q_j} - \sum_{j=1}^{n}\cos(P_j)\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{\sinh^2\left(\frac{x}{2}\right)}{\sinh^2(Q_j-Q_k)}\right]^{\frac{1}{2}}. \tag{C.16}$$

Remark C.1. (*i*) In (C.4) only two of the four external field couplings $\mu_0, \mu_0', \mu_1, \mu_1'$ are scaled with R. However, scaling all four of these parameters also leads to an integrable Ruijsenaars-Schneider type system with a more general 4-parameter external field. For details, see [143, Section II.B]. (*ii*) The connection to Schneider's Hamiltonian was mentioned in [86, Remark 7.1] as well, where a singular limit, similar to (C.15) was taken.

C.2 Proof of Proposition 3.2

In this appendix, we prove Proposition 3.2 that states that the range of the 'position variable' $\hat{p}$ is contained in the closed thick-walled Weyl chamber $\bar{\mathcal{C}}_x$ (3.58).

Proof of Proposition 3.2. According to (3.57) the matrices $e^{2\hat{p}}$ and $e^{2\hat{p}-x\mathbf{1}_n}+\mathrm{sgn}(x)e^{\hat{p}}ww^\dagger e^{\hat{p}}$ are similar and therefore have the same characteristic polynomial. This gives the identity

$$\prod_{j=1}^{n}(e^{2\hat{p}_j}-\lambda) = \prod_{j=1}^{n}(e^{2\hat{p}_j-x}-\lambda) + \mathrm{sgn}(x)\sum_{j=1}^{n}\left[e^{2\hat{p}_j}|w_j|^2\prod_{\substack{k=1\\(k\neq j)}}^{n}(e^{2\hat{p}_k-x}-\lambda)\right], \tag{C.17}$$

where λ is an arbitrary complex parameter. The constraint on $\hat{p}$ arises from the fact that $|w_m|^2$ $(m=1,\dots,n)$ must be non-negative and not all zero, because of the definition (3.56).

Let us assume for a moment that the components of $\hat{p}$ are distinct such that $\hat{p}_1 > \cdots > \hat{p}_n$. This enables us to express $|w_m|^2$ for all $m\in\{1,\dots,n\}$ from the above equation by evaluating it at n different values of λ, viz. $\lambda = e^{2\hat{p}_m - x}$, $m=1,\dots,n$. We obtain the following

$$|w_m|^2 = \mathrm{sgn}(x)(1-e^{-x})\prod_{\substack{j=1\\(j\neq m)}}^{n}\frac{e^{2\hat{p}_j+x}-e^{2\hat{p}_m}}{e^{2\hat{p}_j}-e^{2\hat{p}_m}},\quad m=1,\dots,n. \tag{C.18}$$

For $x>0$ and any $\hat{p}$ with $\hat{p}_1>\cdots>\hat{p}_n$ the formula (C.18) implies that $|w_n|^2>0$ and for $m=1,\dots,n-1$ we have $|w_m|^2\geq 0$ if and only if $\hat{p}_m-\hat{p}_{m+1}\geq x/2$. Similarly, if $x<0$ and $\hat{p}\in\mathbb{R}^n$ with $\hat{p}_1>\cdots>\hat{p}_n$, then (C.18) implies $|w_1|^2>0$ and for $m=2,\dots,n$ we have $|w_m|^2\geq 0$ if and only if $\hat{p}_{m-1}-\hat{p}_m\geq -x/2$. In summary, if

$\hat{p}_1 > \cdots > \hat{p}_n$, then $|w_m|^2 \geq 0 \ \forall m$ implies that $\hat{p} \in \bar{\mathcal{C}}_x$.

Now, let us prove our assumption, that all components of $\hat{p}$ must be different. Indirectly, suppose that some (or maybe all) of the $\hat{p}_j$'s coincide. This can be captured by a partition of the positive integer

$$n = k_1 + \cdots + k_r, \tag{C.19}$$

where $r < n$ (or equivalently, at least one integer $k_1, \ldots, k_r$ must be greater than 1) and the indirect assumption can be written as

$$\hat{p}_1 = \cdots = \hat{p}_{k_1}, \quad \hat{p}_{k_1+1} = \cdots = \hat{p}_{k_1+k_2}, \quad \ldots, \quad \hat{p}_{k_1+\cdots+k_{r-1}+1} = \cdots = \hat{p}_{k_1+\cdots+k_r} \equiv \hat{p}_n. \tag{C.20}$$

Then (C.17) can be reformulated as

$$\prod_{j=1}^{r}(\Delta_j - \lambda)^{k_j} = \prod_{j=1}^{r}(\Delta_j e^{-x} - \lambda)^{k_j} + \mathrm{sgn}(x) \sum_{m=1}^{r} Z_m \Delta_m (\Delta_m e^{-x} - \lambda)^{k_m - 1} \prod_{\substack{j=1 \\ (j \neq m)}}^{r} (\Delta_j e^{-x} - \lambda)^{k_j}, \tag{C.21}$$

where we introduced r distinct variables

$$\Delta_1 = e^{2\hat{p}_{k_1}}, \quad \Delta_2 = e^{2\hat{p}_{k_1+k_2}}, \quad \ldots, \quad \Delta_r = e^{2\hat{p}_{k_1+\cdots+k_r}} \equiv e^{2\hat{p}_n}, \tag{C.22}$$

and r non-negative real variables

$$\begin{gathered} Z_1 = |w_1|^2 + \cdots + |w_{k_1}|^2, \quad Z_2 = |w_{k_1+1}|^2 + \cdots + |w_{k_1+k_2}|^2, \\ \ldots, \quad Z_r = |w_{k_1+\cdots+k_{r-1}+1}|^2 + \cdots + |w_n|^2. \end{gathered} \tag{C.23}$$

Notice that $Z_1 + \cdots + Z_r = |w|^2 = \mathrm{sgn}(x) e^{-x}(e^{nx} - 1) > 0$, therefore at least one of the Z_j's must be positive. Next, we define the rational function of λ

$$Q(\Delta, x, \lambda) = \prod_{j=1}^{r} \frac{(\Delta_j - \lambda)^{k_j}}{(\Delta_j e^{-x} - \lambda)^{k_j - 1}}, \tag{C.24}$$

and use it to rewrite (C.21) as

$$Q(\Delta, x, \lambda) = \prod_{j=1}^{r}(\Delta_j e^{-x} - \lambda) + \mathrm{sgn}(x) \sum_{m=1}^{r} Z_m \Delta_m \prod_{\substack{j=1 \\ (j \neq m)}}^{r} (\Delta_j e^{-x} - \lambda). \tag{C.25}$$

The above equation implies that all poles of Q are apparent, i.e., there must be cancelling factors in its numerator. This observation has a straightforward implication on the Δ's.

$(*)$ For every index $m \in \{1, \dots, r\}$ with $k_m > 1$, there exists an index $s \in \{1, \dots, r\}$ s.t. $\Delta_s = \Delta_m e^{-x}$ and $k_s \geq k_m - 1$.

The quantities $Z_m = Z_m(\Delta, x)$ can be uniquely determined by evaluating (C.25) at r different values of the parameter λ, namely $\lambda_m = \Delta_m e^{-x}$ $(m = 1, \dots, r)$. However, there are 3 disjoint cases which are to be handled separately.
Case 1: $k_m = 1$ and $\nexists s \in \{1, \dots, r\}$: $\Delta_s = \Delta_m e^{-x}$. Then we find

$$Z_m = \operatorname{sgn}(x)(1 - e^{-x})e^{(n-1)x} \prod_{\substack{j=1 \\ (j \neq m)}}^{r} \left(\frac{\Delta_j - \Delta_m e^{-x}}{\Delta_j - \Delta_m} \right)^{k_j} > 0. \tag{C.26}$$

Case 2: $k_m > 1$ and $k_s = k_m - 1$. Then we find

$$Z_m = (-1)^{k_m+1} \operatorname{sgn}(x)(1 - e^{-x})e^{(n-k_m)x} \prod_{\substack{j=1 \\ (j \neq m,s)}}^{r} \left(\frac{\Delta_j - \Delta_m e^{-x}}{\Delta_j - \Delta_m} \right)^{k_j} > 0. \tag{C.27}$$

Case 3: $k_m = 1$ and $\exists s \in \{1, \dots, r\}$: $\Delta_s = \Delta_m e^{-x}$ or $k_m > 1$ and $k_s > k_m - 1$. Then we get

$$Z_m = 0. \tag{C.28}$$

Since there is at least one Z_m which is positive, the set of indices belonging to Case 1 or Case 2 must be non-empty. Introduce a real positive parameter ε and associate to every degenerate configuration (C.20) a continuous family of configurations, denoted by $\hat{p}(\varepsilon)$, with components $\hat{p}(\varepsilon)_1, \dots, \hat{p}(\varepsilon)_n$ defined by the formulae

$$\begin{gathered} \exp(2\hat{p}(\varepsilon)_a + a\varepsilon) = \Delta_1, \quad a = 1, \dots, k_1, \\ \exp(2\hat{p}(\varepsilon)_{\sum_{m=1}^{j-1} k_m + a} + a\varepsilon) = \Delta_j, \quad a = 1, \dots, k_j, \quad j = 2, \dots, r. \end{gathered} \tag{C.29}$$

This way coinciding components of $\hat{p}$ (C.20) are ‘pulled apart’ to points successively separated by $\varepsilon/2$. It is clear that with sufficiently small separation the configuration $\hat{p}(\varepsilon)$ sits in the chamber $\{\hat{x} \in \mathbb{R}^n \mid 0 > \hat{x}_1 > \cdots > \hat{x}_n\}$. For such non-degenerate configurations $\hat{p}(\varepsilon)$, let us consider the expressions

$$|w_\ell(\hat{p}(\varepsilon), x)|^2 = \operatorname{sgn}(x)(1 - e^{-x}) \prod_{\substack{j=1 \\ (j \neq \ell)}}^{n} \frac{e^{2\hat{p}(\varepsilon)_j + x} - e^{2\hat{p}(\varepsilon)_\ell}}{e^{2\hat{p}(\varepsilon)_j} - e^{2\hat{p}(\varepsilon)_\ell}}, \quad \ell = 1, \dots, n, \tag{C.30}$$

which give the unique solution of equation (C.17) at $\hat{p}(\varepsilon)$. The limits $\lim_{\varepsilon \to 0} |w_\ell(\hat{p}(\varepsilon), x)|^2$ exist, and do not vanish for $\ell = k_1 + \cdots + k_m$ if k_m belongs to Case 1 or Case 2. For such $\ell = k_1 + \cdots + k_m$ we must have

$$\lim_{\varepsilon \to 0} |w_{k_1 + \cdots + k_m}(\hat{p}(\varepsilon), x)|^2 = Z_m(\Delta, x) > 0, \tag{C.31}$$

where Z_m is given by (C.26) in Case 1 and by (C.27) in Case 2. It can be also seen that

$$|w_\ell(\hat{p}(\varepsilon), x)|^2 \equiv 0 \quad \iff \quad \begin{cases} \ell \notin \{k_1, k_1 + k_2, \dots, k_1 + \cdots + k_r\} \\ \text{or} \\ \ell = k_1 + \cdots + k_m \text{ with } k_m \text{ from Case 3,} \end{cases} \tag{C.32}$$

i.e. $|w_\ell(\hat{p}(\varepsilon), x)|^2$ vanishes identically except for the components in (C.31). Notice that for a small enough ε some coordinates of $\hat{p}(\varepsilon)$ are separated by less than $|x|/2$. Thus, as it was shown at beginning the proof, we have $|w_\ell(\hat{p}(\varepsilon), x)|^2 < 0$ for some index ℓ, which might depend on ε. Moreover, (C.32) implies that the index in question must have the form $\ell = k_1 + \cdots + k_{m^*}$ for some m^* appearing in (C.31). But since the number of indices is finite, a monotonically decreasing sequence $\{\varepsilon_N\}_{N=1}^\infty$ tending to zero can be chosen such that $|w_{k_1+\cdots+k_{m^*}}(\hat{p}(\varepsilon_N), x)|^2 < 0$ for all N. This together with (C.32) gives the contradiction

$$0 \geq \lim_{N\to\infty} |w_{k_1+\cdots+k_{m^*}}(\hat{p}(\varepsilon_N), x)|^2 = Z_{m^*}(\Delta, x) > 0 \tag{C.33}$$

proving that all components of $\hat{p}$ must be distinct. This concludes the proof. $\square$

The above proof is a straightforward adaptation of the proofs of [37, Lemma 5.2] and [38, Theorem 2]. We presented it since it could be awkward to extract the arguments from those lengthy papers, and also our notations and the ranges of our variables are different.

C.3 Proof of Lemma 3.6

We here prove the following equivalent formulation of Lemma 3.6.

Lemma C.2. *Suppose that* $\frac{\pi}{2} \geq q_1 > \cdots > q_n > 0$ *and*

$$\begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \begin{bmatrix} \eta_R(1)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2)^{-1} \end{bmatrix} = \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \tag{C.34}$$

for $\eta_L, \eta_R \in G_+$. *Then*

$$\eta_L(1) = \eta_R(2) = m_1, \quad \eta_L(2) = \eta_R(1) = m_2 \tag{C.35}$$

with some diagonal matrices $m_1, m_2 \in \mathbb{T}_n$ *having the form*

$$m_1 = \mathrm{diag}(a, \xi), \quad m_2 = \mathrm{diag}(b, \xi), \quad \xi \in \mathbb{T}_{n-1}, \; a, b \in \mathbb{T}_1, \quad \det(m_1 m_2) = 1. \tag{C.36}$$

If in addition $\frac{\pi}{2} > q_1$*, then* $m_1 = m_2$*.*

Proof. The block off-diagonal components of the equality (C.34) give

$$\eta_L(1) = (\sin q)\eta_R(2)(\sin q)^{-1}, \quad \eta_L(2) = (\sin q)\eta_R(1)(\sin q)^{-1}. \tag{C.37}$$

Since $\eta_L(1)^{-1} = \eta_L(1)^\dagger$, the first of these relations implies $\eta_R(2) = (\sin q)^2\eta_R(2)(\sin q)^{-2}$. As the entries of $(\sin q)$ are all different, this entails that $\eta_R(2)$ is diagonal, and consequently we obtain the relations in (C.35) with some diagonal matrices m_1 and m_2. On the other hand, the block-diagonal components of (C.34) require that

$$\cos q = \eta_L(1)(\cos q)\eta_R(1)^{-1}, \quad \cos q = \eta_L(2)(\cos q)\eta_R(2)^{-1}. \tag{C.38}$$

Since $\cos q_k \neq 0$ for $k = 2, \dots, n$, the formula (C.36) follows. If an addition $\cos q_1 \neq 0$, then we also obtain from (C.38) that $a = b$, i.e., $m_1 = m_2 = m$ with some $m \in \mathbb{T}_n$. ☐

C.4 Auxiliary material on Poisson-Lie symmetry

The statements presented here are direct analogues of well-known results [7, 56] about Hamiltonian group actions with zero Poisson bracket on the symmetry group. They are surely familiar to experts, although we could not find them in a reference.

Let us consider a Poisson-Lie group G with dual group G^* and a symplectic manifold P equipped with a left Poisson action of G. Essentially following Lu [80] (cf. Remark C.6), we say that the G-action admits the momentum map $\psi\colon P \to G^*$ if for any $X \in \mathcal{G}$, the Lie algebra of G, and any $f \in C^\infty(P)$ we have

$$(\mathcal{L}_{X_P} f)(p) = \langle X, \{f, \psi\}(p)\psi(p)^{-1}\rangle, \quad \forall p \in P, \tag{C.39}$$

where X_P is the vector field on P corresponding to X, $\langle ., . \rangle$ stands for the canonical pairing between the Lie algebras of G and G^*, and the notation pretends that G^* is a matrix group. Using the Hamiltonian vector field V_f defined by $\mathcal{L}_{V_f} h = -\{f, h\}$ ($\forall h \in C^\infty(P)$), we can spell out equation (C.39) equivalently as

$$(\mathcal{L}_{X_P} f)(p) = -\langle X, \big(D_{\psi(p)} R_{\psi(p)^{-1}}\big)\big((D_p\psi)(V_f(p))\big)\rangle, \quad \forall p \in P, \tag{C.40}$$

where $D_p\psi\colon T_pP \to T_{\psi(p)}G^*$ is the derivative, and $R_{\psi(p)^{-1}}$ denotes the right-translation on G^* by $\psi(p)^{-1}$. Since the vectors of the form $V_f(p)$ span T_pP, we obtain the following characterization of the Lie algebra of the isotropy subgroup $G_p < G$ of $p \in P$.

Lemma C.3. *With the above notations, we have*

$$\mathrm{Lie}(G_p) = \big[\big(D_{\psi(p)} R_{\psi(p)^{-1}}\big)\big(\mathrm{Im}(D_p\psi)\big)\big]^\perp. \tag{C.41}$$

This directly leads to the next statement.

Corollary C.4. *An element $\mu \in G^*$ is a regular value of the momentum map ψ if and only if* $\mathrm{Lie}(G_p) = \{0\}$ *for every* $p \in \psi^{-1}(\mu) = \{p \in P \mid \psi(p) = \mu\}$.

Let us further suppose that $\psi\colon P \to G^*$ is G-equivariant, with respect to the appropriate dressing action of G on G^*. Then we have

$$G_p < G_\mu, \quad \forall p \in \psi^{-1}(\mu). \tag{C.42}$$

Here G_p and G_μ refer to the respective actions of G on P and on G^*. Corollary C.4 and equation (C.42) together imply the following useful result.

Corollary C.5. *If G_μ acts locally freely on $\psi^{-1}(\mu)$, then μ is a regular value of the equivariant momentum map ψ. Consequently, $\psi^{-1}(\mu)$ is an embedded submanifold of P.*

We finish by a clarifying remark concerning the momentum map.

Remark C.6. Let B be the Poisson tensor on P, for which $\{f, h\} = B(df, dh) = \mathcal{L}_{V_h} f$. We can write $V_h = B^\sharp(dh)$ with the corresponding bundle map $B^\sharp\colon T^*P \to TP$. Any $X \in \mathcal{G} = T_eG = (T_{e'}G^*)^*$ extends to a unique right-invariant 1-form ϑ_X on G^* ($e \in G$ and $e' \in G^*$ are the unit elements). With this at hand, equation (C.39) can be reformulated as

$$X_P = B^\sharp(\psi^*(\vartheta_X)), \tag{C.43}$$

which is a slight variation of the defining equation of the momentum map found in [80].

C.5 On the reduced Hamiltonians

In this appendix we prove the claim, made in Section 3.4, that on the momentum surface $\Phi_+^{-1}(\mu)$ the Hamiltonians $\mathcal{H}_j$, $j \in \mathbb{Z}^*$ (3.190) are linear combinations of h_k, $k = 1, \dots, n$ (3.194). This will be achieved by establishing the form of the integer powers of the matrix displayed in (3.193), which we denote here by $\mathcal{L}$, i.e.

$$\mathcal{L} = \begin{bmatrix} e^{-2v}\mathbf{1}_n & -e^{-v}\alpha \\ e^{-v}\alpha^\dagger & e^{2v}\mathbf{1}_n - \alpha^\dagger\alpha \end{bmatrix}. \tag{C.44}$$

Lemma C.7. *For any positive integer j, the j-th power of the $2n \times 2n$ matrix $\mathcal{L}$ (C.44) reads*

$$\mathcal{L}^j = \begin{bmatrix} \mathcal{L}_{11}^j & \mathcal{L}_{12}^j \\ \mathcal{L}_{21}^j & \mathcal{L}_{22}^j \end{bmatrix}, \tag{C.45}$$

where $\mathcal{L}^j_{11}, \mathcal{L}^j_{12}, \mathcal{L}^j_{21}, \mathcal{L}^j_{22}$ *are* $n \times n$ *blocks of the form*

$$\begin{aligned}
\mathcal{L}^j_{11} &= \sum_{m=1}^{j} a^{(j)}_m (\alpha\alpha^\dagger)^{j-m}, \quad \mathcal{L}^j_{12} = \alpha \sum_{m=1}^{j} b^{(j)}_m (\alpha^\dagger\alpha)^{j-m}, \\
\mathcal{L}^j_{21} &= \alpha^\dagger \sum_{m=1}^{j} c^{(j)}_m (\alpha\alpha^\dagger)^{j-m}, \quad \mathcal{L}^j_{22} = (-1)^j (\alpha^\dagger\alpha)^j + \sum_{m=1}^{j} d^{(j)}_m (\alpha^\dagger\alpha)^{j-m},
\end{aligned} \tag{C.46}$$

with the $4j$ *coefficients* $a^{(j)}_m, b^{(j)}_m, c^{(j)}_m, d^{(j)}_m$, $m = 1, \ldots, j$ *depending only on the parameter* v.

Proof. We proceed by induction on j. For $j = 1$ the statement clearly holds, and supposing that (C.45)-(C.46) is valid for some fixed integer $j > 0$ we simply calculate the $(j+1)$-th power $\mathcal{L}^{j+1} = \mathcal{L}\mathcal{L}^j$. This proves the statement. $\square$

Our claim of linear expressibility follows at once, that is for any positive integer j we have

$$\mathcal{H}_j = (-1)^j h_j + \sum_{k=1}^{j-1} \frac{k}{j} (a^{(j)}_{j-k} + d^{(j)}_{j-k}) h_k + \frac{n}{2j} (a^{(j)}_j + d^{(j)}_j). \tag{C.47}$$

Incidentally, one also obtains a recursion for the coefficients $a^{(j)}_m, b^{(j)}_m, c^{(j)}_m, d^{(j)}_m$ from the proof of Lemma C.7. If they are required, this should enable one to establish the values of the constants that occur in (C.47).

As for the negative powers of $\mathcal{L}$, one readily checks that the inverse of $\mathcal{L}$ is

$$\mathcal{L}^{-1} = \begin{bmatrix} e^{2v}\mathbf{1}_n - \alpha\alpha^\dagger & e^{-v}\alpha \\ -e^{-v}\alpha^\dagger & e^{-2v}\mathbf{1}_n \end{bmatrix}, \tag{C.48}$$

which has essentially the same form as $\mathcal{L}$ does, thus the blocks of $\mathcal{L}^{-j}$ $(j > 0)$ can be expressed similarly as in Lemma C.7. In fact, conjugating $\mathcal{L}^{-1}$ with the $2n \times 2n$ involutory block-matrix

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix}, \tag{C.49}$$

leads to the following formula

$$\boldsymbol{C}\mathcal{L}^{-1}\boldsymbol{C} = \begin{bmatrix} e^{-2v}\mathbf{1}_n & -e^{-v}\alpha^\dagger \\ e^{-v}\alpha & e^{2v}\mathbf{1}_n - \alpha\alpha^\dagger \end{bmatrix}, \tag{C.50}$$

which implies that the blocks of $\mathcal{L}^{-j}$ are obtained from those of $\mathcal{L}^j$ by reversing their order and interchanging the role of α and $\alpha^\dagger$. Furthermore, since $\mathrm{tr}((\alpha\alpha^\dagger)^k) = \mathrm{tr}((\alpha^\dagger\alpha)^k)$ we get

$$\mathcal{H}_{-j} = -\mathcal{H}_j \qquad \forall j \in \mathbb{Z}^*. \tag{C.51}$$

D Appendices to Chapter 4

It is clear that our results on the 2-parameter family of hyperbolic systems (4.5) open up a plethora of interesting problems. Besides, based on our numerical calculations, below we also wish to discuss some possible generalizations in two further directions.

D.1 Lax matrix with spectral parameter

First, it is a time-honoured principle that the inclusion of a spectral parameter into the Lax matrix of an integrable system can greatly enrich the analysis by borrowing techniques from complex geometry. Bearing this fact in mind, with the aid of the function

$$\Phi(x \mid \eta) = e^{x \coth(\eta)} \left(\coth(x) - \coth(\eta)\right) \tag{D.1}$$

depending on the complex variables x and η, over the phase space P (4.2) we define the matrix valued smooth function $\mathcal{L} = \mathcal{L}(\lambda, \theta; \mu, \nu \mid \eta)$ with entries

$$\mathcal{L}_{k,l} = \big(\mathrm{i}\sin(\mu) F_k \bar{F}_l + \mathrm{i}\sin(\mu - \nu) C_{k,l}\big)\, \Phi(\mathrm{i}\mu + \Lambda_j - \Lambda_k \mid \eta) \qquad (k, l \in \mathbb{N}_N). \tag{D.2}$$

One of the outcomes of our numerical investigations is that for any values of η the eigenvalues of $\mathcal{L}$ provide a family of first integrals in involution for the van Diejen system (4.5). Thinking of η as a spectral parameter, let us also observe that, in the limit $\mathbb{R} \ni \eta \to \infty$, from $\mathcal{L}$ we can recover our Lax matrix L (4.34); that is, $\mathcal{L} \to L$. Although the spectral parameter dependent matrix $\mathcal{L}$ does not take values in the Lie group $\mathrm{U}(n, n)$ (4.9), we find it interesting that the constituent function Φ (D.1) can be seen as a hyperbolic limit of the elliptic Lamé function, that plays a prominent role in the theory of the elliptic CMS and RS systems (see e.g. the papers [76, 112] and the monograph [11]). Therefore, it is tempting to think that an appropriate elliptic deformation of $\mathcal{L}$ (4.34) may lead to a spectral parameter dependent Lax matrix of the elliptic van Diejen system with coupling parameters μ and ν.

D.2 Lax matrix with three couplings

In Chapter 4 we have studied the van Diejen system (4.5) with only two independent coupling parameters. Though a construction of a Lax matrix for the most general hyperbolic van Diejen system with five independent coupling parameters still seems to be out of reach, we can offer a plausible conjecture for a Lax matrix with *three* independent coupling constants. Simply by generalizing the formulae appearing in the theory of the rational BC_n RSvD systems [108], with the aid of an additional real parameter κ let us define the real valued functions α and β for any $x > 0$ by the formulae

$$\alpha(x) = \frac{1}{\sqrt{2}}\left(1 + \left(1 + \frac{\sin(\kappa)^2}{\sinh(2x)^2}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \quad \text{and} \quad \beta(x) = \frac{\mathrm{i}}{\sqrt{2}}\left(-1 + \left(1 + \frac{\sin(\kappa)^2}{\sinh(2x)^2}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{D.3}$$

Built upon these functions, let us also introduce the Hermitian $N \times N$ matrix

$$h(\lambda) = \begin{bmatrix} \mathrm{diag}(\alpha(\lambda_1), \dots, \alpha(\lambda_n)) & \mathrm{diag}(\beta(\lambda_1), \dots, \beta(\lambda_n)) \\ -\mathrm{diag}(\beta(\lambda_1), \dots, \beta(\lambda_n)) & \mathrm{diag}(\alpha(\lambda_1), \dots, \alpha(\lambda_n)) \end{bmatrix}. \tag{D.4}$$

One can easily show that $hCh = C$, whence the matrix valued function

$$\tilde{L} = h^{-1} L h^{-1} \tag{D.5}$$

also takes values in the Lie group $\mathrm{U}(n, n)$ (4.9). Notice that the rational limit of matrix $\tilde{L}$ gives back the Lax matrix of the rational BC_n RSvD system, that first appeared in equation (4.51) of paper [107]. Moreover, upon setting

$$g = \mu, \quad g_0 = g_1 = \frac{\nu}{2}, \quad g_0' = g_1' = \frac{\kappa}{2}, \tag{D.6}$$

for van Diejen's main Hamiltonian H_1 (4.209) we get that

$$H_1 = 2\sum_{a=1}^{n} \cosh(\theta_a) u_a \left(1 + \frac{\sin(\kappa)^2}{\sinh(2\lambda_a)^2}\right)^{\frac{1}{2}} + 2\sum_{a=1}^{n} \mathrm{Re}\left(z_a \frac{\sinh(\mathrm{i}\kappa + 2\lambda_a)}{\sinh(2\lambda_a)}\right), \tag{D.7}$$

with the functions z_a and u_a defined in the equations (4.31) and (4.32), respectively. The point is that, in complete analogy with (4.215), one can establish the relationship

$$H_1 + 2\cos\left(\nu + \kappa + (n-1)\mu\right) \frac{\sin(n\mu)}{\sin(\mu)} = \mathrm{tr}(\tilde{L}). \tag{D.8}$$

Furthermore, based on numerical calculations for small values of n, it appears that the eigenvalues of $\tilde{L}$ (D.5) provide a commuting family of first integrals for the van

Diejen system (D.7). To sum up, we have numerous evidences that matrix $\tilde{L}$ (D.5) is a Lax matrix for the 3-parameter family of van Diejen systems (D.7), if the pertinent parameters are connected by the relationships displayed in (D.6). As can be seen in [107], the new parameter κ causes many non-trivial technical difficulties even at the level of the rational van Diejen system. Part of the difficulties can be traced back to the fact that for $\sin(\kappa) \neq 0$ the matrix $\tilde{L}$ (D.5) does not belong to the symmetric space $\exp(\mathfrak{p})$ (4.15), whence the diagonalization of $\tilde{L}$ requires a less direct approach than that provided by the canonical form (4.17). We wish to come back to these problems in later publications.

E Appendix to Chapter 5

E.1 Explicit form of the functions $\Lambda^y_{j,\ell}$

In this appendix, we display the building blocks (5.81) of the global elliptic Lax matrix explicitly. Below, ξ varies in the closed simplex $\mathcal{A}_y$ associated with a type (i) coupling y (5.13) for fixed p and M. The function $\mathcal{J}$ was defined in (5.80). The trigonometric case is obtained by simply replacing the s-function (5.72) everywhere by the sine function.

Special components: For $1 \le j \le n-p$

$$\Lambda^y_{j,j+p}(\xi) = -\operatorname{sgn}(M)\,\mathrm{s}(y)\frac{\Big[\prod_{\substack{m=1\\(m\neq p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=j+p}^{j+p+n-m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\Big[\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)\,\mathrm{s}(\sum_{k=j+p}^{j+p+m-1}\xi_k)\Big]^{\frac{1}{2}}}.$$

For $n-p < j \le n$

$$\Lambda^y_{j,j+p-n}(\xi) = \operatorname{sgn}(M)\,\mathrm{s}(y)\frac{\Big[\prod_{\substack{m=1\\(m\neq p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=j+p-n}^{j+p-m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\Big[\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)\,\mathrm{s}(\sum_{k=j+p-n}^{j+p-m-1}\xi_k)\Big]^{\frac{1}{2}}}.$$

Diagonal components: For $1 \le j = \ell \le p$

$$\Lambda^y_{j,j}(\xi) = \Big[\mathcal{J}(|u_j|^2)\mathcal{J}(|u_{j+n-p}|^2)\Big]^{\frac{1}{2}}\frac{\Big[\prod_{\substack{m=1\\(m\neq p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=j}^{j+n-m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)}.$$

For $p < j = \ell \le n$

$$\Lambda^y_{j,j}(\xi) = \Big[\mathcal{J}(|u_j|^2)\mathcal{J}(|u_{j-p}|^2)\Big]^{\frac{1}{2}}\frac{\Big[\prod_{\substack{m=1\\(m\neq p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=j}^{j+n-m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)}.$$

Components above the diagonal: For $1 \le j < \ell \le p$

$$\Lambda^y_{j,\ell}(\xi) = \mathrm{s}(y)\big[\mathcal{J}(|u_j|^2)\mathcal{J}(|u_{\ell+n-p}|^2)\big]^{\frac{1}{2}} \frac{\Big[\prod_{\substack{m=1\\(m\neq \ell-j,p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=\ell}^{\ell+n-m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\big[\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)\,\mathrm{s}(\sum_{k=\ell}^{\ell+m-1}\xi_k)\big]^{\frac{1}{2}}}.$$

For $1 \le j < \ell \le n$ with $p < \ell$ and $\ell \neq j + p$

$$\Lambda^y_{j,\ell}(\xi) = \frac{\mathrm{s}(y)\big[\mathcal{J}(|u_j|^2)\mathcal{J}(|u_{\ell-p}|^2)\big]^{\frac{1}{2}}}{\mathrm{sgn}(j+p-\ell)} \frac{\Big[\prod_{\substack{m=1\\(m\neq \ell-j,p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=\ell}^{\ell+n-m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\big[\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)\,\mathrm{s}(\sum_{k=\ell}^{\ell+m-1}\xi_k)\big]^{\frac{1}{2}}}.$$

Components below the diagonal: For $1 \le \ell < j \le n$ with $\ell \le p$ and $\ell \neq j + p - n$

$$\Lambda^y_{j,\ell}(\xi) = \frac{\mathrm{s}(y)\big[\mathcal{J}(|u_j|^2)\mathcal{J}(|u_{\ell+n-p}|^2)\big]^{\frac{1}{2}}}{\mathrm{sgn}(\ell+n-j-p)} \frac{\Big[\prod_{\substack{m=1\\(m\neq j-\ell,p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+n-m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=\ell}^{\ell+m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\big[\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)\,\mathrm{s}(\sum_{k=\ell}^{\ell+m-1}\xi_k)\big]^{\frac{1}{2}}}.$$

For $p < \ell < j \le n$

$$\Lambda^y_{j,\ell}(\xi) = \mathrm{s}(y)\big[\mathcal{J}(|u_j|^2)\mathcal{J}(|u_{\ell-p}|^2)\big]^{\frac{1}{2}} \frac{\Big[\prod_{\substack{m=1\\(m\neq j-\ell,p)}}^{n-1} \mathrm{s}(\sum_{k=j}^{j+n-m-1}\xi_k - y)\,\mathrm{s}(\sum_{k=\ell}^{\ell+m-1}\xi_k + y)\Big]^{\frac{1}{2}}}{\prod_{m=1}^{n-1}\big[\mathrm{s}(\sum_{k=j}^{j+m-1}\xi_k)\,\mathrm{s}(\sum_{k=\ell}^{\ell+m-1}\xi_k)\big]^{\frac{1}{2}}}.$$

Summary

Integrable many-body systems in one spatial dimension form an important class of exactly solvable Hamiltonian systems with their diverse mathematical structure and widespread applicability in physics [102, 103, 134]. Among these models, the systems of Calogero-Ruijsenaars type occupy a central position, due to their intimate relation with soliton theory [114], and since many other interesting models (e.g. Toda lattice) can be obtained from them by taking various limits and analytic continuations [118]. Calogero-Ruijsenaars systems describe interacting particles moving on a line or circle. They come in different types called rational (I), hyperbolic (II), trigonometric (III), and elliptic (IV) depending on the functional form of their Hamiltonian. They exist in nonrelativistic and relativistic form, and both at the classical and quantum level. This already means sixteen different models (captured by Figure 3). But there are other interesting extensions maintaining integrability, as is exemplified by versions attached to (non-A type) root systems [95] or allowing internal degrees of freedom (spin) [51].

A fascinating aspect of Calogero-Ruijsenaars type systems is their duality relations, which first appeared in the famous paper [68] by Kazhdan, Kostant, and Sternberg, and was extensively explored by Ruijsenaars [113]. Here the word *duality* refers to action-angle duality, which involves two Liouville integrable many-body Hamiltonian systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ with Darboux coordinates q, p and $\tilde{q}, \tilde{p}$. These are said to be duals of each other if there is a global symplectomorphism $\mathcal{R}\colon M \to \tilde{M}$ of the phase spaces, which exchanges the canonical coordinates with the action-angle variables for the Hamiltonians. Practically, this means that $H \circ \mathcal{R}^{-1}$ depends only on $\tilde{q}$, while $\tilde{H} \circ \mathcal{R}$ only on q. In more detail, q are the particle positions for H and action variables for $\tilde{H}$, and similarly, $\tilde{q}$ are the positions of particles modelled by the Hamiltonian $\tilde{H}$ and action variables for H. The idea that dualities can be interpreted in terms of Hamiltonian reduction can be distilled from [68] and was put forward explicitly in several papers in the 1990s, e.g. [48, 54].

During the past ten years, Fehér and collaborators undertook the study of these dualities within the framework of reduction [36, 35, 9, 37, 38, 34, 40]. The list of action-angle dualities was enlarged by Pusztai [105, 106, 107, 108, 109] to dual pairs associated with non-A type root systems. The primary aim of the work presented in this thesis was to further develop these earlier developments. In this effort, our main tool was

the method of Hamiltonian reduction, which belongs to the set of standard toolkits applicable to study a great variety of problems ranging from geometric mechanics to field theory and harmonic analysis [84, 31]. It is especially useful in the theory of integrable Hamiltonian systems [11], where one of the maxims is that one should view the systems of interests as reductions of obviously solvable 'free' systems [98]. This is often advantageous, for example since the reduction produces global phase spaces on which the reduced free flows are automatically complete, which is an indispensable property of any integrable system.

The thesis is divided into two parts. Part I of the thesis takes this reduction approach to Calogero-Ruijsenaars type systems. Part II collects our work that were initiated by reduction results, but were obtained using direct methods.

Results

Here we collect the main results of the thesis, going chapter by chapter. In each title, we cite our related contribution (Publications)

Spectral coordinates of the rational Calogero-Moser system [P6]

Chapter 1 starts by recalling the pivotal work of Kazhdan, Kostant, Sternberg [68]. Subsection 1.1.2 is a recap of their result about the complete integrability and action-angle duality for the rational Calogero-Moser system in the context of Hamiltonian reduction. In Section 1.2, we put these ideas into use, when we identify the canonical variables of [33] in terms of the reduction picture (Lemma 1.1), and prove the relation conjectured in that paper (Theorem 1.2). We attain Sklyanin's formula as a corollary (Corollary 1.3).

Action-angle duality for the trigonometric BC_n Calogero-Moser-Sutherland system [P1, P8, P5]

Chapter 2 is a study on the trigonometric Sutherland system attached to the BC_n root system. We start by providing a physical interpretation of the model in Section 2.1. This is followed by the preparatory Section 2.2, where the group-theoretic ingredients of reduction are introduced together with the unreduced Abelian Poisson algebras and the symplectic reduction to be performed. In Section 2.3, we solve the momentum equations, hence obtaining the first model of the reduced phase space (Theorem 2.1). In a nutshell, this first model of the reduced phase space carries the Sutherland Hamiltonian as the reduction of the free Hamiltonian governing geodesic motion on the Lie group $\mathrm{U}(2n)$ (Corollary 2.2). The content of this section, and even its quantum analogue, is fairly standard [44]. The heart of the second chapter is Subsection 2.3.2, in which we first describe the reduced phase space locally (Theorem 2.3), then extend this construction to a global model (Theorem 2.16). This gives rise to the action-angle dual of the trigonometric BC_n Sutherland system. The main Hamiltonian of the dual system turns out to be a real form of the rational Ruijsenaars-Schneider Hamiltonian with three independent couplings. In Section 2.4, we apply our duality map to various problems, such as finding the equilibrium configuration of the Sutherland model, proving the maximal superintegrability of the dual model, and connecting the Hamiltonians of the hyperbolic analogue with a family of Hamiltonians found by van Diejen.

A Poisson-Lie deformation of the trigonometric BC_n Calogero-Moser-Sutherland system [P2, P3]

Chapter 3 generalises certain parts of the previous chapter as it derives a 1-parameter deformation of the trigonometric BC_n Sutherland system by applying Hamiltonian reduction to the Heisenberg double of the Poisson-Lie group $\mathrm{SU}(2n)$. Here, we were also motivated by the recent work of Marshall [86]. We start in Section 3.1 by defining the reduction of interest. In Section 3.2, we observe that several technical results of [37] can be applied for analysing the reduction at hand, and solve the momentum map constraints by taking advantage of this observation (Proposition 3.2, 3.3). The main result of this chapter is contained in Section 3.3, where we characterize the reduced system. In Subsection 3.3.1, we prove that the reduced phase space is smooth (Theorem 3.9). Then in Subsection 3.3.2 we focus on a dense open submanifold on which the Hamiltonian 'lives' (Theorem 3.11). The demonstration of the Liouville integrability of the reduced free flows is given in Subsection 3.3.3. In particular, we prove the integrability of the completion of the pertinent system carried by the full reduced phase space. Our main result in this chapter is Theorem 3.14 (proved in Subsection 3.3.4), which establishes a globally valid model of the reduced phase space. In Section 3.4, we complete (Theorem 3.22) the recent derivation of the hyperbolic analogue by Marshall [86].

Lax representation of the hyperbolic BC_n Ruijsenaars-Schneider-van Diejen system [P7]

Chapter 4 contains our construction of a Lax pair for the classical hyperbolic van Diejen system with two independent coupling parameters. In Section 4.1, we start with a short overview on some relevant group-theoretic facts and fix notation. In Section 4.2, we define our Lax matrix for the van Diejen system, and also investigate its main algebraic properties (Proposition 4.1, Lemma 4.2). These results can be used to show that the dynamics can be solved by a projection method, which in turn allows us to initiate the study of the scattering properties. This was done by B.G. Pusztai and is included in Subsections 4.2.2, 4.3.1, 4.3.3, 4.3.4, 4.3.5. In Subsection 4.4.1, we elaborate the link between our special 2-parameter family of Hamiltonians and the most general 5-parameter family of hyperbolic van Diejen systems. We affirm the equivalence between van Diejen's commuting family of Hamiltonians and the coefficients of the characteristic polynomial of our Lax matrix (Lemma 4.14). Based on this technical result, we can infer that the eigenvalues of the proposed Lax matrix provide a commuting family of first integrals for the Hamiltonian system (Theorem 4.15).

Trigonometric and elliptic Ruijsenaars-Schneider models on the complex projective space [P4]

Chapter 5 is concerned with the compactified Ruijsenaars-Schneider systems with so-called type (i) couplings [40]. We reconstruct the corresponding compactification on $\mathbb{CP}^{n-1}$ using only direct, elementary methods (Proposition 5.1, Theorem 5.5). Such construction was not known previously except for special type (i) cases. By doing so, we gain a better understanding of the structure of these trigonometric systems. This part of the chapter fills Sections 5.1 and 5.2. In Section 5.3, we explain that the direct method is applicable to obtain type (i) compactifications of the elliptic Ruijsenaars-Schneider system as well (Theorem 5.7). This new result extends the remarks of Ruijsenaars [114, 118].

Összefoglaló

Bevezetés

Az egydimenziós integrálható sokrészecske modellek széleskörû fizikai alkalmazásaik és gazdag matematikai hátterük okán az egzaktul megoldható hamiltoni rendszerek fontos osztályát képezik. A Calogero-Ruijsenaars típusú rendszerek központi helyet foglalnak el ezek között. Ez egyrészt a szolitonok elméletével való kapcsolatuknak, másrészt annak köszönhetõ, hogy számos más érdekes modell (pl. a Toda-molekula) származtatható belõlük, határesetekként és analitikus kiterjesztéssel. A Calogero-Ruijsenaars típusú modellek egyenesen vagy körön mozgó kölcsönható részecskéket írnak le. A kölcsönhatás jellege szerint négy típust különböztetünk meg. Ezek a racionális (I), a hiperbolikus (II), a trigonometrikus (III) és az elliptikus (IV) rendszerek. A modelleknek létezik nemrelativisztikus és relativisztikus, valamint klasszikus- és kvantummechanikai változata is. Integrálható általánosításaik közül kiemelendõk a gyökrendszereken alapuló és a belsõ szabadsági fokot is megengedõ (spin) modellek.

Tudományos előzmények

Tekintsük az (M, ω, H), $(\tilde{M}, \tilde{\omega}, \tilde{H})$ Liouville integrálható rendszereket. A két rendszer *hatás-szög dualitásáról* akkor beszélünk, ha létezik a fázisterek között egy $\mathcal{R}\colon M \to \tilde{M}$ szimplektomorf leképezés, amely az $\tilde{M}$ tér valamely $(\tilde{q}, \tilde{p})$ kanonikus koordinátáit a H Hamilton-függvényhez tartozó rendszer hatás-szög változóiba viszi át, és fordítva, az M térnek léteznek (q, p) kanonikus koordinátái, amelyek a $\tilde{H}$ Hamilton-függvény rendszerének hatás-szög változói lesznek. Ekkor $\mathcal{R}$ az ún. *hatás-szög leképezés*. Ezáltal $H \circ \mathcal{R}^{-1}$ kizárólag $\tilde{q}$-tól, és $\tilde{H} \circ \mathcal{R}$ csakis q-tól függ. Mindemellett az általunk vizsgált rendszerek esetén a H Hamilton-függvény (q, p) koordinátás alakja kölcsönható részecskék egy olyan modelljét adja, amelyben q a részecske-koordináták szerepét játssza, és hasonlóan, a $\tilde{H}$ függvény a $(\tilde{q}, \tilde{p})$ változókkal kifejezve $\tilde{q}$ pozíciókba elhelyezett részecskék kölcsönhatását írja le. Ezen különleges kapcsolat jelentõségét mutatja, hogy a kvantummechanikai tárgyalásban is megjelenik mint a fontos speciális függvényekkel kifejezett hullámfüggvények bispektrális tulajdonsága [30, 114].

Dualitásban álló sokrészecske rendszereket vizsgált Ruijsenaars [113, 115, 117, 118], aki közvetlen úton konstruált hatás-szög leképezéseket az A_{n-1} gyökrendszerhez asszociált Calogero-Ruijsenaars és Toda típusú integrálható rendszerekhez. Ezen dualitási relációk redukciós értelmezésérõl számos cikk született az 1990-es években, pl. [48, 54]. Az ezekben felmerült ötleteket továbbfejlesztve és szisztematikusan kidolgozva az elmúlt évtizedben Fehér és munkatársai ilyen kapcsolatokat vezettek le redukciós keretek között [36, 35, 9, 37, 38, 34, 40] (ld. *Alkalmazott módszerek*). Az a természetes várakozás, hogy hasonló dualitások fennállnak másfajta gyökrendszerek esetén is Pusztai munkájában nyert igazolást [105, 106, 107, 108, 109].

Kutatásunkban során olyan új eredmények elérését tűztük ki célul (ld. Publications), amelyek ezen korábbi fejleményekhez kapcsolódnak.

Célkitűzések

A disszertációban bemutatott doktori munka céljai az alábbi pontokba foglalhatók össze:

I. A racionális A_{n-1} Calogero-Moser modell hatás-szög változóira vonatkozó Sklyanin-formula bizonyítása redukciós módszerrel.

II. A trigonometrikus BC_n Sutherland rendszer hatás-szög duálisának részletes kidolgozása hamiltoni redukciós keretek között.

III. Az elõzõ pont eredményeit és Marshall egy korábbi munkáját általánosítva a trigonometrikus BC_n Sutherland modell egy 1-paraméteres integrálható deformációjának megalkotása.

IV. A Lax formalizmus kiterjesztése az egynél több csatolási állandót tartalmazó általánosított hiperbolikus Ruijsenaars-Schneider rendszerekre.

V. Új elliptikus A_{n-1} Ruijsenaars-Schneider modellek konstruálása az n-dimenziós komplex projektív téren.

A fenti kutatási elképzeléseket sikeresen valósítottuk meg, sõt további, a kezdeti várakozásokon túlmutató elõrelépéseket is tettünk.

Alkalmazott módszerek

A fenti célok eléréséhez az úgynevezett *hamiltoni redukció* módszerét, valamint standard matematikai eszközöket alkalmaztunk.

Dióhéjban összefoglalva, a redukciós eljárás során a levezetendõ rendszerek részecskéinek bonyolult mozgását egy magasabb dimenziós térben mozgó, nagyfokú szimmetriával bíró szabad részecske ‘ügyesen’ választott vetületeként nyerjük.

Pontosabban fogalmazva, a redukció kiindulásaként egy csoportelméleti eredetû fázisteret választunk. Ez lehet egy X mátrix Lie-csoport vagy Lie-algebra $P = T^*X$ koérintõnyalábja. A P nyalábon természetes módon megadható Ω szimplektikus forma és egy $\mathcal{H}\colon P \to \mathbb{R}$ Hamilton-függvény megválasztása egy $(P, \Omega, \mathcal{H})$ hamiltoni rendszert eredményez. Ha a $\mathcal{H}$ Hamilton-függvény kellõen egyszerû alakot ölt, akkor a mozgásegyenletek explicit módon megoldhatók, sõt akár $\{\mathcal{H}_j\}$ Poisson kommutáló elsõ integrálok egy egész serege felírható. Ekkor egy megfelelõen választott G csoport hatása az X (és ezáltal a P) téren, amelyre nézve a $\mathcal{H}_j$ függvények invariánsak, lehetõvé teszi az asszociált $\Phi\colon P \to \mathfrak{g}^*$ momentum leképezés felírását. A Φ momentum leképezés értékének $\mu \in \mathfrak{g}^*$ elemre történõ rögzítése egy $\Phi^{-1}(\mu)$ szintfelületet jelöl ki a P fázistérben. Ez a kényszerfelület a momentum érték $G_\mu \subset G$ izotrópia-részcsoportjának pályáiból áll. Ezen pályák alkotják a $(P_{\mathrm{red}}, \omega_{\mathrm{red}}, H)$ redukált fázistér pontjait. A fenti konstrukciónak köszönhetõen az involúcióban álló $\{\mathcal{H}_j\}$ mozgásállandók hamiltoni folyamai invariánsan hagyják a momentum szintfelületet és a $\{\mathcal{H}_j\}$ függvények állandók G_μ pályái mentén. Következésképpen értelmezhetõk a függvények $H_j\colon P_{\mathrm{red}} \to \mathbb{R}$ redukciói, amelyek Poisson zárójele továbbra eltûnik, és ily módon a származtatott $(P_{\mathrm{red}}, \omega_{\mathrm{red}}, H)$ hamiltoni rendszer Liouville értelemben integrálható. A gyakorlatban jellemzõen a redukált fázisteret a G_μ csoport pályáinak egy S sima szelésével azonosítjuk. Ilyen szelést a $\Phi = \mu$ egyenlet megoldásával nyerünk. Két így kapott $S, \tilde{S}$ modell lehet egymás hatás-szög duálisa.

Új tudományos eredmények

Az alábbiakban röviden ismertetem a disszertációban elért tudományos eredményeket. A kapcsolódó publikációkat a disszertáció végén található Publications lista gyűjti össze. A közleményekre az azoknak megfelelő tézispontok címében, illetve szükség esetén a szövegben hivatkozok.

I. A racionális Calogero-Moser rendszer spektrális koordinátái [P6]

+ A hamiltoni redukció módszerének alkalmazásával azonosítottam a racionális Calogero-Moser rendszer Falqui és Mencattini [33] által felírt kanonikus koordinátáit.

+ Bizonyítottam egy Falqui és Mencattini [33] által megsejtett összefüggést.

+ Igazoltam Sklyanin [130] formuláját, amely spektrális kanonikus koordinátákat szolgáltat a racionális Calogero-Moser rendszerhez.

II. A trigonometrikus BC_n Sutherland rendszer hatás-szög duálisa [P1, P8, P5]

+ Hamiltoni redukció útján származtattam a trigonometrikus BC_n Sutherland modell hatás-szög duálisát, amelyben a racionális BC_n Ruijsenaars-Schneider rendszer egy valós formáját ismertem fel.
+ Bizonyítottam, hogy a duális modell lokális leírásában használt változók kanonikus koordináta-rendszert alkotnak [P8].
+ Felírtam ezen duális rendszer Lax-mátrixát explicit alakban.
+ Megadtam a duális modell fázisterének, valamint Lax-mátrixának globális leírását [P1].
+ Jellemeztem a trigonometrikus BC_n Sutherland modell egyensúlyi konfigurációit a hatás-szög dualitás segítségével.
+ További alkalmazásként igazoltam, hogy a duális rendszer $(n-1)$ extra mozgásállandóval rendelkezik, következésképp maximálisan szuperintegrálható.
+ Végül bizonyítottam, hogy a hiperbolikus BC_n Sutherland modell Pusztai [107] által konstruált involúcióban álló mozgásállandói és a van Diejen [140] által talált Poisson kommutáló elsõ integrálok ekvivalensek, azaz ugyanazt az abeli algebrát generálják [P5]. A két említett függvénycsalád közötti lineáris kapcsolatot explicit formában felírtam és igazoltam.

III. A trigonometrikus BC_n Sutherland rendszer Poisson-Lie deformációja [P2, P3]

+ Marshall korábbi, hiperbolikus esettel foglalkozó munkáját [86] általánosítva levezettem a trigonometrikus BC_n Sutherland rendszer egy 1-paraméteres integrálható deformációját a $2n \times 2n$-es egységnyi determinánsú unitér mátrixok alkotta Poisson-Lie csoport Heisenberg duplájának általánosított Marsden-Weinstein redukciójából.
+ Megoldottam a momentum kényszer-egyenletet, visszavezetve azt egy Fehér és Klimčík [37] által korábban már részletesen vizsgált egyenletre.
+ A fejezet fõ eredményeként globálisan jellemeztem a redukált rendszert [P2]. Igazoltam, hogy a levezetett rendszer Liouville integrálható.

+ Továbbá megmutattam, hogy a modell miként kapható meg van Diejen [140] öt csatolási állandót tartalmazó modelljébõl. Ezáltal a levezetett modellt sikerült beilleszteni a Calogero-Ruijsenaars típusú integrálható rendszerek közé.

+ Végül teljessé tettem a hiperbolikus verzió Marshall [86] által adott származtatását [P3].

IV. A hiperbolikus BC_n Ruijsenaars-Schneider-van Diejen rendszer Lax reprezentációja [P7]

+ Igazoltam, hogy a Lax mátrix eleme az (n, n)-szignatúrájú 'belsõ szorzással' definiált pszeudounitér mátrixok Lie-csoportjának.

+ Pusztai korábbi eredményét [106] felhasználva bizonyítottam, hogy a Lax mátrix pozitív definit.

+ Megmutattam a Pusztai által levezetett szóráselméleti eredmények segítségével, hogy a Lax mátrixból származó spektrális invariánsok és van Diejen [140] öt paramétert tartalmazó Poisson kommutáló függvénycsaládjának megfelelõ specializációja ekvivalensek.

+ Ennek segítségével bebizonyítottam, hogy a Lax mátrix független sajátértékei Poisson kommutáló mozgásállandók teljes rendszerét alkotják.

V. Trigonometrikus és elliptikus Ruijsenaars-Schneider modellek a komplex projektív téren [P4]

+ Megvizsgáltam a Fehér és Kluck [40] által korábban felfedezett ún. *egyes típusú* csatolási állandóval jellemzett kompaktifikált Ruijsenaars-Schneider modelleket, és közvetlen, elemi úton megmutattam, hogy a trigonometrikus esetben ezen rendszerek miként ágyazhatók be a megfelelõ komplex projektív térbe.

+ A trigonometrikus esetben alkalmazott eljárást általánosítottam az elliptikus potenciálok esetére is, ezáltal új elliptikus Ruijsenaars-Schneider modelleket konstruáltam a komplex projektív téren. Ezzel kiterjesztettem Ruijsenaars korábbi eredményeit [114, 118].

Publications

Refereed research papers:

[P1] L. Fehér and **T.F. Görbe**, *Duality between the trigonometric BC_n Sutherland system and a completed rational Ruijsenaars-Schneider-van Diejen system*, J. Math. Phys. **55** (2014) 102704; doi:10.1063/1.4898077; arXiv:1407.2057 [math-ph]

[P2] L. Fehér and **T.F. Görbe**, *On a Poisson-Lie deformation of the BC_n Sutherland system*, Nucl. Phys. B **901** (2015) 85-114; doi:10.1016/j.nuclphysb.2015.10.008; arXiv:1508.04991 [math-ph]

[P3] L. Fehér and **T.F. Görbe**, *The full phase space of a model in the Calogero-Ruijsenaars family*, J. Geom. Phys. (2016); doi:10.1016/j.geomphys.2016.04.018; arXiv:1603.02877 [math-ph]

[P4] L. Fehér and **T.F. Görbe**, *Trigonometric and elliptic Ruijsenaars-Schneider systems on the complex projective space*, Lett. Math. Phys. **106** (2016) 1429-1449; doi:10.1007/s11005-016-0877-z; arXiv:1605.09736 [math-ph]

[P5] **T.F. Görbe** and L. Fehér, *Equivalence of two sets of Hamiltonians associated with the rational BC_n Ruijsenaars-Schneider-van Diejen system*, Phys. Lett. A **379** (2015) 2685-2689; doi:10.1016/j.physleta.2015.08.014; arXiv:1503.01303 [math-ph]

[P6] **T.F. Görbe**, *A simple proof of Sklyanin's formula for canonical spectral coordinates of the rational Calogero-Moser system*, SIGMA **12** (2016) 027; doi:10.3842/SIGMA.2016.027; arXiv:1601.01181 [math-ph]

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[P8] **T.F. Görbe**, *On the derivation of Darboux form for the action-angle dual of trigonometric BC_n Sutherland system*, J. Phys.: Conf. Ser. **563** (2014) 012012; doi:10.1088/1742-6596/563/1/012012; arXiv:1410.0301 [math-ph]

Additional refereed research paper:

[P9] V. Ayadi, L. Fehér, and **T.F. Görbe**, *Superintegrability of rational Ruijsenaars-Schneider systems and their action-angle duals*, J. Geom. Symmetry Phys. **27** (2012) 27-44; doi:10.7546/jgsp-27-2012-27-44; arXiv:1209.1314 [nlin.SI]

Conference posters:

[P10] **T.F. Görbe**, *The trigonometric BC_n Sutherland system: action-angle duality and applications*, 3-rd Conference on Finite Dimensional Integrable Systems in Geometry and Mathematical Physics 2015 (FDIS2015), 12-17 July 2015, Będlewo (Poland)

[P11] **T.F. Görbe**, *Lax representation of the hyperbolic van Diejen system with two coupling parameters*, Integrable Systems Conference 2016, 19-24 June 2016, CSF Ascona (Switzerland)

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