F-inverse covers of E-unitary inverse monoids

Ph.D. thesis

Nóra Szakács

Supervisor: Mária B. Szendrei

Doctoral School of Mathematics and Computer Science
University of Szeged

2016

Contents

Acknowledgments

First and foremost, I would like to express my gratitude to my adviser, Mária Szendrei, who introduced me to semigroup theory and has been guiding my steps as a mathematician for over six years now. She has always been very supportive of my goals and worked hard to help me achieve them. Among many other things, I owe it to her that I found the confidence to truly believe I could be a researcher.

I am also thankful to John Meakin, whom I worked with as a visiting scholar at University of Nebraska-Lincoln in 2013. He inspired me greatly and taught me a lot during my brief stay.

I am sure this thesis could not have been written without the encouragement and support of the teachers of my high school years, especially my mathematics teacher, János Dobcsányi. I am grateful to each and every one of them. I hope I can grow to fulfill the potential they saw in me.

Last but not least, I am sincerely grateful to my family and friends for making me feel loved and appreciated all these years. I would especially like to thank my parents for showing me the joy of thinking at an early age, opening up to me the wonderful and mysterious world of mathematics.

Chapter 1

Introduction

The topic of the dissertation falls in the area of semigroup theory. The definition of a semigroup is quite simple: it is a nonempty set equipped with a multiplication that is associative. Due to the generality of the definition, the class of semigroups is very broad and diverse — e.g., both groups and semilattices fall in here. On one hand, this makes semigroups have connections to more or less any area of mathematics, yielding quite a number of possible applications; on the other hand, different classes of semigroups may require considerably different approaches and apparatus.

The class of semigroups considered in the thesis is called *inverse monoids* (see the monographies of Lawson [12] and Petrich [17] on the topic for uncited results). They are monoids defined by the property that every element x has a unique inverse x^{-1} such that $xx^{-1}x = x$, and $x^{-1}xx^{-1} = x^{-1}$ hold. They are one of the many generalizations of groups. One way they naturally arise is through partial symmetries — to put it informally, inverse monoids are to partial symmetries as what groups are to symmetries. The symmetric inverse monoid $\mathrm{SIM}(X)$ on the set X consists of all partial one-to-one maps on X, that is bijections between subsets of X, equipped with the usual multiplication of partial maps and the usual inverse of bijections between subsets. Analogously to Cayley's theorem, the Wagner–Preston theorem states that all inverse monoids can be embedded into a suitable symmetric inverse monoid.

Unlike in groups, in an inverse monoid, xx^{-1} is not necessarily the identity element, but it is, nevertheless, an idempotent. Idempotents therefore play an important role in the structure, and the set of idempotents of M is denoted by $E(M)$. An important property of inverse monoids is that its idempotens commute, therefore form a semilattice. Inverse monoids also come equipped with a *natural partial order*, which extends the partial order on idempotens induced by the semilattice structure. It is defined by $s \leq t$ if and only if there exists and idempotent e such that $s = te$. For instance, the idempotents of a symmetric inverse monoid are exactly the identical maps on subsets of X, and hence the natural partial order is nothing but the restriction of maps. Observe that the natural partial order is compatible with the multiplication, that is, if $a \leq b$ and $c \leq d$, then $ac \leq bd$.

The idempotents forming a semilattice is such a characteristic feature of inverse monoids that it can be used to describe inverse monoids using only *identities* — equations imposed on all elements. Indeed an inverse monoid is a monoid with an involution $^{-1}$ that satisfies the identities $xx^{-1}x = x$ and $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$. Inverse monoids therefore form a *variety* in the sense of universal algebra, see [3], in particular, free inverse monoids exist on all sets, and every inverse monoid is a homomorphic image of a free one.

It is not hard to see that groups are just inverse monoids with a unique idempotent. Thus factoring an inverse monoid by a congruence which collapses all idempotents yields a group, with the class containing the idempotents as the identity element. Each inverse monoid M has a *smallest group congruence*, denoted by σ, which is generated (as a congruence) by $E(M) \times E(M)$, and a corresponding *greatest group homomorphic image* M/σ. The preimage of the identity element under this homomorphism is, of course, the σ-class containing the semilattice $E(M)$. This hints at the fact that inverse monoids can somehow be 'built' from a group and a semilattice, and this is indeed one of the main tools of investigating inverse monoids, and many of the constructions introduced in the thesis will follow that pattern. We mention two important classes of inverse monoids where the construction is well known and relatively straightforward.

One is the class of *E-unitary inverse monoids*, which is defined by the property that the σ-class containing the idempotents contains nothing but the idempotents. In general, that σ-class coincides with the set $E(M)^\omega = \{s \in M : (\exists e \in E(M))(e \leq s)\}$, therefore the inverse monoid M is E-unitary if and only if its set of idempotents is closed upwards in the natural partial order.

By a famous theorem of McAlister known as the P-theorem, E-unitary inverse monoids can be built using three building blocks: a group G, a partially ordered set X, and a principal order ideal Y of X which is a meet-semilattice with respect to the partial order on X. The group G acts on X by order automorphisms, and, in order to avoid that superfluous elements occur in X or G, it is further assumed that $\{{}^gY : g \in G\} = X$, and, for any $g \in G$, the intersection of the sets ${}^gY$ and Y is not empty. The E-unitary inverse monoid obtained from such a *McAlister triple* (G, X, Y) is $P(G, X, Y) = \{(A, g) \in Y \times G : {}^{g^{-1}}A \in Y\}$, with a semidirect product-like multiplication

$$(A, g)(B, h) = (A \wedge {}^{g}B, gh).$$

The inverse of an element (A, g) is $({}^{g^{-1}}A, g^{-1})$. The semilattice of idempotents of $P(G, X, Y)$ is isomorphic to Y, and the greatest group homomorphic image $P(G, X, Y)/\sigma$ is isomorphic to G.

The P-theorem states that every E-unitary inverse monoid M is isomorphic to one of the form $P(M/\sigma, X, E(M))$, and so E-unitary inverse monoids are, in a way, 'known'. This is what gives particular significance to the McAlister covering theorem stating that every inverse monoid has an *E-unitary cover*, that is, every inverse monoid is a homomorphic image of an E-unitary inverse monoid under a homomorphism which is injective on the idempotens (this property is called *idempotent-separating*). Therefore, if M is an E-unitary cover of the inverse monoid N, then their semilattices of idempotents are isomorphic, making the group M/σ a significant unknown component of McAlister triple. Hence we emphasize its importance by saying that that M is an E-unitary cover *over the group G* if G is isomorphic to M/σ. The simplest proof of the McAlister covering theorem applies the Wagner–Preston theorem and extensions of partial one-to-one maps to permutations. In particular, it shows that finite inverse monoids have finite E-unitary covers.

Another important class of inverse monoids we mention is that of *F-inverse monoids*. An inverse monoid is called F-inverse if its σ-classes have a greatest element with respect to the natural partial order. F-inverse monoids are always E-unitary, they are characterized by a McAlister triple (G, X, Y) where X is also a semilattice. The notion of an F-inverse monoid is among the most important ones in the theory of inverse semigroups, for example, free inverse monoids are F-inverse. Moreover, they play an important role in the theory of partial actions of groups, see Kellendonk and Lawson [9], and in this context they implicitly occur in Dehornoy [4, 5]. In Kaarli and Márki [8], a subclass of finite inverse monoids occurring in the context of universal algebra is proven to have the property that each member has an F-inverse cover within that class. Even in analysis, F-inverse monoids are useful: see Nica [16], Khoshkam and Skandalis [10] and Steinberg [19] for their role in the context of C^*-algebras.

An easy consequence of the fact that each inverse monoid is a homomorphic image of a free one is that every inverse monoid has an *F-inverse cover*, that is, every inverse monoid M is a homomorphic image of an F-inverse monoid by an idempotent-separating homomorphism. Here, we also call F an F-inverse cover of the inverse monoid M *over the group G* if G is isomorphic to M/σ. However, in this case, the proof always produces an

F-inverse cover over a free group, and so it is always infinite. The main motivation of the research described in the dissertation is the following:

Open problem 1.0.1. Does every finite inverse monoid admit a finite F-inverse cover?

The problem has been formulated by Henckell and Rhodes in [7], and a positive answer would have solved an important conjecture connected to the complexity theory of finite semigroups. The latter conjecture has been since proven [1], but the F-inverse cover problem has remained open.

Note that by the McAlister covering theorem, it suffices to restrict our attention to F-inverse covers of E-unitary inverse monoids, as we do throughout the thesis. The most important antecedent to the research presented in the dissertation is the paper of Auinger and Szendrei [2] on the question. They go a step further by applying that it is sufficient to restrict to a special class of E-unitary inverse monoids called Margolis–Meakin expansions, which, as we will see, have a very convenient structure. Thus Auinger and Szendrei are able to reformulate the F-inverse cover problem by means of graphs and locally finite group varieties only. We retell their results in Section 2.3, after the introduction to some basic notions regarding inverse monoids, graphs and categories in Sections 2.1 and 2.2.

The new results of the author and partly of her adviser presented in the dissertation were published in the papers [20] and [21], and are contained in Chapters 3 and 4 respectively. In [20], the condition on graphs and group varieties introduced in [2] is investigated. In Section 3.1, we establish that, when fixing the group variety, the graphs for which the condition is satisfied can be described using forbidden minors. In Section 3.2, we apply this approach to the special case when the variety is Abelian, in which case we are able to give a full description of the graphs and group varieties satisfying the property, as stated in Theorem 3.2.1. Unraveling the details of how the graph condition is related to F-inverse covers of Margolis–Meakin expansions, what we obtain is a description of all Margolis–Meakin expansions M which have an F-inverse cover F such that F/σ is an extension of an Abelian group by M/σ — this we refer to as F being an F-inverse cover *via* a variety of Abelian groups —, presented in Theorem 3.2.4.

In [21], we are motivated by finding *all* finite E-unitary inverse monoids which have an F-inverse cover via a variety of Abelian groups. The first step is introducing a Margolis–Meakin-like structure that describes the much larger class of *finite-above* E-unitary inverse monoids — which, in particular, contains all finite ones —, and generalizing the conditions introduced in [2] accordingly. These results are contained in Section 4.1. Using our framework, in Example 4.1.21, we present a family of finite E-unitary inverse monoids having finite F-inverse covers, for which this fact does not follow by previous techniques. In Section 4.2, we move on to Abelian varieties, and in Theorem 4.2.3, give a sufficient condition for an E-unitary finite-above inverse monoid not to have an F-inverse cover via the variety of Abelian groups, formulated merely by means of the natural partial order and the least group congruence.

Chapter 2

Preliminaries

2.1 Inverse monoids

Let M be an inverse monoid (in particular, a group) and A an arbitrary set. We say that M is an *A-generated inverse monoid* (*A-generated group*) if a map[1] $\epsilon_M\colon A \to M$ is given such that $A\epsilon_M$ generates M as an inverse monoid (as a group). If ϵ_M is injective, then we might assume that A is a subset in M, as usual, i.e., ϵ_M is the inclusion map $A \to M$. If M, N are A-generated inverse monoids, then $\varphi\colon M \to N$ is a *canonical homomorphism* if it is a homomorphism such that $\epsilon_M\varphi = \epsilon_N$. Notice that if ϵ_N is an inclusion, then ϵ_M is injective, and so it also can be chosen to be an inclusion. However, if ϵ_M is injective (in particular, an inclusion), then ϵ_N need not be injective. This is the reason that one cannot suppose in general that $A \subseteq M$ for every A-generated monoid M.

Given an arbitrary set A, the *free monoid* on A, denoted by A^*, is the monoid which consists of all finite sequences of elements of A, called *words*, together with the empty word denoted by 1, and these are multiplied by concatenation. It is well known that for any monoid M and map $\varphi\colon A \to M$, φ extends to a homomorphism $A^* \to M$ uniquely, hence the name 'free'. The first step of the analogous constructions of free groups and free inverse monoids is creating a *free monoid with involution*, the involution being responsible for the inverse. Consider a set A' disjoint from A together with a bijection $'\colon A \to A'$. Put $\overline{A} = A \cup A'$, and consider the free monoid $\overline{A}^*$ on $\overline{A}$, and extend the map $'$ to an involution of $\overline{A}^*$, denoted also by $'$. Notice that this extension is unique, $(a')' = a$ holds for every $a \in A$, and $(b_1b_2\cdots b_n)' = b_n'b_{n-1}'\cdots b_1'$ holds for every word $b_1b_2\cdots b_n \in \overline{A}^*$. The monoid $\overline{A}^*$ together with this involution is the free monoid with involution on A. For simplicity, we do not introduce a new notation for this structure, but throughout the thesis, $\overline{A}^*$ is

[1] As it is customary in semigroup theory, we write maps on the right in this thesis.

meant to denote the free monoid with involution on A.

For any inverse monoid M, there is a unique homomorphism $\varphi\colon \overline{A}^* \to M$ such that $a\varphi = a\epsilon_M$ and $a'\varphi = (a\epsilon_M)^{-1}$ for every $a \in A$, since taking inverse is an involution on M. If M is A-generated, then φ is clearly surjective. For any word $w \in \overline{A}^*$, we denote $w\varphi$ by $[w]_M$. The free inverse monoid and free group on the set A, denoted by FIM(A) and FG(A) respectively, are of course, also homomorphic images of $\overline{A}^*$. The kernel of the homomorphism is just the fully invariant congruence generated by the identities defining inverse monoids and groups, respectively (see [3]). Furthermore, since groups are special inverse monoids, FG(A) is also a factor of FIM(A) — in fact, FIM(A)$/\sigma =$ FG(A).

A variety of inverse monoids is a class of inverse monoids defined by identities, they are denoted by capital bold letters in the sequel. For instance, the variety of groups, the variety **Sl** of semilattices, and the variety **Ab** of Abelian groups are all varieties of inverse monoids. Again, the factor of a free inverse monoid FIM(A) induced by the fully invariant congruence corresponding to the respective defining identities gives rise to the *relatively free inverse monoid*, or, in the case of a group variety, the *relatively free group* on the set A in the variety. If M is the relatively free inverse monoid (or group) on A in a given an inverse monoid (group) variety $\mathbf{U}$, then we write $[w]_{\mathbf{U}}$ for $[w]_M$. Recall that, for every $w, w_1 \in \overline{A}^*$, we have $[w]_{\mathbf{U}} = [w_1]_{\mathbf{U}}$ if and only if the identity $w = w_1$ is satisfied in $\mathbf{U}$. We say that $[w]_{\mathbf{U}}$ *depends on a letter* a if $[w_1]_{\mathbf{U}} \neq [w]_{\mathbf{U}}$ for the word w_1 obtained from w by substituting all occurrences of a by 1. We define the $\mathbf{U}$-*content* $c_{\mathbf{U}}(w)$ of w as the set of elements $a \in A$ that $[w]_{\mathbf{U}}$ depends on.

2.2 Graphs and categories

2.2.1 Edge-labelled graphs

Throughout this thesis, unless otherwise stated, by a *graph* we mean a directed graph, that is, a quadruple $\Delta = (V_\Delta, E_\Delta, \iota, \tau)$, where V_Δ and E_Δ denote the sets of vertices and edges of Δ respectively, and ι, τ are $E_\Delta \to V_\Delta$ maps that assign the initial and the terminal vertices to an edge e. If $\iota e = i$ and $\tau e = j$, then e is called an (i,j)-edge. The set of all (i,j)-edges is denoted by $\Delta(i,j)$, and for our later convenience, we put

$$\Delta(i,-) = \bigcup_{j \in V_\Delta} \Delta(i,j).$$

Connectedness of graphs will, however, be regarded in an undirected sense throughout the thesis, that is, we call a digraph *connected* (*two-edge-connected*) if the underlying undirected graph is connected (two-edge-connected). Recall that an undirected graph is called *two-edge-connected* if it is connected and remains connected whenever an edge is removed. By an *edge-labelled* (or just *labelled*) graph, we mean a graph Δ together with a set A and a map $E_\Delta \to A$ appointing the labels to the edges.

A sequence $p = e_1 e_2 \cdots e_n$ $(n \geq 1)$ of consecutive edges $e_1, e_2, \ldots, e_n$ (i.e., where $\tau e_i = \iota e_{i+1}$ $(i = 1, 2, \ldots, n-1)$) is called a *path* on Δ or, more precisely, an (i, j)-path if $i = \iota e_1$ and $j = \tau e_n$. In particular, if $i = j$ then p is also said to be a *cycle* or, more precisely, an *i-cycle*. Moreover, for any vertex $i \in V_\Delta$, we consider an *empty (i, i)-path* (*i-cycle*) denoted by 1_i. A non-empty path (cycle) $p = e_1 e_2 \cdots e_n$ is called *simple* if the vertices $\iota e_1, \iota e_2, \ldots, \iota e_n$ are pairwise distinct and $\tau e_n \notin \{\iota e_2, \ldots, \iota e_n\}$.

In consistence with the undirected connectedness properties, we do not generally want to restrict to directed paths. For that, we consider paths in a graph extended by the formal reverses of its edges as follows. Given a graph Δ, take a set E' disjoint from E_Δ together with a bijection $'\colon E_\Delta \to E'$, and consider a graph Δ' where $V_{\Delta'} = V_\Delta$ and $E_{\Delta'} = E'$ such that $\iota e' = \tau e$ and $\tau e' = \iota e$ for every $e \in E_\Delta$. Define $\overline{\Delta}$ to be the graph with $V_{\overline{\Delta}} = V_\Delta$ and $E_{\overline{\Delta}} = E_\Delta \cup E_{\Delta'}$. Choosing the set E'_Δ to be $E_{\Delta'}$, the paths on $\overline{\Delta}$ become words in $\overline{E_\Delta}^*$ where $\overline{E_\Delta} = E_\Delta \cup E'_\Delta$.

We can extend the bijection $'$ to paths in a natural way. First, for every edge $f \in E_{\Delta'}$, define $f' = e$ where e is the unique edge in Δ such that $e' = f$. Second, put $1'_i = 1_i$ $(i \in V_\Delta)$ and, for every non-empty path $p = e_1 e_2 \cdots e_n$ on $\overline{\Delta}$, put $p' = e'_n e'_{n-1} \cdots e'_1$. If $p = e_1 e_2 \cdots e_n$ is a non-empty path on $\overline{\Delta}$, then *the subgraph $\langle p \rangle$ of Δ spanned by p* is the subgraph consisting of all vertices and edges p traverses in either direction. Obviously, we have $\langle p' \rangle = \langle p \rangle$ for any path p on $\overline{\Delta}$. The subgraph spanned by the empty path 1_i (consisting of the single vertex i) is denoted by $\emptyset_i$, that is, $\langle 1_i \rangle = \emptyset_i$.

Most of our graphs in this paper have edges of the form (i, a, j), where i is the initial vertex, j the terminal vertex, and a is the label of the edge. For such a graph Δ, choose $\overline{\Delta}$ as follows: consider a set A' disjoint from A together with a bijection $'\colon A \to A'$, and we choose Δ' so that $(i, a, j)' = (j, a', i)$ for any edge (i, a, j) in Δ. Then $\overline{\Delta}$ is labelled by $\overline{A}$, and, given a (possibly empty) path $p = e_1 e_2 \cdots e_n$ on $\overline{\Delta}$, the labels of the edges $e_1, e_2, \ldots, e_n$ determine a word in $\overline{A}^*$.

One particular class of graphs of the type described above is the *Cayley graphs* of groups. If G is an A-generated group by the map $\epsilon_G\colon A \to G$, its Cayley graph is a graph with G as the vertex set and with edges of the form $(g, a, g \cdot a\epsilon_G)$, where $g \in G$ and $a \in A$ are arbitrary. The Cayley graph is, of course, labelled by A, and also has the property that the initial vertex g and the label a determine the edge uniquely, moreover, any word w in $\overline{A}^*$ determines a unique path starting at 1, the terminal vertex being $[w]_G$. Hence, essentially, knowing the Cayley graph of the group means knowing the solution to its word problem and vice versa. In geometric group theory, the word problem and other algorithmic problems in group theory are investigated through geometric properties of the Cayley graph.

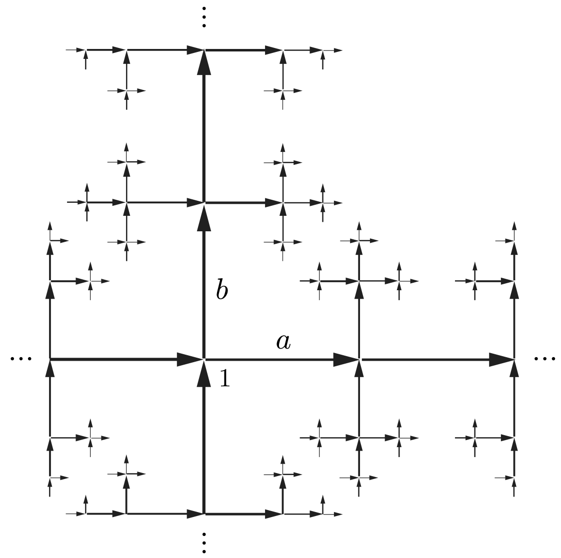


Figure 2.2.1. The Cayley graph of the free group $\mathrm{FG}(a, b)$

In this thesis, Cayley graphs of groups appear as building blocks of certain classes of inverse monoids. For instance, Munn [15] has given a beautiful description of the free inverse monoid $\mathrm{FIM}(A)$ using subtrees of the Cayley graph of the A-generated free group (see Figure 2.2.1). Munn's construction is as follows. The elements of $\mathrm{FIM}(A)$ are pairs of the form (X, g), where $g \in \mathrm{FG}(A)$, and X is a subtree of the Cayley graph containing the vertices 1 and g. The multiplication is given by the rule

$$(\mathrm{X}, g)(\mathrm{Y}, h) = (\mathrm{X} \cup {}^g\mathrm{Y}, gh),$$

where ${}^g\mathrm{Y}$ denotes the subtree obtained by 'translating' Y in the Cayley graph by g, that is, a vertex i is translated to gi, and an edge (i, a, j) to (gi, a, gj). Given a word w in $\overline{A}^*$, $[w]_{\mathrm{FIM}(A)}$ is given by the pair $(\langle p_w \rangle, [w]_{\mathrm{FG}(A)})$, where p_w is the unique path determined by the sequence of labels w.

2.2.2 Small categories

Let Δ be a graph, and suppose that a partial multiplication is given on E_Δ in a way that, for any $e, f \in E_\Delta$, the product ef is defined if and only if e and f are consecutive edges.

If this multiplication is associative in the sense that $(ef)g = e(fg)$ whenever e, f, g are consecutive, and for every $i \in V_\Delta$, there exists a (unique) edge 1_i with the property that $1_i e = e,\ f1_i = f$ for every $e, f \in E_\Delta$ with $\iota e = i = \tau f$, then Δ is called a *(small) category.* Later on, we denote categories in calligraphics. For categories, the usual terminology and notation is different from those for graphs: instead of 'vertex' and 'edge', we use the terms 'object' and 'arrow', respectively, and if $\mathcal{X}$ is a category, then, instead of $V_\mathcal{X}$ and $E_\mathcal{X}$, we write $\mathrm{Ob}\,\mathcal{X}$ and $\mathrm{Arr}\,\mathcal{X}$, respectively. Clearly, each monoid can be considered a one-object category, with the elements playing the roles of the arrows. Therefore, later on, certain definitions and results formulated only for categories will be applied also for monoids.

A category $\mathcal{X}$ is called a *groupoid* if, for each arrow $e \in \mathcal{X}(i, j)$, there exists an arrow $f \in \mathcal{X}(j, i)$ such that $ef = 1_i$ and $fe = 1_j$. Obviously, the one-object groupoids are just the groups, and, as it is well known for groups, the arrow f is uniquely determined, it is called the inverse of e and is denoted e^{-1}. By an *inverse category*, we mean a category $\mathcal{X}$ where, for every arrow $e \in \mathcal{X}(i, j)$, there exists a unique arrow $f \in \mathcal{X}(j, i)$ such that $efe = e$ and $fef = f$. This unique f is also called the *inverse of* e and is denoted e^{-1}. Clearly, each groupoid is an inverse category with the same inverse. Furthermore, the one-object inverse categories are just the inverse monoids. More generally, if $\mathcal{X}$ is an inverse category (in particular, a groupoid), then $\mathcal{X}(i, i)$ is an inverse monoid (a group) for every object i. An inverse category $\mathcal{X}$ is said to be *locally a semilattice* if $\mathcal{X}(i, i)$ is a semilattice for every object i. Similarly, given a group variety $\mathbf{U}$, we say that $\mathcal{X}$ is *locally in* $\mathbf{U}$ if $\mathcal{X}(i, i) \in \mathbf{U}$ for every object i.

Given a graph Δ, we can easily define a category Δ^* as follows: let $\mathrm{Ob}\,\Delta^* = V_\Delta$, let $\Delta^*(i, j)$ $(i, j \in \mathrm{Ob}\,\Delta^*)$ be the set of all (i, j)-paths on Δ, and define the product of consecutive paths by concatenation. The identity arrows will be the empty paths. In the one-object case, this is just the usual construction of a free monoid on a set. In general, Δ^* has a similar universal property among categories, that is, it is the *free category on* Δ. However, as we will mainly be working with inverse categories, the analogue of the free monoid $\overline{A}^*$ with involution $'$ will be more use for us. The category $\overline{\Delta}^*$ together with the bijection $'$ defined for paths on $\overline{\Delta}$ is the *free category with involution* on Δ. For an inverse category $\mathcal{X}$ and a graph Δ, if $\epsilon_\mathcal{X} \colon \Delta \to \mathcal{X}$ is a graph morphism, then there is a unique category morphism $\varphi \colon \overline{\Delta}^* \to \mathcal{X}$ such that $e\varphi = e\epsilon_\mathcal{X}$ and $e'\varphi = (e\epsilon_\mathcal{X})^{-1}$ for every $e \in \mathrm{Arr}\,\mathcal{X}$. We say that $\mathcal{X}$ is Δ-*generated* if φ is surjective. If $\mathcal{X}, \mathcal{Y}$ are Δ-generated inverse categories, then $\psi \colon \mathcal{X} \to \mathcal{Y}$ is called a *canonical category morphism* if it is a category morphism such that $\epsilon_\mathcal{X}\psi = \epsilon_\mathcal{Y}$.

The basic notions and properties known for inverse monoids have their analogues for inverse categories. Given a category $\mathcal{X}$, consider *the subgraph* $\mathrm{E}(\mathcal{X})$ *of idempotents*, where $V_{\mathrm{E}(\mathcal{X})} = \mathrm{Ob}\,\mathcal{X}$ and $E_{\mathrm{E}(\mathcal{X})} = \{h \in \mathrm{Arr}\,\mathcal{X} : hh = h\}$. Obviously, $E_{\mathrm{E}(\mathcal{X})} \subseteq \bigcup_{i \in \mathrm{Ob}\,\mathcal{X}} \mathcal{X}(i,i)$. A category $\mathcal{X}$ is an inverse category if and only if $\mathrm{E}(\mathcal{X})(i,i)$ is a semilattice for every object i, and, for each arrow $e \in \mathcal{X}(i,j)$, there exists an arrow $f \in \mathcal{X}(j,i)$ such that $efe = e$. Thus, given an inverse category $\mathcal{X}$, $\mathrm{E}(\mathcal{X})$ is a subcategory of $\mathcal{X}$, and we define a relation $\leq$ on $\mathcal{X}$ as follows: for any $e, f \in \mathrm{Arr}\,\mathcal{X}$, let $e \leq f$ if $e = fh$ for some $h \in \mathrm{Arr}\,\mathrm{E}(\mathcal{X})$. The relation $\leq$ is a partial order on $\mathrm{Arr}\,\mathcal{X}$ called the *natural partial order* on $\mathcal{X}$, and it is compatible with multiplication. Note that the natural partial order is trivial if and only if $\mathcal{X}$ is a groupoid.

2.2.3 Categories acted upon by groups

Groups acting on graphs come up in several areas. For instance, Bass–Serre theory analyzes groups through their actions on trees. One of the earliest results in the framework is that a group is free if and only if it acts *freely* on a tree, which also yields a proof of the Nielsen–Schreier theorem. In this section, we use groups acting on graphs and categories in order to construct inverse monoids. These results can be found in [14].

Let G be a group and Δ a graph. We say that G *acts on* Δ *(on the left)* if, for every $g \in G$, and for every vertex i and edge e in Δ, a vertex ${}^g i$ and an edge ${}^g e$ is given such that the following are satisfied for any $g, h \in G$ and any $i \in V_\Delta$, $e \in E_\Delta$:

$$ {}^1 i = i, \quad {}^h({}^g i) = {}^{hg} i, \quad {}^1 e = e, \quad {}^h({}^g e) = {}^{hg} e, $$

$$ \iota\, {}^g e = {}^g \iota e, \quad \tau\, {}^g e = {}^g \tau e. $$

An action of G on Δ induces an action on the paths and an action on the subgraphs of Δ in a natural way: if $g \in G$, $i \in V_\Delta$ and $p = e_1 e_2 \cdots e_n$ is a non-empty path, then we put

$$ {}^g p = {}^g e_1 {}^g e_2 \cdots {}^g e_n, $$

and for an empty path, let ${}^g 1_i = 1_{{}^g i}$. For any subgraph X of Δ, define ${}^g\mathrm{X}$ to be the subgraph whose sets of vertices and edges are $\{{}^g i : i \in V_\mathrm{X}\}$ and $\{{}^g e : e \in E_\mathrm{X}\}$ respectively, in particular, ${}^g\emptyset_i = \emptyset_{{}^g i}$. The action of G on Δ can be extended to $\overline{\Delta}$ also in a natural way by setting ${}^g e' = ({}^g e)'$ for every $e \in E_\Delta$. It is easy to check that the equality $\langle {}^g p \rangle = {}^g \langle p \rangle$ holds for every path p on $\overline{\Delta}$.

One example we have already seen is a group acting on its own Cayley graph by translations. In the case of the free group, the induced action on subgraphs is the action used in the construction of free inverse monoids.

By an *action of a group on a category* $\mathcal{X}$ we mean an action of G on the graph $\mathcal{X}$ which has the following additional properties: for any object i and any pair of consecutive arrows e, f, we have

$$^{g}1_i = 1_{^{g}i}, \quad ^{g}(ef) = {^{g}e} \cdot {^{g}f}.$$

In particular, if $\mathcal{X}$ is a one-object category, that is, a monoid, then this defines an action of a group on a monoid. We also mention that if Δ is a graph acted upon by a group G, then the induced action on the paths defines an action of G on the free category $\overline{\Delta}^*$ with involution on Δ. Note that if $\mathcal{X}$ is an inverse category, then $^{g}(e^{-1}) = (^{g}e)^{-1}$ for every $g \in G$ and every arrow e. We say that G acts *transitively* on $\mathcal{X}$ if, for any objects i, j, there exists $g \in G$ with $j = {^{g}i}$, and that G acts on $\mathcal{X}$ *without fixed points* (or freely) if, for any $g \in G$ and any object i, we have $^{g}i = i$ only if $g = 1$. Note that if G acts transitively on $\mathcal{X}$, then the local monoids $\mathcal{X}(i,i)$ $(i \in \mathrm{Ob}\,\mathcal{X})$ are all isomorphic.

Let G be a group acting on a category $\mathcal{X}$. This action determines a category $\mathcal{X}/G$ in a natural way: the objects of $\mathcal{X}/G$ are the orbits of the objects of $\mathcal{X}$, the orbit of i denoted by, as usual, $^{G}i = \{^{g}i : g \in G\}$, and, for every pair $^{G}i, {^{G}j}$ of objects, the $(^{G}i, {^{G}j})$-arrows are the orbits of the (i', j')-arrows of $\mathcal{X}$ where $i' \in {^{G}i}$ and $j' \in {^{G}j}$. The product of consecutive arrows $\tilde{e}, \tilde{f}$ is also defined in a natural way, namely, by considering the orbit of a product ef where e, f are consecutive arrows in $\mathcal{X}$ such that $e \in \tilde{e}$ and $f \in \tilde{f}$. Note that if G acts transitively on $\mathcal{X}$, then $\mathcal{X}/G$ is a one-object category, that is, a monoid. The properties below are proven in [14, Propositions 3.11, 3.14].

Result 2.2.1. *Let G be a group acting transitively and without fixed points on an inverse category $\mathcal{X}$.*

(1) The monoid $\mathcal{X}/G$ is inverse, and it is isomorphic, for every object i, to the monoid $(\mathcal{X}/G)_i$ defined on the set $\{(e,g) : g \in G$ and $e \in \mathcal{X}(i, {^{g}i})\}$ by the multiplication

$$(e,g)(f,h) = (e \cdot {^{g}f}, gh).$$

(2) If $\mathcal{X}$ is connected and it is locally a semilattice, then $\mathcal{X}/G$ is an E-unitary inverse monoid. Moreover, the greatest group homomorphic image of $\mathcal{X}/G$ is G, and its semilattice of idempotents is isomorphic to $\mathcal{X}(i,i)$ for any object i.

(3) If $\mathcal{X}$ is connected, and it is locally in a group variety $\mathbf{U}$, then $\mathcal{X}/G$ is a group which is an extension of $\mathcal{X}(i,i) \in \mathbf{U}$ by G for any object i.

Example 2.2.2. The multiplication in point (1) resembles that seen in Munn's construction, and that is not a coincidence: if $\mathcal{X}$ is the inverse category with the object set $\mathrm{FG}(A)$

and with (i,j)-arrows of the form (i,X,j), where X is a connected subgraph of the Cayley graph of $\mathrm{FG}(A)$ containing vertices i and j, and multiplication is given by

$$(i,\mathrm{X},j)(j,\mathrm{Y},k) = (i,\mathrm{X}\cup\mathrm{Y},k),$$

then $\mathrm{FIM}(A)$ is nothing but $\mathcal{X}/\,\mathrm{FG}(A)$.

For our later convenience, note that the inverse of an element can be obtained in $(\mathcal{X}/G)_i$ in the following manner:

$$(e,g)^{-1} = ({}^{g^{-1}}e^{-1}, g^{-1}).$$

Notice that if a group G acts on an inverse category transitively and without fixed points, then $\mathrm{Ob}\,\mathcal{X}$ is in one-to-one correspondence with G. In the sequel we consider several categories of this kind which have just G as its set of objects. For these categories, we identify $\mathcal{X}/G$ with $(\mathcal{X}/G)_1$.

Any E-unitary inverse monoid can be obtained in the way described in Result 2.2.1(2). To see that, let M be an arbitrary E-unitary inverse monoid, and denote the group M/σ by G. Define the category $\mathcal{I}_M$ in the following way: its set of objects is G, its set of (i,j)-arrows is

$$\mathcal{I}_M(i,j) = \{(i,m,j)\in G\times M\times G : i\cdot m\sigma = j\}\ (i,j\in G),$$

and the product of consecutive arrows $(i,m,j)\in\mathcal{I}_M(i,j)$ and $(j,n,k)\in\mathcal{I}_M(j,k)$ is defined by the rule

$$(i,m,j)(j,n,k) = (i,mn,k).$$

It is easy to see that an arrow (i,m,j) is idempotent if and only if m is idempotent, and since M is E-unitary, this is if and only if $i=j$. Moreover, we have $(i,m,j)^{-1} = (j,m^{-1},i)$ for every arrow (i,m,j). The natural partial order on $\mathcal{I}_M$ is the following: for any arrows $(i,m,j),(k,n,l)$, we have $(i,m,j)\le(k,n,l)$ if and only if $i=k,\ j=l$ and $m\le n$. We remark that $\mathcal{I}_M$ is nothing but the the derived category of the natural homomorphism $\sigma^\natural\colon M\to G$, see [22].

The group G acts naturally on $\mathcal{I}_M$ as follows: ${}^g i = gi$ and ${}^g(i,m,j) = (gi,m,gj)$ for every $g\in G$ and $(i,m,j)\in\mathrm{Arr}\,\mathcal{I}_M$.

The category $\mathcal{I}_M$ and the action of G on it has the following properties [14, Proposition 3.12].

Result 2.2.3. *The category $\mathcal{I}_M$ is a connected inverse category which is locally a semilattice. The group G acts transitively and without fixed points on $\mathcal{I}_M$, and M is isomorphic to $\mathcal{I}_M/G$.*

The isomorphism in the proof is given by $m \mapsto (1, m, m\sigma)$.

2.3 A reformulation of the F-inverse cover problem

2.3.1 Margolis–Meakin expansions

Let G be an A-generated group where $A\epsilon_G \subseteq G \setminus \{1\}$. The *Margolis–Meakin expansion* $M(G)$ *of* G (see [13]) generalizes Munn's construction to arbitrary Cayley graphs. It is defined in the following way: consider the set of all pairs (X, g) where $g \in G$ and X is a finite connected subgraph of the Cayley graph Γ of G containing the vertices 1 and g, and define a multiplication on this set by the rule

$$(\mathrm{X}, g)(\mathrm{Y}, h) = (\mathrm{X} \cup {}^{g}\mathrm{Y}, gh).$$

Then $M(G)$ is an A-generated E-unitary inverse monoid with $\epsilon_{M(G)} \colon A \to M(G)$, $a \mapsto (\langle e_a \rangle, a) = (e_a, a)$ (i.e., for brevity, we identify $\langle e \rangle$ with e for every edge e in Γ), where the identity element is $(\emptyset_1, 1)$ and $(\mathrm{X}, g)^{-1} = ({}^{g^{-1}}\mathrm{X}, g^{-1})$ for every $(\mathrm{X}, g) \in M(G)$. Margolis–Meakin expansions are useful in part because they also have a universal property similar to that of free inverse monoids: A-generated, E-unitary inverse monoids over the group G are homomorphic images of $M(G)$, moreover, an A-generated inverse monoid has an E-unitary cover over the group G if and only if it is a homomorphic image of $M(G)$.

By definition, the arrows in $\mathcal{I}_{M(G)}(i, j)$ are $(i, (\mathrm{X}, g), j)$ where $(\mathrm{X}, g) \in M(G)$ and $ig = j$ in G. Therefore $\mathcal{I}_{M(G)}/G = (\mathcal{I}_{M(G)}/G)_1$ consists of the pairs $((1, (\mathrm{X}, g), g), g)$ which can be identified with (X, g), and this identification is the isomorphism involved in Result 2.2.3. Moreover, notice that the assignment $(i, (\mathrm{X}, g), j) \mapsto (i, {}^{i}\mathrm{X}, j)$ is a bijection from $\mathcal{I}_{M(G)}(i, j)$ onto the set of all triples (i, X, j) where X is a finite connected subgraph of Γ and $i, j \in V_{\mathrm{X}}$. Thus $\mathcal{I}_{M(G)}$ can be identified with the category where the hom-sets are the latter sets, and the multiplication is the following:

$$(i, \mathrm{X}, j)(j, \mathrm{Y}, k) = (i, \mathrm{X} \cup \mathrm{Y}, k).$$

We apply Result 2.2.1 to introduce further structures with some sort of universal property. Recall the notion of a free category $\overline{\Delta}^*$ with involution over the graph Δ. The (i, j)-arrows of $\overline{\Delta}^*$ are the (i, j)-paths in the graph $\overline{\Delta}$, which can be regarded as words in the free monoid $\overline{E}_{\Delta}^{*}$ with involution. Taking that analogy a step further — if we 'evaluate' the (i, j)-paths not in the free monoid with involution, but in a variety of inverse monoids (in particular, of groups), then we are led to the notion of the free inverse category (in particular, free groupoid) in that variety. In the following paragraphs, we introduce this construction precisely in the case when Δ is a Cayley graph.

Consider an inverse monoid (in particular, a group) variety $\mathbf{U}$ and a graph Γ. Denote the relatively free inverse monoid in $\mathbf{U}$ on E_Γ by $F_{\mathbf{U}}(E_\Gamma)$. Any path in $\overline{\Gamma}$, regarded as a word in $\overline{E_\Gamma}^{*}$, determines an element of $F_{\mathbf{U}}(E_\Gamma)$, which is denoted by $[p]_{\mathbf{U}}$, as introduced before.

The *free* $g\mathbf{U}$*-category on* Γ denoted by $F_{g\mathbf{U}}(\Gamma)$, as introduced in [22], is given as follows: its set of objects is V_Γ, and, for any pair of objects i, j, the set of (i, j)-arrows is

$$F_{g\mathbf{U}}(\Gamma)(i, j) = \{(i, [p]_{\mathbf{U}}, j) : p \text{ is an } (i, j)\text{-path in } \overline{\Gamma}\},$$

and the product of consecutive arrows is defined by

$$(i, [p]_{\mathbf{U}}, j)(j, [q]_{\mathbf{U}}, k) = (i, [pq]_{\mathbf{U}}, k).$$

Obviously, the category $F_{g\mathbf{U}}(\Gamma)$ is an inverse category (in particular, a groupoid), and the inverse of an arrow is obtained as follows:

$$(i, [p]_{\mathbf{U}}, j)^{-1} = (j, [p]_{\mathbf{U}}^{-1}, i) = (j, [p']_{\mathbf{U}}, i).$$

Moreover, $F_{g\mathbf{U}}(\Gamma)$ is Γ-generated by the map $\epsilon_{F_{g\mathbf{U}}(\Gamma)} \colon \Gamma \to F_{g\mathbf{U}}(\Gamma)$, $e \mapsto (\iota e, [e]_{\mathbf{U}}, \tau e) = (\iota e, e, \tau e)$ for every edge e in Γ (i.e., as usual, we identify $[e]_{\mathbf{U}}$ with e in $F_{\mathbf{U}}(E_\Gamma)$). If, for example, $\mathbf{U} = \mathbf{Sl}$, the variety of semilattices, and Γ is the Cayley graph of $FG(A)$, then $[p]_{\mathbf{Sl}} = \langle p \rangle$, and $F_{g\mathbf{Sl}}(\Gamma)$ is the category described in Example 2.2.2.

Suppose $\mathbf{U}$ is a group variety, and Γ is the Cayley graph of an A-generated group G. Notice that the action of G on Γ extends to an action of G on $F_{g\mathbf{U}}(\Gamma)$ by ${}^{g}(i, [p]_{\mathbf{U}}, j) = (gi, [{}^{g}p]_{\mathbf{U}}, gj)$, and this action, like the action on Γ, is transitive and has no fixed points. Furthermore, $F_{g\mathbf{U}}(\Gamma)$ is connected since Γ is connected. Thus Result 2.2.1(3) implies that $F_{g\mathbf{U}}(\Gamma)/G$ is a group which is an extension of a member of $\mathbf{U}$ by G. Define the semidirect product $F_{\mathbf{U}}(E_\Gamma) \rtimes G$, where the action of G is the one extended from its action on Γ. It is straightforward to see by Result 2.2.1(1) that the elements of $F_{g\mathbf{U}}(\Gamma)/G = (F_{g\mathbf{U}}(\Gamma)/G)_1$ are exactly the pairs $([p]_{\mathbf{U}}, g) \in F_{\mathbf{U}}(E_\Gamma) \rtimes G$, where p is a $(1, g)$-path in $\overline{\Gamma}$, hence $F_{g\mathbf{U}}(\Gamma)/G$ is a subgroup in the semidirect product $F_{\mathbf{U}}(E_\Gamma) \rtimes G$. Moreover, $F_{g\mathbf{U}}(\Gamma)/G$ is generated by the subset $\{(e_a, a\epsilon_G) : a \in A\}$, and so it is A-generated with $\epsilon_{F_{g\mathbf{U}}(\Gamma)/G} \colon A \to F_{g\mathbf{U}}(\Gamma)/G, a \mapsto (e_a, a\epsilon_G)$. It is well known (cf. the Kaloujnine–Krasner theorem [11]) that $F_{g\mathbf{U}}(\Gamma)/G$ is the 'most general' A-generated group which is an extension of a member of $\mathbf{U}$ by G, that is, it has the universal property that, for each such extension K with $\epsilon_K\colon A \to K$, there exists a surjective homomorphism $\varphi\colon F_{g\mathbf{U}}(\Gamma)/G \to K$ such that $\epsilon_{F_{g\mathbf{U}}(\Gamma)/G}\varphi = \epsilon_K$. For brevity, we denote the group $F_{g\mathbf{U}}(\Gamma)/G$ later on by $G^{\mathbf{U}}$.

2.3.2 Dual premorphisms

A *dual premorphism* $\psi\colon M \to N$ between inverse monoids is a map satisfying $(m\psi)^{-1} = m^{-1}\psi$ and $(mn)\psi \geq m\psi \cdot n\psi$ for all m, n in M (such maps are called dual prehomomorphisms in [12] and prehomomorphisms in [17]). In particular, if M and N are A-generated and $\epsilon_M\psi = \epsilon_N$, then ψ is called a *canonical dual premorphism.* An important class of dual premorphisms from groups to an inverse monoid M is closely related to F-inverse covers of M, as stated in the following well-known result ([17, Theorem VII.6.11]):

Result 2.3.1. *Let H be a group and M be an inverse monoid. If $\psi\colon H \to M$ is a dual premorphism such that*

$$\textit{for every } m \in M, \textit{ there exists } h \in H \textit{ with } m \leq h\psi, \tag{2.3.1}$$

then

$$F = \{(m, h) \in M \times H : m \leq h\psi\}$$

is an inverse submonoid in the direct product $M \times H$, and it is an F-inverse cover of M over H. Conversely, up to isomorphism, every F-inverse cover of M over H can be so constructed.

In the proof of the converse part of Result 2.3.1, the following dual premorphism $\psi\colon F/\sigma \to M$ is constructed for an inverse monoid M, an F-inverse monoid F, and a surjective idempotent-separating homomorphism $\varphi\colon F \to M$: for every $h \in F/\sigma$, let $h\psi = m_h\varphi$, where m_h denotes the maximum element of the σ-class h. It is important to notice that, more generally, this construction gives a dual premorphism with property (2.3.1) for any surjective homomorphism $\varphi\colon F \to M$. In the sequel, we call this map ψ the *dual premorphism induced by φ.*

Notice that, for every group H and inverse monoids M, N, the product of a dual premorphism $\psi\colon H \to M$ with property (2.3.1) and a surjective homomorphism $\varphi\colon M \to N$ is a dual premorphism from H to N with property (2.3.1). As a consequence, notice that if an inverse monoid M has an F-inverse cover over a group H, then so do its homomorphic images.

Dual premorphisms can be defined for inverse categories analogously: it is a graph morphism $\psi\colon \mathcal{X} \to \mathcal{Y}$ such that $1_i\psi = 1_{i\psi}$, $(e^{-1})\psi = (e\psi)^{-1}$ and $(ef)\psi \geq e\psi \cdot f\psi$ for any object i and any consecutive arrows e, f in $\mathcal{X}$.

2.3.3 A graph condition

We are ready to describe the graph condition Auinger and Szendrei have introduced in their paper [2] as a reformulation of the F-inverse cover problem. Their key step is the assertion that every finite inverse monoid admits a finite F-inverse cover if and only if, for every finite connected graph Γ, there exist a locally finite group variety $\mathbf{U}$ and a dual premorphism $\psi\colon F_{g\mathbf{U}}(\Gamma) \to F_{g\mathbf{Sl}}(\Gamma)$ with $\psi|\Gamma = \mathrm{id}_\Gamma$.

We provide a quick run-through of the proof. The first observation is that it is sufficient to try to find finite F-inverse covers for finite Margolis–Meakin expansions, as every inverse monoid is a homomorphic image of one. According to Result 2.3.1, a Margolis–Meakin expansion $M(G)$ has a finite F-inverse cover if and only if there is a dual premorphism $H \to M(G)$ for a finite group H, with property (2.3.1). The second observation is that if G is A-generated, then H can be chosen to be A-generated, and the dual premorphism to be canonical. A canonical dual premorphism $H \to M(G)$ yields a canonical homomorphism $H \to G$, hence H is an A-generated extension of some group K by G. The 'most general' candidates for such a group H are the ones of the form $G^{\mathbf{U}}$ (see Subsection 2.3.1), where the only restriction imposed on K is that it belongs to the variety $\mathbf{U}$. The group $G^{\mathbf{U}}$ is finite if and only if G is finite (which it is, by assumption) and $\mathbf{U}$ is locally finite group variety. Hence the question boils down to finding a locally finite group variety $\mathbf{U}$ for every A-generated group G such that there is a canonical dual premorphism $G^{\mathbf{U}} \to M(G)$, and since $G^{\mathbf{U}} = F_{g\mathbf{U}}(\Gamma)/G$ and $M(G) = F_{g\mathbf{Sl}}(\Gamma)/G$, this translates to finding a canonical dual premorphism $\psi\colon F_{g\mathbf{U}}(\Gamma) \to F_{g\mathbf{Sl}}(\Gamma)$.

Now fix a connected graph Γ and a group variety $\mathbf{U}$. We assign to each arrow x of $F_{g\mathbf{U}}(\Gamma)$ two sequences of finite subgraphs of Γ as follows: let

$$C_0(x) = \bigcap\{\langle p\rangle : (\iota p, [p]_{\mathbf{U}}, \tau p) = x\}, \tag{2.3.2}$$

and let $P_0(x)$ be the connected component of $C_0(x)$ containing ιx. If $C_n(x), P_n(x)$ are already defined for all x, then put

$$C_{n+1}(x) = \bigcap\{P_n(x_1) \cup \cdots \cup P_n(x_k) : k \in \mathbb{N}, x_1 \cdots x_k = x\},$$

and again, let $P_{n+1}(x)$ be the connected component of $C_{n+1}(x)$ containing ιx.

It is easy to see that

$$C_0(x) \supseteq P_0(x) \supseteq \cdots \supseteq C_n(x) \supseteq P_n(x) \supseteq C_{n+1}(x) \supseteq P_{n+1}(x) \supseteq \cdots$$

for all x and n. We define $P(x)$ to be $\bigcap_{n=0}^{\infty} P_n(x)$, which is a connected subgraph of Γ containing ιx. According to [2, Lemma 3.1], there exists a dual premorphism $\psi\colon F_g\mathbf{U}(\Gamma) \to F_g\mathbf{Sl}(\Gamma)$ with $\psi|\Gamma = \mathrm{id}_\Gamma$ if and only if $\tau x \in P(x)$ for all x, and in this case, the assignment $x \mapsto (\iota x, P(x), \tau x)$ gives such a dual premorphism. If $\tau x \notin P(x)$ for some $x = (\iota p, [p]_\mathbf{U}, \tau p)$, then we call p a *breaking path* over $\mathbf{U}$.

The main result [2, Theorem 5.1] is the following:

Result 2.3.2. *The following assertions are equivalent.*

(1) Each finite inverse monoid has an F-inverse cover.

(2) For each finite connected graph Γ, there exists a locally finite group variety $\mathbf{U}$ for which there is a canonical dual premorphism $F_g\mathbf{U}(\Gamma) \to F_g\mathbf{Sl}(\Gamma)$.

(3) For each finite connected graph Γ, there exists a locally finite group variety $\mathbf{U}$ such that, for each arrow x of $F_g\mathbf{U}(\Gamma)$, each of the graphs $P_k(x)$ $(k \geq 1)$ contains the vertex τx.

(4) There exists a prime p such that, for each $n \geq 1$, the inverse monoid $M(C_p^n)$ has a finite F-inverse cover (where C_p denotes the cyclic group of order n).

In [2], $C_0(x)$ is incorrectly defined to be the graph spanned by the $\mathbf{U}$-content of x together with ιx. From the proof of [2, Lemma 3.1] (see the inclusion $\mu(x\psi) \subseteq C_0(x)$), it is clear that the definition of $C_0(x)$ needed is the one in (2.3.2). The following proposition states that in the cases crucial for the main result [2, Theorem 5.1], i.e., where Γ is the Cayley graph of a finite group, these two definitions are equivalent in the sense that $P_0(x)$, and so the sequence $P_n(x)$ does not depend on which definition we use. For our later convenience, let $\hat{C}_0(x)$ denote the graph which is the union of the $\mathbf{U}$-content of x and ιx.

Lemma 2.3.3. *If Γ is two-edge-connected, then for any arrow x of $F_g\mathbf{U}(\Gamma)$, the subgraphs $C_0(x)$ and $\hat{C}_0(x)$ can only differ in isolated vertices (distinct from ιx and τx).*

Proof. Let x be an arrow of $F_g\mathbf{U}(\Gamma)$. It is clear that $\hat{C}_0(x) \subseteq C_0(x)$. For the converse, put $x = (\iota p, [p]_\mathbf{U}, \tau p)$, and suppose e is an edge of $\langle p \rangle$ such that $e \notin \hat{C}_0(x)$. Let s_e be a $(\iota e, \tau e)$-path in $\overline{\Gamma}$ not containing e — such a path exists since Γ is two-edge-connected. Let $p_{e\to s_e}$ be the path obtained from p by replacing all occurrences of e by s_e. Then $p \equiv_\mathbf{U} p_{e\to s_e}$, and $e \notin \langle p_{e\to s_e} \rangle$, hence $e \notin C_0(x)$, which completes the proof. ☐

Remark 2.3.4. We remark that the condition of Γ being two-edge-connected is necessary in Lemma 2.3.3, that is, when Γ is not two-edge-connected, the subgraphs $C_0(x)$ and $\hat{C}_0(x)$ can in fact be different. Put, for example, $\mathbf{U} = \mathbf{Ab}$, the variety of Abelian groups, and let e be an edge of Γ for which $\Gamma\backslash\{e\}$ is disconnected. Let $p = ese'$ be a path in $\overline{\Gamma}$, where $s \not\equiv_{\mathbf{Ab}} 1$ and e, e' do not occur in s. Then the subgraph spanned by the **Ab**-content of p does not contain e, whereas any path q which is co-terminal with and **Ab**-equivalent to p must contain the edge e, as there is no other $(\iota e, \tau e)$-path in $\overline{\Gamma}$.

For a group variety $\mathbf{U}$, we say that a *graph* Γ *satisfies property* $(S_{\mathbf{U}})$, or Γ is $(S_{\mathbf{U}})$ for short, if $\tau x \in P(x)$ holds for any arrow x of $F_g\mathbf{U}(\Gamma)$. By Result 2.3.2, each finite inverse monoid has a finite F-inverse cover if and only if each finite connected graph is $(S_{\mathbf{U}})$ for some locally finite group variety $\mathbf{U}$. This property $(S_{\mathbf{U}})$ for finite connected graphs is our topic for the next section.

We recall that by [2, Lemmas 4.1 and 4.2], the following holds.

Lemma 2.3.5. *If a graph* Γ *is* $(S_{\mathbf{U}})$ *for some group variety* $\mathbf{U}$*, then so is any redirection of* Γ*, and any subgraph of* Γ*.*

However, we remark that the lemma following these observations in [2], namely Lemma 4.3 is false. It states that if a simple graph Γ is $(S_{\mathbf{U}})$, then so is any graph obtained from Γ by adding parallel edges (where both "simple" and "parallel" are meant in the undirected sense). The main result of Chapter 3, Theorem 3.2.1 yields counterexamples.

Lemma 2.3.6. *If* $\mathbf{U}$ *and* $\mathbf{V}$ *are group varieties for which* $\mathbf{U} \subseteq \mathbf{V}$*, then* $(S_{\mathbf{U}})$ *implies* $(S_{\mathbf{V}})$*.*

Proof. Suppose Γ is $(S_{\mathbf{U}})$, let p be any path in $\overline{\Gamma}$. Put $x^{\mathbf{U}} = (\iota p, [p]_{\mathbf{U}}, \tau p) \in F_g\mathbf{U}(\Gamma)$, and similarly let $x^{\mathbf{V}} = (\iota p, [p]_{\mathbf{V}}, \tau p) \in F_g\mathbf{V}(\Gamma)$. Since $\mathbf{U} \subseteq \mathbf{V}$, we have $C_0(x^{\mathbf{U}}) \subseteq C_0(x^{\mathbf{V}})$. Also, since $q \equiv_{\mathbf{V}} q_1 \cdots q_n$ implies $q \equiv_{\mathbf{U}} q_1 \cdots q_n$, we obtain $P_n(x^{\mathbf{U}}) \subseteq P_n(x^{\mathbf{V}})$ by induction. Since $\tau p \in P_n(x^{\mathbf{U}})$ by assumption, this yields $\tau p \in P_n(x^{\mathbf{V}})$, that is, Γ is $(S_{\mathbf{V}})$. □

Chapter 3

F-inverse covers of Margolis–Meakin expansions

This chapter contains the examination of the graph condition $(S_{\mathbf{U}})$ introduced in the previous chapter, as well as the implications of some of our results to F-inverse covers of Margolis–Meakin expansions.

3.1 Forbidden minors

In this section, we prove that, given a group variety $\mathbf{U}$, the class of graphs satisfying $(S_{\mathbf{U}})$ can be described by so-called forbidden minors. Forbidden minors are widely used in mathematics to characterize graphs with a certain property. The most well-known example is Kuratowski's theorem, which characterizes planar graphs as graphs which do not contain K_5, the complete graph on five vertices and $K_{3,3}$, the utility graph as minors.

Let Γ be a graph and let e be a (u, v)-edge of Γ such that $u \neq v$. The operation which removes e and simultaneously merges u and v to one vertex is called *edge-contraction*. We call Δ a *minor* of Γ if it can be obtained from Γ by edge-contraction, omitting vertices and edges, and redirecting edges.

Proposition 3.1.1. *Suppose Γ and Δ are graphs such that Δ is a minor of Γ. Then, if Δ is non-$(S_{\mathbf{U}})$, so is Γ.*

Proof. By Lemma 2.3.5, adding edges and vertices to, or redirecting some edges of a graph does not change the fact that it is non-$(S_{\mathbf{U}})$. Therefore let us suppose that Δ is obtained from Γ by contracting an edge e for which $\iota e \neq \tau e$. Let $x_1, \ldots, x_n$ be the edges of Γ having ιe as their terminal vertex. For a path p in $\overline{\Delta}$, let p_{+e} denote the path in $\overline{\Gamma}$ obtained by replacing all occurrences of x_j $(j = 1, \ldots, n)$ by $x_j e$ (and all occurrences of x'_j by $e'x'_j$). Similarly, for a subgraph Ξ of Δ, let Ξ_{+e} denote the subgraph of Γ obtained from Ξ by taking its preimage under the edge-contraction containing the edge e if Ξ contains some x_j $(j = 1, \ldots, n)$, and its preimage without e otherwise. Obviously, we have $\langle p_{+e} \rangle = \langle p \rangle_{+e}$ for any path p in $\overline{\Delta}$.

Note that if p is a path in $\overline{\Delta}$ traversing the edges $f_1, \ldots, f_k$, then p_{+e}, considered as a word in $\overline{\{e, f_1, \ldots, f_k\}}^*$, is obtained from the word p by substituting $(x_j e)$ for x_j $(j = 1, \ldots, n)$, and leaving the other edges unchanged. Putting $x = (\iota p, [p]_{\mathbf{U}}, \tau p)$ and $x_{+e} = (\iota p_{+e}, [p_{+e}]_{\mathbf{U}}, \tau p_{+e})$, this implies $(C_0(x))_{+e} \supseteq C_0(x_{+e})$ for any path p is $\overline{\Delta}$. Moreover, we also see that, for any paths $q, q_1, \ldots, q_k$ in $\overline{\Delta}$, we have $q \equiv_{\mathbf{U}} q_1 \cdots q_n$ if and only if $q_{+e} \equiv_{\mathbf{U}} (q_1)_{+e} \cdots (q_n)_{+e}$. Using that for any subgraph $\Xi \subseteq \Delta$, the connected components of Ξ and Ξ_{+e} are in one-one correspondence, an induction shows that $(P_n(x))_{+e} \supseteq P_n(x_{+e})$ for every n. In particular, $P_n(x)$ contains τp if and only if $(P_n(x))_{+e}$ contains τp_{+e}. Therefore if p is a breaking path in Δ over $\mathbf{U}$, then $\tau p_{+e} \notin (P_n(x))_{+e}$ and hence $\tau p_{+e} \notin P_n(x_{+e})$, that is, p_{+e} is a breaking path in Γ over $\mathbf{U}$, which proves our statement. □

By the previous proposition, the class of all graphs containing a breaking path over $\mathbf{U}$ (that is, of all non-$(S_{\mathbf{U}})$ graphs) is closed upwards in the minor ordering, hence, it is determined by its minimal elements. This enables us to characterize $(S_{\mathbf{U}})$-graphs by these minimal elements — these are precisely the graphs which are forbidden minors for graphs with property $(S_{\mathbf{U}})$. According to the theorem of Robertson and Seymour [18], there is no infinite anti-chain in the minor ordering, that is, the set of minimal non-$(S_{\mathbf{U}})$ graphs must be finite.

These observations are summarized in the following theorem:

Theorem 3.1.2. *For any group variety* $\mathbf{U}$*, there exists a finite set of graphs* $\Gamma_1, \ldots, \Gamma_n$ *such that the graphs containing a breaking path over* $\mathbf{U}$ *are exactly those having one of* $\Gamma_1, \ldots, \Gamma_n$ *as a minor.*

By Lemma 2.3.6, if $\mathbf{U}$ and $\mathbf{V}$ are group varieties with $\mathbf{U} \subseteq \mathbf{V}$, the forbidden minors for $\mathbf{U}$ are smaller (in the minor ordering) then the ones for $\mathbf{V}$.

The next statement contains simple observations regarding the nature of forbidden minors.

Proposition 3.1.3. *For any group variety* $\mathbf{U}$*, the set of minimal non-*$(S_{\mathbf{U}})$ *graphs are two-edge-connected graphs without loops.*

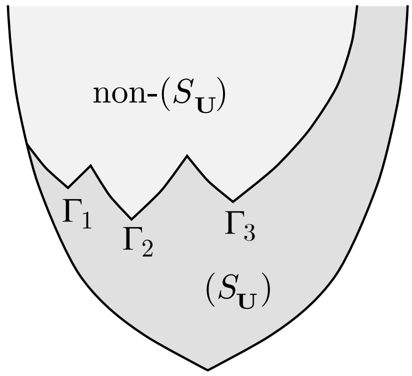


Figure 3.1.1. The partially ordered set of graphs and the forbidden minors

Proof. We show that if Γ is a non-$(S_{\mathbf{U}})$ graph which has loops or is not two-edge-connected, then there exists a graph below Γ in the minor ordering which is also non-$(S_{\mathbf{U}})$. Indeed, suppose that Γ has a loop e, and take $\Gamma\backslash\{e\}$. For a path p in $\overline{\Gamma}$, let p_{-e} denote the corresponding path in $\overline{\Gamma}$ obtained by omitting all occurrences of e, and for an arrow $x = (\iota p, [p]_{\mathbf{U}}, \tau p) \in F_g\mathbf{U}(\Gamma)$, put $x_{-e} = (\iota p, [p_{-e}]_{\mathbf{U}}, \tau p) \in F_g\mathbf{U}(\Gamma\backslash\{e\})$. Then it is easy to see by induction that $C_n(x_{-e}) \subseteq C_n(x)\backslash\{e\}$ and $P_n(x_{-e}) \subseteq P_n(x)\backslash\{e\}$ for every x and n, and hence $\tau p \in P_n(x_{-e})$ implies $\tau p \in P_n(x)\backslash\{e\}$.

Now suppose Γ is not two-edge-connected, that is, there is a (u, v)-edge e of Γ for which $\Gamma\backslash\{e\}$ is disconnected. Then let $\Gamma_{u=v}$ denote the graph which we obtain from Γ by contracting e. For a path p in $\overline{\Gamma}$, let $p_{u=v}$ denote the path in $\overline{\Gamma_{u=v}}$ which we obtain by omitting all occurrences of e from p, and for an arrow $x = (\iota p, [p]_{\mathbf{U}}, \tau p) \in F_g\mathbf{U}(\Gamma)$, put $x_{u=v} = (\iota p_{u=v}, [p_{u=v}]_{\mathbf{U}}, \tau p_{u=v}) \in F_g\mathbf{U}(\Gamma_{u=v})$. Observe that for co-terminal paths s, t in $\overline{\Gamma}$, $s \equiv_{\mathbf{U}} t$ implies $s_{u=v} \equiv_{\mathbf{U}} t_{u=v}$. This, by induction yields $C_n(x_{u=v}) \subseteq C_n(x)_{u=v}$ and $P_n(x_{u=v}) \subseteq P_n(x)_{u=v}$ for all n, and hence $\tau p \in P_n(x_{u=v})$ implies $\tau p \in P_n(x)_{u=v}$. □

3.2 F-inverse covers via Abelian groups

In this section, we describe the forbidden minors (in the sense of the previous section) for all non-trivial varieties of Abelian groups. Recall that the variety of all Abelian groups is denoted by **Ab**.

Theorem 3.2.1. *A graph contains a breaking path over* **Ab** *if and only if its minors contain at least one of the graphs in Figure 3.2.1.*

Proof. First, suppose Γ is a graph which does not have either graph in Figure 3.2.1 as a minor. Then Γ's connected components are either a cycles of length n for some $n \in \mathbb{N}_0$ with

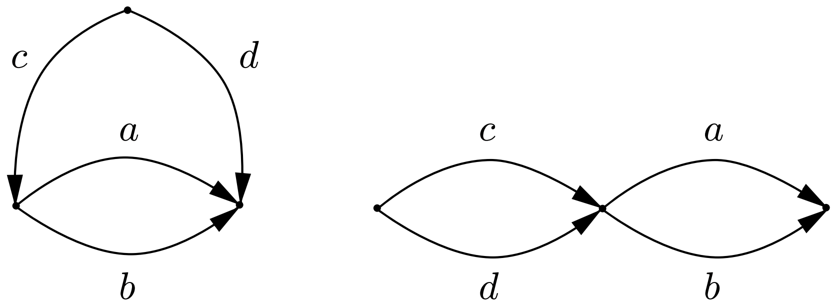


Figure 3.2.1. The forbidden minors for **Ab**

possibly some trees and loops attached, or graphs with at most 2 vertices. According to Proposition 3.1.3, Γ contains a breaking path if and only if its two-edge-connected minors do, which are then either cycles Γ_n of length n, or two-edge-connected graphs on at most 2 vertices. It is easy to see that in both cases, for any path p, the **Ab**-content $\hat{C}_0(x)$ with $x = (\iota p, [p]_{\mathbf{Ab}}, \tau p)$ is connected, therefore by Lemma 2.3.3, these graphs do not contain a breaking path over **Ab**.

For the converse part, we prove that both graphs in Figure 3.2.1 contain a breaking path over **Ab** — namely, the path a. For brevity, denote $\iota a, \tau a$ and ιc by u, v and w respectively, and put $x = (u, [a]_{\mathbf{Ab}}, v)$. Since both graphs are two-edge-connected, Lemma 2.3.3 implies that $C_0(x)$ and the **Ab**-content $\hat{C}_0(x) = \langle a \rangle$ are (almost) the same, that is, $P_0(x) = \langle a \rangle$ in both cases. Now put $x_1 = (u, [c']_{\mathbf{Ab}}, w)$, $x_2 = (w, [cab'c']_{\mathbf{Ab}}, w)$, $x_3 = (w, [cb]_{\mathbf{Ab}}, v)$, and note that $x = x_1x_2x_3$, that is, $C_1(x) \subseteq P_0(x) \cap (P_0(x_1) \cup P_0(x_2) \cup P_0(x_3))$. Again, using $\hat{C}_0$ and Lemma 2.3.3, we obtain that $\hat{C}_0(x_1) = \langle c \rangle$, $\hat{C}_0(x_2) = \{w\} \cup \langle ab' \rangle$, $\hat{C}_0(x_3) = \langle cb \rangle$, and so $P_0(x_1) \cup P_0(x_2) \cup P_0(x_3) = \langle c \rangle \cup \{w\} \cup \langle cb \rangle = \langle cb \rangle$ for both graphs in Figure 3.2.1. Therefore $C_1(x) \subseteq \langle a \rangle \cap \langle cb \rangle = \{u, v\}$ and so $v \notin P_1(x) \subseteq \{u\}$. Hence a is, indeed, a breaking path over **Ab** in both graphs. □

Corollary 3.2.2. *For any non-trivial variety* **U** *of Abelian groups, a graph contains a breaking path over* **U** *if and only if its minors contain at least one of the graphs in Figure 3.2.1.*

Proof. The statement is proven in Theorem 3.2.1 if $\mathbf{U} = \mathbf{Ab}$. Now let **U** be a proper subvariety of **Ab**. Then **U** is the variety of Abelian groups of exponent n for some positive integer $n \geq 2$. By Lemma 2.3.6, the forbidden minors for **U** must be minors of one of the forbidden minors of **Ab**, that is, by Proposition 3.1.3, they are either the same, or the only forbidden minor is the cycle Γ_2 of length two. However, it is clear that Γ_2 contains no breaking path over $\mathbf{U}$ for the same reason as in the case of $\mathbf{Ab}$, which proves our statement.

□

Remark 3.2.3. For the variety $\mathbf{1}$ of trivial groups, a connected graph is $(S_{\mathbf{1}})$ if and only if it is a tree with some loops attached. That is, even the smallest two-edge-connected graph in the minor ordering, the cycle of length two contains a breaking path over $\mathbf{1}$.

Let us examine what these imply for F-inverse covers of Margolis–Meakin expansions. As described in Section 2.3, the Margolis–Meakin expansion $M(G)$ has an F-inverse cover via $\mathbf{U}$ if and only if there is no breaking path in the Cayley graph of G over the variety $\mathbf{U}$. According to 3.2.1, a Cayley graph will contain no breaking path over a non-trivial Abelian variety if and only if it is a cycle or a tree, that is, G is either cyclic or free. Of course, if G is a free group generated by A, then $M(G)$ is nothing but the free inverse monoid generated by A, which is itself F-inverse, which is why it also has an F-inverse cover via the trivial variety. This is consistent with the fact that trees contain no breaking path even over $\mathbf{1}$.

We sum up our observations in the following theorem:

Theorem 3.2.4. *A Margolis–Meakin expansion of a group admits an F-inverse cover via an Abelian group if and only if the group is cyclic or free.*

3.3 Outlook

Let us go back to the original question of the F-inverse cover problem, and discuss where are results stand. Recall that by Result 2.3.2, an affirmative answer to the F-inverse cover problem is equivalent to the existence of a locally finite group variety $\mathbf{U}$ for every graph Γ such that Γ is $(S_{\mathbf{U}})$. So far, we have seen that locally finite Abelian varieties only suffice for a very narrow class of graphs, in which there is nothing surprising. A step up from Abelian varieties would be locally finite varieties of *meta-Abelian* groups $\mathbf{Ab}_s\mathbf{Ab}_r$: groups G in which there is a normal series $\{1\} \lhd N \lhd G$ for which the factors N and G/N are in $\mathbf{Ab}_s$ and $\mathbf{Ab}_r$, respectively. The relatively free meta-Abelian groups have an easy-to-solve word problem, which makes them ideal candidates, however, almost nothing is known about $(S_{\mathbf{Ab}_s\mathbf{Ab}_r})$. We do not currently know of a breaking path over these varieties. Some of what is known is implied by the following fact, which can be found in [2, page 502]:

Result 3.3.1. *If a graph Γ contains a breaking path over the variety $\mathbf{U}$, then there is an arrow x in $F_{g\mathbf{U}}(\Gamma)$ such that $C_0(x)$ is not connected.*

The smallest graph (in the minor ordering) in which such an arrow x can occur over a non-Abelian variety is the one in Figure 3.3.1. For meta-Abelian group varieties $\mathbf{Ab}_s\mathbf{Ab}_r$, where r, s are co-primes or both 2, such a path is known: $bcb'a^rbc'b'$

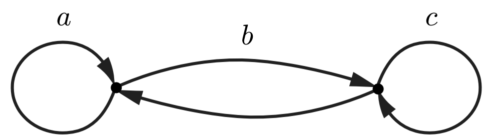


Figure 3.3.1. Smallest graph where $C_0(x)$ is not connected for some x

One possible direction for further research in the matter is to describe forbidden minors for some locally finite meta-Abelian varieties. By Lemma 2.3.6, these must be greater than the Abelian forbidden minors and the graph above. Another approach would be to try and find a locally finite group variety $\mathbf{U}$ such that for every arrow x of $F_{g\mathbf{U}}(\Gamma)$, $C_0(x)$ is connected.

This thesis continues with different generalization of Chapter 3. One way the results of Section 3.2 can be interpreted is as characterizing Margolis–Meakin expansions $M(G)$ which have an F-inverse cover over a group which is an extension of some Abelian group by G. One could formulate the very same question for general inverse monoids. In the following chapter, we develop a framework analogous to the one in [2], which allows us to investigate the proposed problem for a large class of E-unitary inverse monoids.

Chapter 4

F-inverse covers of finite-above inverse monoids

4.1 Conditions on the existence of F-inverse covers

4.1.1 Finite-above inverse monoids

In this section, the framework introduced in [2] and Section 2.3 for Margolis–Meakin expansions resulting in the graph condition is generalized for a class of E-unitary inverse monoids which also contains all finite ones. Analogously, we formulate necessary and sufficient conditions for any member of this class to have an F-inverse cover via a given variety of groups.

First, we define the class of E-unitary inverse monoids we intend to consider. In [2], when F-inverse covers are built from dual premorphisms, condition (2.3.1) is ensured by considering canonical dual premorphisms which respect the distinguished generating elements of the inverse monoids in question. The key lemma [2, Lemma 2.3] states that if M is A-generated with A consisting of maximal elements with respect to the natural partial order, then dual premorphisms satisfying (2.3.1) can be assumed to be canonical. A key idea to the class of inverse monoids to be defined comes from the observation that [2, Lemma 2.3] remains valid under an assumption weaker than M being A-generated where the elements of A are maximal. We introduce the appropriate notion more generally for inverse categories.

Let $\mathcal{X}$ be an inverse category and Δ an arbitrary graph. We say that $\mathcal{X}$ is *quasi-Δ-generated* if a graph morphism $\epsilon_{\mathcal{X}}\colon \Delta \to \mathcal{X}$ is given such that the subgraph $\Delta\epsilon_{\mathcal{X}} \cup \mathrm{E}(\mathcal{X})$ generates $\mathcal{X}$, where $\mathrm{E}(\mathcal{X})$ is the subgraph of the idempotents of $\mathcal{X}$. Clearly, a Δ-generated inverse category is quasi-Δ-generated. Furthermore, notice that a groupoid is quasi-Δ-

generated if and only if it is Δ-generated. If $\epsilon_{\mathcal{X}}$ is injective, then we might assume that Δ is a subgraph in $\mathcal{X}$, i.e., $\epsilon_{\mathcal{X}}$ is the inclusion graph morphism $\Delta \to \mathcal{X}$.

A dual premorphism $\psi\colon \mathcal{Y} \to \mathcal{X}$ between quasi-Δ-generated inverse categories is called *canonical* if $\epsilon_{\mathcal{Y}}\psi = \epsilon_{\mathcal{X}}$. Again, if $\epsilon_{\mathcal{X}}$ is an inclusion, then $\epsilon_{\mathcal{Y}}$ is necessarily injective, and so it also can be chosen to be an inclusion. However, if $\epsilon_{\mathcal{Y}}$ is injective (in particular, an inclusion), then $\epsilon_{\mathcal{X}}$ need not be injective, and so one cannot suppose in general that $\epsilon_{\mathcal{X}}$ is an inclusion.

In particular, if $\mathcal{X}, \mathcal{Y}$ are one-object inverse categories, that is, inverse monoids, and Δ is a one-vertex graph, that is, a set, then this defines a quasi-A-generated inverse monoid and a canonical dual premorphism between inverse monoids. We also point out that a group is quasi-A-generated if and only if A-generated.

An inverse monoid M is called *finite-above* if the set $m^\omega = \{n \in M : n \geq m\}$ is finite for every $m \in M$. For example, finite inverse monoids and the Margolis–Meakin expansions of A-generated groups are finite-above. The class we investigate in this section is that of all finite-above E-unitary inverse monoids.

Notice that if M is a finite-above inverse monoid, then, for every element $m \in M$, there exists $m' \in M$ such that $m' \geq m$ and m' is maximal in M with respect to the natural partial order. Denoting by $\max M^-$ the set of all elements of M distinct from 1 which are maximal with respect to the natural partial order, we obtain that M is quasi-$\max M^-$-generated. Hence the following is straightforward.

Lemma 4.1.1. *Every finite-above inverse monoid is quasi-A-generated for some $A \subseteq \max M^-$.*

What is more, the following lemma shows that each quasi-generating set of a finite-above inverse monoid can be replaced in a natural way by one contained in $\max M^-$. As usual, the set of idempotents $E(M)$ of M is simply denoted by E. Note that if $A \subseteq \max M^-$, then $A \cap E = \emptyset$. Here and later on, we need the following notation. If M is quasi-A-generated and w is a word in $\overline{A \cup E}^*$, then the word in $\overline{A \setminus E}^* \subseteq \overline{A}^*$ obtained from w by deleting all letters from $\overline{E}$ is denoted by w^-. Obviously, we have $[w]_M \leq [w^-]_M$.

Lemma 4.1.2. *Let M be a finite-above inverse monoid, and assume that $A \subseteq M$ is a quasi-generating set in M. For every $a \in A$, let us choose and fix a maximal element $\tilde{a}$ such that $a \leq \tilde{a}$. Then $\tilde{A} = \{\tilde{a} : a \in A\} \setminus \{1\}$ is a quasi-generating set in M such that $\tilde{A} \subseteq \max M^-$.*

Proof. Since A is a quasi-generating set, for every $m \in M$, there exists a word $w \in \overline{A \cup E}^*$ such that $m = [w]_M$, whence $m \leq [w^-]_M$ follows. Moreover, the word $\tilde{u}$ obtained from $u = w^-$ by substituting $\tilde{a}$ for every $a \in A \setminus E$ has the property that $[u]_M \leq [\tilde{u}]_M$, and so $m \leq [\tilde{u}]_M$ holds. Thus m belongs to the inverse submonoid of M generated by $\tilde{A} \cup E$. □

This observation establishes that, within the class of finite-above inverse monoids, it is natural to restrict our consideration to quasi-generating sets contained in $\max M^-$. Now we present a statement on the E-unitary covers of finite-above inverse monoids.

Lemma 4.1.3. *Let M be an inverse monoid.*

(1) If M is finite-above, then so are its E-unitary covers.

(2) If M is quasi-A-generated for some $A \subseteq \max M^-$, then every E-unitary cover of M contains a quasi-A-generated inverse submonoid T with $A\epsilon_T \subseteq \max T^-$ such that T is an E-unitary cover of M.

Proof. Let U be any E-unitary cover of M, and let $\varphi\colon U \to M$ be an idempotent separating and surjective homomorphism.

(1) Since φ is order preserving, we have $t^\omega\varphi \subseteq (t\varphi)^\omega$ for every $t \in U$, and the latter set is finite by assumption. To complete the proof, we verify that $\varphi|_{t^\omega}$ $(t \in U)$ is injective. Let $t \in U$ and $y, y_1 \in t^\omega$ such that $y\varphi = y_1\varphi$. This equality implies $yy^{-1} = y_1y_1^{-1}$, since φ is idempotent separating. Moreover, the relation $y, y_1 \geq t$ implies $y \,\sigma\, t \,\sigma\, y_1$, and so we deduce $y = y_1$, since U is E-unitary.

(2) For every $a \in A$, let us choose and fix an element $u_a \in U$ such that $u_a\varphi = a$, consider the inverse submonoid T of U generated by the set $\{u_a : a \in A\} \cup E(U)$, and put $\epsilon_T\colon A \to T,\ a \mapsto u_a$ which is clearly injective. Obviously, T is a quasi-A-generated E-unitary inverse monoid, and the restriction $\varphi|_T\colon T \to M$ of φ is an idempotent separating and surjective homomorphism. It remains to verify that $A\epsilon_T \subseteq \max T^-$. Observe that an element $m \in M$ is maximal if and only if the set m^ω is a singleton, and similarly for T. Thus the last part of the proof of (1) shows that $A\epsilon_T \subseteq \max T$. Since, for every $a \in A$, the relation $a \neq 1$ implies $u_a \neq 1$, the proof is complete. □

This implies the following statement.

Corollary 4.1.4. *Each quasi-A-generated finite-above inverse monoid M with $A \subseteq \max M^-$ has an E-unitary cover with the same properties.*

This shows that the study of the F-inverse covers of finite-above inverse monoids can be reduced to the study of the F-inverse covers of finite-above E-unitary inverse monoids in the same way as in the case of finite inverse monoids generated by their maximal elements, see [2]. Furthermore, the fundamental observations [2, Lemmas 2.3 and 2.4] can be easily adapted to quasi-A-generated finite-above inverse monoids.

Lemma 4.1.5. *Let H be an A-generated group and M a quasi-A-generated inverse monoid. Then any canonical dual premorphism from H to M has property* (2.3.1).

Proof. Consider a canonical dual premorphism $\psi\colon H \to M$, and let $m \in M$. Since M is quasi-A-generated, we have $m = [w]_M$ for some $w \in \overline{A \cup E}^*$, and so $m \leq [w^-]_M$ where $w^- \in \overline{A}^*$. Since ψ is a canonical dual premorphism, we obtain that $[w^-]_H\psi \geq [w^-]_M \geq m$. □

Lemma 4.1.6. *Let M be a quasi-A-generated inverse monoid such that $A \subseteq \max M^-$. If M has an F-inverse cover over a group H, then there exists an A-generated subgroup H' of H and a canonical dual premorphism from H' to M.*

Proof. Let F be an F-inverse monoid and $\varphi\colon F \to M$ a surjective homomorphism. Put $H = F/\sigma$, and consider the dual premorphism $\psi\colon H \to M,\ h \mapsto m_h\varphi$ induced by φ. Since ψ has property (2.3.1), for any $a \in A$, there exists $h_a \in H$ such that $a \leq h_a\psi$. However, since a is maximal in M, this implies $a = h_a\psi$. Now let H' be the subgroup of H generated by $\{h_a : a \in A\}$. Then the restriction $\psi|_{H'}\colon H' \to M$ of ψ is obviously a dual premorphism. Moreover, the subgroup H' is A-generated with $\epsilon_{H'}\colon A \to H', a \mapsto h_a$, so $\psi|_{H'}$ is also canonical. □

So far, the question of whether a finite-above inverse monoid M has an F-inverse cover over the class of groups $\mathbf{C}$ closed under taking subgroups has been reduced to the question of whether there is a canonical dual premorphism from an A-generated group in $\mathbf{C}$ to M, where $A \subseteq \max M^-$ is a quasi-generating set in M. The answer to this question does not depend on the choice of A.

Let M be a quasi-A-generated inverse monoid with $A \subseteq \max M^-$, H an A-generated group in $\mathbf{C}$, and let $\psi\colon H \to M$ be a canonical dual premorphism. Denote the A-generated group M/σ by G, and note that $\sigma^\natural\colon M \to G$ is canonical. The product $\kappa = \psi\sigma^\natural$ is a canonical dual premorphism from H to G. However, a dual premorphism between groups is necessarily a homomorphism. Consequently, $\kappa\colon H \to G$ is a canonical, and therefore surjective, homomorphism. Hence H is an A-generated extension of a group N by the A-generated group G. If F is an F-inverse cover of M over H then, to simplify our terminology, we also say that F is an F-inverse cover of M *via* N or *via* a class $\mathbf{D}$ of groups if $N \in \mathbf{D}$. If we are only interested in whether M has an F-inverse cover via a member of a given group variety $\mathbf{U}$, then we may replace H by the 'most general' A-generated extension $G^{\mathbf{U}}$ of a member of $\mathbf{U}$ by G. Thus Lemma 4.1.6 implies the following assertion.

Proposition 4.1.7. *Let M be a quasi-A-generated inverse monoid with $A \subseteq \max M^-$, put $G = M/\sigma$, and let $\mathbf{U}$ be a group variety. Then M has an F-inverse cover via the group variety $\mathbf{U}$ if and only if there exists a canonical dual premorphism $G^{\mathbf{U}} \to M$.*

Therefore our question to be studied is reduced to the question of whether there exists a canonical dual premorphism $G^{\mathbf{U}} \to M$ with $G = M/\sigma$ for a given group variety $\mathbf{U}$ and for a given quasi-A-generated inverse monoid M with $A \subseteq \max M^-$. In the sequel, we deal with this question in the case where M is finite-above and E-unitary.

4.1.2 Closed subgraphs

Let M be an E-unitary inverse monoid, denote M/σ by G, and consider the inverse category $\mathcal{I}_M$ acted upon by G. Recall from Subsection 2.2.3 that the set of objects of $\mathcal{I}_M$ is G, and the set of (i,j) arrows are of the form (i,m,j), where $m \in M$ and $i \cdot m\sigma = j$. Given a path $p = e_1 e_2 \cdots e_n$ in $\overline{\mathcal{I}_M}$ where $e_j = (\iota e_j, m_j, \tau e_j)$ with $m_j \in \overline{M}$ for every j $(j = 1, 2, \ldots, n)$, consider the word $w = m_1 m_2 \cdots m_n \in \overline{M}^*$ determined by the labels of the arrows in p, and let us assign an element of M to the path p by defining $\lambda(p) = [w]_M$ — this is a key definition of the section. Notice that, for every path p, we have $\lambda(p) = \lambda(pp'p)$, and $\lambda(p)$ is just the label of the arrow $p\varphi$, where $\varphi\colon \overline{\mathcal{I}_M}^* \to \mathcal{I}_M$ is the unique category morphism such that $e\varphi = e$ and $e'\varphi = e^{-1}$ for every $e \in \operatorname{Arr}\mathcal{I}_M$. Since the local monoids of the category $\mathcal{I}_M$ are semilattices by Result 2.2.3, the following property follows from [13, Lemma 2.6] (see also [6, Chapter VII] and [22, Section 12]), or can be proven from the definition itself quite straightforwardly.

Lemma 4.1.8. *For any co-terminal paths p, q in $\overline{\mathcal{I}_M}$, if $\langle p\rangle = \langle q\rangle$, then $\lambda(p) = \lambda(q)$.*

This allows us to assign an element of M to any birooted finite connected subgraph: if X is a finite connected subgraph in $\mathcal{I}_M$ and $i, j \in V_{\mathrm{X}}$, then let $\lambda_{(i,j)}(\mathrm{X})$ be $\lambda(p)$, where p is an (i,j)-path in $\overline{\mathcal{I}_M}$ with $\langle p\rangle = \mathrm{X}$.

Now assume that M is a quasi-A-generated E-unitary inverse monoid with $A \subseteq \max M^-$, and recall that in this case, $G = M/\sigma$ is an A-generated group. Based on the ideas in [13], we now give a model for $\mathcal{I}_M$ as a quasi-Γ-generated inverse category where Γ is the Cayley graph of G. Choose and fix a subset I of E such that $A \cup I$ generates M. In particular, if M is A-generated, then I can be chosen to be empty. Consider the subgraphs Γ and Γ^I of $\mathcal{I}_M$ consisting of all edges with labels from A and from $A \cup I$, respectively. Notice that Γ is, in fact, the Cayley graph of the A-generated group G, and Γ^I is obtained from Γ by adding loops to it (with labels from I).

We are going to introduce a closure operator on the set $\mathrm{Sub}(\Gamma^I)$ of all subgraphs of Γ^I. We need to make a few observations before.

Lemma 4.1.9. *Let* X, Y *be finite connected subgraphs in* Γ^I*, and let* $i, j \in V_\mathrm{X} \cap V_\mathrm{Y}$*. If* $\lambda_{(i,j)}(\mathrm{X}) \leq \lambda_{(i,j)}(\mathrm{Y})$*, then* $\lambda_{(i,j)}(\mathrm{X}) = \lambda_{(i,j)}(\mathrm{X} \cup \mathrm{Y})$*.*

Proof. Let r and s be arbitrary (i,j)-paths spanning X and Y, respectively. Then $rr's$ is an (i,j)-path spanning $\mathrm{X} \cup \mathrm{Y}$. According to the assumption, $\lambda(r) \leq \lambda(s)$, so $\lambda(rr's) = \lambda(r)$. □

Lemma 4.1.10. *Let* X, Y *be finite connected subgraphs in* Γ^I*, and let* $i, j \in V_\mathrm{X} \cap V_\mathrm{Y}$*. If* $\lambda_{(i,j)}(\mathrm{X}) \leq \lambda_{(i,j)}(\mathrm{Y})$*, then* $\lambda_{(k,l)}(\mathrm{X}) \leq \lambda_{(k,l)}(\mathrm{Y})$ *for every* $k, l \in V_\mathrm{X} \cap V_\mathrm{Y}$*.*

Proof. Let r and s be (i,j)-paths spanning X and Y, respectively, and let p_1 and q_1 be (k,i)-paths in X and Y, and let p_2 and q_2 be (j,l)-paths in X and Y, respectively. Then $p_1 r p_2$ and $q_1 s q_2$ are (k,l)-paths spanning X and Y, respectively. Therefore, by applying Lemmas 4.1.8 and 4.1.9, we obtain that

$$\begin{aligned} \lambda_{(k,l)}(\mathrm{X}) &= \lambda(p_1 r p_2) = \lambda(p_1)\lambda_{(i,j)}(\mathrm{X})\lambda(p_2) = \lambda(p_1)\lambda_{(i,j)}(\mathrm{X} \cup \mathrm{Y})\lambda(p_2) \\ &= \lambda(p_1)\lambda(rr's)\lambda(p_2) = \lambda(p_1 rr's p_2) = \lambda(q_1 rr's q_2) \\ &\leq \lambda(q_1 s q_2) = \lambda_{(k,l)}(\mathrm{Y}). \end{aligned}$$

□

Given a finite connected subgraph X in Γ^I with vertices $i, j \in V_\mathrm{X}$, consider the subgraph

$$\mathrm{X}^{\mathrm{cl}} = \bigcup \{\mathrm{Y} \in \mathrm{Sub}(\Gamma^I) : \mathrm{Y} \text{ is finite and connected, } i, j \in V_\mathrm{Y}, \text{ and } \lambda_{(i,j)}(\mathrm{Y}) \geq \lambda_{(i,j)}(\mathrm{X})\}$$

of Γ^I which is clearly connected. Note that, by Lemma 4.1.10, the graph X^{cl} is independent of the choice of i, j. Moreover, by Lemma 4.1.9, the same subgraph is obtained if the relation '$\geq$' is replaced by '$=$' in the definition of X^{cl}. More generally, for any $\mathrm{X} \in \mathrm{Sub}(\Gamma^I)$, let us define the subgraph X^{cl} in the following manner:

$$\mathrm{X}^{\mathrm{cl}} = \bigcup \{\mathrm{Y}^{\mathrm{cl}} : \mathrm{Y} \text{ is a finite and connected subgraph of X}\}.$$

It is routine to check that $\mathrm{X} \to \mathrm{X}^{\mathrm{cl}}$ is a closure operator on $\mathrm{Sub}(\Gamma^I)$, that is, $\mathrm{X} \subseteq \mathrm{X}^{\mathrm{cl}}$, $(\mathrm{X}^{\mathrm{cl}})^{\mathrm{cl}} = \mathrm{X}^{\mathrm{cl}}$, and $\mathrm{X} \subseteq \mathrm{X}_1$ implies $\mathrm{X}^{\mathrm{cl}} \subseteq {\mathrm{X}_1}^{\mathrm{cl}}$ for any $\mathrm{X}, \mathrm{X}_1 \in \mathrm{Sub}(\Gamma^I)$. As usual, a subgraph X of Γ^I is said to be *closed* if $\mathrm{X} = \mathrm{X}^{\mathrm{cl}}$. Note that, in particular, we have

$$\emptyset_i^{\mathrm{cl}} = \bigcup \left\{ \langle h \rangle : h \text{ is an } i\text{-cycle in } \Gamma^I \text{ such that } \lambda(h) = 1 \right\},$$

and so $\emptyset_i$ is closed if and only if there is no $a \in A$ such that $a \,\mathcal{R}\, 1$ or $a \,\mathcal{L}\, 1$. Furthermore, we have $\mathrm{X}^{\mathrm{cl}} \supseteq \emptyset_i^{\mathrm{cl}}$ for every $\mathrm{X} \in \mathrm{Sub}(\Gamma^I)$ and $i \in V_{\mathrm{X}^{\mathrm{cl}}}$. In particular, we see that the closure of a finite subgraph need not be finite. For example, if M is the bicyclic inverse monoid generated by $A = \{a\}$ where $aa^{-1} = 1$, then a is a maximal element in M, M/σ is the infinite cyclic group generated by $a\sigma$, and we have $\emptyset_1^{\mathrm{cl}} = \{((a\sigma)^n, a, (a\sigma)^{n+1}) : n \in \mathbb{N}_0\}$.

Denote the set of all closed subgraphs of Γ^I by $\mathrm{ClSub}(\Gamma^I)$, and its subset consisting of the closures of all finite connected subgraphs by $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$. Moreover, for any family X_j ($j \in J$) of subgraphs of Γ^I, define $\bigvee_{j \in J} \mathrm{X}_j = \left(\bigcup_{j \in J} \mathrm{X}_j\right)^{\mathrm{cl}}$. The following lemmas formulate important properties of closed subgraphs which can be easily checked.

Lemma 4.1.11. *For every quasi-A-generated E-unitary inverse monoid M with $A \subseteq \max M^-$, the following statements hold.*

(1) Each component of a closed subgraph is closed.

(2) The partially ordered set $(\mathrm{ClSub}(\Gamma^I); \subseteq)$ forms a complete lattice with respect to the usual intersection and the operation $\bigvee$ defined above.

(3) For any $\mathrm{X}, \mathrm{Y} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$ with $V_{\mathrm{X}} \cap V_{\mathrm{Y}} \neq \emptyset$, we have $\mathrm{X} \vee \mathrm{Y} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$.

(4) For any finite connected subgraph in Γ^I and for any $g \in G$, we have ${}^g(\mathrm{X}^{\mathrm{cl}}) = ({}^g\mathrm{X})^{\mathrm{cl}}$. Consequently, the action of G on $\mathrm{Sub}(\Gamma^I)$ restricts to an action on $\mathrm{ClSub}(\Gamma^I)$ and to an action on $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$, respectively.

We prove that the descending chain condition holds for $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$ if M is finite-above.

Lemma 4.1.12. *If M is a quasi-A-generated finite-above E-unitary inverse monoid with $A \subseteq \max M^-$, then, for every $\mathrm{X} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$ and $i \in V_{\mathrm{X}}$, there are only finitely many closed connected subgraphs in X containing the vertex i, and all belong to $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$.*

Proof. Let $\mathrm{X} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$, whence $\mathrm{X} = \mathrm{Y}^{\mathrm{cl}}$ for some finite connected subgraph Y, and let $i \in V_{\mathrm{Y}}$. If Z is any finite connected subgraph such that $\mathrm{X} \supseteq \mathrm{Z}^{\mathrm{cl}}$ and $i \in V_{\mathrm{Z}}$, then $\lambda_{(i,i)}(\mathrm{Y}) \leq \lambda_{(i,i)}(\mathrm{Z})$. Since M is finite-above, the set $\Lambda = \{\mathrm{X}_0 \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I) : \mathrm{X}_0 \subseteq \mathrm{X} \text{ and } i \in V_{\mathrm{X}_0}\}$ is finite. If $\mathrm{X}_1 \in \mathrm{ClSub}(\Gamma^I)$ is connected with $\mathrm{X}_1 \subseteq \mathrm{X}$ and $i \in V_{\mathrm{X}_1}$, then, by definition, X_1 is a join of a subset of the finite set Λ which is closed under $\vee$. Hence it follows that X_1 belongs to Λ, i.e., $X_1 \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$. $\square$

We define an inverse category $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ in the following way: its set of objects is G, its set of (i,j)-arrows $(i,j \in G)$ is

$$\mathcal{X}_{\mathrm{cl}}(\Gamma^I)(i,j) = \{(i, \mathrm{X}, j) : \mathrm{X} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma) \text{ and } i,j \in V_{\mathrm{X}}\},$$

and the product of two consecutive arrows is defined by

$$(i, \mathrm{X}, j)(j, \mathrm{Y}, k) = (i, \mathrm{X} \vee \mathrm{Y}, k).$$

It can be checked directly (see also [13]) that $\mathcal{X}_{\mathrm{cl}}(\Gamma^I) \to \mathcal{I}_M,\ (i, \mathrm{X}, j) \mapsto (i, \lambda_{(i,j)}(\mathrm{X}), j)$ is a category isomorphism. Hence $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ is an inverse category with $(i, \mathrm{X}, j)^{-1} = (j, \mathrm{X}, i)$, it is locally a semilattice, and the natural partial order on it is the following: $(i, \mathrm{X}, j) \leq (k, \mathrm{Y}, l)$ if and only if $i = k, j = l$ and $\mathrm{X} \supseteq \mathrm{Y}$. Moreover, the group G acts on it by the rule ${}^g(i, \mathrm{X}, j) = (gi, {}^g\mathrm{X}, gj)$ transitively and without fixed points. The inverse category $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ is Γ^I-generated with $\epsilon^I_{\mathcal{X}_{\mathrm{cl}}(\Gamma^I)} \colon \Gamma^I \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I), e \mapsto (\iota e, e^{\mathrm{cl}}, \tau e)$. Therefore $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ is also quasi-Γ-generated with $\epsilon_{\mathcal{X}_{\mathrm{cl}}(\Gamma^I)} = \epsilon^I_{\mathcal{X}_{\mathrm{cl}}(\Gamma^I)}|_\Gamma \colon \Gamma \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$. By Results 2.2.1 and 2.2.3, hence we deduce the following proposition.

Proposition 4.1.13. *(1) The E-unitary inverse monoid $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$ can be described, up to isomorphism, in the following way: its underlying set is*

$$\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G = \{(\mathrm{X}, g) : \mathrm{X} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I),\ 1, g \in V_{\mathrm{X}}\},$$

and the multiplication is defined by

$$(\mathrm{X}, g)(\mathrm{Y}, h) = (\mathrm{X} \vee {}^g\mathrm{Y}, gh).$$

(2) The monoid $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$ is quasi-A-generated with

$$\epsilon_{\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G} \colon A \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G, \quad a \mapsto (e_a^{\mathrm{cl}}, a\sigma).$$

(3) The map $\varphi \colon \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G \to M,\ (\mathrm{X}, g) \mapsto \lambda_{(1,g)}(\mathrm{X})$ is a canonical isomorphism.

Remark 4.1.14. Proposition 4.1.13 provides a representation of M as a P-semigroup. The McAlister triple involved consists of G, the partially ordered set $(\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I); \subseteq)$ and its order ideal and subsemilattice $(\{\mathrm{X} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I),\ 1 \in V_{\mathrm{X}}\}; \vee)$.

The inverse category $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ can very clearly be seen as an analogue of $F_{g\mathbf{Sl}}(\Gamma)$, and $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$ of $M(G)$. In the sequel, further generalizing [2], the fact that $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$ is isomorphic to M will enable us to find F-inverse covers directly to M. Also, notice that if we apply the construction before Proposition 4.1.13 for M being the Margolis–Meakin expansion $M(G)$ of an A-generated group G with $A \subseteq G \setminus \{1\}$, then $\Gamma^I = \Gamma$, the Cayley graph of G, the closure operator $\mathrm{X} \to \mathrm{X}^{\mathrm{cl}}$ is identical on $\mathrm{Sub}(\Gamma)$, and the operation $\vee$ coincides with the usual $\cup$. Thus the category $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ is just the category isomorphic to $\mathcal{I}_{M(G)}$ which is presented after Result 2.2.3, and the map φ given in the last statement of the proposition is, in fact, identical.

The goal of this section is to give equivalent conditions for the existence of a canonical dual premorphism $G^{\mathbf{U}} \to M$. The previous proposition reformulates it by replacing M with $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$. Since $G^{\mathbf{U}} = F_{g\mathbf{U}}(\Gamma)/G$, it is natural to study the connection between the canonical dual premorphisms $F_{g\mathbf{U}}(\Gamma)/G \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$ and the canonical dual premorphisms $F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$. As one expects, there is a natural correspondence between these formulated in the next lemma in a more general setting. The proof is straightforward, it is left to the reader.

Lemma 4.1.15. *Let Δ be any graph, and let $\mathcal{Y}$ be a Δ-generated, and $\mathcal{X}$ a quasi-Δ-generated inverse category containing Δ. Suppose that G is a group acting on both $\mathcal{X}$ and $\mathcal{Y}$ transitively and without fixed points in a way that Δ is invariant with respect to both actions, and the two actions coincide on Δ. Let i be a vertex in Δ.*

(1) We have $\mathrm{Ob}\,\mathcal{X} = V_\Delta = \mathrm{Ob}\,\mathcal{Y}$, and so the actions of G on $\mathrm{Ob}\,\mathcal{X}$ and $\mathrm{Ob}\,\mathcal{Y}$ coincide.

(2) The inverse monoid $\mathcal{Y}_i$ is $\Delta(i,-)$-generated, and the inverse monoid $\mathcal{X}_i$ is quasi-$\Delta(i,-)$-generated with the maps

$$\epsilon_{\mathcal{Y}_i}\colon \Delta(i,-) \to \mathcal{Y}_i,\ e \mapsto (e,g), \quad \textit{provided } e \in \mathcal{Y}(i, {}^g i),$$

and

$$\epsilon_{\mathcal{X}_i}\colon \Delta(i,-) \to \mathcal{X}_i,\ e \mapsto (e,g), \quad \textit{provided } e \in \mathcal{X}(i, {}^g i),$$

respectively.

(3) If $\Psi\colon \mathcal{Y} \to \mathcal{X}$ is a canonical dual premorphism such that

$$({}^g y)\Psi = {}^g(y\Psi) \quad \textit{for every } g \in G \textit{ and } y \in \mathrm{Arr}\,\mathcal{Y}, \tag{4.1.1}$$

then $\iota(y\Psi) = \iota y$, $\tau(y\Psi) = \tau y$, and the map $\psi\colon \mathcal{Y}_i \to \mathcal{X}_i,\ (e,g) \mapsto (e\Psi, g)$ is a canonical dual premorphism.

(4) If $\psi\colon \mathcal{Y}_i \to \mathcal{X}_i$ is a canonical dual premorphism and $(e,g)\psi = (\tilde{e},\tilde{g})$ for some $(e,g) \in \mathcal{Y}_i$ and $(\tilde{e},\tilde{g}) \in \mathcal{X}_i$, then $g = \tilde{g}$, $\iota e = \iota\tilde{e}$ and $\tau e = \tau\tilde{e}$. Thus a graph morphism $\Psi\colon \mathcal{Y} \to \mathcal{X}$ can be defined such that, for any arrow $y \in \mathcal{Y}({}^{g}i, {}^{h}i)$, we set $y\Psi$ to be the unique arrow $x \in \mathcal{X}({}^{g}i, {}^{h}i)$ such that $({}^{g^{-1}}y, g^{-1}h)\psi = ({}^{g^{-1}}x, g^{-1}h)$. This Ψ is a canonical dual premorphism satisfying (4.1.1).

From now on, let M be a quasi-A-generated finite-above E-unitary inverse monoid with $A \subseteq \max M^{-}$, and let $\mathbf{U}$ be an arbitrary group variety. Motivated by Lemma 4.1.15, we intend to find a necessary and sufficient condition in order that a canonical dual premorphism $F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ exists fulfilling condition (4.1.1).

4.1.3 A graph condition

Analogously to [2], we are going to assign two sequences of subgraphs of Γ^I to any arrow x of $F_{g\mathbf{U}}(\Gamma)$. Let

$$C_0^{\mathrm{cl}}(x) = \bigcap\{\langle p\rangle^{\mathrm{cl}} : p \text{ is a } (\iota x, \tau x)\text{-path in } \overline{\Gamma} \text{ such that } x = (\iota x, [p]_{\mathbf{U}}, \tau x)\},$$

and let $P_0^{\mathrm{cl}}(x)$ be the component of $C_0^{\mathrm{cl}}(x)$ containing ιx. Suppose that, for some n ($n \geq 0$), the subgraphs $C_n^{\mathrm{cl}}(x)$ and $P_n^{\mathrm{cl}}(x)$ are defined for every arrow x of $F_{g\mathbf{U}}(\Gamma)$. Then let

$$C_{n+1}^{\mathrm{cl}}(x) = \bigcap\{P_n^{\mathrm{cl}}(x_1) \vee \cdots \vee P_n^{\mathrm{cl}}(x_k) : k \in \mathbb{N}_0,\ x_1, \ldots, x_k \in F_{g\mathbf{U}}(\Gamma) \text{ are consecutive arrows, and } x = x_1 \cdots x_k\},$$

and again, let $P_{n+1}^{\mathrm{cl}}(x)$ be the component of $C_{n+1}^{\mathrm{cl}}(x)$ containing ιx. Applying Lemma 4.1.11 we see that, for every n, the subgraph $P_n^{\mathrm{cl}}(x)$ of Γ^I is a component of an intersection of closed subgraphs, so $P_n^{\mathrm{cl}}(x) \in \mathrm{ClSub}(\Gamma^I)$ and is connected. Also, $P_n^{\mathrm{cl}}(x)$ contains ιx for all n. Moreover, observe that

$$C_0^{\mathrm{cl}}(x) \supseteq P_0^{\mathrm{cl}}(x) \supseteq \cdots \supseteq C_n^{\mathrm{cl}}(x) \supseteq P_n^{\mathrm{cl}}(x) \supseteq C_{n+1}^{\mathrm{cl}}(x) \supseteq P_{n+1}^{\mathrm{cl}}(x) \supseteq \cdots$$

for all x and n. By Lemma 4.1.12 we deduce that, for every x, all these subgraphs belong to $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$, and there exists $n_x \in \mathbb{N}_0$ such that $P_{n_x}^{\mathrm{cl}}(x) = P_{n_x+k}^{\mathrm{cl}}(x)$ for every $k \in \mathbb{N}_0$. For brevity, denote $P_{n_x}^{\mathrm{cl}}(x)$ by $P^{\mathrm{cl}}(x)$. Furthermore, for any consecutive arrows x and y, we have

$$P_{n+1}^{\mathrm{cl}}(xy) \subseteq C_{n+1}^{\mathrm{cl}}(xy) \subseteq P_n^{\mathrm{cl}}(x) \vee P_n^{\mathrm{cl}}(y),$$

and so

$$P^{\mathrm{cl}}(xy) \subseteq P^{\mathrm{cl}}(x) \vee P^{\mathrm{cl}}(y)$$

is implied.

Proposition 4.1.16. *There exists a canonical dual premorphism $\psi\colon F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ if and only if $P_n^{\mathrm{cl}}(x)$ contains τx for every $n \in \mathbb{N}_0$ and for every $x \in F_{g\mathbf{U}}(\Gamma)$, or, equivalently, if and only if $P^{\mathrm{cl}}(x)$ contains τx for every $x \in F_{g\mathbf{U}}(\Gamma)$.*

Proof. Let $\psi\colon F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ be a canonical dual premorphism. We denote the middle entry of $x\psi$ by $\mu(x\psi)$, which belongs to $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$ and contains ιx and τx. The fact that ψ is a dual premorphism means that $\mu((xy)\psi) \subseteq \mu(x\psi) \vee \mu(y\psi)$. Moreover, ψ is canonical, therefore we have $(\iota e, [e]_{\mathbf{U}}, \tau e)\psi = (\iota e, e^{\mathrm{cl}}, \tau e)$ for every $e \in E_\Gamma$. Hence for an arbitrary representation of an arrow $x = (\iota x, [p]_{\mathbf{U}}, \tau x)$, where $p = e_1 \cdots e_n$ is a $(\iota x, \tau x)$-path in $\overline{\Gamma}$ and $e_1, \ldots, e_n \in E_{\overline{\Gamma}}$, we have

$$\begin{aligned} \mu(x\psi) &\subseteq \mu((\iota e_1, [e_1]_{\mathbf{U}}, \tau e_1)\psi) \vee \cdots \vee \mu((\iota e_n, [e_n]_{\mathbf{U}}, \tau e_n)\psi) \\ &= e_1^{\mathrm{cl}} \vee \cdots \vee e_n^{\mathrm{cl}} = \langle p \rangle^{\mathrm{cl}}, \end{aligned}$$

which implies $\mu(x\psi) \subseteq C_0^{\mathrm{cl}}(x)$. Since $\mu(x\psi)$ is connected and contains ιx, $\mu(x\psi) \subseteq P_0^{\mathrm{cl}}(x)$, and this implies $\tau x \in P_0^{\mathrm{cl}}(x)$.

Now suppose $n \geq 0$ and $\mu(y\psi) \subseteq P_n^{\mathrm{cl}}(y)$ for any arrow y. Let $x = x_1 \cdots x_k$ be an arbitrary decomposition in $F_{g\mathbf{U}}(\Gamma)$. Then

$$\mu(x\psi) \subseteq \mu(x_1\psi) \vee \cdots \vee \mu(x_k\psi) \subseteq P_n^{\mathrm{cl}}(x_1) \vee \cdots \vee P_n^{\mathrm{cl}}(x_k)$$

holds, whence $\mu(x\psi) \subseteq C_{n+1}^{\mathrm{cl}}(x)$. As before, $\mu(x\psi)$ is connected and contains both ιx and τx, so we see that $\mu(x\psi) \subseteq P_{n+1}^{\mathrm{cl}}(x)$ and $\tau x \in P_{n+1}^{\mathrm{cl}}(x)$. This proves the 'only if' part of the statement.

For the converse, suppose that for any arrow x in $F_{g\mathbf{U}}(\Gamma)$, we have $\tau x \in P_n^{\mathrm{cl}}(x)$ for all $n \in \mathbb{N}_0$. We have seen above that $P^{\mathrm{cl}}(x) \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$, and $P^{\mathrm{cl}}(xy) \subseteq P^{\mathrm{cl}}(x) \vee P^{\mathrm{cl}}(y)$ for any arrows x, y. Furthermore, the equality $P^{\mathrm{cl}}(x) = P^{\mathrm{cl}}(x^{-1})$ can be easily checked for all arrows x by definition. Now consider the map P^{cl} which assigns the arrow $(\iota x, P^{\mathrm{cl}}(x), \tau x)$ of $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ to the arrow x of $F_{g\mathbf{U}}(\Gamma)$. By the previous observations, this is a dual premorphism from $F_{g\mathbf{U}}(\Gamma)$ to $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$, and the image of $(\iota e, [e]_{\mathbf{U}}, \tau e)$ is $(\iota e, e^{\mathrm{cl}}, \tau e)$, hence it is also canonical. $\square$

The next lemma states that the canonical dual premorphism P^{cl} constructed in the previous proof has property (4.1.1).

Lemma 4.1.17. *For every $g \in G$ and for any arrow x of $F_{g\mathbf{U}}(\Gamma)$, we have $P^{\mathrm{cl}}({}^g x) = {}^g P^{\mathrm{cl}}(x)$.*

Proof. One can see by definition that $C_0^{\mathrm{cl}}({}^g x) = {}^g C_0^{\mathrm{cl}}(x)$ for all $x \in F_{g\mathbf{U}}(\Gamma)$, and so $P_0^{\mathrm{cl}}({}^g x) = {}^g P_0^{\mathrm{cl}}(x)$ also holds. By making use of Lemma 4.1.11(4), an easy induction shows that $C_n^{\mathrm{cl}}({}^g x) = {}^g C_n^{\mathrm{cl}}(x)$ and $P_n^{\mathrm{cl}}({}^g x) = {}^g P_n^{\mathrm{cl}}(x)$ for all n. □

Recall that the categories $F_{g\mathbf{U}}(\Gamma)$ and $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ satisfy the assumptions of Lemma 4.1.15. Combining this lemma with Proposition 4.1.16 and Lemma 4.1.17, we obtain the following.

Proposition 4.1.18. *There exists a canonical dual premorphism $F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ if and only if there exists a canonical dual premorphism $G^{\mathbf{U}} = F_{g\mathbf{U}}(\Gamma)/G \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$.*

The main results of the section, see Propositions 4.1.7, 4.1.13, 4.1.16 and 4.1.18, are summed up in the following theorem.

Theorem 4.1.19. *Let M be a quasi-A-generated finite-above E-unitary inverse monoid with $A \subseteq \max M^-$, put $G = M/\sigma$, and let $\mathbf{U}$ be a group variety. Let Γ be the Cayley graph of G. The following statements are equivalent.*

(1) M has an F-inverse cover via the group variety $\mathbf{U}$.

(2) There exists a canonical dual premorphism $G^{\mathbf{U}} \to M$.

(3) There exists a canonical dual premorphism $G^{\mathbf{U}} \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$.

(4) There exists a canonical dual premorphism $F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$.

(5) For any arrow x in $F_{g\mathbf{U}}(\Gamma)$ and for any $n \in \mathbb{N}_0$, the graph $P_n^{\mathrm{cl}}(x)$ contains τx.

As an example, we describe a class of non-F-inverse finite-above inverse monoids for which Theorem 4.1.19 yields F-inverse covers via any non-trivial group variety in a straightforward way. The following observation on the series $C_0^{\mathrm{cl}}(x), C_1^{\mathrm{cl}}(x), \ldots$ and $P_0^{\mathrm{cl}}(x), P_1^{\mathrm{cl}}(x), \ldots$ of subgraphs plays a crucial role in our argument. Recall that, given a group variety $\mathbf{U}$ and a word $w \in \overline{A}^*$, the $\mathbf{U}$-content $c_{\mathbf{U}}(w)$ of w consists of the elements $a \in A$ such that $[w]_{\mathbf{U}}$ depends on a.

Proposition 4.1.20. *(1) If $x = (\iota x, [p]_{\mathbf{U}}, \tau x)$ for some $(\iota x, \tau x)$-path p in $\overline{\Gamma}$ then $C_0^{\mathrm{cl}}(x) = \langle c_{\mathbf{U}}(p)\rangle^{\mathrm{cl}}$.*

(2) If $C_0^{\mathrm{cl}}(x)$ is connected for every arrow $x \in F_{g\mathbf{U}}(\Gamma)$ then $C_0^{\mathrm{cl}}(x) = P^{\mathrm{cl}}(x)$ for every $x \in F_{g\mathbf{U}}(\Gamma)$.

Proof. The proof of 2.3.3 can be easily adapted to show Lemma (1). By assumption in (2), we have $P_0^{\mathrm{cl}}(x) = C_0^{\mathrm{cl}}(x)$ for any $x \in F_{g\mathbf{U}}(\Gamma)$. Applying (1), an easy induction implies that $C_{n+1}^{\mathrm{cl}}(x) = P_n^{\mathrm{cl}}(x)$ and $P_{n+1}^{\mathrm{cl}}(x) = C_{n+1}^{\mathrm{cl}}(x)$ for every $n \in \mathbb{N}_0$ and $x \in F_{g\mathbf{U}}(\Gamma)$. This verifies statement (2). □

Example 4.1.21. Let G be a group acting on a semilattice S where S has no greatest element, and for every $s \in S$, the set of elements greater than s is finite. Consider a semidirect product $S \rtimes G$ of S by G, and let $M = (S \rtimes G)^1$, the inverse monoid obtained from $S \rtimes G$ by adjoining an identity element 1. Then M is a finite-above E-unitary inverse monoid which is not F-inverse, but it has an F-inverse cover via any non-trivial group variety.

Notice that $S \rtimes G$ has no identity element, therefore $M \setminus \{1\} = S \rtimes G$. Recall that the rules of multiplication and taking inverse in $M \setminus \{1\}$ are as follows:

$$(s, g)(t, h) = (s \cdot {}^{g}t, gh) \quad \text{and} \quad (s, g)^{-1} = ({}^{g^{-1}}s, g^{-1}).$$

The semilattice of idempotents of M is $(S \times \{1_G\}) \cup \{1\}$, and the natural partial order on $M \setminus \{1\}$ is given by

$$(s, g) \leq (t, h) \quad \text{if and only if} \quad s \leq t \text{ and } g = h.$$

The kernel of the projection of $M \setminus \{1\}$ onto G, which is clearly a homomorphism, is the least group congruence on $M \setminus \{1\}$. Hence $M \setminus \{1\}$, and therefore M also is E-unitary. Moreover, M is finite-above and non-F-inverse due to the conditions imposed on S. By Lemma 4.1.1, M is quasi-A-generated with $A = \max M^-$, and it is easy to check that $\max M^- = \max S \times (G \setminus \{1_G\})$ where $\max S$ denotes the maximal elements of S.

Now that all conditions of Theorem 4.1.19 are satisfied, construct the graph Γ: its set of vertices is $V_\Gamma = G$ and set of edges is

$$\begin{aligned} E_\Gamma = \{(g_1, (s', g), g_2) : s' \in \max S \text{ and } g_1, g_2, g \in G \\ \text{such that } g \neq 1_G \text{ and } g_1 g = g_2\}, \end{aligned}$$

where $\iota(g_1, (s', g), g_2) = g_1$ and $\tau(g_1, (s', g), g_2) = g_2$. (This is essentially the Cayley graph of the A-generated group G with $\epsilon_G \colon A \to G$, $(s', g) \mapsto g$, and it is obtained from the Cayley graph of G, considered as a $(G \setminus \{1_G\})$-generated group, by replacing each edge with $|\max S|$ copies.) Let $\mathbf{U}$ be a non-trivial group variety. By Proposition 4.1.20, it suffices to prove that, for each edge e of Γ, the set of vertices of the graph e^{cl} is G. For, in this case, statement (1) obviously shows that $C_0^{\mathrm{cl}}(x)$ is connected for every arrow x in $F_{g\mathbf{U}}(\Gamma)$, and so statement (2) implies that Theorem 4.1.19(5) holds for M. Our statement for M follows by the equivalence of Theorem 4.1.19(1) and (5).

Consider an arbitrary edge $e = (g_1, (s', g), g_2) \in E_\Gamma$ and an arbitrary element $h \in G$, and prove that h is a vertex of e^{cl}. Since g_1 is obviously a vertex of e^{cl}, we can assume that $h \neq g_1$. Then we have $h = g_1 u$ for some $u \in G \setminus \{1_G\}$, and $\lambda(e) = (s', g) = (s', u)(s', u)^{-1}(s', g)$. This implies that $(g_1, (s', u), h)$ is an edge in Γ belonging to e^{cl}, and so h is, indeed, a vertex of e^{cl}.

This example sheds light on the generality of our construction in contrast with that in [2]. By the main result of Chapter 3, the Margolis–Meakin expansion of a group admits an F-inverse cover via an Abelian group if and only if the group is cyclic or free. The previous example shows that, for any group G, there exist finite-above E-unitary inverse monoids with greatest group homomorphic image G that fail to be F-inverse but admit F-inverse covers via Abelian groups.

4.2 F-inverse covers via Abelian groups

In this section, we make further inquiries on how Theorem 3.2.4 generalizes for finite-above E-unitary inverse monoids. The main result of the section gives a sufficient condition for such an F-inverse cover not to exist, formulated merely by means of the natural partial order and the least group congruence.

An easy consequence of Theorem 4.1.19 is the following:

Proposition 4.2.1. *If M is a finite-above E-unitary inverse monoid with $|M/\sigma| \leq 2$, then M has an F-inverse cover via any non-trivial group variety. In particular, M has an F-inverse cover via an elementary Abelian p-group for any prime p.*

Proof. If $|M/\sigma| = 1$, that is, M is a semilattice monoid, then M is itself F-inverse, and the statement holds for any group variety, including the trivial one.

Now we consider the case $|M/\sigma| = 2$. Let $A \subseteq \max M^-$ such that M is quasi-A-generated. Then the graph Γ and the inverse category $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ has two vertices and objects, say, 1 and u. If $\mathbf{U}$ is a non-trivial group variety, and q is a $(1, u)$-path in $\overline{\Gamma}$, then $u \neq 1$ implies that $c_{\mathbf{U}}(q)$ is non-empty. Thus $C_0^{\mathrm{cl}}(x)$ is connected for every arrow x in $F_g\mathbf{U}(\Gamma)$, and Proposition 4.1.20 shows that condition (5) in Theorem 4.1.19 is satisfied, completing the proof. □

This proposition shows that if a finite-above E-unitary inverse monoid M has no F-inverse cover via an Abelian group (and consequently, M itself is not F-inverse), then M/σ

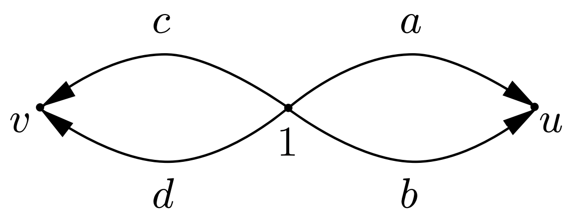


Figure 4.2.1. The most general constellation of a, b, and v in $\mathcal{I}_M$

has at least two elements distinct from 1, and there exists a σ-class in M containing at least two maximal elements.

From now on, let M be a finite-above E-unitary inverse monoid. Let us choose elements $a, b \in M$ with $a \,\sigma\, b$, and a σ-class $v \in M/\sigma$. Denote by $\max v$ the set of maximal elements of the σ-class v. Notice that $\max 1 = 1_M$, and if $v \neq 1$, then $\max v = v \cap \max M^-$.

Consider the following set of idempotents:

$$H(a, b; v) = \{d^{-1}ab^{-1}d : d \in \max v\}.$$

The set of all upper bounds of $H(a, b; v)$ is clearly $\bigcap\{h^\omega : h \in H(a, b; v)\}$. Since M is finite-above, h^ω is a finite subsemilattice of E for every $h \in H(a, b; v)$ which contains 1_M. Therefore $\bigcap\{h^\omega : h \in H(a, b; v)\}$ is also a finite subsemilattice of E containing 1_M. This implies that $H(a, b; v)$ has a least upper bound which we denote by $h(a, b; v)$. The following condition will play a crucial role in this section:

(C) $\quad c \cdot h(a, b; v) \cdot c^{-1}b \not\leq a$ for some $c \in \max v$.

Note that if (C) is satisfied, then it is not difficult to check that 1, $u = a\sigma = b\sigma$, v are pairwise distinct elements of M/σ. Moreover, a and b are distinct, and $\max v$ contains an element d different from c. Figure 1 shows the arrows of $\mathcal{I}_M$ related to condition (C).

Denote the variety of Abelian groups by **Ab**. The main result of the section is based on the following statement.

Proposition 4.2.2. *Let M be a finite-above E-unitary inverse monoid such that condition* (C) *is satisfied for some $a, b \in \max M^-$ with $a \,\sigma\, b$ and for some $v \in M/\sigma$, and consider an appropriate $c \in \max v$. Let A be a quasi-generating set in M such that $A \subseteq \max M^-$ and $a, b, c \in A$, and consider M as a quasi-A-generated inverse monoid. Then there exists an arrow x in $F_{g\mathbf{Ab}}(\Gamma)$ such that $P_1^{\mathrm{cl}}(x)$ does not contain τx.*

Proof. For the proof, we adapt the proof of the converse part of Theorem 3.2.1 to the framework of the chapter. For every $d \in A$, denote the edge $(1, d, d\sigma)$ of Γ by $\underline{d}$, and put

$u = a\sigma = b\sigma$. Furthermore, consider the following arrows in $F_{g\mathbf{Ab}}(\Gamma)$:

$$x = (1, [\underline{a}]_{\mathbf{Ab}}, u), \quad y = (1, [\underline{b}]_{\mathbf{Ab}}, u), \quad z = (1, [\underline{c}]_{\mathbf{Ab}}, v).$$

Then we have $z^{-1}xy^{-1}z = (v, [\underline{c}'\underline{a}\underline{b}'\underline{c}]_{\mathbf{Ab}}, v) \in F_{g\mathbf{Ab}}(\Gamma)$, where $[\underline{c}'\underline{a}\underline{b}'\underline{c}]_{\mathbf{Ab}} = [\underline{a}\underline{b}']_{\mathbf{Ab}}$ in $F_{\mathbf{Ab}}(E_\Gamma)$. For brevity, put $h = h(a, b; v)$, and let o be a v-cycle in Γ^I such that $\lambda(o) = h$. It suffices to verify the following two statements:

$$P_0^{\mathrm{cl}}(z^{-1}xy^{-1}z) \subseteq \langle o\rangle^{\mathrm{cl}}, \tag{4.2.1}$$

$$\langle \underline{a}\rangle^{\mathrm{cl}} \cap \langle \underline{c}o\underline{c}'\underline{b}\rangle^{\mathrm{cl}} \text{ contains no } (1, u)\text{-path.} \tag{4.2.2}$$

For, we have $[\underline{c}(\underline{c}'\underline{a}\underline{b}'\underline{c})\underline{c}'\underline{b}]_{\mathbf{Ab}} = [\underline{a}]_{\mathbf{Ab}}$, whence $z(z^{-1}xy^{-1}z)z^{-1}y = x$, and so

$$C_1^{\mathrm{cl}}(x) \subseteq \langle \underline{a}\rangle^{\mathrm{cl}} \cap (\langle \underline{c}\rangle^{\mathrm{cl}} \vee P_0^{\mathrm{cl}}(z^{-1}xy^{-1}z) \vee \langle \underline{c}'\underline{b}\rangle^{\mathrm{cl}}).$$

Here (4.2.1) implies

$$\langle \underline{c}\rangle^{\mathrm{cl}} \vee P_0^{\mathrm{cl}}(z^{-1}xy^{-1}z) \vee \langle \underline{c}'\underline{b}\rangle^{\mathrm{cl}} \subseteq \langle \underline{c}\rangle^{\mathrm{cl}} \vee \langle o\rangle^{\mathrm{cl}} \vee \langle \underline{c}'\underline{b}\rangle^{\mathrm{cl}} = \langle \underline{c}o\underline{c}'\underline{b}\rangle^{\mathrm{cl}},$$

and so it follows by (4.2.2) that $C_1^{\mathrm{cl}}(x)$ contains no $(1, u)$-path.

Contrary to (4.2.2), assume that the graph $\langle \underline{a}\rangle^{\mathrm{cl}} \cap \langle \underline{c}o\underline{c}'\underline{b}\rangle^{\mathrm{cl}}$ contains a $(1, u)$-path, say s. Then $\lambda(s) \geq \lambda(\underline{a}) = a$ and $\lambda(s) \geq \lambda(\underline{c}o\underline{c}'\underline{b}) = chc^{-1}b$. Since a is a maximal element in M, the first inequality implies $\lambda(s) = a$, and so the second contradicts (C). This shows that (4.2.2) holds.

To prove (4.2.1), first we verify that

$$C_0^{\mathrm{cl}}(z^{-1}xy^{-1}z) = \bigcap\big\{\langle t'\underline{a}\underline{b}'t\rangle^{\mathrm{cl}} : t \text{ is a } (1, v)\text{-path}\big\}. \tag{4.2.3}$$

It suffices to show that, for every v-cycle s with $[s]_{\mathbf{Ab}} = [\underline{c}'\underline{a}\underline{b}'\underline{c}]_{\mathbf{Ab}} = [\underline{a}\underline{b}']_{\mathbf{Ab}}$, there exists a $(1, v)$-path t such that $\langle s\rangle = \langle t'\underline{a}\underline{b}'t\rangle$.

Let s be a v-cycle such that $[s]_{\mathbf{Ab}} = [\underline{a}\underline{b}']_{\mathbf{Ab}}$. Since $\underline{a}\underline{b}'$ is a non-trivial simple cycle, the former equality implies that s necessarily contains both $\underline{a}$ and $\underline{b}'$. Independently of the occurrences of $\underline{a}$ and $\underline{b}$ in s, the edges $\underline{a}$ and $\underline{b}'$ appear somewhere in the v-cycle $\tilde{s} = ss's$ in this order, that is, $\tilde{s} = t_0\underline{a}t_1\underline{b}'t_2$ for appropriate paths t_0, t_1, t_2. Moreover, we obviously have $\langle\tilde{s}\rangle = \langle s\rangle$ and $[\tilde{s}]_{\mathbf{Ab}} = [s]_{\mathbf{Ab}}$. Putting $\bar{s} = t_0\underline{a}\underline{b}'t$, where $t = \underline{b}t_1\underline{b}'t_2s's$, we easily see that $\langle\bar{s}\rangle = \langle s\rangle = \langle t\rangle$ and $[\bar{s}]_{\mathbf{Ab}} = [s]_{\mathbf{Ab}}$. Finally, the equalities $[t_0\underline{a}\underline{b}'t]_{\mathbf{Ab}} = [s]_{\mathbf{Ab}} = [\underline{a}\underline{b}']_{\mathbf{Ab}}$ imply that $[t_0]_{\mathbf{Ab}} = [t']_{\mathbf{Ab}}$, and so $[s]_{\mathbf{Ab}} = [t'\underline{a}\underline{b}'t]_{\mathbf{Ab}}$ and $\langle s\rangle = \langle t'\underline{a}\underline{b}'t\rangle$ follow. This completes the proof of (4.2.3).

Turning to the proof of (4.2.1), assume that k is a v-cycle in $C_0^{\mathrm{cl}}(z^{-1}xy^{-1}z)$. By (4.2.3) we see that

$$\lambda(k) \geq \lambda(t'\underline{a}\underline{b}'t) = \lambda(t)^{-1}ab^{-1}\lambda(t)$$

for every $(1, v)$-path t. Since there exists a $(1, v)$-path t with $\lambda(t) = d$ for every $d \in \max v$, we obtain that $\lambda(k)$ is an upper bound of $H(a, b; v)$, and so $\lambda(k) \geq h = \lambda(o)$ and $P_0^{\mathrm{cl}}(z^{-1}xy^{-1}z) \subseteq C_0^{\mathrm{cl}}(z^{-1}xy^{-1}z) \subseteq \langle o \rangle^{\mathrm{cl}}$. This verifies (4.2.1), and the proof of the proposition is complete. □

Combining Proposition 4.2.2 and Theorem 4.1.19(1) and (5), we obtain the following sufficient condition for a finite-above E-unitary inverse monoid to have no F-inverse cover via Abelian groups.

Theorem 4.2.3. *If M is a finite-above E-unitary inverse monoid such that for some $a, b \in \max M$ with $a \,\sigma\, b$ and for some $v \in M/\sigma$, condition* (C) *is satisfied, then M has no F-inverse cover via Abelian groups.*

Summary

The topic of the thesis falls in the area of semigroup theory, the class of semigroups considered is called *inverse monoids.* They are monoids defined by the property that every element x has a unique inverse x^{-1} such that $xx^{-1}x = x$, and $x^{-1}xx^{-1} = x^{-1}$ hold. They are one of the many generalizations of groups. One way they naturally arise is through partial symmetries — to put it informally, inverse monoids are to partial symmetries as what groups are to symmetries.

An important property of inverse monoids is that its idempotens commute, therefore form a semilattice. Inverse monoids also come equipped with a *natural partial order*, which extends the partial order on idempotens induced by the semilattice structure. It is defined by $s \leq t$ if and only if there exists and idempotent e such that $s = te$. It is not hard to see that factoring an inverse monoid by a congruence which collapses all idempotents yields a group, with the class containing all the idempotents as the identity element. Each inverse monoid M has a *smallest group congruence*, denoted by σ, and a corresponding *greatest group homomorphic image* M/σ.

A class of inverse monoids which play an important role in the thesis is called *E-unitary inverse monoids*, which is defined by the property that the σ-class containing the idempotents contains nothing but the idempotents. By a famous theorem of McAlister known as the P-theorem, each E-unitary inverse monoid can be assembled from a group, a semilattice and a partially ordered set. Hence, E-unitary inverse monoids are, in a way, 'known'. This is what gives particular significance to the McAlister covering theorem stating that every inverse monoid has an *E-unitary cover*, that is, every inverse monoid is a homomorphic image of an E-unitary inverse monoid under a homomorphism which is injective on the idempotens (this property is called *idempotent-separating*). It has also been shown that finite inverse monoids have finite E-unitary covers.

The other class of inverse monoids specified in the title is the one of *F-inverse monoids.* An inverse monoid is called F-inverse if its σ-classes have a greatest element with respect to the natural partial order. F-inverse monoids are always E-unitary. It is a well-known

folklore result that every inverse monoid has an *F-inverse cover*, that is, every inverse monoid M is a homomorphic image of an F-inverse monoid by an idempotent-separating homomorphism. We also call F an F-inverse cover of the inverse monoid M *over the group* G if G is isomorphic to M/σ. However, in this case, the proof always produces an F-inverse cover over a free group, and so it is always infinite. The main motivation of the research presented in this dissertation is the following:

Open problem (Henckell and Rhodes, [7])**.** Does every finite inverse monoid admit a finite F-inverse cover?

By the McAlister covering theorem, it suffices to restrict our attention to F-inverse covers of E-unitary inverse monoids, as we do throughout the thesis. The most important antecedent to the research presented in the dissertation is the paper of Auinger and Szendrei [2] on the question. They go a step further by applying that it is sufficient to restrict to a special class of E-unitary inverse monoids called Margolis–Meakin expansions, which have a very convenient structure. Thus Auinger and Szendrei are able to reformulate the F-inverse cover problem in the following way.

Let Γ be (directed) graph. There is an evident notion of paths in directed graphs, however, paths in this thesis are regarded in the larger graph $\overline{\Gamma}$ where Γ is extended by formal reverses of edges of Γ. Hence the path p, as a sequence of edges and reverse edges, represents a word in the free monoid $\overline{E_\Gamma}^*$ with involution $'$. If $\mathbf{U}$ is a variety of inverse monoids, then the value of p in the relatively free inverse monoid $F_\mathbf{U}(E_\Gamma)$ is denoted by $[p]_\mathbf{U}$.

The *free $g\mathbf{U}$-category on* Γ denoted by $F_{g\mathbf{U}}(\Gamma)$, as introduced in [22], is given as follows: its set of objects is V_Γ, and, for any pair of objects i, j, the set of (i, j)-arrows is

$$F_{g\mathbf{U}}(\Gamma)(i,j) = \{(i, [p]_\mathbf{U}, j) : p \text{ is an } (i,j)\text{-path in } \overline{\Gamma}\},$$

and the product of consecutive arrows is defined by

$$(i, [p]_\mathbf{U}, j)(j, [q]_\mathbf{U}, k) = (i, [pq]_\mathbf{U}, k).$$

We assign to each arrow x of $F_{g\mathbf{U}}(\Gamma)$ two sequences of finite subgraphs of Γ as follows: let

$$C_0(x) = \bigcap\{\langle p\rangle : (\iota p, [p]_\mathbf{U}, \tau p) = x\}, \tag{4.2.4}$$

and let $P_0(x)$ be the connected component of $C_0(x)$ containing ιx. If $C_n(x), P_n(x)$ are already defined for all x, then put

$$C_{n+1}(x) = \bigcap\{P_n(x_1) \cup \cdots \cup P_n(x_k) : k \in \mathbb{N}, x_1 \cdots x_k = x\},$$

and again, let $P_{n+1}(x)$ be the connected component of $C_{n+1}(x)$ containing ιx.

It is easy to see that

$$C_0(x) \supseteq P_0(x) \supseteq \cdots \supseteq C_n(x) \supseteq P_n(x) \supseteq C_{n+1}(x) \supseteq P_{n+1}(x) \supseteq \cdots$$

for all x and n. We define $P(x)$ to be $\bigcap_{n=0}^{\infty} P_n(x)$, which is a connected subgraph of Γ containing ιx. According to [2, Lemma 3.1], finite inverse monoids admit a finite F-inverse cover if and only if for any finite graph Γ, there exists a locally finite group variety $\mathbf{U}$ such that if $\tau x \in P(x)$ for all x. In this case, we say that Γ *satisfies property* $(S_{\mathbf{U}})$. In particular, the Cayley graph of a group G satisfies property $(S_{\mathbf{U}})$ if and only if the Margolis–Meakin expansion $M(G)$ has an F-inverse cover over a group which is an extension of some group in $\mathbf{U}$ by G — an F-inverse cover *via* $\mathbf{U}$, for short. If $\tau x \notin P(x)$ for some $x = (\iota p, [p]_{\mathbf{U}}, \tau p)$, then we call p a *breaking path* over $\mathbf{U}$.

In [20] and Chapter 3 of the thesis, we examine the property $(S_{\mathbf{U}})$. A main observation is that for a fixed group variety $\mathbf{U}$, non-$(S_{\mathbf{U}})$ graphs are closed upwards in the minor ordering, and can therefore be described by their minimal elements, called *forbidden minors.* The following theorem is main result of the chapter, and consists of the characterization of forbidden minors for non-trivial Abelian varieties.

Theorem ([20])**.** *A graph contains a breaking path over a non-trivial Abelian group variety of and only if its minors contain contain one of the graphs below:*

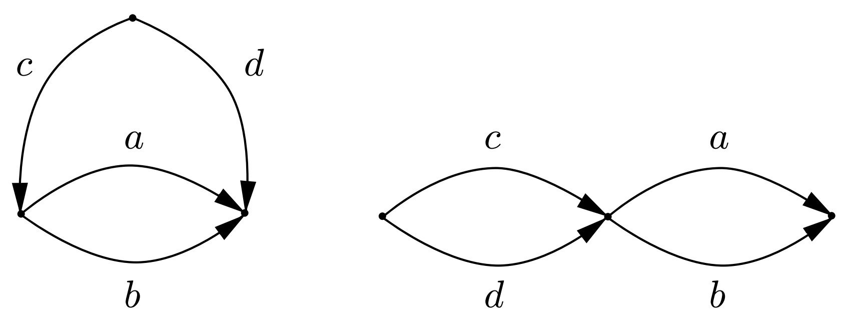


Figure 4.2.2. The forbidden minors for **Ab**

We also have the following consequence:

Theorem ([20])**.** *A Margolis–Meakin expansion of a group admits an F-inverse cover via an Abelian group if and only if the group is cyclic or free.*

In [21] and Chapter 4, we are looking to describe *all* finite E-unitary inverse monoids which admit an F-inverse cover via an Abelian group. The first step is introducing a

Margolis–Meakin-like structure that describes the much larger class of *finite-above* E-unitary inverse monoids — which, in particular, contains all finite ones —, and generalizing the conditions introduced in [2] accordingly.

Let M be an arbitrary E-unitary inverse monoid, and denote the group M/σ by G. The category $\mathcal{I}_M$ defined in the following way plays a crucial role in our construction: its set of objects is G, its set of (i,j)-arrows is

$$\mathcal{I}_M(i,j) = \{(i,m,j) \in G \times M \times G : i \cdot m\sigma = j\} \ (i,j \in G),$$

and the product of consecutive arrows $(i,m,j) \in \mathcal{I}_M(i,j)$ and $(j,n,k) \in \mathcal{I}_M(j,k)$ is defined by the rule

$$(i,m,j)(j,n,k) = (i,mn,k).$$

Finite-above E-unitary inverse monoids M have the property that they are generated by a set A consisting of maximal elements of M together with a set I of idempotents of M. We refer to this fact by saying that M is *quasi-A-generated*. The subgraph of $\mathcal{I}_M$ spanned by edges with middle components from $A \cup I$ is denoted by Γ^I. We introduce a closure operator on the set $\mathrm{Sub}(\Gamma^I)$ of all subgraphs of Γ^I. Given a path $p = e_1e_2\cdots e_n$ in $\overline{\mathcal{I}_M}$ where $e_j = (\iota e_j, m_j, \tau e_j)$ with $m_j \in \overline{M}$ for every j $(j = 1,2,\ldots,n)$, consider the word $w = m_1m_2\cdots m_n \in \overline{M}^*$ determined by the labels of the arrows in p, and let us assign an element of M to the path p by defining $\lambda(p) = [w]_M$. For a finite connected subgraph X in $\mathcal{I}_M$ and for $i,j \in V_{\mathrm{X}}$, let $\lambda_{(i,j)}(\mathrm{X})$ be $\lambda(p)$, where p is an (i,j)-path in $\overline{\mathcal{I}_M}$ with $\langle p\rangle = \mathrm{X}$, which can be seen to be well defined.

Given a finite connected subgraph X in Γ^I with vertices $i,j \in V_{\mathrm{X}}$, consider the subgraph

$$\mathrm{X}^{\mathrm{cl}} = \bigcup\{\mathrm{Y} \in \mathrm{Sub}(\Gamma^I) : \mathrm{Y} \text{ is finite and connected, } i,j \in V_{\mathrm{Y}}, \\ \text{and } \lambda_{(i,j)}(\mathrm{Y}) \geq \lambda_{(i,j)}(\mathrm{X})\},$$

which, again, is well defined. More generally, for any $\mathrm{X} \in \mathrm{Sub}(\Gamma^I)$, let us define the subgraph X^{cl} in the following manner:

$$\mathrm{X}^{\mathrm{cl}} = \bigcup\{\mathrm{Y}^{\mathrm{cl}} : \mathrm{Y} \text{ is a finite and connected subgraph of X}\}.$$

It is routine to check that $\mathrm{X} \to \mathrm{X}^{\mathrm{cl}}$ is a closure operator on $\mathrm{Sub}(\Gamma^I)$, and, as usual, a subgraph X of Γ^I is said to be *closed* if $\mathrm{X} = \mathrm{X}^{\mathrm{cl}}$. For any family X_j $(j \in J)$ of subgraphs of Γ^I, define $\bigvee_{j\in J} \mathrm{X}_j = \left(\bigcup_{j\in J} \mathrm{X}_j\right)^{\mathrm{cl}}$. The partially ordered set $(\mathrm{ClSub}(\Gamma^I); \subseteq)$ forms a complete lattice with respect to the usual intersection and the operation $\bigvee$ defined above.

Analogously to [2], we assign two sequences of subgraphs of Γ^I to any arrow x of $F_{g\mathbf{U}}(\Gamma)$. Let

$$C_0^{\mathrm{cl}}(x) = \bigcap\{\langle p\rangle^{\mathrm{cl}} : p \text{ is a } (\iota x, \tau x)\text{-path in } \overline{\Gamma} \text{ such that } x = (\iota x, [p]_{\mathbf{U}}, \tau x)\},$$

and let $P_0^{\mathrm{cl}}(x)$ be the component of $C_0^{\mathrm{cl}}(x)$ containing ιx. Suppose that, for some n ($n \geq 0$), the subgraphs $C_n^{\mathrm{cl}}(x)$ and $P_n^{\mathrm{cl}}(x)$ are defined for every arrow x of $F_{g\mathbf{U}}(\Gamma)$. Then let

$$C_{n+1}^{\mathrm{cl}}(x) = \bigcap\{P_n^{\mathrm{cl}}(x_1) \vee \cdots \vee P_n^{\mathrm{cl}}(x_k) : k \in \mathbb{N}_0,\ x_1, \ldots, x_k \in F_{g\mathbf{U}}(\Gamma) \text{ are consecutive arrows, and } x = x_1 \cdots x_k\},$$

and again, let $P_{n+1}^{\mathrm{cl}}(x)$ be the component of $C_{n+1}^{\mathrm{cl}}(x)$ containing ιx.

A main result of [21] and Chapter 4 states that the quasi-A-generated finite-above E-unitary inverse monoid M has an F-inverse cover via a group variety $\mathbf{U}$ if and only if for any arrow x in $F_{g\mathbf{U}}(\Gamma)$ and for any $n \in \mathbb{N}_0$, the graph $P_n^{\mathrm{cl}}(x)$ contains τx. Using this theorem, an example of a family of finite E-unitary inverse monoids is presented which have finite F-inverse cover, and this fact does not follow by previous techniques.

In Section 4.2, we concentrate on the variety $\mathbf{Ab}$ of Abelian groups. Let M be a finite-above E-unitary inverse monoid. Let us choose elements $a, b \in M$ with $a\,\sigma\,b$, and a σ-class $v \in M/\sigma$. Denote by $\max v$ the set of maximal elements of the σ-class v, and consider the following set of idempotents:

$$H(a, b; v) = \{d^{-1}ab^{-1}d : d \in \max v\}.$$

This set has a least upper bound in $E(M)$ which we denote by $h(a, b; v)$. The following condition plays a crucial role:

(C) $\quad c \cdot h(a, b; v) \cdot c^{-1}b \not\leq a$ for some $c \in \max v$.

We close the thesis with the following theorem on F-inverse covers of finite-above E-unitary monoids via $\mathbf{Ab}$:

Theorem ([21])**.** *If M is a finite-above E-unitary inverse monoid such that for some $a, b \in \max M$ with $a\,\sigma\,b$ and for some $v \in M/\sigma$, condition* (C) *is satisfied, then M has no F-inverse cover via Abelian groups.*

Összefoglaló

A disszertáció témája a félcsoportelmélet témaköréhez tartozik, a tárgyalt félcsoportok az úgynevezett *inverz monoidok*, vagyis olyan monoidok, amelyek bármely x elemének létezik olyan egyértelmű x^{-1} inverze, melyre $xx^{-1}x = x$ és $x^{-1}xx^{-1} = x^{-1}$ teljesül. Az inverz félcsoportok a csoportok általánosításai. Többek között parciális szimmetriák absztraktciójaként jönnek elő — az inverz monoidok olyan szerepet játszanak a parciális szimmetriák elméletében, mint a csoportok a szimmetriákéban.

Az inverz monoidok fontos tulajdonsága, hogy idempotensei felcserélhetőek, azaz félhálót alkotnak. Minden inverz monoidon adott egy *természetes részbenrendezés*, amely a félháló struktúrából adódó részbenrendezést terjeszti ki. Formálisan $s \leq t$ pontosan akkor, ha létezik olyan e idempotens, melyre $s = te$. Nem nehéz látni, hogy egy inverz monoidot olyan kongurenciával faktorizálva, amely minden idempotenst egybeejt, csoportot kapunk, melynek egységeleme az idempotenseket tartalmazó osztály. Ezen kongruenciák közül σ jelöli a *legkisebb csoportkongurenciát*, és így M/σ az M inverz monoid legnagyobb csoport homomorf képe.

A disszertáció címében is említett, úgynevezett *E-unitér inverz monoidok* definíciója az, hogy az idempotenseket tartalmazó σ-osztály csak az idemptenseket tartalmazza. A McAlister-féle P-tételként ismert híres eredmény szerint minden E-unitér inverz monoid felépíthető egy csoportból, egy félhálóból és egy részbenrendezett halmazból. Emiatt az E-unitér inverz monoidok bizonyos értelemben ismertek. Ez ad különleges jelentőséget a McAlister-féle fedési tételnek, mely azt mondja ki, hogy minden inverz monoidnak van E-unitér fedője, azaz minden inverz monoid homomorf képe valamely E-unitér inverz monoidnak, mégpedig olyan homomorfizmus mellett, mely az idempotenseken injektív (*idempotens-szétválasztó* homomorfizmus). Szintén ismert, hogy minden véges inverz monoidnak van véges E-unitér fedője.

A másik, a disszertációban fontos szerepet játszó félcsoportosztály az *F-inverz monoidok* osztálya. Egy inverz monoidot F-inverznek nevezünk, ha minden σ-osztálya tartalmaz legnagyobb elemet a természetes részbenrendezésre nézve. Az F-inverz monoidok mindig

E-unitérek. Jól ismert eredmény, hogy minden inverz monoidnak van *F-inverz fedője*, azaz minden inverz monoid idempotens-szétválasztó homomorf képe egy F-inverz monoidnak. Azt mondjuk, hogy az F inverz monoid F-inverz fedője M-nek a *G csoport felett*, ha G izomorf M/σ-val. A bizonyítás azonban ez esetben mindig szabad csoport feletti F-inverz fedőt eredményez, és ez mindig végtelen. A disszertáció fő motivációja a következő probléma:

Nyitott kérdés (Henckell és Rhodes, [7])**.** Létezik-e bármely véges inverz monoidnak véges F-inverz fedője?

A McAlister-féle fedési tétel alapján elég E-unitér inverz monoidok esetén vizsgálnunk a kérdést, ahogyan a disszertáció során is tesszük. A kutatásunk fő előzménye Auinger és Szendrei [2] cikke erről a kérdéskörről. Ebben még egy lépéssel tovább mennek azt alkalmazva, hogy elegendő speciális E-unitér inverz monoidok, úgynevezett Margolis–Meakin-kiterjesztések esetében megválaszolni a kérdést. Ennek segítségével Auinger és Szendrei a következőképp fogalmazzák át az F-inverz fedési problémát.

Legyen Γ (irányított) gráf. A továbbiakban nem szorítkozunk az irányított gráfokon megszokott irányított sétákra, ezért a Γ gráfot kiegészítjük az élek (vesszővel jelölt) fordítottjaival, és a sétákat az így kapott $\overline{\Gamma}$ gráfban tekintjük. Így egy p séta, mint élek és fordított élek formális sorozata, az $\overline{E}_\Gamma^*$ szabad involúciós monoid egy elemét határozza meg. Ha $\mathbf{U}$ inverz monoidok varietása, akkor p értékét az $F_\mathbf{U}(E_\Gamma)$ relatívan szabad inverz monoidban $[p]_\mathbf{U}$ jelöli.

Jelölje $F_{g\mathbf{U}}(\Gamma)$ a Γ-n értelmezett *szabad $g\mathbf{U}$-kategórát* [22], amelyet a következőképp adunk meg: az objektumok halmaza V_Γ, és bármely két i, j objektum esetén az (i, j)-morfizmusok halmaza

$$F_{g\mathbf{U}}(\Gamma)(i,j) = \{(i, [p]_\mathbf{U}, j) : p\ (i,j)\text{-séta } \overline{\Gamma}\text{-n}\},$$

csatlakozó morfizmusok szorzata pedig a következőképp definiált:

$$(i, [p]_\mathbf{U}, j)(j, [q]_\mathbf{U}, k) = (i, [pq]_\mathbf{U}, k).$$

Az $F_{g\mathbf{U}}(\Gamma)$ kategória minden x morfizmusához hozzárendeljük Γ részgráfjainak a következő két sorozatát: legyen

$$C_0(x) = \bigcap\{\langle p\rangle : (\iota p, [p]_\mathbf{U}, \tau p) = x\}, \tag{4.2.5}$$

és legyen $P_0(x)$ a $C_0(x)$ gráf ιx-et tartalmazó összefüggő komponense. Ha $C_n(x), P_n(x)$ minden x esetén definiált, akkor legyen

$$C_{n+1}(x) = \bigcap\{P_n(x_1) \cup \dots \cup P_n(x_k) : k \in \mathbb{N}, x_1 \cdots x_k = x\},$$

és legyen $P_{n+1}(x)$ ismét $C_{n+1}(x)$-nek a ιx-et tartalmazó összefüggő komponense.

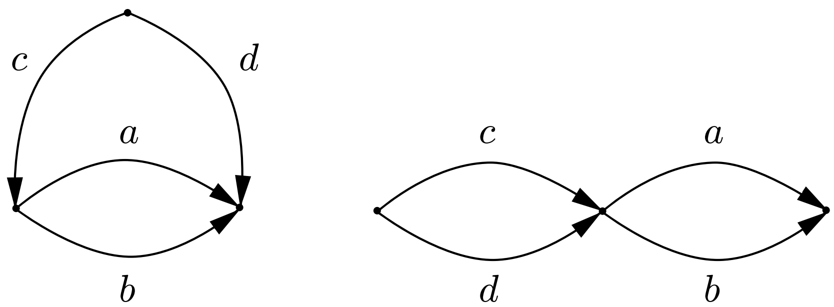
Könnyen látható, hogy bármely x és n esetén

$$C_0(x) \supseteq P_0(x) \supseteq \cdots \supseteq C_n(x) \supseteq P_n(x) \supseteq C_{n+1}(x) \supseteq P_{n+1}(x) \supseteq \cdots .$$

Jelölje $P(x)$ a $\bigcap_{n=0}^{\infty} P_n(x)$ metszetet, ez Γ-nak egy ιx-et tartalmazó részgráfja. Pontosan akkor van minden véges inverz monoidnak véges F-inverz fedője, ha bármely véges gráfhoz létezik olyan lokálisan véges **U** csoportvarietás, melyre $\tau x \in P(x)$ bármely x esetén [2, Lemma 3.1]. Ez esetben azt mondjuk, hogy Γ rendelkezik az $(S_{\mathbf{U}})$ *tulajdonsággal.* A G csoport Cayley-gráfja pontosan akkor rendelkezik az $(S_{\mathbf{U}})$ tulajdonsággal, ha az $M(G)$ Margolis–Meakin-kiterjesztésnek van F-inverz fedője olyan csoport felett, mely valamely **U**-beli csoport G-vel vett bővítése — röviden **U**-n *keresztüli* F-inverz fedője. Ha $\tau x \notin P(x)$ teljesül valamely $x = (\iota p, [p]_{\mathbf{U}}, \tau p)$ morfizmusra, akkor p-t *szakadó sétának* nevezzük **U** felett.

A disszertáció 3. fejezetében és [20]-ban az $(S_{\mathbf{U}})$ tulajdonságot vizsgáljuk. Fontos észrevétel, hogy rögzített **U** csoportvarietás esetén a nem-$(S_{\mathbf{U}})$ gráfok felfelé zártak a természetes részbenrendezésben, így leírhatók a minimális elemeikkel, úgynevezett *kizárt minorokkal.* A következő tétel a fejezet fő eredménye, és az Abel-féle varietásokhoz tartozó kizárt minorokat írja le.

Tétel ([20])**.** *Egy gráf pontosan akkor tartalmaz szakadó sétát nemtriviális Abel-féle csoportvarietás felett, ha minorként tartalmazza az alábbi gráfok valamelyikét:*



4.2.3. ábra. A kizárt minorok **Ab** esetén

Ebből megkapható, hogy mely Margolis–Meakin-kiterjesztéseknek van F-inverz fedője Abel-csoportokon keresztül:

Tétel ([20])**.** *Egy G csoport $M(G)$ Margolis–Meakin-kiterjesztésének pontosan akkor van F-inverz fedője Abel-csoporton keresztül, ha G szabad vagy ciklikus.*

A 4. fejezetben és [21]-ben az a célunk, hogy leírjunk *minden* olyan véges, E-unitér inverz monoidot, amelynek van F-inverz fedője Abel-csoporton keresztül. Az első lépés olyan Margolis–Meakin-kiterjesztéshez hasonló struktúra bevezetése, mely az ezeknél jóval szélesebb, úgynevezett *felfele véges* E-unitér inverz monoidok osztályát írja le, a második pedig a [2]-beli feltételek általánosítása ezen keretek között.

Legyen M tetszőleges E-unitér inverz monoid, jelölje az M/σ csoportot G. A konstrukciónkban kulcsszerepet játszik a következőképp definiált $\mathcal{I}_M$ kategória: az objektumok halmaza G, az (i,j)-morfizmusok halmaza

$$\mathcal{I}_M(i,j) = \{(i,m,j) \in G \times M \times G : i \cdot m\sigma = j\} \ (i,j \in G),$$

két csatlakozó morfizmus, $(i,m,j) \in \mathcal{I}_M(i,j)$ és $(j,n,k) \in \mathcal{I}_M(j,k)$ szorzata pedig

$$(i,m,j)(j,n,k) = (i,mn,k).$$

A felfele véges E-unitér inverz monoidokban választható olyan $A \cup I$ alakú generátorrendszer, ahol A elemei maximálisak M-ben, I pedig idempotensekből áll. Röviden úgy fogalmazzuk ezt meg, hogy M kvázi-A-generált. Jelölje Γ^I az $\mathcal{I}_M$ gráf azon élei által feszített részgráfját, melyek középső komponense $A \cup I$-ből való. Bevezetünk egy lezárási operátort Γ^I összes részgráfjának $\mathrm{Sub}(\Gamma^I)$ részbenrendezett halmazán. Legyen $p = e_1 e_2 \cdots e_n$ séta $\overline{\mathcal{I}_M}$-on, ahol $e_j = (\iota e_j, m_j, \tau e_j)$ és $m_j \in \overline{M}$ minden j $(j = 1,2,\ldots,n)$ esetén, és tekintsük a $w = m_1 m_2 \cdots m_n \in \overline{M}^*$ szót. A p sétához hozzárendeljük M-nek a $\lambda(p) = [w]_M$ elemét. Tetszőleges X véges, összefüggő, $\mathcal{I}_M$-beli részgráf és $i,j \in V_{\mathrm{X}}$ esetén legyen $\lambda_{(i,j)}(\mathrm{X}) = \lambda(p)$, ahol p egy X-et feszítő (i,j)-séta. Belátható, hogy ez jóldefiniált.

Tekintsük Γ^I egy X részgráfját, melyre $i,j \in V_{\mathrm{X}}$, és legyen

$$\begin{aligned}\mathrm{X}^{\mathrm{cl}} = \bigcup \{\mathrm{Y} \in \mathrm{Sub}(\Gamma^I) : \mathrm{Y} \text{ véges, összefüggő, } i,j \in V_{\mathrm{Y}},\\ \text{és } \lambda_{(i,j)}(\mathrm{Y}) \geq \lambda_{(i,j)}(\mathrm{X})\},\end{aligned}$$

amelyről ismét belátható, hogy jóldefiniált. Általánosabban, tetszőleges $\mathrm{X} \in \mathrm{Sub}(\Gamma^I)$ részgráf esetén a következőképp definiáljuk az X^{cl} gráfot:

$$\mathrm{X}^{\mathrm{cl}} = \bigcup \{\mathrm{Y}^{\mathrm{cl}} : \mathrm{Y} \text{ véges összefüggő részgráfja X-nek}\}.$$

Könnyen ellenőrizhető, hogy $\mathrm{X} \to \mathrm{X}^{\mathrm{cl}}$ lezárási operátor $\mathrm{Sub}(\Gamma^I)$-n, és a szokott módon az X részgráfot *zártnak* nevezzük, ha $\mathrm{X} = \mathrm{X}^{\mathrm{cl}}$. Részgráfok bármely $\mathrm{X}_j (j \in J)$ halmaza esetén legyen $\bigvee_{j \in J} \mathrm{X}_j = \left(\bigcup_{j \in J} \mathrm{X}_j\right)^{\mathrm{cl}}$. A $(\mathrm{ClSub}(\Gamma^I); \subseteq)$ részbenrendezett halmaz teljes hálót alkot a megszokott metszetre és a fent definiált $\bigvee$ egyesítésre.

Hasonlóan ahhoz, ahogyan [2]-ben láthattuk, az $F_{g\mathbf{U}}(\Gamma)$ kategória minden x morfizmusához hozzárendeljük Γ^I részgráfjainak két sorozatát. Legyen

$$C_0^{\mathrm{cl}}(x) = \bigcap\{\langle p\rangle^{\mathrm{cl}} : p\ (\iota x, \tau x)\text{-séta } \overline{\Gamma}\text{-ban, melyre } x = (\iota x, [p]_{\mathbf{U}}, \tau x)\},$$

és legyen $P_0^{\mathrm{cl}}(x)$ a $C_0^{\mathrm{cl}}(x)$ gráf ιx-et tartalmazó összefüggő komponense. Ha $C_n^{\mathrm{cl}}(x), P_n^{\mathrm{cl}}(x)$ minden x esetén definiált, legyen

$$C_{n+1}^{\mathrm{cl}}(x) = \bigcap\{P_n^{\mathrm{cl}}(x_1) \vee \cdots \vee P_n^{\mathrm{cl}}(x_k) : k \in \mathbb{N}_0,\ x_1, \ldots, x_k \in F_{g\mathbf{U}}(\Gamma) \text{ csatlakozó morfizmusok, és } x = x_1 \cdots x_k\},$$

és legyen $P_{n+1}^{\mathrm{cl}}(x)$ ismét $C_{n+1}^{\mathrm{cl}}(x)$-nek a ιx-et tartalmazó összefüggő komponense.

A 4. fejezet és [21] egyik fő eredménye azt mondja ki, hogy egy kvázi-A-generált felfele véges E-unitér inverz monoidnak pontosan akkor van F-inverz fedője az $\mathbf{U}$ csoportvarietáson keresztül, ha bármely $x \in F_{g\mathbf{U}}(\Gamma)$ morfizmus és $n \in \mathbb{N}_0$ esetén $P_n^{\mathrm{cl}}(x)$ tartalmazza τx-et. Ezen tétel segítségével megadunk véges E-unitér inverz monoidoknak olyan családját, amelyeknek van véges F-inverz fedője, és ez a korábbi eredményekből nem következik.

A 4.2 alfejezetben az Abel-csoportok varietására koncentrálunk. Tekintsük az M felfele véges E-unitér inverz monoidot, legyenek $a, b \in M$ olyan elemek, melyekre $a\ \sigma\ b$, és legyen $v \in M/\sigma$ egy σ-osztály. Jelölje $\max v$ a v σ-osztály maximális elemeinek halmazát, és tekintsük idempotenseknek a következő halmazát:

$$H(a, b; v) = \{d^{-1}ab^{-1}d : d \in \max v\}.$$

Ennek a halmaznak létezik legkisebb felső korlátja $E(M)$-ben, melyet $h(a, b; v)$ jelöl. A következő tulajdonság fontos szerepet játszik a disszertáció utolsó tételében:

(C) $\quad c \cdot h(a, b; v) \cdot c^{-1}b \not\leq a$ valamely $c \in \max v$ esetén.

Tétel ([21]). *Ha M olyan felfele véges E-unitér inverz monoid, melynek léteznek olyan $a, b \in M$, $a\ \sigma\ b$ elemei és olyan v σ-osztálya, amelyekre a (C) feltétel teljesül, akkor M-nek nincs F-inverz fedője Abel-csoporton keresztül.*

Bibliography

[1] C. J. Ash. Inevitable graphs: A proof of the Type II conjecture and some related decision procedures. *Internat. J. Algebra Comput.*, 1:127–146, 1991.

[2] K. Auinger and M. B. Szendrei. On F-inverse covers of inverse monoids. *J. Pure Appl. Algebra*, 204:493–506, 2006.

[3] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra.* Springer-Verlag, New York, 1981.

[4] P. Dehornoy. *Braids and self-distributivity.* Progress in Mathematics, 192, Birkhäuser Verlag, Basel, 2000.

[5] P. Dehornoy. The geometry monoid of left self-distributivity. *J. Pure Appl. Algebra*, 160:123–156, 2001.

[6] S. Eilenberg. *Automata, Languages and Machines, Vol. B.* Academic Press, New York, 1976.

[7] K. Henckell and J. Rhodes. The theorem of Knast, the $\mathbf{PG} = \mathbf{BG}$ and type II conjectures. In *Monoids and Semigroups with Applications*, pages 453–463, (Berkeley, CA, 1989), 1991. World Scientific, River Edge.

[8] K. Kaarli and L. Márki. A characterization of the inverse monoid of bi-congruences of certain algebras. *Internat. J. Algebra Comput.*, 19:791–808, 2009.

[9] J. Kellendonk and M.V. Lawson. Partial actions of groups. *Internat. J. Algebra Comput.*, 14:87–114, 2004.

[10] M. Koshkam and G. Skandalis. Regular representation of groupoid C^*-algebras and applications to inverse semigroups. *J. Reine Angew. Math.*, 546:47–72, 2002.

[11] M. Krasner and L. Kaloujnine. Produit complet des groupes de permutations et problème d'extensions de groupes II. *Acta Sci. Math. (Szeged)*, 13:208–230, 1950.

[12] M. V. Lawson. *Inverse Semigroups: The Theory of Partial Symmetries.* World Scientific, Singapore, 1998.

[13] S. W. Margolis and J. C. Meakin. E-unitary inverse monoids and the Cayley graph of a group presentation. *J. Pure Appl. Algebra*, 58:45–76, 1989.

[14] S. W. Margolis and J.-E. Pin. Inverse semigroups and varieties of finite semigroups. *J. Algebra*, 110:306–323, 1987.

[15] W. D. Munn. Free inverse semigroups. *Proc. London Math. Soc.*, 29:385–404, 1974.

[16] A. Nica. On a groupoid construction for actions of certain inverse semigroups. *Internat. J. Math.*, 5:249–372, 1994.

[17] M. Petrich. *Inverse Semigroups.* Wiley & Sons, New York, 1984.

[18] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 98:325–357, 2004.

[19] B. Steinberg. Strong Morita equivalence of inverse semigroups. *Houston J. Math.*, 37:895–927, 2011.

[20] N. Szakács. On the graph condition regarding the F-inverse cover problem. *Semigroup Forum.* DOI: 10.1007/s00233-015-9713-5.

[21] N. Szakács and M. B. Szendrei. On F-inverse covers of finite-above inverse monoids. *J. Algebra*, 452:42–65, 2016.

[22] B. Tilson. Categories as algebra: an essential ingredient in the theory of monoids. *J. Pure Appl. Algebra*, 48:83–198, 1987.