F-inverse covers of E-unitary inverse monoids

Outline of Ph.D. thesis

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1 Introduction

The topic of the thesis falls in the area of semigroup theory, the class of semigroups considered is called *inverse monoids* (see the monographies of Lawson [4] and Petrich [7] on the topic). They are monoids defined by the property that every element x has a unique inverse x^{-1} such that $xx^{-1}x = x$, and $x^{-1}xx^{-1} = x^{-1}$ hold. They are one of the many generalizations of groups. One way they naturally arise is through partial symmetries — to put it informally, inverse monoids are to partial symmetries as what groups are to symmetries.

Unlike in groups, in an inverse monoid, xx^{-1} is not necessarily the identity element, but it is, nevertheless, an idempotent. Idempotents therefore play an important role in the structure, and the set of idempotents of M is denoted by $E(M)$. An important property of inverse monoids is that its idempotens commute, therefore form a semilattice. Inverse monoids also come equipped with a *natural partial order*, which extends the partial order on idempotens induced by the semilattice structure. It is defined by $s \leq t$ if and only if there exists and idempotent e such that $s = te$. It is not hard to see that factoring an inverse monoid by a congruence which collapses all idempotents yields a group, with the class containing all the idempotents as the identity element. Each inverse monoid M has a *smallest group congruence*, denoted by σ, and a corresponding *greatest group homomorphic image* M/σ.

A class of inverse monoids which play an important role in the thesis is called *E-unitary inverse monoids*, which is defined by the property that the σ-class containing the idempotents contains nothing but the idempotents. By a famous theorem, known as the P-theorem of McAlister, each E-unitary inverse monoid is isomorphic to one of a special structure, built using three building blocks: a group G, a partially ordered set X, and a meet-semilattice Y which is a principal order ideal of X. By the P-theorem, E-unitary inverse monoids are, in a way, 'known'. This is what gives particular significance to the McAlister covering theorem stating that every inverse monoid has an *E-unitary cover*, that is, every inverse monoid is a homomorphic image of an E-unitary inverse monoid under a homomorphism which is injective on the idempotens (this property is called *idempotent-separating*). It has also been shown that finite inverse monoids have finite E-unitary covers.

The other class of inverse monoids specified in the title is the one of *F-inverse monoids.* An inverse monoid is called F-inverse if its σ-classes have a greatest element with respect to the natural partial order. F-inverse monoids are always E-unitary. It is a well-known folklore result that every inverse monoid has an *F-inverse cover*, that is, every inverse monoid M is a homomorphic image of an F-inverse monoid by an idempotent-separating homomorphism. We also call F an F-inverse cover of the inverse monoid M *over the group* G if G is isomorphic to M/σ. However, in this case, the proof always produces an F-inverse cover over a free group, and so it is always infinite. The main motivation of the research described in the dissertation is the following:

Open problem 1.1. Does every finite inverse monoid admit a finite F-inverse cover?

The problem has been formulated by Henckell and Rhodes in [3], and a positive answer would have solved an important conjecture connected to the complexity theory of finite semigroups. The latter conjecture has been since proven [1], but the F-inverse cover problem has remained open.

Note that by the McAlister covering theorem, it suffices to restrict our attention to F-inverse covers of E-unitary inverse monoids, as we do throughout the thesis. The most important antecedent to the research presented in the dissertation is the paper of Auinger and Szendrei [2] on the question. They go a step further by applying that it is sufficient to restrict to a special class of E-unitary inverse monoids called Margolis–Meakin expansions, which, as we will see, have a very convenient structure. Thus Auinger and Szendrei are able to reformulate the F-inverse cover problem by means of graphs and locally finite group varieties only.

The new results of the author and her adviser presented in the dissertation were published in the papers [9] and [10]. In [9], the condition on graphs and group varieties introduced in [2] is investigated. In [10], we generalize the results of both

[2] and [9] to a much larger class of inverse monoids.

2 Preliminaries

Throughout the thesis, by a *graph* we mean a directed graph. Given a graph Δ, its set of vertices and set of edges are denoted by V_Δ and E_Δ respectively. If $e \in E_\Delta$, then ιe and τe are used to denote the initial and terminal vertices of e. By an *edge-labelled* (or just *labelled*) graph, we mean a graph Δ together with a set A and a map $E_\Delta \to A$ appointing the labels to the edges.

There is an evident notion of paths on directed graphs, however, we do not generally want to restrict to directed paths. For that, we consider paths in a graph extended by the formal reverses of its edges as usual in such settings: let e' be the reverse of the edge e, and given a graph Δ, define $\overline{\Delta}$ to be the graph with $V_{\overline{\Delta}} = V_\Delta$ and $E_{\overline{\Delta}} = E_\Delta \cup E_{\Delta'}$, where $E_{\Delta'} = \{e' : e \in E_\Delta\}$. It is important to notice that paths in $\overline{\Delta}$ can be regarded as words in the free monoid $\overline{E_\Delta}^*$ with involution $'$ where $\overline{E_\Delta} = E_\Delta \cup E'_\Delta$. Connectedness of graphs, in consistency with these generalized paths, are regarded in an undirected sense throughout the thesis. The operation $'$ responsible for 'reversing' can be extended to all edges and paths of $\overline{\Delta}$ in a natural way. For a path p, *the subgraph* $\langle p \rangle$ *of* Δ *spanned by* p is the subgraph consisting of all vertices and edges p traverses in either direction.

A *(small) category* is a graph Δ with an associative, partial multiplication given on E_Δ in a way that, for any $e, f \in E_\Delta$, the product ef is defined if and only if e and f are consecutive edges, and a unique loop 1_i is given around every $i \in V_\Delta$ which acts as a local identity. For categories, the usual terminology and notation is different from those for graphs: instead of 'vertex' and 'edge', we use the terms 'object' and 'arrow', respectively, and if $\mathcal{X}$ is a category, then, instead of $V_\mathcal{X}$ and $E_\mathcal{X}$, we write $\operatorname{Ob}\mathcal{X}$ and $\operatorname{Arr}\mathcal{X}$, respectively.

By an *inverse category*, we mean a category $\mathcal{X}$ where, for every arrow $e \in \mathcal{X}(i,j)$, there exists a unique arrow $f \in \mathcal{X}(j,i)$ such that $efe = e$ and $fef = f$. This unique f is also called the *inverse of* e and is denoted e^{-1}. Inverse monoids are just inverse categories with one vertex, hence any statement about inverse categories also applies to inverse monoids. Furthermore, the basic notions for inverse monoids, such as idempotents and the natural partial order, have their evident analogues for inverse categories.

Let $\mathbf{U}$ be a variety of inverse monoids (in particular, a group variety), let Γ be a graph, and let $[p]_{\mathbf{U}}$ be the element of the relatively free inverse monoid $F_{\mathbf{U}}(E_\Gamma)$ determined by p. The *free* $g\mathbf{U}$*-category on* Γ denoted by $F_{g\mathbf{U}}(\Gamma)$, as introduced in [11], is given as follows: its set of objects is V_Γ, and, for any pair of objects i, j, the set of (i,j)-arrows is

$$F_{g\mathbf{U}}(\Gamma)(i,j) = \{(i, [p]_{\mathbf{U}}, j) : p \text{ is an } (i,j)\text{-path in } \overline{\Gamma}\},$$

and the product of consecutive arrows is defined by

$$(i, [p]_{\mathbf{U}}, j)(j, [q]_{\mathbf{U}}, k) = (i, [pq]_{\mathbf{U}}, k).$$

In the thesis, we construct several families of inverse monoids using groups acting on graphs and categories by graph and category morphisms. In particular, suppose G is a group acting on a category $\mathcal{X}$. This action determines a category $\mathcal{X}/G$ in a natural way: the objects of $\mathcal{X}/G$ are the orbits of the objects of $\mathcal{X}$, the orbit of i denoted by, as usual, ${}^G i = \{{}^g i : g \in G\}$, and, for every pair ${}^G i, {}^G j$ of objects, the $({}^G i, {}^G j)$-arrows are the orbits of the (i', j')-arrows of $\mathcal{X}$ where $i' \in {}^G i$ and $j' \in {}^G j$. Note that if G acts transitively on $\mathcal{X}$, then $\mathcal{X}/G$ is a one-object category, that is, a monoid. According to [6, Propositions 3.11, 3.14], if G is a group acting transitively and without fixed points on an inverse category $\mathcal{X}$, then the monoid $\mathcal{X}/G$ is inverse, and it is isomorphic, for every object i, to the monoid $(\mathcal{X}/G)_i$ defined on the set $\{(e, g) : g \in G \text{ and } e \in \mathcal{X}(i, {}^g i)\}$ by the multiplication

$$(e, g)(f, h) = (e \cdot {}^g f, gh).$$

For example, the *Margolis–Meakin expansion* $M(G)$ of an A-generated group G can be obtained this way as follows. Let Γ be the Cayley-graph of G, and consider the category $F_{g\mathbf{Sl}}(\Gamma)$, where $\mathbf{Sl}$ is the variety of semilattices. The group G acts on its Cayley graph Γ by left multiplication transitively and without fixed points, and this action can be extended to $F_{g\mathbf{Sl}}(\Gamma)$ naturally. Then $M(G)$ is exactly $F_{g\mathbf{Sl}}(\Gamma)/G$, with elements of the form (X, g) where $g \in G$ and X is a finite connected subgraph of Γ containing the vertices 1 and g. The multiplication is, of course, given by the rule

$$(\mathrm{X}, g)(\mathrm{Y}, h) = (\mathrm{X} \cup {}^{g}\mathrm{Y}, gh).$$

If $\mathbf{U}$ is a group variety, then $F_{g\mathbf{U}}(\Gamma)/G$ is a group, and is denoted by $G^{\mathbf{U}}$. It is the 'most general' A-generated group which is an extension of some group in $\mathbf{U}$ by G.

3 F-inverse covers of Margolis–Meakin expansions

In [2], Auinger and Szendrei reformulate the F-inverse cover problem using graphs and group varieties. They observe that it is sufficient to restrict their attention to the aforementioned Margolis–Meakin expansions of finite groups, and study their F-inverse covers by means of dual premorphisms.

A *dual premorphism* $\psi \colon M \to N$ between inverse monoids is a map satisfying $(m\psi)^{-1} = m^{-1}\psi$ and $(mn)\psi \geq m\psi \cdot n\psi$ for all m, n in M (such maps are called dual prehomomorphisms in [4] and prehomomorphisms in [7]). An important class of dual premorphisms from groups to an inverse monoid M is closely related to F-inverse covers of M, as stated in the following well-known result ([7, Theorem VII.6.11]):

Result 3.1. *Let H be a group and M be an inverse monoid. If $\psi \colon H \to M$ is a dual premorphism such that*

$$\text{for every } m \in M, \text{ there exists } h \in H \text{ with } m \leq h\psi, \tag{3.1}$$

then

$$F = \{(m, h) \in M \times H : m \leq h\psi\}$$

is an inverse submonoid in the direct product $M \times H$, and it is an F-inverse cover of M over H. Conversely, up to isomorphism, every F-inverse cover of M over H can be so constructed.

Such dual premorphisms into $M(G)$ can be studied through dual premorphisms $F_g\mathbf{U}(\Gamma)/G \to F_g\mathbf{Sl}(\Gamma)$ with $\psi|\Gamma = \mathrm{id}_\Gamma$. Fix a connected graph Γ and a group variety $\mathbf{U}$. Assign to each arrow x of $F_g\mathbf{U}(\Gamma)$ two sequences of finite subgraphs of Γ as follows: let

$$C_0(x) = \bigcap\{\langle p\rangle : (\iota p, [p]_\mathbf{U}, \tau p) = x\}, \tag{3.2}$$

and let $P_0(x)$ be the connected component of $C_0(x)$ containing ιx. If $C_n(x), P_n(x)$ are already defined for all x, then put

$$C_{n+1}(x) = \bigcap\{P_n(x_1) \cup \cdots \cup P_n(x_k) : k \in \mathbb{N}, x_1 \cdots x_k = x\},$$

and again, let $P_{n+1}(x)$ be the connected component of $C_{n+1}(x)$ containing ιx.

According to [2, Lemma 3.1], there exists a dual premorphism $\psi\colon F_g\mathbf{U}(\Gamma) \to F_g\mathbf{Sl}(\Gamma)$ with $\psi|\Gamma = \mathrm{id}_\Gamma$ if and only if $\tau x \in P_n(x)$ for all x and n. If $\tau x \notin P_n(x)$ for some $x = (\iota p, [p]_\mathbf{U}, \tau p)$ and $n \in \mathbb{N}$, then we call p a *breaking path* over $\mathbf{U}$.

For a group variety $\mathbf{U}$, we say that a *graph* Γ *satisfies property* $(S_\mathbf{U})$, or Γ is $(S_\mathbf{U})$ for short, if there is no breaking path in Γ over $\mathbf{U}$. In particular, the Cayley graph of a group G satisfies property $(S_\mathbf{U})$ if and only if the Margolis–Meakin expansion $M(G)$ has an F-inverse cover over a group which is an extension of some group in $\mathbf{U}$ by G — an F-inverse cover *via* $\mathbf{U}$, for short. The following consequence is a main result of [2].

Theorem 3.2 ([2]). *Each finite inverse monoid has a finite F-inverse cover if and only if each finite connected graph is $(S_\mathbf{U})$ for some locally finite group variety $\mathbf{U}$.*

This property $(S_\mathbf{U})$ for finite connected graphs is our topic in [9], and our findings are presented in Chapter 3 of the thesis. In [9], we prove that, given a group variety $\mathbf{U}$, the class of graphs satisfying $(S_\mathbf{U})$ can be described by so-called *forbidden minors*. Let Γ be a graph and let e be a (u, v)-edge of Γ such that $u \neq v$. The operation which removes e and simultaneously merges u and v to one vertex is called *edge-contraction.* We call Δ a *minor* of Γ if it can be obtained from Γ by edge-contraction, omitting vertices and edges, and redirecting edges.

A key lemma is the following:

Proposition 3.3. *Suppose Γ and Δ are graphs such that Δ is a minor of Γ. Then, if Δ is non-$(S_{\mathbf{U}})$, so is Γ.*

According to the previous proposition, the set of non-$(S_{\mathbf{U}})$ graphs is closed upwards in the minor ordering, hence, it is determined by its minimal elements, as illustrated in Figure 3.1. It is a consequence of a theorem of Robertson and Seymour [8] that the set of minimal elements must be finite.

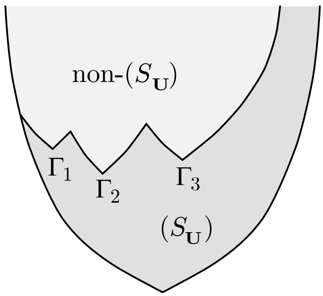

Figure 3.1: The partially ordered set of graphs and the forbidden minors

Theorem 3.4. *For any group variety $\mathbf{U}$, there exists a finite set of double-edge connected graphs $\Gamma_1, \ldots, \Gamma_n$ such that the graphs containing a breaking path over $\mathbf{U}$ are exactly those having one of $\Gamma_1, \ldots, \Gamma_n$ as a minor.*

The following is the main theorem of [9], which describes the forbidden minors for all non-trivial varieties of Abelian groups.

Theorem 3.5. *A graph contains a breaking path over $\mathbf{Ab}$ if and only if its minors contain at least one of the graphs in Figure 3.2.*

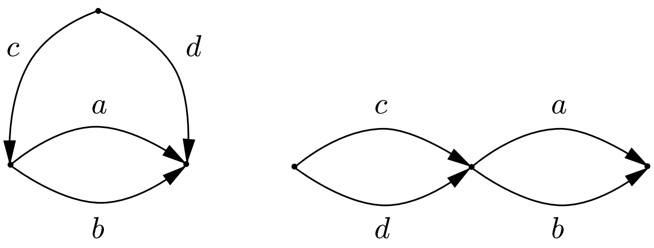

Figure 3.2: The forbidden minors for **Ab**

Unraveling the details of how the graph condition is related to F-inverse covers of Margolis–Meakin expansions, we get the following consequence:

Theorem 3.6. *A Margolis–Meakin expansion of a group admits an F-inverse cover via an Abelian group if and only if the group is cyclic or free.*

The previous theorem characterizes all Margolis–Meakin expansions $M(G)$ which have an F-inverse cover over a group which is an extension of some Abelian group by G. One could formulate the very same question for general inverse monoids. In [10] and Chapter 4 of the dissertation, we introduce a framework analogous to the one in [2], which allows us to investigate the proposed problem for a large class of E-unitary inverse monoids.

4 F-inverse covers of finite-above inverse monoids

In [10], the framework introduced in [2] for Margolis–Meakin expansions resulting in the graph condition is generalized for a class of E-unitary inverse monoids called finite-above. An inverse monoid M is called *finite-above* if the set $m^\omega = \{n \in M : n \geq m\}$ is finite for every $m \in M$. For example, finite inverse monoids and the Margolis–Meakin expansions of A-generated groups are finite-above.

Let $\mathcal{X}$ be an inverse category and Δ an arbitrary graph. We say that $\mathcal{X}$ is *quasi-Δ-generated* if a graph morphism $\epsilon_{\mathcal{X}} \colon \Delta \to \mathcal{X}$ is given such that the subgraph

$\Delta\epsilon_{\mathcal{X}} \cup \mathrm{E}(\mathcal{X})$ generates $\mathcal{X}$, where $\mathrm{E}(\mathcal{X})$ is the subgraph of the idempotents of $\mathcal{X}$.

Lemma 4.1. *Every finite-above inverse monoid is quasi-A-generated for some $A \subseteq \max M^-$.*

A dual premorphism $\psi\colon \mathcal{Y} \to \mathcal{X}$ between quasi-Δ-generated inverse categories is called *canonical* if $\epsilon_{\mathcal{Y}}\psi = \epsilon_{\mathcal{X}}$. An analogous argument to that found in [2] regarding F-inverse covers of Margolis–Meakin expansions yields the following:

Proposition 4.2. *Let M be a quasi-A-generated inverse monoid with $A \subseteq \max M^-$, put $G = M/\sigma$, and let $\mathbf{U}$ be a group variety. Then $G^{\mathbf{U}}$ is an A-generated group, and M has an F-inverse cover via the group variety $\mathbf{U}$ if and only if there exists a canonical dual premorphism $G^{\mathbf{U}} \to M$.*

We study canonical dual premorphisms $G^{\mathbf{U}} \to M$ by introducing a Margolis–Meakin-like structure for M. Let M be an arbitrary E-unitary inverse monoid, and again, put $G = M/\sigma$. The category $\mathcal{I}_M$ defined in the following way plays a crucial role in our construction: its set of objects is G, its set of (i,j)-arrows is

$$\mathcal{I}_M(i,j) = \{(i,m,j) \in G \times M \times G : i \cdot m\sigma = j\} \ (i,j \in G),$$

and the product of consecutive arrows $(i,m,j) \in \mathcal{I}_M(i,j)$ and $(j,n,k) \in \mathcal{I}_M(j,k)$ is defined by the rule

$$(i,m,j)(j,n,k) = (i,mn,k).$$

Given a path $p = e_1e_2\cdots e_n$ in $\overline{\mathcal{I}_M}$ where $e_j = (\iota e_j, m_j, \tau e_j)$ with $m_j \in \overline{M}$ for every j ($j = 1,2,\ldots,n$), consider the word $w = m_1m_2\cdots m_n \in \overline{M}^*$ determined by the labels of the arrows in p, and let us assign an element of M to the path p by defining $\lambda(p) = [w]_M$. For a finite connected subgraph X in $\mathcal{I}_M$ and for $i,j \in V_{\mathrm{X}}$, let $\lambda_{(i,j)}(\mathrm{X})$ be $\lambda(p)$, where p is an (i,j)-path in $\overline{\mathcal{I}_M}$ with $\langle p\rangle = \mathrm{X}$, which can be seen to be well defined.

Let M be a quasi-A-generated E-unitary inverse monoid with $A \subseteq \max M^-$. We give a model for $\mathcal{I}_M$ as a quasi-Γ-generated inverse category where Γ is the Cayley graph of G. Choose and fix a subset I of $E(M)$ such that $A \cup I$ generates

M. Consider the subgraphs Γ and Γ^I of $\mathcal{I}_M$ consisting of all edges with labels from A and from $A \cup I$, respectively. We introduce a closure operator on the set $\mathrm{Sub}(\Gamma^I)$ of all subgraphs of Γ^I.

Given a finite connected subgraph X in Γ^I with vertices $i, j \in V_{\mathrm{X}}$, consider the subgraph

$$\mathrm{X}^{\mathrm{cl}} = \bigcup \{\mathrm{Y} \in \mathrm{Sub}(\Gamma^I) : \mathrm{Y} \text{ is finite and connected, } i, j \in V_{\mathrm{Y}}, \\ \text{and } \lambda_{(i,j)}(\mathrm{Y}) \geq \lambda_{(i,j)}(\mathrm{X})\},$$

which, again, is well defined. More generally, for any $\mathrm{X} \in \mathrm{Sub}(\Gamma^I)$, let us define the subgraph X^{cl} in the following manner:

$$\mathrm{X}^{\mathrm{cl}} = \bigcup \{\mathrm{Y}^{\mathrm{cl}} : \mathrm{Y} \text{ is a finite and connected subgraph of X}\}.$$

It is routine to check that $\mathrm{X} \to \mathrm{X}^{\mathrm{cl}}$ is a closure operator on $\mathrm{Sub}(\Gamma^I)$, and, as usual, a subgraph X of Γ^I is said to be *closed* if $\mathrm{X} = \mathrm{X}^{\mathrm{cl}}$. We remark that the closure of a finite subgraph need not be finite. Denote the set of all closed subgraphs of Γ^I by $\mathrm{ClSub}(\Gamma^I)$, and its subset consisting of the closures of all finite connected subgraphs by $\mathrm{ClSub}_{\mathrm{fc}}(\Gamma^I)$. For any family X_j $(j \in J)$ of subgraphs of Γ^I, define $\bigvee_{j \in J} \mathrm{X}_j = \left(\bigcup_{j \in J} \mathrm{X}_j\right)^{\mathrm{cl}}$. The partially ordered set $(\mathrm{ClSub}(\Gamma^I); \subseteq)$ forms a complete lattice with respect to the usual intersection and the operation $\bigvee$ defined above.

We define an inverse category $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)$ in the following way: its set of objects is G, its set of (i,j)-arrows $(i, j \in G)$ is

$$\mathcal{X}_{\mathrm{cl}}(\Gamma^I)(i,j) = \{(i, \mathrm{X}, j) : \mathrm{X} \in \mathrm{ClSub}_{\mathrm{fc}}(\Gamma) \text{ and } i, j \in V_{\mathrm{X}}\},$$

and the product of two consecutive arrows is defined by

$$(i, \mathrm{X}, j)(j, \mathrm{Y}, k) = (i, \mathrm{X} \vee \mathrm{Y}, k).$$

It can be checked directly (see also [5]) that $\mathcal{X}_{\mathrm{cl}}(\Gamma^I) \to \mathcal{I}_M,\ (i, \mathrm{X}, j) \mapsto (i, \lambda_{(i,j)}(\mathrm{X}), j)$ is a category isomorphism, and therefore $\mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$ is canonically isomorphic to the inverse monoid M by $(X, g) \mapsto \lambda_{(1,g)}(\mathrm{X})$. This provides a representation of M as a P-semigroup.

The goal of this section is to give equivalent conditions for the existence of a canonical dual premorphism $G^{\mathbf{U}} \to M$. Since $G^{\mathbf{U}} = F_{g\mathbf{U}}(\Gamma)/G$ and $M \cong \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$, this naturally corresponds to canonical dual premorphisms $F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$.

Analogously to [2], we assign two sequences of subgraphs of Γ^I to any arrow x of $F_{g\mathbf{U}}(\Gamma)$. Let

$$C_0^{\mathrm{cl}}(x) = \bigcap\{\langle p\rangle^{\mathrm{cl}} : p \text{ is a } (\iota x, \tau x)\text{-path in } \overline{\Gamma} \text{ such that } x = (\iota x, [p]_{\mathbf{U}}, \tau x)\},$$

and let $P_0^{\mathrm{cl}}(x)$ be the component of $C_0^{\mathrm{cl}}(x)$ containing ιx. Suppose that, for some n ($n \geq 0$), the subgraphs $C_n^{\mathrm{cl}}(x)$ and $P_n^{\mathrm{cl}}(x)$ are defined for every arrow x of $F_{g\mathbf{U}}(\Gamma)$. Then let

$$C_{n+1}^{\mathrm{cl}}(x) = \bigcap\{P_n^{\mathrm{cl}}(x_1) \vee \cdots \vee P_n^{\mathrm{cl}}(x_k) : k \in \mathbb{N}_0,\ x_1, \ldots, x_k \in F_{g\mathbf{U}}(\Gamma) \text{ are consecutive arrows, and } x = x_1 \cdots x_k\},$$

and again, let $P_{n+1}^{\mathrm{cl}}(x)$ be the component of $C_{n+1}^{\mathrm{cl}}(x)$ containing ιx.

For any arrow x and index n, $C_n^{\mathrm{cl}}(x)$ and $P_n^{\mathrm{cl}}(x)$ are closed subgraphs, furthermore, $P_n^{\mathrm{cl}}(x)$ is connected and contains ιx. The following is a main theorem of [10]:

Theorem 4.3. *Let M be a quasi-A-generated finite-above E-unitary inverse monoid with $A \subseteq \max M^-$, put $G = M/\sigma$, and let $\mathbf{U}$ be a group variety. Let Γ be the Cayley graph of M/σ. The following statements are equivalent.*

(1) M has an F-inverse cover via the group variety $\mathbf{U}$.

(2) There exists a canonical dual premorphism $G^{\mathbf{U}} \to M$.

(3) There exists a canonical dual premorphism $G^{\mathbf{U}} \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)/G$.

(4) There exists a canonical dual premorphism $F_{g\mathbf{U}}(\Gamma) \to \mathcal{X}_{\mathrm{cl}}(\Gamma^I)$.

(5) For any arrow x in $F_{g\mathbf{U}}(\Gamma)$ and for any $n \in \mathbb{N}_0$, the graph $P_n^{\mathrm{cl}}(x)$ contains τx.

As an example, we describe a class of non-F-inverse finite-above inverse monoids for which Theorem 4.3 yields F-inverse covers via any non-trivial group variety.

Example 4.4. Let G be a group acting on a semilattice S where S has no greatest element, and for every $s \in S$, the set of elements greater than s is finite. Consider a semidirect product $S \rtimes G$ of S by G, and let $M = (S \rtimes G)^1$, the inverse monoid obtained from $S \rtimes G$ by adjoining an identity element 1. Then M is a finite-above E-unitary inverse monoid which is not F-inverse, but it has an F-inverse cover via any non-trivial group variety.

This example sheds light on the generality of our construction in contrast with that in [2]. By Theorem 3.6, the Margolis–Meakin expansion of a group admits an F-inverse cover via an Abelian group if and only if the group is cyclic or free. The previous example shows that, for any group G, there exist finite-above E-unitary inverse monoids with greatest group homomorphic image G that fail to be F-inverse but admit F-inverse covers via Abelian groups.

In the following, our aim is to generalize Theorem 3.6 to finite-above, E-unitary inverse monoids. An easy consequence of Theorem 4.3 is the following:

Proposition 4.5. *If M is a finite-above E-unitary inverse monoid with $|M/\sigma| \leq 2$, then M has an F-inverse cover via any non-trivial group variety. In particular, M has an F-inverse cover via an elementary Abelian p-group for any prime p.*

Now suppose M is a finite-above E-unitary inverse monoid such that M/σ has at least two elements distinct from 1, and there exists a σ-class in M containing at least two maximal elements. Let us choose such elements $a, b \in M$ with $a \,\sigma\, b$, and σ-class $v \in M/\sigma$. Denote by $\max v$ the set of maximal elements of the σ-class v, and consider the following set of idempotents:

$$H(a, b; v) = \{d^{-1}ab^{-1}d : d \in \max v\}.$$

This set has a least upper bound in $E(M)$ which we denote by $h(a, b; v)$. The following condition plays a crucial role:

(C) $\quad c \cdot h(a, b; v) \cdot c^{-1} b \not\leq a$ for some $c \in \max v$.

Using Theorem 4.3, we obtain the following sufficient condition for M not to have a F-inverse cover via the variety of Abelian groups.

Theorem 4.6. *If M is a finite-above E-unitary inverse monoid such that for some $a, b \in \max M$ with $a \,\sigma\, b$ and for some $v \in M/\sigma$, condition* (C) *is satisfied, then M has no F-inverse cover via Abelian groups.*

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