

Smoothness of the sum and Riemann  
summability of double trigonometric series,  
Lebesgue summability of double  
trigonometric integrals

Abstract of Ph.D. Thesis

LÍVIA KRIZSÁN

Supervisor:

FERENC MÓRICZ D.Sc.  
professor emeritus

Doctoral School in Mathematics and Computer Science  
University of Szeged  
Faculty of Science and Informatics  
Bolyai Institute

Szeged  
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# Introduction

In the first part of the disertation we deal with single and double trigonometric series which are absolutely and uniformly convergent, consequently the sum function of the series is uniformly continuous. It is also well known that if a trigonometric series is uniformly convergent, then it is the Fourier series of its sum. In the first two chapters we study the smoothness of the sum of single and double trigonometric series. We give sufficient conditions under which the sum function of the series belongs to one of the Lipschitz  $\text{lip}(\alpha)$ ,  $\text{Lip}(\alpha)$  or Zygmund classes  $\text{zyg}(\alpha)$ ,  $\text{Zyg}(\alpha)$  in the single-variable cases, or belongs to one of the multiplicative Lipschitz  $\text{lip}(\alpha, \beta)$ ,  $\text{Lip}(\alpha, \beta)$  or Zygmund classes  $\text{zyg}(\alpha, \beta)$ ,  $\text{Zyg}(\alpha, \beta)$  in the two-variable cases.

In the second part of the disertation we define two new summation methods: the Riemann summability of double trigonometric series, and Lebesgue summability of double trigonometric integrals. In the third chapter we extend the concept of the Riemann summability from single to double trigonometric series. The Riemann summability of trigonometric series is defined in terms of the second symmetric differentiability of the sum of the formally twice integrated series. We give sufficient conditions which guarantee that if the double series converges regularly at some point, then it is also Riemann summable to the same limit. The proof is based on Robison's theorem (see in [16]) which describes the properties of bounded-regular matrices.

In the fourth chapter we extend the concept of the Lebesgue summability from single to double trigonometric integrals. The Lebesgue summability of trigonometric series is defined in terms of the symmetric differentiability of the sum of the formally integrated series (see in [21, Vol I, pp. 321.]). We give sufficient conditions under which the double integral is Lebesgue summable at some point if and only if it converges in Pringsheim's sense at to the same limit.

The disertation is based on the following papers of the author:

L. Krizsán and F. Móricz, *Generalization of Zygmund's theorem on the smoothness of the sum of trigonometric series*, Acta Sci. Math. (Szeged), **78** (2012), 155–164.

L. Krizsán and F. Móricz, *A two-dimensional extension of Zygmund's theorem on the smoothness of the sum of trigonometric series*, Acta Sci. Math. (Szeged), **79** (2013), 49–62.

L. Krizsán and F. Móricz, *The Lebesgue summability of double trigonometric integrals*, Math. Inequal. Appl., **17** (2014), 1543–1550.

L. Krizsán, *On the Riemann summability of double trigonometric series*, Math. Pannonica, **25/1** (2014-2015), 135–145.

## 1. Smoothness of the sum of single trigonometric series

**Definition.** Let  $\{c_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$  be a sequence of complex numbers such that

$$\sum_{n \in \mathbb{Z}} |c_n| < \infty.$$

Then the trigonometric series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

converges absolutely and uniformly. Denote the sum of it by  $f(x)$ :

$$(1.1) \quad f(x) := \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad x \in \mathbb{T} := [-\pi, \pi) .$$

Due to the uniform convergence the function  $f(x)$  is continuous.

**Definition.** We consider periodic functions  $f : \mathbb{T} := [-\pi, \pi) \rightarrow \mathbb{C}$ . The function  $f$  is said to belong to the Lipschitz class  $\text{Lip}(\alpha)$  for some  $\alpha > 0$  if there exists a constant  $C$  depending on  $f$  and  $\alpha$  such that

$$|\Delta f(x, h)| := |f(x+h) - f(x)| \leq Ch^\alpha$$

for all  $x$  and  $h > 0$ .

We say that  $f$  belongs to the little Lipschitz class  $\text{lip}(\alpha)$  if

$$\lim_{h \rightarrow 0} h^{-\alpha} |f(x+h) - f(x)| = 0$$

uniformly in  $x$ .

A continuous function  $f$  is said to belong to the Zygmund class  $\text{Zyg}(\alpha)$  for some  $\alpha > 0$  if there exists a constant  $C$  depending on  $f$  and  $\alpha$  such that

$$|\Delta^2 f(x, h)| := |f(x+h) - 2f(x) + f(x-h)| \leq Ch^\alpha$$

for all  $x$  and  $h > 0$ .

We say that a continuous function  $f$  belongs to the little Zygmund class  $\text{zyg}(\alpha)$  for some  $\alpha > 0$  if

$$\lim_{h \rightarrow 0} h^{-\alpha} |f(x+h) - 2f(x) + f(x-h)| = 0$$

uniformly in  $x$ .

Zygmund gave sufficient conditions to ensure that the sum  $f(x)$  of series (1.1) belongs to the class  $\text{Zyg}(1)$  or  $\text{zyg}(1)$  (see in [21, Vol. I, p. 320.]). We generalized these theorems for any arbitrary  $0 < \alpha \leq 2$ . Our new results are formulated in the following two theorems.

**Theorem 1.1.** *Let  $\{c_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$ . If for some  $0 < \alpha \leq 2$  we have*

$$(1.2) \quad \frac{1}{N^{2-\alpha}} \sum_{|n| \leq N} n^2 |c_n| \leq C_\alpha \quad (N = 1, 2, \dots),$$

*where  $C_\alpha$  is a constant, then the series (1.1) converges absolutely and uniformly, and its sum  $f(x) \in \text{Zyg}(\alpha)$ .*

**Theorem 1.2.** *Let  $\{c_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$ . If for some  $0 < \alpha < 2$  we have*

$$(1.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{2-\alpha}} \sum_{|n| \leq N} n^2 |c_n| = 0,$$

*then  $f(x) \in \text{zyg}(\alpha)$ .*

**Remark.** If  $0 < \alpha < 1$ , then

$$\text{Lip}(\alpha) = \text{Zyg}(\alpha) \quad \text{and} \quad \text{lip}(\alpha) = \text{zyg}(\alpha).$$

Consequently, under condition (1.2) and (1.3) the sum function  $f(x)$  also belongs to the Lipschitz classes  $\text{Lip}(\alpha)$  or  $\text{lip}(\alpha)$ .

In the case  $\alpha = 1$  we have  $\text{Lip}(1) \subset \text{Zyg}(1)$ . We were able to prove only the following.

**Theorem 1.3.** *If  $\{c_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$  is such that*

$$\sum_{n \in \mathbb{Z}} |nc_n| < \infty,$$

*then the sum  $f(x) \in \text{Lip}(1)$ .*

## 2. Smoothness of the sum of double trigonometric series

**Definition.** Let  $\{c_{m,n}\}_{(m,n) \in \mathbb{Z}^2} \subset \mathbb{C}$  be a double sequence of complex numbers such that

$$(2.1) \quad \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |c_{m,n}| < \infty.$$

Then the double trigonometric series

$$(2.2) \quad f(x, y) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} c_{m,n} e^{i(mx+ny)}, \quad (x, y) \in \mathbb{T}^2$$

converges absolutely and uniformly. Consequently, its sum  $f(x, y)$  is uniformly continuous.

**Definition.** We define the difference operator  $\Delta$  as follows

$$\Delta f(x, y; h, k) := f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y), \quad h, k > 0.$$

**Definition.** We recall that a continuous function  $f$  is said to belong to the multiplicative Lipschitz class  $\text{Lip}(\alpha, \beta)$  for some  $\alpha, \beta > 0$  (see in [11]) if

$$|\Delta f(x, y; h, k)| \leq Ch^\alpha k^\beta \quad \text{for all } x, y \quad \text{and } h, k > 0,$$

where the constant  $C$  depends only on  $f$ ,  $\alpha$  and  $\beta$ .

We say that  $f$  belongs to the little Lipschitz class  $\text{lip}(\alpha, \beta)$  if

$$\lim_{h, k \rightarrow 0} h^{-\alpha} k^{-\beta} \Delta f(x, y; h, k) = 0$$

uniformly in  $x, y$ .

We recall that a continuous function  $f$  is said to belong to the multiplicative Zygmund class  $\text{Zyg}(\alpha, \beta)$  for some  $\alpha, \beta > 0$  (see in [2]) if

$$|\Delta^2 f(x, y; h, k)| \leq Ch^\alpha k^\beta \quad \text{for all } x, y \quad \text{and } h, k > 0,$$

where the constant  $C$  depends only on  $f$ ,  $\alpha$  and  $\beta$ .

We say that  $f$  belongs to the little Zygmund class  $\text{zyg}(\alpha, \beta)$  if

$$\lim_{h, k \rightarrow 0} h^{-\alpha} k^{-\beta} \Delta^2 f(x, y; h, k) = 0$$

uniformly in  $x$  and  $y$ .

Our main new results are stated in the following theorems.

**Theorem 2.1.** *Suppose  $\{c_{m,n}\} \subset \mathbb{C}$  is such that*

$$(2.3) \quad \sum_{m \in \mathbb{Z}} |c_{m,0}| < \infty \quad \text{and} \quad \sum_{n \in \mathbb{Z}} |c_{0,n}| < \infty.$$

*If for some  $0 < \alpha, \beta \leq 2$*

$$(2.4) \quad \frac{1}{M^{2-\alpha} N^{2-\beta}} \sum_{|m| \leq M} \sum_{|n| \leq N} m^2 n^2 |c_{m,n}| \leq C_{\alpha,\beta} \quad (M, N = 1, 2, \dots),$$

*where  $C_{\alpha,\beta}$  is a constant depending only on  $\alpha$  and  $\beta$ , then condition (2.1) is satisfied and the function  $f$  defined in (2.2) belongs to the class  $\text{Zyg}(\alpha, \beta)$ .*

**Theorem 2.2.** *Suppose  $\{c_{m,n}\} \subset \mathbb{C}$  is such that conditions (2.3) and (2.4) are satisfied for some  $0 < \alpha, \beta < 2$ , and, in addition, if*

$$\lim_{M, N \rightarrow \infty} \frac{1}{M^{2-\alpha} N^{2-\beta}} \sum_{|m| \leq M} \sum_{|n| \leq N} m^2 n^2 |c_{m,n}| = 0,$$

*then the function  $f$  defined in (2.2) belongs to the class  $\text{zyg}(\alpha, \beta)$ .*

Under stronger conditions analogous theorems can also be proved for the Lipschitz classes  $\text{Lip}(\alpha, \beta)$  and  $\text{lip}(\alpha, \beta)$ , where  $0 < \alpha, \beta \leq 1$ .

**Theorem 2.3.** *Suppose  $\{c_{m,n}\} \subset \mathbb{C}$  is such that condition (2.3) is satisfied. If for some  $0 < \alpha, \beta \leq 1$*

$$(2.5) \quad \frac{1}{M^{1-\alpha} N^{1-\beta}} \sum_{|m| \leq M} \sum_{|n| \leq N} |mnc_{m,n}| \leq C_{\alpha,\beta}^{(3)} \quad (M, N = 1, 2, \dots),$$

*where  $C_{\alpha,\beta}^{(3)}$  is a constant depending only on  $\alpha$  and  $\beta$ , then condition (2.1) is satisfied and the function  $f$  defined in (2.2) belongs to the class  $\text{Lip}(\alpha, \beta)$ .*

**Theorem 2.4.** *Suppose  $\{c_{m,n}\} \subset \mathbb{C}$  is such that conditions (2.3) and (2.5) are satisfied for some  $0 < \alpha, \beta < 1$ , and, in addition, if*

$$\lim_{M, N \rightarrow \infty} \frac{1}{M^{1-\alpha} N^{1-\beta}} \sum_{|m| \leq M} \sum_{|n| \leq N} |mnc_{m,n}| = 0$$

*then the function  $f$  defined in (2.2) belongs to the class  $\text{lip}(\alpha, \beta)$ .*

### 3. Riemann summability of double trigonometric series

Let  $\{c_{m,n}\} \subset \mathbb{C}$  be a double sequence of complex numbers. We consider the double trigonometric series

$$(3.1) \quad \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} c_{m,n} e^{i(mx+ny)}$$

with the symmetric rectangular partial sums

$$s_{M,N}(x, y) := \sum_{|m| \leq M} \sum_{|n| \leq N} c_{m,n} e^{i(mx+ny)} \quad (M, N = 0, 1, 2, \dots).$$

**Definition.** We recall that the double series (3.1) is said to converge in Pringsheim's sense to the finite sum  $S$  at some point  $(x, y)$  if for every  $\varepsilon > 0$  there exists a natural number  $n_0$  such that

$$|s_{M,N}(x, y) - S| < \varepsilon \quad \text{if } M, N > n_0.$$

The convergence of a double series in Pringsheim's sense does not imply the boundedness of its terms, and also does not involve the convergence of any of its row or column series defined respectively by

$$(3.2) \quad \begin{aligned} & \sum_{m \in \mathbb{Z}} e^{imx} (c_{m,-n} e^{-iny} + c_{m,n} e^{iny}) \quad (n \in \mathbb{Z}) \\ & \text{and} \\ & \sum_{n \in \mathbb{Z}} e^{inx} (c_{-m,n} e^{-imx} + c_{m,n} e^{imx}) \quad (m \in \mathbb{Z}). \end{aligned}$$

These are the reasons why Hardy [4] introduced a stronger notion of convergence, namely the regular convergence of double series.

**Definition.** The series (3.1) is said to converge regularly to the sum  $S$  if it converges to  $S$  in Pringsheim's sense and each of its row and column series defined in (3.2) also converge as single series.

**Definition.** Let  $\{c_{m,n}\} \subset \mathbb{C}$  be a double sequence of complex numbers. Integrating the double series (3.1) formally twice with respect to both  $x$  and  $y$ , we obtain the double series

$$(3.3) \quad \begin{aligned} R(x, y) := & c_{0,0} \frac{x^2 y^2}{4} - \frac{y^2}{2} \sum_{|m| \geq 1} c_{m,0} \frac{e^{imx}}{m^2} - \frac{x^2}{2} \sum_{|n| \geq 1} c_{0,n} \frac{e^{inx}}{n^2} + \\ & + \sum_{|m| \geq 1} \sum_{|n| \geq 1} c_{m,n} \frac{e^{i(mx+ny)}}{m^2 n^2}, \quad (x, y) \in \mathbb{T}^2. \end{aligned}$$

If the sequence  $\{c_{m,n}\}$  is bounded, then the double series in (3.3) converges absolutely and uniformly. Consequently, the function  $R$  is defined at every  $(x, y) \in \mathbb{T}^2$ , and it is continuous.

**Definition.** We introduce the notation

$$\begin{aligned} \Delta^2 R(x, y; 2u, 2v) := & R(x+2u, y+2v) + R(x-2u, y+2v) + R(x+2u, y-2v) + \\ & + R(x-2u, y-2v) - 2R(x+2u, y) - 2R(x, y+2v) - \\ & - 2R(x-2u, y) - 2R(x, y-2v) + 4R(x, y) \quad (u, v > 0). \end{aligned}$$

If the limit

$$\lim_{u,v \rightarrow 0} \frac{\Delta^2 R(x, y; 2u, 2v)}{16u^2v^2} = s$$

exists, then the double series (3.1) is said to be summable at the point  $(x, y)$  by the Riemann method of summation (or briefly: Riemann summable) to the sum  $s$ .

The next two theorems are counterparts of Riemann's theorems published in [15].

**Theorem 3.1.** *Suppose that  $\{c_{m,n}\} \subset \mathbb{C}$  is such that*

$$(3.4) \quad \lim_{|m|+|n| \rightarrow \infty} c_{m,n} = 0.$$

*If the double series (3.1) converges regularly at some point  $(x, y)$  to a finite sum  $s$ , then it is also Riemann summable to  $s$ .*

**Theorem 3.2.** *If condition (3.4) is satisfied, then uniformly in  $(x, y)$  we have*

$$\frac{\Delta^2 R(x, y; 2u, 2v)}{16uv} \rightarrow 0 \quad (u, v \rightarrow 0).$$



## 4. Lebesgue summability of double trigonometric integrals

The Lebesgue summability of trigonometric series is defined in terms of the symmetric differentiability of the sum of the formally integrated series (see in [21, Vol I, pp. 321.]). We note that M. Bagota and F. Móricz extended the concept of Lebesgue summability from single to double trigonometric series in [1]. The most recent results on Lebesgue summability of trigonometric integrals are published in Móricz[13].

**Definition.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  be such that it is integrable in Lebesgue's sense over any bounded rectangle  $[a, b] \times [c, d]$  of  $\mathbb{R}^2$ , in symbols:  $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ . We consider the double trigonometric integral

$$(4.1) \quad \iint_{\mathbb{R}^2} f(s, t) e^{i(sx+ty)} ds dt, \quad (x, y) \in \mathbb{T}^2$$

with its symmetric rectangular partial integrals

$$I_{S,T}(x, y) := \int_{|s| < S} \int_{|t| < T} f(s, t) e^{i(sx+ty)} ds dt \quad (S, T > 0).$$

We say that the double integral (4.1) converges in Pringsheim's sense at a point  $(x, y) \in \mathbb{T}^2$  to the limit  $l$ , if for every  $\varepsilon > 0$  there exists  $\rho = \rho(\varepsilon) > 0$  such that

$$|I_{S,T}(x, y) - l| < \varepsilon, \quad \text{if } S, T > \rho.$$

**Definition.** A formal integration of the integrand in (4.1) with respect to both  $x$  and  $y$  gives

$$(4.2) \quad L(x, y) := \iint_{\mathbb{R}^2} f(s, t) \frac{e^{i(sx+ty)}}{i^2 st} ds dt, \quad (x, y) \in \mathbb{R}^2.$$

The definition of  $L(x, y)$  is interpreted formally, since the double integral in (4.2) may not exist in Lebesgue's sense.

We say that the integral (4.1) is Lebesgue summable at some point  $(x, y) \in \mathbb{T}^2$  to the finite limit  $l$  if

$$\begin{aligned} \frac{\Delta L(x, y; h, k)}{4hk} &:= \frac{1}{4hk} (L(x+h, y+k) - L(x-h, y+k) \\ &\quad - L(x+h, y-k) + L(x-h, y-k)) \rightarrow l \quad (0 < h, k \rightarrow 0). \end{aligned}$$

Our main result is formulated in the following theorem.

**Theorem 4.1.** *If  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  is such that*

$$s \int_{\mathbb{R}} f(s, t) dt \in L_{loc}^1(\mathbb{R}, ds)$$

*and*

$$(4.3) \quad \lim_{S, T \rightarrow \infty} \frac{1}{S} \int_{|s| < S} \int_{|t| < T} |sf(s, t)| ds dt = 0$$

*as well as*

$$t \int_{\mathbb{R}} f(s, t) ds \in L_{loc}^1(\mathbb{R}, dt)$$

*and*

$$(4.4) \quad \lim_{S, T \rightarrow \infty} \frac{1}{T} \int_{|s| < S} \int_{|t| < T} |tf(s, t)| ds dt = 0,$$

*then the double integral in (4.1) exists in Lebesgue's sense and we have uniformly in  $(x, y)$  that*

$$\lim_{h, k \rightarrow 0} \left( \frac{\Delta L(x, y; h, k)}{4hk} - I_{1/h, 1/k}(x, y) \right) = 0 \quad (h, k > 0).$$

*In other words, under conditions (4.3) and (4.4) the double integral (4.1) is Lebesgue summable at some point  $(x, y)$  to a finite limit if and only if (4.1) converges in Pringsheim's sense at  $(x, y)$  to the same limit.*

**Remark.** We recall that the Fourier transform  $\hat{f}$  of a function  $f \in L_{loc}^1(\mathbb{R}^2)$  is defined by

$$(4.5) \quad \hat{f}(x, y) := \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} f(s, t) e^{-i(sx+ty)} ds dt, \quad (x, y) \in \mathbb{R}^2.$$

Clearly, Theorem 4.1. can be reformulated in terms of the Lebesgue summability of the double integral in (4.5) under the same conditions (4.3) and (4.4).

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