Observation problems related to string vibrations

Outline of Ph.D. Thesis

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1 Introduction

The subject of the thesis is the solvability of observation problems related to vibrating strings, and the smoothness of their solutions.

Observation problems have origins in control theory, which has a wide literature where attainability results were gained to various control conditions.

We investigated observation problems in the following setting: the partial state of the vibrating string is known (observed) at two time instants, and we attempt to describe the whole vibration process, or at least determine an initial position and speed, from which the observed states are attained. The investigations of this type of problems is quite new, maybe the work [3] of L. N. Znamenskaya is the closest to our setting, where she studied observation problems related to the standard vibrating string with homogeneous boundary conditions. Further works similar to the subject of this thesis are [4], [5] and [6], which discuss observation problems related to vibrating beams, plates and membranes.

In Chapter 2 of the thesis, we investigate observation problems posed in terms of generalized functions. Our starting point is a vibration described by the Klein-Gordon equation, but later we managed to generalize our corresponding result with respect to the equation, to the boundary conditions and to the observed states.

Chapter 3 generalize Duhamel's principle concerning the loaded infinite vibrating string. We gained this result in order to formulate the statements in the next chapter as sharp as possible, but it may be of interest also in itself.

Chapter 4 focus on the observation problem of the infinite vibrating string utilizing classical tools. Our result can be extended to half-infinite strings with the help of the reflection method.

The thesis is based on the [7]–[10] publications of the author. We use the same notations and numbering (except for the references) in the outline as in the thesis.

2 Observation problems posed for vibrating strings in terms of generalized functions

In this chapter of the dissertation, we use the definition of the spaces $D^s(S)$, $s \in \mathbb{R}$ given in [1].

Let the system $\{X_n(x)\}_{n=0}^{\infty}$ be a complete orthonormal basis in $L_2(S)$. Given arbitrary real number s, we consider on the linear span D of the functions $X_n(x)$, $n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $x \in \overline{S}$, the following Euclidean norm:

$$\left\| \sum_{n=0}^{\infty} c_n X_n(x) \right\|_s := \left(\sum_{n=0}^{\infty} n^{2s} |c_n|^2 \right)^{\frac{1}{2}} .$$

Completing D with respect to this norm, we obtain a Hilbert space denoted by D^s. We use the notation $S = (0, l)$ associated with string vibrations. Further information about the spaces D^s can be found in [2].

The subject of Section 2.1 is a somewhat simple observation problem related to the Klein-Gordon equation. Our statement is the following:

Theorem 2.1. *Let the given observed states* $f_1,\ f_2$ *and the observation instants* $t_1,\ t_2$ *be such that*

$$f_1 \in D^{s+2}, \quad f_2 \in D^{s+2}, \qquad s \in \mathbb{R},$$

and

$$t_2 - t_1 = \frac{p}{q} \frac{2l}{a}, \qquad p,\ q \in \mathbb{N},$$

where p *and* q *are relative primes. Moreover let us suppose, that*

$$\sin \left((t_2 - t_1) \sqrt{\left(\frac{n\pi}{l} a \right)^2 + c} \right) \neq 0, \qquad \forall n \in \mathbb{N}.$$

Then the initial functions

$$(\varphi, \psi) = (u(x,0), u_t(x,0)) \in D^{s+1} \times D^s$$

can be uniquely determined, such that the corresponding solution of the equation

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - cu(x,t), \qquad (x,t) \in [0,l] \times \mathbb{R}, \quad 0 < a, c \in \mathbb{R}$$

with boundary conditions

$$u(0,t) = 0,\ u(l,t) = 0, \qquad t \in \mathbb{R}$$

will satisfy the observation condition

$$u(x,t_1) = f_1(x), \quad u(x,t_2) = f_2(x), \qquad 0 \leq x \leq l.$$

The proof is constructive, the Fourier series representations of the initial functions can be found in the proof.

Similar statements can be made in the cases when the observation condition gives the speed of the string both at t_1 and at t_2; or when the observation condition gives the position at one of the time instants, and the speed at the another.

During our research, we managed to generalize this result in terms of each the equation describing the vibration, the boundary conditions and the observed states. This generalization can be found in Section 2.2, and it solves the following observation problem (which is the main result of Chapter 2):

Consider the following mixed problem:

$$\begin{gathered} u_{tt} = (p(x)u_x)_x - q(x)u \equiv Lu, \\ (x,t) \in [0,l] \times \mathbb{R}, \quad 0 < p, q \in C^\infty([0,l]), \end{gathered} \tag{2.29}$$

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \tag{2.30}$$

$$\mathcal{U}_i[u] \equiv \mathcal{U}_i(u|_{x=0},\ u|_{x=l},\ u_x|_{x=0},\ u_x|_{x=l}) = 0, \qquad i = 1,2, \tag{2.31}$$

where $\mathcal{U}_1,\ \mathcal{U}_2$ independent, self-adjoint linear expressions, and the functions $u,\ \varphi,\ \psi$ are from the generalized function space D^s.

Let us suppose, that for every $s \in \mathbb{R}$ and for every $(\varphi, \psi) \in D^{s+1}(0,l) \times D^s(0,l)$, this mixed problem possesses the following good properties:

$$\exists! u \text{ solution and } u \in C(D^{s+1}, \mathbb{R}) \cap C^1(D^s, \mathbb{R}) \cap C^2(D^{s-1}, \mathbb{R}), \tag{2.33}$$

and u can be written in the following form:

$$\begin{gathered} u(x,t) = \sum_{n=0}^{\infty} \left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\right] X_n(x), \quad (x,t) \in [0,l] \times \mathbb{R}, \\ LX_n = -\omega_n^2 X_n, \qquad \mathcal{U}_i X_n = 0, \quad i = 1,2. \end{gathered} \tag{2.34}$$

Our observation conditions are some known linear combination of the position and the speed of the string at the time instants t_1 and t_2:

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f_1, \qquad |A_1| + |B_1| > 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= f_2, \qquad |A_2| + |B_2| > 0, \end{aligned} \tag{2.32}$$

where the coefficients $A_1,\ A_2,\ B_1,\ B_2$ and the functions $f_1,\ f_2$ are given.

Let

$$f_1 \in D^{s+2},\ f_2 \in D^{s+2}, \qquad s \in \mathbb{R}, \tag{2.35}$$

and assume, that there are constants $0 < A \in \mathbb{Q},\ B \in \mathbb{R},\ 0 < M_1 \in \mathbb{R}$ and a series $C_n \in \mathbb{R}\backslash\{0\}$ such that

$$\begin{gathered} \omega_n(t_2 - t_1) + \gamma_n - \delta_n = n\pi A + B + C_n, \\ 0 < \frac{M_1}{n} < |C_n|,\ \forall n \in \mathbb{N} \quad \text{and} \quad C_n \to 0 \text{ as } n \to \infty, \end{gathered} \tag{2.36}$$

and

$$\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \neq 0, \qquad n \in \mathbb{N}_0. \tag{2.37}$$

We use Lemma 2.10 to estimate the values of the above sine function for n large enough:

Lemma 2.10. *Let the series r_n such that $0 < M/n < |r_n| \to 0$ with a positive constant M and let $x_0 > 0$ a rational number. Then there exists a threshold number N for every fixed $d \in \mathbb{R}$, such that*

$$|\sin(n\pi x_0 + d + r_n)| > \frac{M}{2n}, \qquad \forall n > N.$$

This estimation will ensure, that the Fourier series of the initial functions will converge and will construate functions with the required smoothness.

The angles $\gamma_n,\ \delta_n \in [0, 2\pi)$ in conditions (2.36), (2.37) are uniquely determined by the following relationships:

$$\begin{aligned} \sin\gamma_n &= \frac{A_1}{\sqrt{A_1^2 + B_1^2\omega_n^2}}, \quad & \cos\gamma_n &= \frac{B_1\omega_n}{\sqrt{A_1^2 + B_1^2\omega_n^2}}, \\ \sin\delta_n &= \frac{A_2}{\sqrt{A_2^2 + B_2^2\omega_n^2}}, \quad & \cos\delta_n &= \frac{B_2\omega_n}{\sqrt{A_2^2 + B_2^2\omega_n^2}}. \end{aligned}$$

Our statement (which is the main result of Chapter 2) is the following:

Theorem 2.11. *The observation problem* (2.29)–(2.37) *described above has unique solution $(\varphi, \psi) \in D^{s+1} \times D^s$, that is, the initial functions can be uniquely determined from which the observed states described by condition* (2.32) *are attained during the string vibration.*

In Section 2.3, we give three examples to illustrate the applicability of Theorem 2.11 in the case of the Klein-Gordon equation with various boundary conditions. We consider the vibrating string with fixed ends in Subsection 2.3.1, with free ends in Subsection 2.3.2, and with Sturm-Liouville boundary condition in Subsection 2.3.3. During these considerations we can see, that the rather technical conditions of Theorem 2.11 can be significantly simplified or they even automatically hold in certain cases. In Subsection 2.3.4 we investigate the case when we doesn't make any restrictions to the observation instants t_1 and t_2 in Theorem 2.11.

3 A new version of Duhamel's principle for the infinite vibrating string

In Chapter 3 of the thesis we introduce a new version of Duhamel's principle for the infinite vibrating string, which is the following:

Theorem 3.1. *If $f(x,t) \in C(\mathbb{R}^2)$ and the directional derivative f_t of f along t exists and $f_t \in C(\mathbb{R}^2)$, then the problem*

$$v(x,t) \in C^2(\mathbb{R}^2),$$

$$v_{tt}(x,t) - a^2 v_{xx}(x,t) = f(x,t), \qquad (x,t) \in \mathbb{R}^2,$$

$$v|_{t=0} = v_t|_{t=0} \equiv 0$$

can be uniquely solved and the solution v can be written as

$$v(x,t) = \frac{1}{2a} \int_0^t \left(\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi \right) d\tau, \qquad (x,t) \in \mathbb{R}^2.$$

This theorem sharpens the smoothness condition $f(x,t) \in C^1(\mathbb{R}^2)$ of the classical result, and this representation of the solution v corresponds to the one in the usual setting. The base of the proof is that the function defined by this expression is well-defined and from C^2 even with our weaker condition for the function f.

Comment. We mention, that Theorem 3.1 (and accordingly the results of Chapter 4) remains valid, even if

$$f(x,t) \in C(\mathbb{R}^2), \quad f_\nu(x,t) := \frac{\partial f}{\partial \nu} \in C(\mathbb{R}^2),$$

where f_ν is the directional derivative of f along the vector $\nu = (\nu_1, \nu_2)$, provided that ν is transversal to the characteristics. The proof of this statement requires a tedious computation with several cases depending on the direction ν, hence we would like to communicate this proof in a subsequent work, which is in preparation.

4 Classically posed observation problems

In Chapter 4, we use classical tools for investigating observation problems. To formulate as sharp statements as possible, we use the result of the previous chapter. The main result of the chapter is the following:

Theorem 4.5. *Consider the problem of the infinite vibrating string given by*

$$u(x,t) \in C^2(\mathbb{R}^2)$$

$$u_{tt}(x,t) - a^2 u_{xx}(x,t) = f(x,t), \quad (x,t) \in \mathbb{R}^2, \quad a > 0,$$

where $f(x,t)$ is continuous, and the directional derivative f_t exists and $f_t \in C(\mathbb{R}^2)$. The observed partial states of the string are described by the conditions

$$\begin{aligned} A_1(x)u|_{t=t_1} + B_1(x)u_t|_{t=t_1} &= f_1(x), \qquad && x \in \mathbb{R}, \\ A_2(x)u|_{t=t_2} + B_2(x)u_t|_{t=t_2} &= f_2(x), \qquad && x \in \mathbb{R}, \end{aligned}$$

where the given coefficients and right-hand sides satisfy

$$A_1(x), A_2(x), B_1(x), B_2(x) \neq 0,\ x \in \mathbb{R}, \quad A_1,\ A_2,\ B_1,\ B_2,\ f_1,\ f_2 \in C^1(\mathbb{R}).$$

This observation problem can be solved, namely there can be found initial functions

$$u|_{t=t_0}(x) = \varphi(x), \quad u_t|_{t=t_0}(x) = \psi(x), \qquad \varphi \in C^2(\mathbb{R}),\ \psi \in C^1(\mathbb{R})$$

such that the corresponding vibration described by $u(x,t)$ produce the observed states. The solution is not unique.

This result can be extended to half-infinite strings with the help of the reflection method. Let us consider the equation of the half-infinite ($x \geq 0$) vibrating string:

$$u(x,t) \in C^2([0,\infty) \times \mathbb{R}), \tag{4.19}$$

$$u_{tt}(x,t) - a^2 u_{xx}(x,t) = g(x,t), \qquad (x,t) \in [0,\infty) \times \mathbb{R}, \quad a > 0, \tag{4.20}$$

where

$$g(x,t) \in C([0,\infty) \times \mathbb{R}), \quad g_t(x,t) \in C([0,\infty) \times \mathbb{R}). \tag{4.21}$$

The endpoint is fixed, namely

$$u(0,t) = 0, \qquad t \in \mathbb{R}, \tag{4.22}$$

and the observed partial states are the following:

$$u|_{t=t_1} = g_1(x), \quad u|_{t=t_2} = g_2(x), \qquad g_1, g_2 \in C^2([0,\infty)). \tag{4.23}$$

Theorem 4.3.

The observation problem (4.19)–(4.23) *has a solution u for any functions g_1, g_2 and g, which satisfy*

$$g(0,t) = g_1(0) = g_1''(0) = g_2(0) = g_2''(0) = 0, \qquad t \in \mathbb{R}.$$

The solution is not unique, but every solution u can be written in the following form:

$$u(x,t) = \frac{f_1(x-a(t-t_1)) + f_1(x+a(t-t_1))}{2} + \frac{1}{2a} \int\limits_{x-a(t-t_1)}^{x+a(t-t_1)} \psi(s)ds +$$

$$+\frac{1}{2a} \int\limits_{t_1}^{t} \left(\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi \right) d\tau,$$

where the right-hand side is restricted to $x \geq 0$. The functions f_1 and f are the odd extensions of the functions g_1 and g to the real numbers respect to the $x = 0$ point/line, and the function

$$\Psi(x) = \int\limits_{0}^{x} \psi(s)ds \in C^2(\mathbb{R})$$

can be chosen arbitrarily inside the interval $[0,T]$.

In the case of the Neumann boundary condition

$$u_x(0,t) = 0, \qquad t \in \mathbb{R}, \tag{4.26}$$

using even extensions we get the following results:

Theorem 4.4. *The observation problem* (4.19), (4.20), (4.21), (4.23), (4.26) *has a solution u for any functions g_1 and g_2, which satisfy*

$$g_1'(0) = g_2'(0) = 0.$$

The solution is not unique, but it can written as

$$u(x,t) = \frac{f_1(x-a(t-t_1)) + f_1(x+a(t-t_1))}{2} + \frac{1}{2a} \int\limits_{x-a(t-t_1)}^{x+a(t-t_1)} \psi(s)ds,$$

$$+\frac{1}{2a} \int\limits_{t_1}^{t} \left(\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi \right) d\tau,$$

where the right-hand side is restricted to $x \geq 0$. *Here the function* f_1 *is the even extension of* g_1, *the function* $f(x,t)$ *is the even extension of* $g(x,t)$ *with respect to the* $x = 0$ *point/line. The function*

$$\Psi(x) = \int\limits_0^x \psi(s)ds \in C^2(\mathbb{R})$$

can be chosen arbitrary inside the interval $[0,T]$.

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