

Asymptotic behaviour of Hilbert space contractions

Outline of Ph.D. thesis

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1 Introduction

The theory of contractions, which was developed by Béla Sz.-Nagy and Ciprian Foias from the dilation theorem of Sz.-Nagy, is one of the main methods of examining non-normal operators. Our basic reference in connection with this theory is [NFBK]. The dissertation contains two different topics concerning absolutely continuous contractions on a complex, separable Hilbert space. In the first part we study the stability of contractions, while in the bigger second part we investigate quasianalytic contractions.

The dissertation is based on the following three papers of the author.

- L. KÉRCHY and A. SZALAI, Characterization of stability of contractions, *Acta Sci. Math. (Szeged)*, **79** (2013), 325–332.
- L. KÉRCHY and A. SZALAI, Asymptotically cyclic quasianalytic contractions, *Studia Math.*, **223** (2014), 53–75.
- L. KÉRCHY and A. SZALAI, Spectral behaviour of quasianalytic contractions, *Proc. Amer. Math. Soc.*, accepted.

In this outline we use the same numbering, labeling, and notations as in the thesis.

The investigations included in this dissertation were mainly motivated by the most challenging open problems in the theory of Hilbert spaces and bounded linear operators on them, namely, the well-known invariant and hyperinvariant subspace problems. In what follows let $\mathcal{L}(\mathcal{H})$ stand for the C^* -algebra of all bounded linear operators acting on the complex, separable Hilbert space \mathcal{H} . The Invariant Subspace Problem (ISP) asks the existence of a non-trivial invariant subspace $\mathcal{M} \subset \mathcal{H}$ of an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$, while the Hyperinvariant Subspace Problem (HSP) asks whether there exists a non-trivial hyperinvariant subspace $\mathcal{N} \subset \mathcal{H}$ of an arbitrary non-scalar $T \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}I$, i.e., an operator which is not a scalar multiple of the identity operator. The subspace (closed linear manifold) $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} = \{Tx : x \in \mathcal{M}\} \subset \mathcal{M}$ holds, and it is *non-trivial* if $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$. The subspace $\mathcal{N} \subset \mathcal{H}$ is *hyperinvariant* for T if it is invariant for

every operator commuting with T . Considering these problems we can suppose that the operator T in question is an absolutely continuous contraction (see [NFBK, Theorem II.2.3] and [Dou60, Corollary 5.1 and Theorem 3]). We recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called a *contraction* if $\|T\| \leq 1$, and that any contraction can be uniquely decomposed into the orthogonal sum $T = T_1 \oplus U_a \oplus U_s$ of a completely non-unitary (c.n.u.) contraction T_1 , an absolutely continuous (a.c.) unitary operator U_a , and a singular unitary operator U_s (see [NFBK], Theorem I.3.2 and [Hal51]). We recall that a contraction is *c.n.u.* if it is not unitary on any of its non-zero reducing subspaces, and that a unitary operator is *a.c.* or *singular* if its spectral measure is a.c. or singular with respect to Lebesgue measure on the unit circle. T is *absolutely continuous* if its singular unitary part is zero. For such contractions, by the aid of the usual functional calculus (operating with the bounded measurable functions on the unit circle \mathbb{T}) for the minimal unitary dilation U of T , we can introduce the so-called *Sz.-Nagy–Foias functional calculus* Φ_T for T , which plays a crucial role in the theory of contractions:

$$\Phi_T: H^\infty \rightarrow \mathcal{L}(\mathcal{H}), f \mapsto f(T) := P_{\mathcal{H}}f(U)|_{\mathcal{H}},$$

where H^∞ denotes the Hardy space of all bounded analytic functions on the open unit disc \mathbb{D} (what can be identified with the space of bounded measurable functions on the unit circle \mathbb{T} with zero Fourier coefficients of negative indices). This Φ_T is a contractive, unital algebra-homomorphism, which is continuous in the weak-* topologies and satisfies the condition $T = \Phi_T(\chi) = \chi(T)$, where $\chi(z) = z$ denotes the identical function.

Another important tool in the study of a contraction T is its unitary asymptote. The pair (X, V) is a *unitary asymptote* of T if V is a unitary operator acting on a Hilbert space \mathcal{K} and $X: \mathcal{H} \rightarrow \mathcal{K}$ is a linear transformation satisfying the conditions $\bigvee_{n=1}^{\infty} V^{-n} X \mathcal{H} = \mathcal{K}$, $\|Xh\| = \lim_{n \rightarrow \infty} \|T^n h\|$ for every $h \in \mathcal{H}$, and $XT = VX$. For further properties of unitary asymptotes we refer to [Kér13] and [NFBK, Chapter IX]. It is easy to see that the nullspace of X is hyperinvariant for T , it is the so-called *stable subspace* of T :

$$\mathcal{H}_0(T) = \left\{ h \in \mathcal{H} : \lim_{n \rightarrow \infty} \|T^n h\| = 0 \right\}.$$

Considering the asymptotic behaviour of contractions Sz.-Nagy and Foias introduced the following classes:

- $T \in C_0$. if $\mathcal{H}_0(T) = \mathcal{H}$, that is, when $T^n \rightarrow 0$ in the strong operator topology (SOT). In this case T is called *stable*, while a non-stable T is usually called *asymptotically non-vanishing*.
- $T \in C_1$. if $\mathcal{H}_0(T) = \{0\}$. Contractions of this type are called *asymptotically strongly non-vanishing*.
- $T \in C_{.0}$ if $T^* \in C_0$;
- $T \in C_{.1}$ if $T^* \in C_1$;
- $C_{ij} = C_i \cap C_{.j}$ ($i, j = 0, 1$).

In Chapter 2 we characterize the stability of contractions, while in Chapter 3-5 we examine asymptotically non-vanishing contractions.

2 Stability results

In Chapter 2 we study some stability properties of contractions and polynomially bounded operators. In Section 2.1 we characterize those sequences $\{h_n\}_{n=1}^\infty$ of bounded analytic functions, which can serve to test the stability of an a.c. contraction. This answers a question of M. Dritschel.

Definition 2.4. A sequence of bounded analytic functions $\{h_n\}_{n=1}^\infty \subset H^\infty$ is a *test sequence of stability for a.c. contractions* if for every a.c. contraction T the condition $T^n \rightarrow 0$ (SOT) holds exactly when $h_n(T) \rightarrow 0$ (SOT).

The main result of this chapter is the following statement.

Theorem 2.5. A sequence of bounded analytic functions $\{h_n\}_{n=1}^\infty \subset H^\infty$ is a test sequence of stability for a.c. contractions if and only if it converges to zero exclusively on \mathbb{D} , i.e., if

- (i) $\lim_{n \rightarrow \infty} h_n(z) = 0$ for all $z \in \mathbb{D}$,
- (ii) $\sup \{\|h_n\|_\infty : n \in \mathbb{N}\} < \infty$,

(iii) $\limsup_{n \rightarrow \infty} \|\chi_\alpha h_n\|_2 > 0$ for every Borel set $\alpha \subset \mathbb{T}$ of positive measure, where χ_α is the characteristic function of α .

For the proof of necessity we deal with the unilateral shift operator $S \in \mathcal{L}(H^2)$, $Sf = \chi f$ and with some other multiplication operators. At the sufficiency part we use Rota's universal model for contractions, the triangular matrix form of a non-stable contraction, and the concept of unitary asymptote. A part of the proof yields the following proposition.

Proposition 2.6. *Let $\{h_n\}_{n=1}^\infty \subset H^\infty$. Then $h_n(T) \rightarrow 0$ (SOT) for every stable contraction T if and only if $\{h_n\}_{n=1}^\infty$ satisfies the conditions (i) and (ii).*

A well-known property of the Sz.-Nagy–Foias functional calculus is that $h_n(T) \rightarrow 0$ (SOT) for every a.c. contraction T , whenever $\{h_n\}_{n=1}^\infty$ boundedly converges to zero a.e. on \mathbb{T} (see [NFBK, Chapter III]). Our following proposition shows that the necessary and sufficient condition is weaker.

Proposition 2.7. *Let $\{h_n\}_{n=1}^\infty \subset H^\infty$. Then $h_n(T) \rightarrow 0$ (SOT) for every a.c. contraction T exactly when $\{h_n\}_{n=1}^\infty$ is a bounded sequence in H^∞ and $\lim_{n \rightarrow \infty} \|h_n\|_2 = 0$.*

In Section 2.2 we examine analogous questions for polynomially bounded operators. Let us denote by $\mathcal{P}(\mathbb{T})$ the set of analytic polynomials on \mathbb{T} , and by $A = A(\mathbb{T})$ the disc algebra, i.e., the closure of $\mathcal{P}(\mathbb{T})$ in the Banach space $C(\mathbb{T})$ of continuous functions on \mathbb{T} . An operator $T \in \mathcal{L}(\mathcal{H})$ is called *polynomially bounded* if there exists a real number K_T such that $\|p(T)\| \leq K_T \|p\|$ for all $p \in \mathcal{P}(\mathbb{T})$, where $\|p\| = \max\{|p(\zeta)| : \zeta \in \mathbb{T}\}$. For a polynomially bounded operator T , the mapping $\Phi_{T,0}: \mathcal{P}(\mathbb{T}) \rightarrow \mathcal{L}(\mathcal{H})$, $p \mapsto p(T)$ is a bounded algebra-homomorphism which extends continuously to the disc algebra: $\Phi_{T,1}: A \rightarrow \mathcal{L}(\mathcal{H})$, $f \mapsto f(T)$. Polynomially bounded operators were studied by W. Mlak in a series of papers using ‘elementary measures’. He defined absolute continuity and singularity of polynomially bounded operators, and it turned out that T admits an H^∞ -functional calculus, i.e., a weak-* continuous, unital algebra-homomorphism $\Phi_T: H^\infty \rightarrow \mathcal{L}(\mathcal{H})$ such that $\Phi_T(\chi) = T$, exactly when T is an a.c. polynomially bounded operator (see p. 68 in [Mla74a]). Thus the following definition makes sense.

Definition 2.9. A sequence of bounded analytic functions $\{h_n\}_{n=1}^\infty \subset H^\infty$ is a *test sequence of stability for a.c. polynomially bounded operators* if for every a.c. polynomially bounded operator $T \in \mathcal{L}(\mathcal{H})$ the condition $T^n \rightarrow 0$ (SOT) holds exactly when $h_n(T) \rightarrow 0$ (SOT).

Note that, in principle, a test sequence for a.c. contractions is not necessarily a test sequence for a.c. polynomially bounded operators. Nevertheless, we can prove the following theorem.

Theorem 2.10. *A sequence of bounded analytic functions $\{h_n\}_{n=1}^\infty \subset H^\infty$ is a test sequence of stability for a.c. polynomially bounded operators if and only if $\{h_n\}_{n=1}^\infty$ converges to zero exclusively on \mathbb{D} .*

We conclude this chapter by formulating a statement about the stability of singular polynomially bounded operators.

Proposition 2.11. *Let $\{h_n\}_{n=1}^\infty \subset A$ be a bounded sequence in the disc algebra. Then $h_n(T) \rightarrow 0$ (SOT) for every singular polynomially bounded operator T if and only if $\lim_{n \rightarrow \infty} h_n(\zeta) = 0$ for every $\zeta \in \mathbb{T}$. In that case $h_n(T) \rightarrow 0$ (SOT) for every polynomially bounded operator T .*

3 Hyperinvariant subspaces of quasianalytic contractions

Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction and let (X, V) be a unitary asymptote of T . It is known that $V \in \mathcal{L}(\mathcal{K})$ is an a.c. unitary operator. The *residual set* $\omega(T)$ of T is the measurable support of the spectral measure E of V , which means that $E(\alpha) = 0$ if and only if $m(\alpha \cap \omega(T)) = 0$. For any $x, y \in \mathcal{H}$, $w_{x,y} \in L^1(\mathbb{T})$ is the *asymptotic density function* of T at x and y : $E_{Xx, Xy} = w_{x,y} dm$. The measurable $\omega(T, x) = \{\zeta \in \mathbb{T} : w_{x,x}(\zeta) > 0\}$ is the *local residual set* of T at x .

We recall the notion of another spectral invariant, defined by the Sz.-Nagy–Foias functional calculus Φ_T for T . This calculus is monotone in the sense that $\|f(T)x\| \leq \|g(T)x\|$ for every $x \in \mathcal{H}$ (in notation: $f(T) \stackrel{a}{\prec} g(T)$) whenever $|f(z)| \leq |g(z)|$ for every z in the unit disc \mathbb{D} (in notation: $f \stackrel{a}{\prec} g$).

Given a decreasing sequence $F = \{f_n\}_{n=1}^\infty$ in H^∞ ($f_{n+1} \stackrel{a}{\prec} f_n$ for every n), let us consider the limit function φ_F on \mathbb{T} , defined by $\varphi_F(\zeta) = \lim_{n \rightarrow \infty} |f_n(\zeta)|$ for a.e. $\zeta \in \mathbb{T}$, and the measurable set $N_F = \{\zeta \in \mathbb{T} : \varphi_F(\zeta) > 0\}$. Then the sequence $F(T) = \{f_n(T)\}_{n=1}^\infty$ of operators is also decreasing ($f_{n+1}(T) \stackrel{a}{\prec} f_n(T)$ for every n) and the set

$$\mathcal{H}_0(T, F) = \left\{ x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|f_n(T)x\| = 0 \right\}$$

of stable vectors for $F(T)$ is a hyperinvariant subspace of T . For measurable subsets α and β of \mathbb{T} , we write $\alpha = \beta$, $\alpha \neq \beta$ and $\alpha \subset \beta$ if $m(\alpha \Delta \beta) = 0$, $m(\alpha \Delta \beta) > 0$ and $m(\alpha \setminus \beta) = 0$ respectively, that is when $\chi_\alpha = \chi_\beta$, $\chi_\alpha \neq \chi_\beta$ and $\chi_\alpha \leq \chi_\beta$ hold for the corresponding characteristic functions as elements of the Banach space $L^1(\mathbb{T})$. We say that T is *quasianalytic on a measurable subset α of \mathbb{T} at a vector $x \in \mathcal{H}$* if $x \notin \mathcal{H}_0(T, F)$ whenever F is non-vanishing on α , that is $N_F \cap \alpha \neq \emptyset$. Let $\mathcal{A}(T, x)$ be the system of sets α with this property. The *local quasianalytic spectral set* of T at x is the largest element $\pi(T, x)$ of $\mathcal{A}(T, x)$. (Note that $\pi(T, x)$ is uniquely determined up to sets of measure 0.) We recall from [Kér11] that T is quasianalytic on α if $\mathcal{H}_0(T, F) = \{0\}$ whenever $N_F \cap \alpha \neq \emptyset$; the (global) quasianalytic spectral set $\pi(T)$ is the largest element of $\mathcal{A}(T)$, the system of sets where T is quasianalytic.

The next lemma claims that local stability is determined by the asymptotic density function.

Lemma 3.2. *Let $F = \{f_n\}_{n=1}^\infty$ be a decreasing sequence in H^∞ and $x \in \mathcal{H}$.*

- (a) *If $\lim_{n \rightarrow \infty} \|f_n(T)x\| = 0$ then $\varphi_F w_{x,x} = 0$.*
- (b) *If $\varphi_F w_{x,x} = 0$ then there exists an increasing mapping $\tau: \mathbb{N} \rightarrow \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \|T^{\tau(n)} f_n(T)x\| = 0$.*

The following theorem establishes connection among the local and global spectral invariants introduced before.

Theorem 3.3. *For every non-zero $x \in \mathcal{H}$ we have*

$$\pi(T) \subset \pi(T, x) = \omega(T, x) \subset \omega(T).$$

As a consequence we obtain conditions for the existence of a non-trivial hyperinvariant subspace. (Statement (b) below appears already in [Kér01].)

Corollary 3.4.

- (a) *If $\omega(T, x) \neq \omega(T)$ for some non-zero $x \in \mathcal{H}$ and $F = \{f_n\}_{n=1}^\infty$ is a decreasing sequence with $N_F = \omega(T) \setminus \omega(T, x)$, then there exists an increasing mapping $\tau: \mathbb{N} \rightarrow \mathbb{N}$, such that $G = \{\chi^{\tau(n)} f_n\}_{n=1}^\infty$ is also a decreasing sequence with $\varphi_G = \varphi_F$, $x \in \mathcal{H}_0(T, G)$, and $\mathcal{H}_0(T, G) \cap \mathcal{H}_\omega(T) = \emptyset$. Therefore $\mathcal{H}_0(T, G)$ is a non-trivial hyperinvariant subspace of T .*
- (b) *If $\pi(T) \neq \omega(T)$ then $\text{Hlat } T$ is non-trivial.*

The a.c. contraction $T \in \mathcal{L}(\mathcal{H})$ is *quasianalytic* if $\pi(T) = \omega(T) \neq \emptyset$. In view of Corollary 3.4, in the setting of asymptotically non-vanishing contractions (HSP) can be reduced to the case when T is quasianalytic.

In Theorem 3.8 we show that quasianalyticity determines the asymptotic behaviour, namely, if T is a quasianalytic contraction, then $T \in C_{10}$.

We say that an a.c. contraction T is *asymptotically cyclic*, if its unitary asymptote $V \in \mathcal{L}(\mathcal{K})$ is cyclic, that is $\bigvee_{n=0}^\infty V^n y = \mathcal{K}$ holds for some $y \in \mathcal{K}$. The set of asymptotically cyclic, quasianalytic contractions acting on the Hilbert space \mathcal{H} is denoted by $\mathcal{L}_0(\mathcal{H})$. If T is cyclic then so is V (but not conversely), hence (ISP) in the setting of quasianalytic contractions can be reduced to the class $\mathcal{L}_0(\mathcal{H})$. We have a lot of information on the structure of a contraction if its residual set covers the unit circle. Hence it is worth considering the special class $\mathcal{L}_1(\mathcal{H}) = \{T \in \mathcal{L}_0(\mathcal{H}) : \pi(T) = \mathbb{T}\}$. For such contractions $\bigvee \text{Lat}_s T = \mathcal{H}$, where $\text{Lat}_s T$ stands for the set of those invariant subspaces \mathcal{M} , where the restriction $T|_{\mathcal{M}}$ is similar to S , the simple unilateral shift.

Examples of operators in $\mathcal{L}_1(\mathcal{H})$ are provided by the following proposition. First we fix some notation. The operator $A \in \mathcal{L}(\mathcal{H})$ is a *quasiaffine transform* of the operator $B \in \mathcal{L}(\mathcal{K})$, in notation: $A \prec B$, if there exists a *quasiaffinity* (i.e. an injective transformation with dense range) $Q \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $QA = BQ$.

Proposition 3.13. *If $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that $T \prec S$, then $T \in \mathcal{L}_1(\mathcal{H})$ and $H^\infty(T) = \{T\}'$.*

By Theorem 1 in [KT12], (HSP) in $\mathcal{L}_0(\mathcal{H})$ can be reduced to $\mathcal{L}_1(\mathcal{H})$. If $\{T\}' = H^\infty(T)$, then $\text{Hlat } T = \text{Lat } T$ is non-trivial. However, if $\{T\}' \neq H^\infty(T)$ then the shift-type invariant subspaces are not hyperinvariant.

Proposition 3.16. *Let $T \in \mathcal{L}_1(\mathcal{H})$ be such that $\{T\}' \neq H^\infty(T)$. Then, for every $C \in \{T\}' \setminus H^\infty(T)$, we have $\text{Lat } C \cap \text{Lat}_s T = \emptyset$.*

However, we proved that if non-trivial hyperinvariant subspaces exist, then such subspaces can be derived from shift-type invariant subspaces.

Theorem 3.18. *Let $T \in \mathcal{L}_1(\mathcal{H})$ be such that $\{T\}' \neq H^\infty(T)$. Then the following statements are equivalent:*

- (i) $\text{Hlat } T$ is non-trivial;
- (ii) there exists $\mathcal{M} \in \text{Lat}_s T$ such that $\vee \{C\mathcal{M} : C \in \{T\}'\} \neq \mathcal{H}$;
- (iii) there exists $\mathcal{S} \subset \text{Lat}_s T$ such that $\mathcal{H} \neq \vee \mathcal{S} \in \text{Hlat } T$.

For $T \in \mathcal{L}_1(\mathcal{H})$, by Proposition 3.19, T is the quasiaffine transform of S if and only if T is not quasiunitary, hence in this class (HSP) can be reduced to the quasiunitary case. If $T \in \mathcal{L}_1(\mathcal{H})$ is quasiunitary, then there exist $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}_s T$ such that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$ (see Proposition 3.20).

Let $T \in \mathcal{L}(\mathcal{H})$ be an asymptotically cyclic a.c. contraction, and assume that $T \in C_1$. and $\omega(T) = \mathbb{T}$. The universal property of the unitary asymptote (X, V) implies that for every $C \in \{T\}'$ there is a unique $D \in \{V\}'$ such that $XC = DX$, and the mapping $\gamma: \{T\}' \rightarrow \{V\}', C \mapsto D$ is a contractive, unital algebra-homomorphism, which is injective because of $T \in C_1$. The functional calculus $\Phi: L^\infty(\mathbb{T}) \rightarrow \{V\}', f \mapsto f(V)$ is an isomorphism between the corresponding Banach algebras. The composition $\widehat{\gamma}_T = \Phi^{-1} \circ \gamma: \{T\}' \rightarrow L^\infty(\mathbb{T})$ is also an injective, contractive, unital algebra-homomorphism. It can be easily checked that $\widehat{\gamma}_T$ is independent of the special choice of (X, V) .

The uniquely determined $\widehat{\gamma}_T$ is called the *functional mapping* of T , and its range $\mathcal{F}(T)$ is called the *functional commutant* of T . Since $\widehat{\gamma}_T(f(T)) = f$ holds

for every $f \in H^\infty$, we obtain that $\mathcal{F}(T)$ is a subalgebra of $L^\infty(\mathbb{T})$ containing H^∞ . It is natural to ask the following questions. Which function algebras $H^\infty \subset \mathcal{A} \subset L^\infty(\mathbb{T})$ are attainable as a functional commutant: $\mathcal{A} = \mathcal{F}(T)$, and what kind of information can be derived from the properties of $\widehat{\gamma}_T$ and $\mathcal{F}(T)$ on the behaviour of T ? We recall that the *function algebra* \mathcal{A} is called *quasianalytic*, if $f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$ whenever f is a non-zero element of \mathcal{A} . Proposition 4.2 in [Kér11] shows that if $T \in \mathcal{L}_1(\mathcal{H})$, then $\mathcal{F}(T)$ is quasianalytic.

It is clear that $\mathcal{F}(T) = H^\infty$ exactly when $\{T\}' = H^\infty(T)$, and this happens in particular if $T \prec S$. (For a more complete characterization of this case see Theorem 5.2 in [Kér11].)

If $T \in \mathcal{L}_1(\mathcal{H})$ and $\mathcal{F}(T) \neq H^\infty$, then the closure $\mathcal{F}(T)^-$ contains $H^\infty + C(\mathbb{T})$ (see Theorems IX.1.4 and IX.2.2 in [Gar07]); thus $\mathcal{F}(T)^-$ is not quasianalytic, and so $\mathcal{F}(T)$ is not closed, or equivalently, $\widehat{\gamma}_T$ is not bounded from below.

We recall that $\eta \in H^\infty$ is an *inner function*, if $|\eta(\zeta)| = 1$ holds for a.e. $\zeta \in \mathbb{T}$. Let H_i^∞ stand for the multiplicative semigroup of all inner functions. Given a subsemigroup \mathcal{B} of H_i^∞ , the algebra $\overline{\mathcal{B}} \cdot H^\infty$ generated by $\overline{\mathcal{B}}$ (set of conjugates of functions in \mathcal{B}) and H^∞ is clearly quasianalytic. The closure $(\overline{\mathcal{B}} \cdot H^\infty)^-$ is called the *Douglas algebra* induced by \mathcal{B} . By the celebrated Chang–Marshall theorem every closed subalgebra \mathcal{A} of $L^\infty(\mathbb{T})$, containing H^∞ , is a Douglas algebra (see Theorem IX.3.1 in [Gar07]). Therefore, $\mathcal{F}(T)^- = (\overline{\mathcal{B}} \cdot H^\infty)^-$ holds with $\mathcal{B} = \{\eta \in \mathcal{F}(T)^- \cap H_i^\infty : \overline{\eta} \in \mathcal{F}(T)^-\}$. The question which pre-Douglas algebras $\overline{\mathcal{B}} \cdot H^\infty$ arise as functional commutants of contractions of class $\mathcal{L}_1(\mathcal{H})$ was posed in [Kér11]. Our next theorem settles this problem.

Theorem 3.22. *The only attainable pre-Douglas algebra is H^∞ .*

Special case of the following property of the functional commutant has been exploited in the proof of the previous theorem.

Proposition 3.23. *If $f \in \mathcal{F}(T)$, $r > \|\widehat{\gamma}_T^{-1}(f)\|$ and φ is analytic on $r\mathbb{D}$, then $\varphi \circ f \in \mathcal{F}(T)$.*

We show that the functional commutant is a similarity invariant. Actually, the following theorem contains a more general statement.

Theorem 3.25. *For $j = 1, 2$, let $T_j \in \mathcal{L}_1(\mathcal{H}_j)$ be given with unitary asymptote (X_j, V_j) . Let us assume that there exist $Y \in \mathcal{I}(T_1, T_2)$ and $Z \in \mathcal{I}(T_2, T_1)$ such that $ZY \neq 0$. Then*

- (a) *Y and Z are injective;*
- (b) *$0 \neq \widehat{\gamma}_{T_1}(ZY) = \widehat{\gamma}_{T_2}(YZ) =: g$ belongs to $\mathcal{F}(T_1) \cap \mathcal{F}(T_2)$ and $g\mathcal{F}(T_1) \subset \mathcal{F}(T_2)$, $g\mathcal{F}(T_2) \subset \mathcal{F}(T_1)$;*
- (c) *in particular, if $ZY = I$, that is when $T_1 \approx T_2$, then $g = \mathbb{1}$ and $\mathcal{F}(T_1) = \mathcal{F}(T_2)$.*

We concluded Chapter 3 by providing representation of the functional mapping in the functional model. Here we omit the details.

4 Quasianalytic contractions in special classes

Chapter 4 was devoted to special classes of operators, where quasianalytic contractions naturally arise. Namely, we studied analytic contractions and bilateral weighted shifts.

In [ARS07], on a general Hilbert space \mathcal{H}_a of analytic functions defined on the unit disc \mathbb{D} , the analytic multiplication operator $M_a \in \mathcal{L}(\mathcal{H}_a)$, $M_a f = \chi f$ has been studied. The boundary behaviour of functions in \mathcal{H}_a is governed by the set

$$\Delta(\mathcal{H}_a) = \{\zeta \in \mathbb{T} : \text{nt-}\lim_{\lambda \rightarrow \zeta} (1 - |\lambda|^2)^{-1} \|k_\lambda\|^{-2} > 0\},$$

where $k_\lambda \in \mathcal{H}_a$ is the unique reproducing kernel with the property $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in \mathcal{H}_a$. We proved in Section 4.1 that the measurable set $\Delta(\mathcal{H}_a)$ is always contained in the quasianalytic spectral set of M_a (see Proposition 4.1). Therefore, the conditions for the equality $\Delta(\mathcal{H}_a) = \omega(M_a)$ given in [ARS07] ensures the quasianalyticity of M_a . It is not transparent how to identify the unitary asymptote of a general analytic multiplication operator M_a . We carry out this identification in the special case when \mathcal{H}_a is induced by a measure satisfying particular conditions considered in [ARS09].

In Section 4.2 we deal with bilateral weighted shifts, which are C_{10} contractions, mainly applying the ideas of [Shi74], but working with actual functions

instead of formal series. Without restricting the generality, we can suppose that the bilateral weighted shift in consideration is asymptotically cyclic and quasiunitary. We realize a bilateral weighted shift T_β as multiplication by the identical function on a function space $L^2(\beta)$. Up to our knowledge, (HSP) for bilateral weighted shifts, which are C_{10} contractions, is open in the case when

$$0 < \delta_\beta \leq r_\beta < R_\beta = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\log \beta(-n)}{n^2} = \infty.$$

Here $\delta_\beta > 0$ means that T_β is invertible, r_β denotes the inner spectral radius of T_β , while the growth condition on $\beta(-n)$ ensures the quasianalyticity of the function algebra $L^2(\beta)$. Under these conditions we relate the functional commutant to bounded analytic functions defined on an annulus.

5 Spectral behaviour of quasianalytic contractions

Though (ISP) and (HSP) are open for asymptotically non-vanishing (a.n.v.) contractions, Corollary 3.4 shows that these questions are settled in the non-quasianalytic case. By this fact it becomes crucial to determine the spectral behaviour of quasianalytic contractions. Namely, if an a.n.v. contraction T does not meet this behaviour, then T is not quasianalytic, and so $\text{Hlat } T$ is non-trivial.

If the contraction T is quasianalytic, then it is of class C_{10} ; see Theorem 3.8. Under this asymptotic behaviour there is a connection between the spectrum $\sigma(T)$ of T and the spectrum $\sigma(V)$ of its unitary asymptote V . First we note that $\sigma(V)$ is the essential support of $\omega(T)$: $\sigma(V) = \text{es}(\omega(T))$, which is the complement of the largest open subset \mathcal{O} of \mathbb{T} with the property $m(\mathcal{O} \cap \omega(T)) = 0$. It can be easily proved that $\sigma(V)$ is neatly contained in $\sigma(T)$, that is $\sigma(V) \subset \sigma(T)$ and $m(\sigma(V) \cap \sigma') > 0$ holds for every non-empty closed subset σ' of $\sigma(T)$ with the property that $\sigma(T) \setminus \sigma'$ is also closed. More importantly, this is the only constraint on the spectrum of a C_{10} -contraction, even in the cyclic case; see Chapter IX in [NFBK]. In Chapter 5 we posed the question whether there are any other constraints if T is quasianalytic.

Question 1. Given a measurable set $\omega_0 \subset \mathbb{T}$ of positive measure and a compact subset σ of the closed unit disc \mathbb{D}^- such that $\text{es}(\omega_0)$ is neatly contained in σ , does a quasianalytic contraction T exist with the properties $\sigma(T) = \sigma$ and $\omega(T) = \omega_0$?

In the C_{10} class the construction starts by producing a C_{10} -contraction T satisfying the conditions $\omega(T) = \omega_0$ and $\sigma(T) = \text{es}(\omega_0)$, as a restriction of a bilateral weighted shift W to an appropriately chosen invariant subspace. The contraction T , obtained this way, cannot be quasianalytic, therefore, we have to find another approach to provide a quasianalytic contraction T , if it exists at all, such that its spectrum $\sigma(T)$ is a proper subset of \mathbb{T} . First of all the following simpler question should be answered.

Question 2. Do we have for every closed arc J of positive measure on \mathbb{T} and for every $c > 0$ a quasianalytic contraction T satisfying the conditions $\sigma(T) = \pi(T) = J$ and $\|T^{-1}\| > c$?

We know that the a.c. contraction T has shift-type invariant subspaces if $\omega(T) = \mathbb{T}$. Moreover, $\mathcal{H} = \vee \text{Lat}_s T$ in this case. Any quasianalytic contraction can be related to such a contraction having a rich invariant subspace lattice.

Theorem 5.1. *For every quasianalytic contraction T_1 , there exists a quasianalytic contraction T_2 with $\pi(T_2) = \mathbb{T}$ such that $\{T_2\}' \supset \{T_1\}'$ and so $\text{Hlat } T_2 \subset \text{Hlat } T_1$.*

Therefore, the (HSP) for a.n.v. contractions can be reduced to the case, when T is quasianalytic and $\pi(T) = \mathbb{T}$. Clearly, \mathbb{T} is neatly contained in $\sigma(T)$ exactly when $\sigma(T)$ is connected. Thus, in this particular class Question 1 has the following modified form.

Question 3. Given a connected, compact subset σ of \mathbb{D}^- , containing \mathbb{T} , does there exist a quasianalytic contraction T satisfying the conditions $\sigma(T) = \sigma$ and $\pi(T) = \mathbb{T}$?

Our next result shows that the preceding two questions are related. Let $\mathbb{D}_+ := \{z \in \mathbb{D} : \text{Im } z > 0\}$, $\mathbb{T}_+ := \{\zeta \in \mathbb{T} : \text{Im } \zeta \geq 0\}$, and for any $K \subset \mathbb{C}$ let $K^2 := \{z^2 : z \in K\}$.

Theorem 5.2. *A positive answer for Question 2 implies an affirmative answer for Question 3 in the special case, when $\sigma = K^2$ for some connected, compact set K such that $\mathbb{T}_+ \subset K \subset \mathbb{D}_+^-$.*

We prove it applying the technique used in Section IX.2 of [NFBK] to obtain a quasianalytic contraction \tilde{T} satisfying the conditions $\sigma(\tilde{T}) = K$ and $\pi(\tilde{T}) = \mathbb{T}_+$. In Remark 5.3 we show that unfortunately not every connected, compact set $\mathbb{T} \subset \sigma \subset \mathbb{D}^-$ can be represented as $\sigma = K^2$ with a connected, compact set $\mathbb{T}_+ \subset K \subset \mathbb{D}_+^-$.

Clearly, the (ISP) can be reduced to the case when T is asymptotically cyclic. Therefore, it is important to know the spectral behaviour in this setting too. In the class $\mathcal{L}_0(\mathcal{H})$ of asymptotically cyclic quasianalytic contractions and $\mathcal{L}_1(\mathcal{H}) = \{T \in \mathcal{L}_0(\mathcal{H}) : \pi(T) = \mathbb{T}\}$ the same commutants arise and (HSP) can be reduced to $\mathcal{L}_1(\mathcal{H})$ by Theorem 1 in [KT12]. This fact makes it especially important to answer the following question.

Question 4. What are the possible spectra of the contractions belonging to $\mathcal{L}_1(\mathcal{H})$?

We know that for every $0 \leq \delta < 1$ there is a contraction $T_\delta \in \mathcal{L}_1(\mathcal{H})$ such that $\sigma(T_\delta) = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$; see Example 3.24. Our next theorem shows that the spectrum can be the unit circle \mathbb{T} too, and it also gives a positive answer for Question 2 in the special case, when the arc J is the whole circle \mathbb{T} .

Theorem 5.4. *For every $c > 1$, there is a contraction $T \in \mathcal{L}_1(\mathcal{H})$ such that $\sigma(T) = \mathbb{T}$ and $\|T^{-1}\| \geq c$.*

The proof is a presentation of a bilateral weighted shift for every $c > 1$ with the prescribed properties.

Relying on this statement we can provide contractions in $\mathcal{L}_1(\mathcal{H})$ with more sophisticated spectra. T_δ given in Example 5.5.(a) has spectrum $\sigma(T_\delta) = \mathbb{T} \cup \delta\mathbb{T} \cup [\delta, 1]$. Observe that $\mathbb{D} \setminus \sigma(T_\delta)$ is not connected. Example 5.5.(b) provides a contraction $\tilde{T} \in \mathcal{L}_1(\mathcal{H})$ such that $\sigma(\tilde{T}) = \mathbb{T} \cup \{r\zeta : \zeta \in H, \rho(\zeta) \leq r < 1\}$ for some $\rho: H \rightarrow (0, 1)$ and dense subset H of \mathbb{T} .

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