Asymptotic behaviour of Hilbert space operators with applications

Theses of Ph. D. dissertation

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1. Introduction

One of the main methods of examining non-normal operators, acting on a complex Hilbert space, is the theory of contractions. We say that a bounded, linear operator $T \in \mathcal{B}(\mathcal{H})$ is a contraction if $\|T\| \leq 1$ is satisfied where $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators acting on the complex Hilbert space $\mathcal{H}$. This area of operator theory was developed by B. Sz.-Nagy and C. Foias from the dilatation theorem of Sz.-Nagy.

Sz.-Nagy and Foias classified the contractions according to their asymptotic behaviour. This classification can be done in a more general setting, namely when we consider the class of power bounded operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is called power bounded if $\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty$ holds. We call a vector $x \in \mathcal{H}$ stable for T if $\lim_{n\to\infty} \|T^n x\| = 0$, and the set of all stable vectors is denoted by $\mathcal{H}_0 = \mathcal{H}_0(T)$. It can be derived quite easily that $\mathcal{H}_0$ is a hyperinvariant subspace for T (see [**11**]), which means that $\mathcal{H}_0$ is a subspace that is invariant for every operator which commutes with T. Therefore we call it the stable subspace of T. The natural classification of power bounded operators given by Sz.-Nagy and Foias is the following:

- T is said to be of class $C_{1\cdot}$ or asymptotically non-vanishing if $\mathcal{H}_0(T) = \{0\}$;
- T is said to be of class $C_{0\cdot}$ or stable if $\mathcal{H}_0(T) = \mathcal{H}$, i.e. when $T^n \to 0$ holds in the strong operator topology (SOT);
- T is said to be of class $C_{\cdot j}$ $(j \in \{0,1\})$ whenever T^* is of class $C_{j\cdot}$;
- the class C_{jk} $(j,k \in \{0,1\})$ consists of those operators that are of class $C_{j\cdot}$ and $C_{\cdot k}$, simultaneously.

In 1947, Sz.-Nagy characterized those operators that are similar to unitary operators. This theorem belongs to the best known results concerning the study of Hilbert space operators that are similar to normal operators. Now we state the result.

THEOREM 1.1 (Sz.-Nagy [**15**]). *An operator T is similar to a unitary operator if and only if it is invertible and both T and T^{-1} are power bounded.*

In order to prove the theorem, Sz.-Nagy defined the so-called L-asymptotic limit of a power bounded $T \in \mathcal{B}(\mathcal{H})$ which usually depends on the chosen Banach limit L. In case when T is a contraction, this definition is independent of L, and it is simply the SOT limit of the self-adjoint iterates $\{T^{*n}T^n\}_{n=1}^{\infty}$. This SOT limit is denoted by A_T and it is called the asymptotic limit of T. However, when T is a general power bounded operator, usually the previous sequence does not converge. In this case Sz.-Nagy considered the following sesqui-linear form:

$$w_{T,L}\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}, \quad w_{T,L}(x,y) := \operatorname*{L-lim}_{n\to\infty}\langle T^{*n}T^n x, y\rangle.$$

Since $w_{T,L}$ is bounded and positive, there exists a unique representing positive operator $A_{T,L}\in\mathcal{B}(\mathcal{H})$ such that

$$w_{T,L}(x,y) = \langle A_{T,L}x, y\rangle \quad (x,y\in\mathcal{H}).$$

The operator $A_{T,L}$ is called the L-asymptotic limit of the power bounded operator T. It is quite straightforward to show that when T and T^{-1} are both power bounded, then $A_{T,L}$ is invertible and there exists a unique unitary operator $U\in\mathcal{B}(\mathcal{H})$ such that $A_{T,L}^{1/2}T = UA_{T,L}^{1/2}$ is satisfied. It is also easy to see that $\ker A_{T,L} = \mathcal{H}_0(T)$ is satisfied for every Banach limit L.

We note that there is a certain reformulation of Theorem 1.1 which reads as follows (see e. g. [**13**]).

THEOREM 1.2 (Sz.-Nagy). *Consider an arbitrary operator $T\in\mathcal{B}(\mathcal{H})$ and fix a Banach limit L. The following six conditions are equivalent:*

(i) *T is similar to a unitary operator;*
(ii) *T is onto and similar to an isometry;*
(iii) *T is power bounded and there exists a constant $c>0$ for which the inequalities $\|T^n x\|\geq c\|x\|$ and $\|T^{*n}x\|\geq c\|x\|$ hold for every $n\in\mathbb{N}$ and $x\in\mathcal{H}$;*
(iv) *T is onto, power bounded and there exists a constant $c>0$ for which the inequality $\|T^n x\|\geq c\|x\|$ holds for every $n\in\mathbb{N}$ and $x\in\mathcal{H}$;*
(v) *T is power bounded and the L-asymptotic limits $A_{T,L}$ and $A_{T^*,L}$ are invertible;*
(vi) *T has bounded inverse and both T^{-1} and T are power bounded.*

Moreover, if we have an arbitrary power bounded operator $T\in\mathcal{B}(\mathcal{H})$, then the next three conditions are also equivalent:

(i') *T is similar to an isometry;*
(ii') *there exists a constant $c>0$ for which the inequality $\|T^n x\|\geq c\|x\|$ holds for every vector $x\in\mathcal{H}$;*
(iii') *the L-asymptotic limit $A_{T,L}$ is invertible.*

Sz.-Nagy's method naturally leads us to a more general definition, the so-called isometric and unitary asymptote of a power bounded operator. We consider the operator $X_{T,L}^+ \in \mathcal{B}(\mathcal{H}, \mathcal{H}_T^+)$ where $\mathcal{H}_T^+ = (\operatorname{ran} A_{T,L})^- = (\ker A_{T,L})^\perp = \mathcal{H}_0^\perp$ and $X_{T,L}^+ x = A_{T,L}^{1/2} x$ holds for every $x \in \mathcal{H}$. Since $\|X_{T,L}^+ Tx\| = \|X_{T,L}^+ x\|$ is satisfied ($x \in \mathcal{H}$), there exists a unique isometry $V_{T,L} \in \mathcal{B}(\mathcal{H}_T^+)$ such that $X_{T,L}^+ T = V_{T,L} X_{T,L}^+$ holds. The operator $V_{T,L}$ (or sometimes the pair $(V_{T,L}, X_{T,L}^+)$) is called the isometric asymptote of T. Let $W_{T,L} \in \mathcal{B}(\mathcal{H}_{T,L})$ be the minimal unitary dilation of $V_{T,L}$ and $X_{T,L} \in \mathcal{B}(\mathcal{H}, \mathcal{H}_{T,L}), X_{T,L}x = X_{T,L}^+ x$ for every $x \in \mathcal{H}$. Obviously we have $X_{T,L}T = W_{T,L}X_{T,L}$. The operator $W_{T,L}$ (or sometimes the pair $(W_{T,L}, X_{T,L})$) is said to be the unitary asymptote of T. These asymptotes play an important role in the hyperinvariant subspace problem, similarity problems and operator models.

When $T \notin C_{1\cdot}(\mathcal{H}) \cup C_{0\cdot}(\mathcal{H})$, we have the following result which was first proven by Sz.-Nagy and Foias for contractions and by L. Kérchy for power bounded operators.

LEMMA 1.3 (Kérchy [**12**]). *Consider a power bounded operator $T \notin C_{1\cdot}(\mathcal{H}) \cup C_{0\cdot}(\mathcal{H})$ and the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$. The block-matrix form of T in this decomposition is the following:*

$$T = \begin{pmatrix} T_0 & R \\ 0 & T_1 \end{pmatrix} \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp), \tag{1.1}$$

where the elements T_0 and T_1 are of class $C_{0\cdot}$ and $C_{1\cdot}$, respectively.

This is a very important structural result which is used several times throughout the dissertation.

Our aim in this dissertation is to explore the asymptotic behaviour of power bounded operators. We present some applications, namely we investigate similarity to normal operators, the commutant of non-stable contractions and cyclic properties of a recently introduced operator class, the weighted shifts on directed trees. The dissertation contains the results of five papers [**3, 4, 5, 6, 7**]. In the next sections we present them.

2. Positive operators arising asymptotically from contractions

In Chapter 2 we have characterized those positive operators $A \in \mathcal{B}(\mathcal{H})$ that arise from a contraction asymptotically, i.e. there is such a contraction $T \in \mathcal{B}(\mathcal{H})$ for which $A_T = A$ holds. First we proved that the norm of the L-asymptotic limit of any power bounded operator is either 0 or at least 1.

THEOREM 2.1. *Suppose L is a Banach limit and T is a power bounded operator for which $A_{T,L} \neq 0$ holds. Then we have*

$$\|A_{T,L}\| \geq 1. \tag{2.1}$$

In particular, $\|A_T\| = 1$ holds whenever T is a contraction.

Next, the characterization in finite dimensions was provided which reads as follows.

THEOREM 2.2. *Let $T \in \mathcal{B}(\mathbb{C}^d)$ be a contraction. Then $A_T = A_T^2 = A_{T^*}$, i.e. A_T is simply the orthogonal projection onto the subspace $\mathcal{H}_0(T)^\perp$, and $\mathcal{H}_0(T) = \mathcal{H}_0(T^*)$ is satisfied.*

Turning to the infinite dimensional case, we say that the positive operator $A \in \mathcal{B}(\mathcal{H})$ arises asymptotically from a contraction in uniform convergence if $\lim_{n\to\infty} \|T^{*n}T^n - A\| = 0$ holds. Of course in this case $A = A_T$. On the other hand, it is easy to see that usually for a contraction $T \in \mathcal{B}(\mathcal{H})$ the equation $\lim_{n\to\infty} T^{*n}T^n = A$ holds only in SOT. The symbols σ_e and r_e denote the essential spectrum and the essential spectral radius. Our characterization concerning the separable, infinite dimensional case reads as follows.

THEOREM 2.3. *Let $\dim \mathcal{H} = \aleph_0$ and let A be a positive contraction acting on $\mathcal{H}$. The following four conditions are equivalent:*

- (i) *A arises asymptotically from a contraction;*
- (ii) *A arises asymptotically from a contraction in uniform convergence;*
- (iii) *$r_e(A) = 1$ or A is a projection of finite rank;*
- (iv) *$\dim \mathcal{H}(]0,1]) = \dim \mathcal{H}(]\delta,1])$ holds for every $0 \leq \delta < 1$, where $\mathcal{H}(\omega)$ denotes the spectral subspace of A associated with the Borel subset $\omega \subseteq \mathbb{R}$.*

Moreover, if one of these conditions holds and $\dim\ker(A - I) \in \{0, \aleph_0\}$, then T can be chosen to be a $C_{\cdot 0}$-contraction such that it satisfies (ii).

We have also given the characterization in non-separable spaces. Let κ be an infinite cardinal number, satisfying $\kappa \leq \dim \mathcal{H}$, then the closure of the set $\mathcal{E}_\kappa := \{S \in \mathcal{B}(\mathcal{H})\colon \dim(\mathcal{R}(S))^- < \kappa\}$ is a proper two-sided ideal, denoted by $\mathcal{C}_\kappa$. Let $\mathcal{F}_\kappa := \mathcal{B}(\mathcal{H})/\mathcal{C}_\kappa$ be the quotient algebra, the mapping $\pi_\kappa\colon \mathcal{B}(\mathcal{H}) \to \mathcal{F}_\kappa$ be the quotient map and $\|\cdot\|_\kappa$ the quotient norm on $\mathcal{F}_\kappa$. For an operator $A \in \mathcal{B}(\mathcal{H})$ we use the notations $\|A\|_\kappa := \|\pi_\kappa(A)\|_\kappa$, $\sigma_\kappa(A) := \sigma(\pi_\kappa(A))$ and $r_\kappa(A) := r(\pi_\kappa(A))$. (For $\kappa = \aleph_0$ we get the ideal of compact operators, $\|A\|_{\aleph_0} = \|A\|_e$ is the essential norm, $\sigma_{\aleph_0}(A) = \sigma_e(A)$ and $r_{\aleph_0}(A) = r_e(A)$.) For more details see [**2**] or [**14**]. Now we state our result in the non-separable case.

THEOREM 2.4. *Let* $\dim \mathcal{H} > \aleph_0$ *and* $A \in \mathcal{B}(\mathcal{H})$ *be a positive contraction. Then the following four conditions are equivalent:*

(i) *A arises asymptotically from a contraction;*
(ii) *A arises asymptotically from a contraction in uniform convergence;*
(iii) *A is a finite rank projection, or* $\kappa = \dim \mathcal{H}(]0,1]) \geq \aleph_0$ *and* $r_\kappa(A) = 1$ *holds;*
(iv) $\dim \mathcal{H}\,(]0,1]) = \dim \mathcal{H}\,(]\delta,1])$ *for any* $0 \leq \delta < 1$.

Moreover, when $\dim\ker(A-I) \in \{0,\infty\}$ *and* (i) *holds, then we can choose a $C_{\cdot 0}$-contraction T such that A is the uniform asymptotic limit of T.*

It is natural to ask what conditions on two contractions $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ imply $A_{T_1} = A_{T_2}$, or reversely what connection between T_1 and T_2 follows from the equation $A_{T_1} = A_{T_2}$. The last theorem of Chapter 2 concerns this problem.

THEOREM 2.5. *Let $\mathcal{H}$ be an arbitrary Hilbert space and* $T, T_1, T_2 \in \mathcal{B}(\mathcal{H})$ *contractions. The following statements hold:*

(i) *if T_1, T_2 commute, then* $A_{T_1T_2} \leq A_{T_1}$ *and* $A_{T_1T_2} \leq A_{T_2}$*;*
(ii) *if $u \in H^\infty$ is a non-constant inner function and T is a c.n.u. contraction, then* $A_T = A_{u(T)}$*;*
(iii) $A_{T_1} = A_{T_2} = A$ *implies* $A \leq A_{T_1T_2}$*;*
(iv) *if T_1 and T_2 commute and* $A_{T_1} = A_{T_2}$, *then we have* $A_{T_1T_2} = A_{T_1} = A_{T_2}$.

3. Cesàro asymptotic limits of power bounded matrices

Chapter 3 was devoted to the characterization of all possible L-asymptotic limits of power bounded matrices which has been given in [**5**]. The first important step towards the desired characterization was to show that Banach limits can be replaced by usual limits if we consider Cesáro means of the self-adjoint iterates of T.

THEOREM 3.1. *Let* $T \in \mathcal{B}(\mathbb{C}^d)$ *be power bounded, then*

$$A_{T,L} = A_{T,C} := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} T^{*j}T^j$$

holds for all Banach limits L.

The matrix $A_{T,C} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} T^{*j}T^{j}$ is called the Cesàro asymptotic limit of T. In order to prove Theorem 3.1 we derived some properties of the Jordan decomposition of T. Then we gave the characterization of C_{11} matrices which we state now.

THEOREM 3.2. *The following statements are equivalent for a positive definite $A \in \mathcal{B}(\mathbb{C}^d)$:*

(i) *A is the Cesàro asymptotic limit of a power bounded matrix $T \in C_{11}(\mathbb{C}^d)$;*

(ii) *if the eigenvalues of A are $t_1, \dots, t_d > 0$, each of them is counted according to their multiplicities, then*

$$\frac{1}{t_1} + \cdots + \frac{1}{t_d} = d \tag{3.1}$$

holds;

(iii) *there is an invertible $S \in \mathcal{B}(\mathbb{C}^d)$ with unit columnvectors such that*

$$A = S^{*-1}S^{-1} = (SS^*)^{-1}.$$

The proof of the general case uses the C_{11} case, Lemma 1.3 and a block-diagonalization of a special type of block matrices. We call a power bounded matrix $T \in \mathcal{B}(\mathbb{C}^d)$ l-stable ($0 \le l \le d$) if $\dim \mathcal{H}_0 = l$. In the following theorem the symbol I_l stands for the identity matrix on $\mathbb{C}^l$ and $0_k \in \mathcal{B}(\mathbb{C}^k)$ is the zero matrix.

THEOREM 3.3. *The following three conditions are equivalent for a non-invertible positive semidefinite $A \in \mathcal{B}(\mathbb{C}^d)$ and a number $1 \le l < d$:*

(i) *there exists a power bounded, l-stable $T \in \mathcal{B}(\mathbb{C}^d)$ such that $A_{T,C} = A$;*

(ii) *let $k = d - l$, if $t_1, \dots t_k$ denote the non-zero eigenvalues of A counted with their multiplicities, then*

$$\frac{1}{t_1} + \cdots + \frac{1}{t_k} \le k;$$

(iii) *there exists an invertible $S \in \mathcal{B}(\mathbb{C}^d)$ such that it has unit columnvectors and*

$$A = S^{*-1}(I_l \oplus 0_k)S^{-1}.$$

One could ask whether there is any connection between the Cesàro asymptotic limit of a matrix and the Cesàro-asymptotic limit of its adjoint. If $T \in \mathcal{B}(\mathbb{C}^d)$ is contractive, then $A_{T^*} = A_T$ is valid (see Theorem 2.2). In the case of power bounded matrices usually the subspaces $\mathcal{H}_0(T)$ and

$\mathcal{H}_0(T^*)$ are different and hence $A_{T^*,C}$ and $A_{T,C}$ are different, too. However, we have provided the next connection for C_{11} class 2×2 power bounded matrices.

THEOREM 3.4. *For each $T \in C_{11}(\mathbb{C}^2)$ the harmonic mean of the Cesàro asymptotic limits $A_{T,C}$ and $A_{T^*,C}$ is exactly the identity I, i.e.*

$$A_{T,C}^{-1} + A_{T^*,C}^{-1} = 2I_2. \tag{3.2}$$

Theorem 3.4 is not true in three-dimension any more which was justified by a concrete 3×3 example.

4. Asymptotic limits of operators similar to normal operators

Chapter 4 contains our results from [**6**]. First we have given a generalization of the necessity part in Sz.-Nagy's similarity result (Theorems 1.1 and 1.2). Let us consider a normal power bounded operator $N \in \mathcal{B}(\mathcal{H})$. Since $r(N)^n = r(N^n) = \|N^n\|$ holds ($n \in \mathbb{N}$), we obtain that N is a contraction. A quite straightforward application of the functional model of normal operators gives us that $A_N = I_{\mathcal{H}_0(N)^\perp} \oplus 0_{\mathcal{H}_0(N)}$ holds where $N|(\operatorname{ran} A_N)^-$ is the unitary part of N. In view of Theorem 1.2 it is reasonable to conjecture that when a power bounded operator T is similar to a normal operator then $A_{T,L}|(\operatorname{ran} A_{T,L})^-$ should be invertible. We recall that if the operator $A \in \mathcal{B}(\mathcal{H})$ is not zero, then the reduced minimum modulus of A is the quantity $\gamma(A) := \inf\{\|Ax\| : x \in (\ker A)^\perp, \|x\| = 1\}$. In particular if A is a positive operator, then $\gamma(A) > 0$ holds exactly when $A|(\operatorname{ran} A)^-$ is invertible. Our first result concerns two similar power bounded operators and it reads as follows.

THEOREM 4.1. *Let us consider two power bounded operators $T, S \notin C_{0\cdot}(\mathcal{H})$ which are similar to each other. Then $\gamma(A_{T,L}) > 0$ holds for some (and then for all) Banach limits L if and only if $\gamma(A_{S,L}) > 0$ is valid.*

Moreover, $\gamma(A_{T,L}) > 0$ holds if and only if the powers of T are bounded from below uniformly on $\mathcal{H}_0(T)^\perp$, i.e. there exists a constant $c > 0$ such that

$$c\|x\| \leq \|T^n x\| \quad (x \in \mathcal{H}_0(T)^\perp, n \in \mathbb{N}).$$

In particular, if T is similar to a normal operator, then $\gamma(A_{T,L}) > 0$ and $\gamma(A_{T^*,L}) > 0$ are satisfied.

The proof of Theorem 4.1 uses Lemma 1.3. This theorem can be considered as a generalization of the necessity part in Sz.-Nagy's theorem and it can help in deciding whether an operator is similar to e.g. a normal operator. However, the last statement of Theorem 4.1 is clearly not reversible which can be seen by easy counterexamples.

As we have seen, Sz.-Nagy's theorem says that $A_{T,L}$ is invertible when T is similar to a unitary. If T is a contraction, then we can state more than simply the invertibility of A_T. Namely we will prove the following theorem where (i) is only an implication, but (ii) is its converse in the separable case. We recall that the minimal element of the spectrum of a self-adjoint operator A is denoted by $\underline{r}(A)$.

THEOREM 4.2.

(i) *Let* $\dim \mathcal{H} \geq \aleph_0$ *and* $T \in \mathcal{B}(\mathcal{H})$ *be a contraction which is similar to a unitary operator. Then* $\dim \ker(A_T - \underline{r}(A_T)I) \in \{0, \infty\}$. *Consequently, the condition* $\underline{r}(A_T) \in \sigma_e(A_T)$ *holds.*

(ii) *Let* $\dim \mathcal{H} = \aleph_0$, $A \in \mathcal{B}(\mathcal{H})$ *be a positive, invertible contraction and suppose that the conditions* $1 \in \sigma_e(A)$ *and* $\dim \ker(A - \underline{r}(A)I) \in \{0, \aleph_0\}$ *are fulfilled. Then there exists a contraction* $T \in \mathcal{B}(\mathcal{H})$ *which is similar to a unitary operator and the asymptotic limit of* T *is exactly* A.

This theorem can be considered as a strengthening of the necessity part in Sz.-Nagy's similarity theorem. In particular, in the separable case it provides a characterization of asymptotic limits for those contractions which are similar to unitary operators. We note that it is an open problem to describe the L asymptotic limits of those operators that are similar to unitary operators and act on infinite dimensional spaces.

5. Injectivity of the commutant mapping

Let us consider a contraction $T \in \mathcal{B}(\mathcal{H})$. Given any $C \in \{T\}'$ there exists exactly one $D \in \{W_T\}'$ such that $X_T C = D X_T$. This enables us to define the commutant mapping γ_T of T in the following way:

$$\gamma = \gamma_T \colon \{T\}' \to \{W_T\}',\ C \mapsto D, \text{ where } X_T C = D X_T.$$

It can be shown that γ is a contractive algebra-homomorphism (see [**16**, Section IX.1] for further details). This commutant mapping is among the few links which relate the contraction to a well-understood operator. It can be exploited to get structure theorems or stability results. Hence it is of interest to study its properties. Our purpose in [**3**] was to examine the injectivity of γ, and in Chapter 5 we gave the results obtained in that publication. If T is asymptotically non-vanishing, then X_T and hence γ_T

are clearly injective. A natural and non-trivial question is that if the commutant mapping is injective, is necessarily T of class $C_{1\cdot}$? However, this is not true. In Chapter 5 a counterexample have been given. We have also provided four necessary conditions of injectivity. For a contraction T let P_0 denote the orthogonal projection onto the stable subspace. The compressions $P_0 T|\mathcal{H}_0$ and $(I-P_0)T|\mathcal{H}_0^\perp$ are denoted by T_{00} and T_{11}, respectively.

THEOREM 5.1. *If the commutant mapping γ of the contraction $T \in \mathcal{B}(\mathcal{H})$ is injective, then*

(i) *the equation $XT_{11} = XT_{00}$ is satisfied only with $X = 0$,*
(ii) $\overline{\sigma_{ap}(T_{00}^*)} \cap \sigma_{ap}(T_{11}) \neq \emptyset$,
(iii) $\sigma_p(T) \cap \overline{\sigma_p(T^*)} \cap \mathbb{D} = \emptyset$, *and*
(iv) *there is no direct decomposition $\mathcal{H} = \mathcal{M}_0 \dot{+} \mathcal{M}_1$ such that $\mathcal{M}_0, \mathcal{M}_1$ are invariant subspaces of T and $\{0\} \neq \mathcal{M}_0 \subset \mathcal{H}_0$.*

For an operator A on a Hilbert space $\mathcal{F}$, and for a complex number λ, the root subspace of A corresponding to λ is defined by $\widetilde{\ker}(A-\lambda I) := \vee_{j=1}^{\infty} \ker(A-\lambda I)^j$. We say that the operator A has a generating root subspace system if $\mathcal{F} = \vee \left\{ \widetilde{\ker}(A-\lambda I) : \lambda \in \sigma_p(A) \right\}$.

Our second result was about sufficiency. Namely, it shows that under certain circumstances (iii) of Theorem 5.1 is sufficient for the injectivity of γ.

THEOREM 5.2. *Let us assume that the stable component T_{00} of the contraction T satisfies the following conditions:*

(i) $\sigma_p(T_{00}^*) \subset \overline{\sigma_p(T_{00})}$;
(ii) T_{00}^* *has a generating root subspace system.*

Then γ is injective if and only if $\sigma_p(T) \cap \overline{\sigma_p(T^*)} \cap \mathbb{D} = \emptyset$.

We note that the conditions of the previous theorem are satisfied in a large class of C_0-contractions.

After that we were able to present our example for a non-$C_{1\cdot}$ contraction which has injective commutant mapping. This was followed by investigating the injectivity of commutant mappings for quasisimilar contractions and orthogonal sum of contractions. Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are quasisimilar if there exist quasiaffinities (i.e. operators with trivial kernel and dense range) $X \in \mathcal{B}(\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H})$ such that $XB = AX$ and $YA = BY$. We say that $T \in \mathcal{B}(\mathcal{H})$ is in stable relation with $T' \in \mathcal{B}(\mathcal{H}')$, if $CT = T'C$ and $\operatorname{ran} C \subset \mathcal{H}_0'(T')$ imply $C = 0$, and if $C'T' = TC'$ and $\operatorname{ran} C' \subset \mathcal{H}_0(T)$ imply $C' = 0$. Our result concerning quasisimilar contractions and orthogonal sum of contractions reads as follows.

THEOREM 5.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $T' \in \mathcal{B}(\mathcal{H}')$ be contractions.*

(i) *If T and T' are quasisimilar, then γ_T is injective if and only if $\gamma_{T'}$ is injective.*
(ii) *The commutant mapping $\widetilde{\gamma} := \gamma_{T\oplus T'}$ is injective if and only if γ_T and $\gamma_{T'}$ are injective, and T is in stable relation with T'.*
(iii) *Assuming $T = T'$, the commutant mapping $\widetilde{\gamma} = \gamma_{T\oplus T}$ is injective if and only if γ_T is injective.*

We would like to point out that (ii)-(iii) of the above theorem can be extended quite easily for orthogonal sum of countably many contractions. We also note that the injectivity of γ_T and $\gamma_{T'}$ does not imply the injectivity of $\widetilde{\gamma}$. This was verified by a concrete example.

Finally, using certain properties of shift operators, we have provided an example for which (i)-(iv) of Theorem 5.1 are satisfied but still γ_T is not injective. This shows that these four conditions together are not enough to provide a characterization of injectivity of γ_T.

6. Cyclic properties of weighted shifts on directed trees

The classes of the so-called weighted bilateral, unilateral or backward shift operators are very useful for an operator theorist. Besides normal operators these are the next natural classes on which conjectures could be tested. Recently Z. J. Jabłonski, I. B. Jung and J. Stochel defined a natural generalization of these classes in [**10**]. They were interested among others in hyponormality, co-hyponormality, subnormality etc., and they provided many examples for several unanswered questions using this new class.

In the last chapter we have studied cyclic properties of bounded (mainly contractive) weighted shift operators on directed trees. Namely first we have explored their asymptotic behaviour, and as an application we have obtained some results concerning cyclicity. These theorems are from [**7**].

We recall some definitions from [**10**]. The pair $\mathcal{T} = (V, E)$ is a directed graph if V is an arbitrary (usually infinite) set and $E \subseteq (V \times V) \setminus \{(v, v) : v \in V\}$. We call an element of V and E a vertex and a (directed) edge of $\mathcal{T}$, respectively. We say that $\mathcal{T}$ is connected if for any two distinct vertices $u, v \in V$ there exists an undirected path between them, i.e. there are finitely many vertices: $u = v_0, v_1, \ldots v_n = v \in V$ $(n \in \mathbb{N})$ such that (v_{j-1}, v_j) or $(v_j, v_{j-1}) \in E$ for every $1 \le j \le n$. The finite sequence of distinct vertices $v_0, v_1, \ldots v_n \in V$ $(n \in \mathbb{N})$ is called a (directed) circuit if

$(v_{j-1}, v_j) \in E$ for all $1 \le j \le n$ and $(v_n, v_0) \in E$. The directed graph $\mathcal{T} = (V, E)$ is a directed tree if the following three conditions are satisfied:

(i) $\mathcal{T}$ is connected;
(ii) for each $v \in V$ there exists at most one $u \in V$ such that $(u, v) \in E$;
(iii) $\mathcal{T}$ has no circuit.

From now on $\mathcal{T}$ always denotes a directed tree. In the directed tree a vertex v is called a child of $u \in V$ if $(u, v) \in E$. The set of all children of u is denoted by $\mathrm{Chi}_{\mathcal{T}}(u) = \mathrm{Chi}(u)$. Conversely, if for a given vertex v we can find a (unique) vertex u such that $(u, v) \in E$, then we shall say that u is the parent of v. We denote u by $\mathrm{par}_{\mathcal{T}}(v) = \mathrm{par}(v)$. We also use the notation $\mathrm{par}^k(v) = \underbrace{\mathrm{par}(\dots(\mathrm{par}}_{k\text{-times}}(v))\dots)$ if it makes sense, and par^0 is the identity map.

If a vertex has no parent, then we call it a root of $\mathcal{T}$. A directed tree is either rootless or has a unique root (see [**10**, Proposition 2.1.1.]). We denote this unique root by $\mathrm{root}_{\mathcal{T}} = \mathrm{root}$, if it exists. A subgraph of a directed tree which is itself a directed tree is called a subtree. We use the notation $V^\circ = V \setminus \{\mathrm{root}\}$. If a vertex has no children, then we call it a leaf, and $\mathcal{T}$ is leafless if it has no leaves. The set of all leaves of $\mathcal{T}$ is denoted by $\mathrm{Lea}(\mathcal{T})$. Given a subset $W \subseteq V$ of vertices, we put $\mathrm{Chi}(W) = \cup_{v \in W} \mathrm{Chi}(v)$, $\mathrm{Chi}^0(W) = W$, $\mathrm{Chi}^{n+1}(W) = \mathrm{Chi}(\mathrm{Chi}^n(W))$ for all $n \in \mathbb{N}$ and $\mathrm{Des}_{\mathcal{T}}(W) = \mathrm{Des}(W) = \bigcup_{n=0}^{\infty} \mathrm{Chi}^n(W)$, where $\mathrm{Des}(W)$ is called the descendants of the subset W, and if $W = \{u\}$, then we simply write $\mathrm{Des}(u)$.

If $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then the set $\mathrm{Gen}_{n,\mathcal{T}}(u) = \mathrm{Gen}_n(u) = \bigcup_{j=0}^{n} \mathrm{Chi}^j(\mathrm{par}^j(u))$ is called the nth generation of u (i.e. we can go up at most n levels and then down the same amount of levels) and $\mathrm{Gen}_{\mathcal{T}}(u) = \mathrm{Gen}(u) = \bigcup_{n=0}^{\infty} \mathrm{Gen}_n(u)$ is the (whole) generation or the level of u. From the equation (see [**10**, Proposition 2.1.6])

$$V = \bigcup_{n=0}^{\infty} \mathrm{Des}(\mathrm{par}^n(u)) \tag{6.1}$$

one can easily see that the different levels can be indexed by the integer numbers (or with a subset of the integers) in such a way that if a vertex v is in the kth level, then the children of v are in the $(k+1)$th level and $\mathrm{par}(v)$ is in the $(k-1)$th level whenever $\mathrm{par}(v)$ makes sense.

The complex Hilbert space $\ell^2(V)$ is the usual space of all square summable complex functions on V with the standard inner product

$$\langle f, g\rangle = \sum_{u\in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

For $u \in V$ we define $e_u(v) = \delta_{u,v} \in \ell^2(V)$, where $\delta_{u,v}$ is the Kronecker-delta symbol. Obviously the set $\{e_u : u \in V\}$ is an orthonormal basis. We usually refer to $\ell^2(W)$ as the subspace $\vee\{e_w : w \in W\}$, for any subset $W \subseteq V$.

Let $\underline{\lambda} = \{\lambda_v : v \in V^\circ\} \subseteq \mathbb{C}$ be a set of weights satisfying

$$\sup\left\{\sqrt{\sum_{v\in \mathrm{Chi}(u)} |\lambda_v|^2} : u \in V\right\} < \infty.$$

Then the weighted shift operator on the directed tree $\mathcal{T}$ is the operator defined by

$$S_{\underline{\lambda}} : \ell^2(V) \to \ell^2(V), \quad e_u \mapsto \sum_{v\in \mathrm{Chi}(u)} \lambda_v e_v.$$

By [**10**, Proposition 3.1.8.] this defines a bounded linear operator with norm $\|S_{\underline{\lambda}}\| = \sup\left\{\sqrt{\sum_{v\in \mathrm{Chi}(u)} |\lambda_v|^2} : u \in V\right\}$.

We considered only bounded weighted shifts on directed trees, especially contractions. We recall that every $S_{\underline{\lambda}}$ is unitarily equivalent to $S_{|\underline{\lambda}|}$ where $|\underline{\lambda}| := \{|\lambda_v| : v \in V^\circ\} \subseteq [0, \infty[$. If a weight λ_v is zero, then the weighted shift operator on this directed tree is a direct sum of two other weighted shift operators on directed trees (see [**10**, Theorem 3.2.1 and Proposition 3.1.6]. In view of these facts, we exclusively considered weighted shift operators on directed trees with positive weights.

An operator $T \in \mathcal{B}(\mathcal{H})$ is cyclic if there is a vector h such that

$$\mathcal{H}_{T,h} = \mathcal{H}_h := \vee\{T^n h : n \in \mathbb{N}_0\} = \{p(T)h : p \in \mathcal{P}_\mathbb{C}\}^- = \mathcal{H},$$

where $\mathcal{P}_\mathbb{C}$ denotes the set of all complex polynomials. Such an $h \in \mathcal{H}$ is called a cyclic vector for T. The first theorem of the chapter concerned a countable orthogonal sum of weighted backward shift operators.

THEOREM 6.1. *Suppose that* $\{e_{j,k} : j \in \mathcal{J}, k \in \mathbb{N}_0\}$ *is an orthonormal basis in the Hilbert space* $\mathcal{H}$ *where* $\mathcal{J} \neq \emptyset$ *is a countable set and* $\{w_{j,k} : j \in \mathcal{J}, k \in \mathbb{N}_0\} \subset \mathbb{C}$ *is a bounded set of weights. Consider the following operator:*

$$Be_{j,k} = \begin{cases} 0 & \textit{if } k = 0 \\ w_{j,k-1}e_{j,k-1} & \textit{otherwise} \end{cases}.$$

(i) *If there is no zero weight, then* B *is cyclic.*

(ii) *Assume there is no zero weight and there exists a vector* $g \in \cap_{n=1}^{\infty} \operatorname{ran}(B^n)$ *such that for every fixed* $j \in \mathcal{J}$, $\langle g, e_{j,k} \rangle \neq 0$ *is fulfilled for infinitely many* $k \in \mathbb{N}_0$. *Then we can find a cyclic vector from the linear manifold* $\cap_{n=1}^{\infty} \operatorname{ran}(B^n)$.

(iii) *The operator* B *is cyclic if and only if* B *has at most one zero weight.*

The proof of Theorem 6.1 was motivated by the solution of [**8**, Problem 160]. We would like to note that the case in (iii) of Theorem 6.1 when $\#\mathcal{J} = 1$ was done by Z. G. Hua in [**9**]. However this article was written in Chinese, therefore we were not able to read the proof. The above theorem can be considered as a generalization of Hua's result.

After this we gave our results concerning cyclicity of weighted shifts on directed trees. The quantity

$$\operatorname{Br}(\mathcal{T}) = \sum_{u \in V \setminus \operatorname{Lea}(\mathcal{T})} (\#\operatorname{Chi}(u) - 1)$$

is called the branching index of $\mathcal{T}$. By (ii) of [**10**, Proposition 3.5.1] we have

$$\dim(\operatorname{ran}(S_{\underline{\lambda}})^{\perp}) = \begin{cases} 1 + \operatorname{Br}(\mathcal{T}) & \text{if } \mathcal{T} \text{ has a root,} \\ \operatorname{Br}(\mathcal{T}) & \text{if } \mathcal{T} \text{ has no root.} \end{cases} \tag{6.2}$$

It is easy to see that for every cyclic operator $T \in \mathcal{B}(\mathcal{H})$ we have $\dim(\operatorname{ran}(T)^{\perp}) \leq 1$. Therefore the only interesting case concerning the cyclicity of $S_{\underline{\lambda}}$ is when $\mathcal{T}$ has no root and $\operatorname{Br}(\mathcal{T}) = 1$ (if the branching index is zero, we obtain a usual weighted shift operator).

If $\#\operatorname{Lea}(\mathcal{T}) = 2$, then $\mathcal{T}$ is represented by the graph (V, E) with vertices $V = \{j \in \mathbb{Z} \colon j \leq j_0\} \cup \{k' \colon 1 \leq k \leq k_0\}$ where we assume $1 \leq k_0 \leq j_0 < \infty$, and edges $E = \{(j-1, j) \colon j \leq j_0\} \cup \{(0, 1')\} \cup \{((j-1)', j') \colon 1 < j \leq k_0\}$. The corresponding weights are $\underline{\lambda} = \{\lambda_v \colon v \in V\}$. If $\#\operatorname{Lea}(\mathcal{T}) = 1$ or 0, then the representative graphs are very similar. Our next result is about the cyclicity of $S_{\underline{\lambda}}$.

Theorem 6.2. *Suppose that the directed tree* $\mathcal{T} = (V, E)$ *has no root and* $\operatorname{Br}(\mathcal{T}) = 1$.

(i) *If* $\#\operatorname{Lea}(\mathcal{T}) = 2$, *then every bounded weighted shift on* $\mathcal{T}$ *is cyclic.*

(ii) *Suppose* $\mathcal{T} = (V, E)$ *has a unique leaf. A weighted shift* $S_{\underline{\lambda}}$ *on* $\mathcal{T}$ *is cyclic if and only if the bilateral shift operator* W *on the subtree* $\mathcal{T}' := (\mathbb{Z}, E \cap (\mathbb{Z} \times \mathbb{Z}))$ *with weights* $\{\lambda_n\}_{n=-\infty}^{\infty} = \{\lambda_v \colon \in V \cap \mathbb{Z}\}$

is cyclic. In particular, if $S_{\underline{\lambda}} \notin C_{\cdot 0}(\ell^2(V))$ *is contractive, then* $S_{\underline{\lambda}}$ *is cyclic.*

(iii) *Assume that* $S_{\underline{\lambda}}$ *is contractive and of class* $C_{1\cdot}$ *(consequently* $\mathcal{T}$ *has no leaf). Then it has no cyclic vectors.*

(iv) *If* $\mathcal{T}$ *as no leaf, then there exists a cyclic weighted shift operator* $S_{\underline{\lambda}}$ *on* $\mathcal{T}$.

As far as we know complete characterization of cyclicity of bilateral weighted shift operators is still open. Our last result concerns the cyclicity of the adjoint of weighted shift operators on directed trees.

THEOREM 6.3.

(i) *If* $\mathcal{T}$ *has a root and the contractive weighted shift operator* $S_{\underline{\lambda}}$ *on* $\mathcal{T}$ *is of class* $C_{1\cdot}$*, then* $S_{\underline{\lambda}}^*$ *is cyclic.*

(ii) *If* $\mathcal{T}$ *is rootless,* $\mathrm{Br}(\mathcal{T}) < \infty$ *and the weighted shift contraction* $S_{\underline{\lambda}}$ *on* $\mathcal{T}$ *is of class* $C_{1\cdot}$*, then* $S_{\underline{\lambda}}^*$ *is cyclic.*

In order to prove the above theorem we have verified that $S \oplus (S_k^+)^*$ ($k \in \mathbb{N}$) is cyclic where S denotes a simple bilateral shift operator and S_k^+ denotes the orthogonal sum of k pieces of simple unilateral shift operators. It is interesting to ask what happens if in (ii) of Theorem 6.3 the condition $\mathrm{Br}(\mathcal{T}) < \infty$ is dropped. If $S \oplus (S_{\aleph_0}^+)^*$ were cyclic, we could drop this condition. These questions were left open in the dissertation.

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