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PhD School in Computer Science

Topology preserving image operators and new methods in thinning

Theses of PhD Dissertation

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1 Introduction

Digital topology deals with the topological properties of binary digital pictures [18,20]. Elements of binary pictures have value "0" or "1". There are three types of image operators [3]:

- reductions which never change (delete) a "0" to "1",
- additions which never turn a "1" into "0", and
- general operators which are neither reductions nor additions.

The most important problem in digital topology is to ensure topological preservation of image operators. There have been proposed several sufficient conditions for reductions on pictures sampled on the square and the cubic grid. [19, 21, 24, 26].

Thinning is an iterative object reduction technique for producing skeleton-like shape features (i.e., centerline and topological kernel in 2D and 3D, and medial surface in 3D) [2, 28]. It has the advantages that it is the fastest skeletonization method, it can be efficiently parallelized, and topology preservation can be guaranteed.

Thinning algorithms can be either parallel or sequential [2, 28]. In an iteration step of parallel algorithms some deletable border points are deleted simultaneously. Sequential algorithms use contour tracing, and they remove only one border point at a time. In the parallel case, ensuring topology preservation is a more difficult problem than in sequential thinning, however sequential algorithms have the drawback that their results usually depend on the visiting order of border points.

A general problem of skeletonization methods is that they are rather sensitive to coarse object boundaries, hence the produced skeletons generally contain some false segments. In order to overcome this problem, unwanted skeletal parts are usually removed by a pruning process as a post-processing step [27]. The main disadvantage of this method is that it also preserves the distortions of "substantial" parts of objects.

In my thesis I discuss the following research results:

- I give some general sufficient conditions for topology preserving 2D reductions, which hold for pictures sampled on triangular grid, square grid or hexagonal grid. Based on these new conditions and on the three kinds of parallel thinning strategies, I also introduce some new hexagonal and triangular thinning algorithms, whose topological correctness is guaranteed. Furthermore I show a duality theorem between additions and reductions, which can be applied for deriving new conditions for additions from the existing sufficient conditions on reductions.
- I introduce a topology preserving 3D contour smoothing algorithm and I also discuss a new thinning scheme that uses iteration-by-iteration

smoothing for reducing the number of unwanted segments in skeleton-like shape features produced by thinning algorithms.

- I prove some necessary and sufficient conditions for verifying if a sequential thinning algorithm is order-independent, i.e., if it is sensitive for the visiting order of border points or not. After that I introduce some 2D and 3D sequential thinning algorithms whose order-independency is proved by the mentioned conditions. These algorithms are capable of producing 2D and 3D centerlines and 3D medial surfaces. One of them solves the problem of order-independency in arbitrary dimensions.

2 Definitions and basic notions

In the dissertation, I examine the topological properties of the triangular, square, hexagonal, and cubic grids, which are denoted by $\mathcal{T}$, $\mathcal{S}$, $\mathcal{H}$, and $\mathcal{C}$, respectively. In pictures sampled on the regular 2D grids two polygons (pixels) are 1-*adjacent*, if they share an edge, and they are 2-*adjacent*, if they share an edge or a vertex. Two voxels of the cubic grid are 1-, 2-, or 3-*adjacent*, if they share a face, a face or an edge, or a face, an edge, or a vertex, respectively. I refer to the set of elements being m-adjacent to an element $p \in V$ by the notion $N_m^V(p)$, furthermore $N_m^{*V}(p) = N_m^V(p) \setminus \{p\}$.

An n-dimensional *digital binary picture* is given by the quadruple $\mathcal{P} = (V, k, \overline{k}, B)$ [18], where:

- V is the set of picture elements,
- $B \subseteq V$ is the set of black picture elements, whose complementary set $\overline{B} = V \setminus B$ is the set of white picture elements,
- k is the adjacency relation assigned to the black elements,
- $\overline{k}$ is the adjacency relation assigned to the white elements.

A picture $(V, k, \overline{k}, B)$ is also called shortly as picture $(V, k, \overline{k})$.

In the thesis I mainly deal with $(\mathcal{S}, 2, 1)$, $(\mathcal{H}, 1, 1)$, $(\mathcal{T}, 2, 1)$ and $(\mathcal{C}, 3, 1)$ pictures. The adjacency relations applied on these grids are illustrated in Fig. 1.

All the considered adjacency relations k and $\overline{k}$ are reflexive and symmetric, hence their transitive closure are equivalence relations, which means that they partition the sets B and $V \setminus B$ into black k-components and white $\overline{k}$-components. The black k-components are called as *object*s, the only infinite white $\overline{k}$-component on finite pictures is referred to as the *background*, and the finite white $\overline{k}$-components are called as *cavities.* The picture element $p \in B$ is a *border element* in the picture $(V, k, \overline{k}, B)$, if p is $\overline{k}$-adjacent to at least one white element.

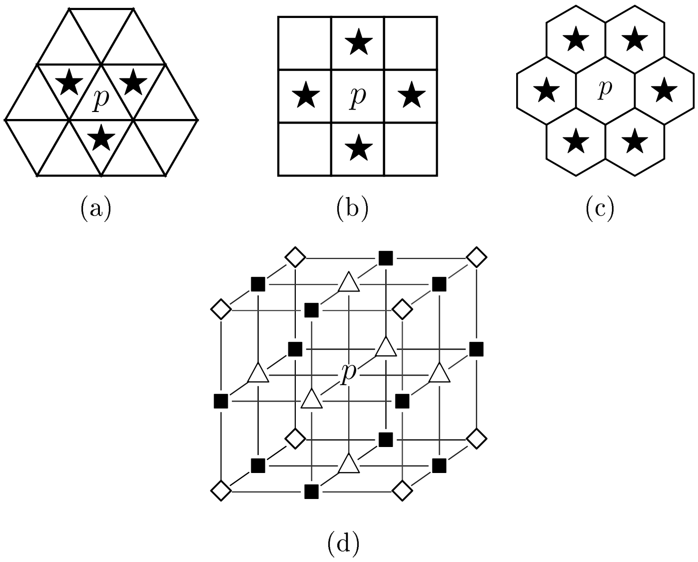

Figure 1: Adjacency relations in the triangular (a), square (b), and hexagonal grid (c), and in the cubic lattice dual to the cubic grid (d). In figures (a)-(c), pixels 1-adjacent to p are depicted "★", the other pixels are 2-adjacent but not 1-adjacent to p. In figure (d), points 1-adjacent to point p are denoted by "△". The points being 2-, but not 1-adjacent to p are the ones depicted "■", and the points being 3-, but not 2-adjacent to p are denoted by "◇".

For 3D pictures, besides objects and cavities, we must also deal with the topological notion of *holes* (or *tunnels*) [18].

A 2D reduction is *topology preserving*, if any object in the input picture contains exactly one object of the output picture, and every white component in the output picture contains exactly one white component of the input picture [18].

A 2D addition is topology preserving, if any white component in the original picture contains exactly one white component of the output picture, and every object in the output picture contains exactly one object of the input picture.

In 3D pictures, "preservation" of holes must be also ensured.

A (black or white) picture element is *simple* in the picture $(V, k, \overline{k}, B)$, if the modification of its color is a topology preserving operation [18]. In pictures $(\mathcal{S}, 2, 1)$ and $(\mathcal{C}, 3, 1)$ simplicity is a local property, which can be decided by the examination of the 3×3 or $3 \times 3 \times 3$ neighborhood of the given picture element.

Thinning, as the method for skeletonization, is an iterative object reduction, which is capable of generating all the three skeleton-like shape features (i.e., the topological kernel, the centerline, and the medial surface) [2, 28].

A thinning algorithm is topology preserving, if the reduction of any of its phases is topology preserving. For the pictures $(\mathcal{S}, 2, 1)$ and $(\mathcal{C}, 3, 1)$, there were already proposed some sufficient conditions for topology preservation of reductions [19, 21, 24, 26], however, for the cases $(\mathcal{H}, 1, 1)$ and $(\mathcal{T}, 2, 1)$ no such criteria have been given yet, furthermore, topology preservation of additions and general operators is still an unsolved question.

The centerlines and medial surfaces extracted by the known skeletonization algorithms often contain unwanted branches or surface segments. Methods for removing such parts are mainly based on *pruning* [27] which, however have the drawback that the false segments resulting from coarse object boundaries also distort the substantial parts of objects, and these deformations are preserved by pruning as a post-processing step.

Another approach is based on the smoothing of the object contour before each phase of the thinning process. There exist numerous approaches for smoothing curves and surfaces [1, 4, 29, 30]. Unfortunately, among them only the method of Couprie and Bertrand [1] is capable of smoothing 3D binary pictures, however, due to its complexity, it cannot be combined with 3D thinning algorithms.

The central problem of sequential thinning is to ensure *order independency*, as the results of sequential thinning algorithms usually depend on the visiting order of object point.

The first order-independent sequential thinning algorithms (i.e., algorithms that produce the same results for any visiting orders) have been proposed by Ranwez and Soille and later by Iwanowski and Soille [5, 25]. The main disadvantage of these attempts lies in the fact that they do not retain endpoints which are important in the view of shape-preservation. This means that, in a pre-processing step, endpoints must be previously detected as anchors (i.e., those detected points are not deleted during the entire object reduction process).

3 Results of the dissertation

The results belonging to the three key points of the dissertation are summarized in the following three subsections.

3.1 New sufficient conditions for topology preserving operators, hexagonal and triangular topology preserving thinning algorithms

In Section 3 of my dissertation, first I introduce some necessary and sufficient conditions for characterizing simple pixels in $(\mathcal{H}, 1, 1)$ and in $(\mathcal{T}, 2, 1)$ pictures. Then, I give the general form of those criteria, which holds for all the three discussed types of grids. Here I only emphasize the latter result.

Theorem 1. (Theorem 3.1.6)
A pixel $p \in B$ in the picture $(V, k, \overline{k}, B)$ $(V \in \{\mathcal{S}, \mathcal{H}, \mathcal{T}\}, (k, \overline{k}) = (2, 1), (1, 2))$ is simple if and only if the following conditions hold:

1. p is k-adjacent to exactly one k-component of $N_k^{*V}(p) \cap B$.

2. p is $\overline{k}$-adjacent to exactly one $\overline{k}$-component of $N_k^V(p) \setminus B$.

In my thesis, I also formulate another general theorem for the examination of simplicity, which is based on the model of so-called attached sets [17]. However the formulation of this would require the explanation of some further notions.

After this, I introduce some sufficient conditions for topology-preserving reductions on $(\mathcal{H}, 1, 1)$, $(\mathcal{T}, 2, 1)$, and $(\mathcal{T}, 1, 2)$ pictures. I also give the general form of these conditions, which holds for all the three regular 2D grids. The conditions of the following theorem consider the examination of pixel-configurations.

Theorem 2. (Theorem 3.2.4)
Reduction $\mathcal{R}$ is topology preserving if for any picture $(V, k, \overline{k}, B)$ $(V \in \{\mathcal{S}, \mathcal{H}, \mathcal{T}\}; (k, \overline{k}) = (2,1), (1,2))$, the following conditions hold:

1. Any pixel removed by $\mathcal{R}$ is simple.

2. For any pair of k-adjacent pixels $\{p, q\}$ removed by $\mathcal{R}$, p is simple in picture $(V, k, \overline{k}, B \setminus \{q\})$.

3. If $(k, \overline{k}) = (2,1)$ then $\mathcal{R}$ does not completely remove any small object (see Figures 1.6-1.8 in the dissertation).

The following conditions consider individual pixels instead of pixel configurations. We make a distinction between symmetric and asymmetric conditions: the symmetric ones do not make a distinction between the pixels of some pixel configurations (see Theorem 3), while asymmetric ones designate a point as "marked" in each configuration (see Theorem 4).

Theorem 3. (Theorem 3.2.5)
Reduction $\mathcal{R}$ is topology preserving if for any picture $(V, k, \overline{k}, B)$ $(V \in \{\mathcal{S}, \mathcal{H}, \mathcal{T}\}; (k, \overline{k}) = (2,1), (1,2))$, and any pixel $p \in B$ removed by $\mathcal{R}$ the following conditions hold:

1. p is a simple pixel.

2. For any simple pixel $q \in N_{\overline{k}}^{*V}(p) \cap B$, p is simple in picture $(V, k, \overline{k}, B \setminus \{q\})$.

3. If $(k, \overline{k}) = (2,1)$, then pixel p is not a member of any small object (see Figures 1.6-1.8 in the dissertation).

Theorem 4. (Theorem 3.2.6)
Reduction $\mathcal{R}$ is topology preserving if for any picture $(V, k, \overline{k}, B)$ ($V \in \{\mathcal{S}, \mathcal{H}, \mathcal{T}\}$; $(k, \overline{k}) = (2,1), (1,2)$), and any pixel $p \in B$ removed by $\mathcal{R}$ the following conditions hold:

1. p is a simple pixel.
2. For any simple pixel $q \in N_{\overline{k}}^{*V}(p) \cap B$, the following holds:
   - If $(k, \overline{k}) = (2,1)$, then q is the designated element in any configuration of Fig. 1.4 in the dissertation.
   - If $(k, \overline{k}) = (1,2)$, then q is the designated element in any configuration of Fig. 1.5 in the dissertation.
3. If $(k, \overline{k}) = (2,1)$, then pixel p is not the designated element of any small object (see Figs. 1.6-1.8 in the dissertation).

For the verification of the topological correctness of additions, I formulate a duality rule between additions and reductions in Section 3.3. Using this rule some general sufficient conditions for additions can be derived from the criteria for reductions.

Theorem 5. (Theorem 3.3.2)
Addition $\mathcal{A}$ is topology preserving, if for any picture $(V, k, \overline{k}, B)$ ($V \in \{\mathcal{S}, \mathcal{H}, \mathcal{T}\}$; $(k, \overline{k}) = (2,1), (1,2)$) the following conditions hold:

1. Any pixel added by $\mathcal{A}$ is simple.
2. For any pair of k-adjacent white pixels $\{p, q\}$ added by $\mathcal{A}$, p is simple in picture $(V, k, \overline{k}, B \cup \{q\})$.
3. If $(k, \overline{k}) = (1,2)$ then $\mathcal{A}$ does not fill any small cavity (see Figures 1.6-1.8 of the dissertation).

Thanks to our new sufficient conditions, the topological correctness of the thinning algorithms working on square, hexagonal, or triangular grids can be guaranteed. In the first key point of the thesis, I also introduced hexagonal and triangular algorithms, whose deleting conditions combine the parallel thinning strategies with some various geometric constraints, hence their topology preserving is ensured.

3.2 Thinning combined with iteration-by-iteration smoothing for 3D binary images

In Chapter 5 of my thesis, I introduce a parallel 3D contour smoothing algorithm, which preserves the topology of pictures, and it is easy to build into thinning algorithms. The algorithm constitutes of two parallel reductions (see

Algorithm 1), that are denoted by R_1 and R_2. Deletable points by R_1 are given by $3 \times 3 \times 3$ matching templates (see Figs. 5.1-5.5 of the thesis). Operator R_2 considers the masks of R_1 reflected to the point p.

We combined our smoothing algorithm with the conventional thinning algorithms in order to reduce the number of unwanted skeletal segments produced in thinning. Algorithm 2 introduces the scheme of an arbitrary thinning algorithm A combined with iteration-by-iteration smoothing.

Algorithm 1 Parallel smoothing algorithm.

Input: picture $(\mathcal{C}, 3, 1, X)$
Output: picture $(\mathcal{C}, 3, 1, Y)$
$Y = X$
// Phase 1
$Y \ = \ Y \setminus \{ \ p \mid p$ is deletable by R_1 in picture $(\mathcal{C}, 3, 1, Y)\}$
// Phase 2
$Y \ = \ Y \setminus \{ \ p \mid p$ is deletable by R_2 in picture $(\mathcal{C}, 3, 1, Y)\}$

Algorithm 2 Thinning combined with iteration-by-iteration smoothing.

$Y \ = \ X$
repeat
    Input: picture $(\mathcal{C}, 3, 1, X)$
    Output: picture $(\mathcal{C}, 3, 1, Y)$
    $Y = X$
    // two-phase smoothing
    $Y \ = \ Y \setminus \{ \ p \mid p$ is deletable by R_1 in picture $(\mathcal{C}, 3, 1, Y)\}$
    $Y \ = \ Y \setminus \{ \ p \mid p$ is deletable by R_2 in picture $(\mathcal{C}, 3, 1, Y)\}$
    // an iteration step of thinning
    $D \ = \ \{ \ p \mid p$ is deletable by A in picture $(\mathcal{C}, 3, 1, Y)\}$
    $Y \ = \ Y \setminus D$
until $D \ = \ \emptyset$

For the implementation of the smoothing operator, we have proposed an efficient method. This uses a previously generated look-up table for encoding deletable points, which requires only 8 MB storage size. For the further acceleration of the algorithm two lists are suggested to use for storing the actual border points and the actual deletable point, respectively.

3.3 Order-independent sequential thinning

My third main research topic was designing sequential thinning algorithms that are not sensitive to the visiting order of border points.

In Section 6.1, I introduce some necessary and sufficient conditions that ensure order-independency. In my first theorem regarding to this I formulate some sufficient conditions that consider sequential thinning algorithms whose deleting conditions are given by matching templates (their scheme is shown by Algorithm 3). Those conditions state that if we consider the matching templates of the algorithm $SM(\mathcal{M})$ as “picture regions”, and a template $M \in \mathcal{M}$ does not contain any other element than p such that it matches a template of the algorithm, then order-independency holds. The formulation of my theorem proven in [6] would require the introduction of some further definitions.

Algorithm 3 ST($\mathcal{M}$)

Input: a $(\mathcal{C}^n, n, 1, X, \emptyset)$ and the set $\mathcal{M}$ of matching templates
Output: a $(\mathcal{C}^n, n, 1, Y, Y^+)$
$Y = X$
$Y^+ = \emptyset$
repeat
    // *Phase 1: contour-tracing*
    for all p in Y **do**
        if p is a border pixel **then**
            $Y^+ = Y^+ \cup \{p\}$
    // *Phase 2: reduction*
    modified=**false**
    for all $p \in Y^+$ **do**
        if p is $\mathcal{M}$-deletable **then**
            $Y = Y \setminus \{p\}$
            modified=**true**
until modified=**false**

Furthermore, I also propose some general criteria, which are not constrained to algorithms given by matching templates, and which are both necessary and sufficient. These are based on the following notion: Let us consider a sequential thinning algorithm ST (see Algorithm 3). A parallel thinning algorithm ST^* is the *parallel version* of ST if it has exactly the same deletion conditions as ST with the exception that in Phase 2, object points are examined simultaneously (i.e., unlike in the case of ST, the second phase of ST^* is considered as a parallel reduction).

Theorem 6. (Theorem 6.1.2)
Let ST be a sequential thinning algorithm, and let ST^* be its parallel version. Let us consider an iteration step of ST and ST^*, let $\mathcal{P} = (C^n, n, 1, B)$ an arbitrary picture ($n = 2, 3$), and let $D \subseteq B$ the set of points deleted by ST^* for the input picture $\mathcal{P}$ in the examined iteration. ST is order-independent if and only if for any $p \in B$, one of the following two conditions hold:

1. $p \in D$, and for each $Q \subseteq D \setminus \{p\}$, ST^* deletes p from the input picture $(\mathcal{C}^n, n, 1, B \setminus Q)$, or

2. $p \notin D$, and for each $Q \subseteq D \setminus \{p\}$, ST^* does not delete p from the input picture $(\mathcal{C}^n, n, 1, B \setminus Q)$.

In Section 6.2, first I gave some sets of 2D matching templates $\mathcal{M}_1$ and $\mathcal{M}_2$ that satisfy the sufficient conditions proposed in [6], hence algorithms $\mathrm{ST}(\mathcal{M}_1)$ and $\mathrm{ST}(\mathcal{M}_2)$ are order-independent. These two algorithms examine different endpoint-characterizations for producing 2D centerlines.

I also introduced four further order-independent algorithms on $(\mathcal{S}, 2, 1)$ pictures and $(\mathcal{C}, 3, 1)$ pictures that are based on the following idea: We examine in advance if the removal of any set of simple points in the neighborhood of p would affect the simplicity of the visited point p, and if this is not the case, then we can safely remove p. However, if there is at least one set Q of black points in the examined neighborhood such that p is no more simple after the removal of Q, then the deletion of p could result in order-dependence, therefore p must be preserved.

One of these four algorithms works on arbitrary dimensions, i.e., on $(\mathcal{C}^n, n, 1)$ pictures, the second one only works on $(\mathcal{S}, 2, 1)$ pictures while the remaining two consider $(\mathcal{C}, 3, 1)$ pictures. The algorithms also differ in their geometric constraint.

4 Summary of the thesis points

4.1 New sufficient conditions for topology preserving operators, hexagonal and triangular topology preserving thinning algorithms

Results belonging to the first thesis point are:

- We gave the characterizations of simple pixels for $(\mathcal{S}, 2, 1)$, $(\mathcal{H}, 1, 1)$, and $(\mathcal{T}, 2, 1)$ pictures

- We proposed some sufficient conditions for topology preserving reductions for $(\mathcal{H}, 1, 1)$, $(\mathcal{T}, 2, 1)$ and $(\mathcal{T}, 1, 2)$ pictures.

- For the additions, I formulated a duality rule which can be applied for deriving from sufficient conditions on reductions some similar criteria for additions. Using this rule I gave sufficient conditions for pictures $(V, k, \overline{k})$ $(V \in \{\mathcal{S}, \mathcal{H}, \mathcal{T}\}; (k, \overline{k}) = (2, 1), (1, 2))$ and also for $(\mathcal{C}, 3, 1)$ pictures.

- I introduced various fully parallel, directional, and subfield-based hexagonal and triangular thinning algorithms that combine the sufficient conditions on reductions with three parallel thinning strategies and with some geometric constraints. The topology-preserving algorithms are both capable of producing topology kernels and centerlines.

We presented our results regarding the examination of the conditions of topology preservation in a journal paper [9] and in three conference papers [10, 14, 16]. Furthermore, we published our hexagonal thinning algorithms in a further conference paper [12].

4.2 Thinning combined with iteration-by-iteration smoothing for 3D binary images

Results belonging to the second thesis point are:

- We proposed a contour smoothing algorithm for $(\mathcal{C}, 3, 1)$ pictures, which is capable of removing some extremities.
- We presented a new thinning scheme, which proposes contour smoothing before each iteration step of thinning algorithms.
- We gave an efficient implementation of our 3D contour smoothing algorithm which can be also generally applied for thinning algorithms.
- The efficiency of the new scheme were tested for several thinning algorithms. The results confirm that by using the proposed method, we get skeleton-like shape features with significantly less unwanted surface- and line-segments

We presented the first version of our 3D contour smoothing algorithm in a conference paper [22], the improved method, and the new thinning scheme combined with iteration-by-iteration smoothing were published in a journal paper [23].

4.3 Order-independent sequential thinning

Results concerning the third thesis point are listed as follows:

- I introduced some necessary and sufficient conditions to verify the order-independency of sequential thinning algorithms.
- I also proposed some special sufficient conditions for algorithms given by with matching templates.
- I gave two template-based order-independent sequential thinning algorithms for producing 2D centerlines.
- We presented four further order-independent sequential algorithms, too, which apply special (non-template based) deletion conditions. One of them can be applied in arbitrary dimensions, but in 3D, it can only produce medial surface. On the other hand, the other three algorithms are based on the preservation of isthmus points, and they are capable of producing 2D centerlines, 3D medial surfaces, and 3D centerlines, as well.

We presented our results in two journal papers [6, 8] and in four conference papers [7, 11, 13, 15].

Author's publications related to the theses

The following table shows the relations between the publications and the thesis points. This table contains only those publications that are accepted by the PhD School in Computer Science, University of Szeged.

Publications	Thesis points			Type	Number of authors	Score
	1.	2.	3.			
[9][1]	•			Journal paper	2	0,75
[10]	•			Conference paper	2	0,75
[14]	•			Conference paper	2	0,75
[16]	•			Conference paper	2	0,75
[12]	•			Conference paper	2	0,75
[23][2]		•		Journal paper	3	0,60
[22]		•		Conference paper	3	0,60
[6]			•	Journal paper	1	1,00
[8]			•	Journal paper	3	0,60
[11]			•	Conference paper	2	0,75
[13]			•	Conference paper	2	0,75
[15]			•	Conference paper	2	0,75
[7]			•	Conference paper	3	0,60
Total:						9,40

[1] Impact Factor: 0,499 (2011)
[2] Impact Factor: 1,00 (2011)

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