Modular and semimodular lattices

Ph.D. Dissertation

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Introduction

The concept of modular lattices is as old as that of lattices itself. Both are due to Richard Dedekind, although there were others, e.g., Charles S. Pierce or Ernst Schröder, who also found the concept of lattices, cf. also the Foreword of George Grätzer [42]. A lattice is said to be *modular* if it satisfies the following identity:

$$x \vee \big(y \wedge (x \vee z)\big) = (x \vee y) \wedge (x \vee z).$$

Dedekind showed around 1900 that the submodules of a module form a modular lattice with respect to set inclusion. Many other algebraic structures are closely related to modular lattices: both normal subgroups of groups and ideals of rings form modular lattices; distributive lattices (thus also Boolean algebras) are special modular lattices. Later, it turned out that, in addition to algebra, modular lattices appear in other areas of mathematics as well, such as geometry and combinatorics.

The first nontrivial result for modular lattices was proved by R. Dedekind [31] as well, who found the free modular lattice on three generators, which has 28 elements. Comparing this with a result of Garrett Birkhoff, see ,e.g., [8], which proves that the free lattice on three generators is infinite, one can think that modular lattices are less complicated structures than lattices itself. However, the situation is just the opposite. Birkhoff also showed that the free modular lattice on four generators is already infinite. Furthermore this comparison becomes more interesting if we consider the respective word problems. Philip M. Whitman [85, 86] proved that the word problem for any free lattice is solvable, that is, there is an algorithm which can decide for arbitrary lattice terms p and q whether $p = q$ holds identically in all lattices. (Note that Thoralf Skolem gave a much more effective algorithm more than 20 years earlier, but it was forgotten till Stan Burris found it, see Freese, Ježek, Nation [37, page 14] for details.) In view of Whitman's result, it is quite astonishing that the word problem for free modular lattices is unsolvable, which was independently proved by George Hutchinson [57] and Leonard M. Lipshitz [66].

Later, Ralph Freese [35] improved Hutchinson's and Lipshitz's result by showing that the word problem is unsolvable even for the free modular lattice on five generators. One can derive easily from Dedekind's result that the word problem for the free modular lattice on three generators is solvable. Thus Christian Herrmann [51] reached the lower bound when he proved that the word problem for the free modular lattice on four generators is unsolvable. Therefore, many of the computations in modular lattices with at least four variables need particular ideas; they are not automatic at all.

In *Chapter 1*, we introduce the concept of von Neumann frames, which is the basic concept of coordinatization theory, one of the deepest and most amazing part of modular lattice theory. After introducing frames and recalling the most important examples and lemmas, we define the concept of product frames, which is due to Gábor Czédli [18]. He used product frames in connection with fractal lattices, cf. [18]. Finally we present a theorem that shows that modular lattices with product frames and matrix rings are closely related. This part is based on a joint paper with Czédli [27].

One of the most fruitful generalization of modularity is the so-called semimodularity. A lattice is said to be (upper) *semimodular* if it satisfies the following Horn formula

$$x \prec y \Rightarrow x \vee z \preceq y \vee z.$$

In contrast to modular lattices, the class of semimodular lattices cannot be characterized by identities. In the preface of his book titled Semimodular Lattices, Manfred Stern [79] attributes the abstract concept of semimodularity to Birkhoff [8]. He also mentions that classically semimodular lattices came from closure operators that satisfies the nowadays usually called Steinitz-Mac Lane Exchange Property, cf. [79, page ix, 2 and 40]. One of the most important class of semimodular lattices that was systematically studied at first is the class of *geometric lattices*, which are semimodular, atomistic algebraic lattices, cf. Birkhoff [8, Chapter IV] and Crawley and Dilworth [13, Chapter 14]. Since one can think of finite geometric lattices as (simple) *matroids*, it is not surprising that the theory of semimodular lattices has been developing simultaneously with matroid theory since the beginning, cf. the Preface of Stern [79].

The dual concept of semimodularity, called *lower semimodularity*, has also a strong connection to certain closure operators that satisfies the so-called Antiexchange Property. An important class of lower semimodular lattices, the class of meet distributive lattices, was introduced by Robert P. Dilworth under the name of (lower) locally distributive lattices. The corresponding combinatorial objects are *convex geometries* and their combinatorial duals, the *antimatroids*.

One can say that semimodular lattices provide a bridge between lattice theory and combinatorics. For getting a better picture about them, we refer to the already mentioned book of Stern, who offers his work as "a supplement to certain aspects to vol. 26 (Theory of Matroids), vol. 29 (Combinatorial Geometries), and vol. 40 (Matroid Applications) of Encyclopedia of Mathematics and its Applications".

It seems that the research of semimodular lattices have recently come again into focus of several lattice theorists. Let us pick out some of the latest results. We already mentioned that the normal subgroups of a group form a modular lattice. By a classical result of Helmut Wielandt [88], subnormal subgroups of a group form a (lower) semimodular lattice. Lattice theoretic formulation of the classical Jordan-Hölder theorem for groups is well-known, cf. Rotman [76, Theorem 5.12] and Grätzer [41, Theorem 1 in Section IV.2]. It is less-known that the original theorem follows from the lattice theoretic one. Grätzer and James Bryant Nation [47] pointed out that there is a stronger version for semimodular lattices. Using their technique, Czédli and E. Tamás Schmidt [24] strengthened these Jordan-Hölder theorems both for groups and semimodular lattices. For more details, see also Grätzer [41, Section V.2].

On the other hand, it is worth mentioning that several new results have appeared recently about some geometric aspects of semimodular lattices; only to refer to planar semimodular lattices, see Grätzer and Edward Knapp [44, 45, 46] and Czédli and Schmidt [25, 26], or to semimodular lattices that can be "represented" in higher dimensional spaces, see the web site of Schmidt (`http://www.math.bme.hu/~schmidt/`) for more details. As for convex geometries, there has been some recent improvement as well, see, e.g., the papers of Adaricheva, Gorbunov, Tumanov and Czédli [2, 1, 19].

In *Chapter 2*, we deal with lattice embeddings into geometric lattices, which also have nice consequences for semimodular lattices. After that we formulate a classical result of Dilworth [13, Theorem 14.1] and its generalization by Grätzer and Emil W. Kiss [43] for finite lattices, we present our extension of Grätzer and Kiss' theorem for some class of infinite lattices. This part is based on [78].

Congruence lattices of algebras play a central role in universal algebra. Many lattice theorists have also studied congruence lattices and congruence lattice varieties, only to mention Bjarni Jónsson, Grätzer or Schmidt, cf. ,e.g., [64] and [48]. This concept was generalized by Ivan Chajda [11], who dealt with algebras with a constant operation symbol in their type. He studied lattices formed by those classes of congruences that contains the constant.

In *Chapter 3*, we return to modularity. After defining the concept of a Mal'cev condition, we show that a classical result of Alan Day [28], which says that congruence modular varieties can be defined by a Mal'cev condition, can be generalized for lattices observed by Chajda. This part is based on [77].

Chapter 1

Von Neumann frames

Von Neumann normalized frames, *frames* shortly, are due to von Neumann [69]. Although he worked in lattice theory just for two years between 1935 and 1937, many lattice theorists, including Grätzer [41, p. 292], say that his results belong to the deepest part of lattice theory. For instance, Birkhoff, the founder and pioneer of universal algebra and lattice theory, wrote in a paper [7] about him: "John von Neumann's brilliant mind blazed over lattice theory like a meteor." Also, it was Birkhoff who turned von Neumann's attention to lattice theory. Then von Neumann began to think that he could probably use lattice theory as a tool. At that time he was trying to find an appropriate concept of space for modern physics. In contrast to the usual concept of dimension, where the dimension function has a discrete range $(0, 1, 2, \dots)$, he was looking for a dimension function with a continuous range. In full extent, his work was published much later, see [69]. It is centered around the concept of *continuous geometries*, which are special complemented modular lattices.

On his way to continuous geometries, von Neumann introduced the concept of frames, and he used them to extend the classical Veblen-Young coordinatization theorem of projective spaces [82, 83] to arbitrary complemented modular lattices with frames. Note that the best known method for the classical coordinatization is "von Staudt's algebra of throws", cf. Grätzer [42, p. 384]. As a first step, von Neumann associated a ring, the so-called *coordinate ring*, with each frame. It turned out that in case of a complemented modular lattice with a frame, the coordinate ring satisfies some additional property, which is nowadays called *von Neumann regularity*. It is worth mentioning that this property has proved to be particularly useful. The theory of von Neumann regular rings has become an independent discipline later.

Since there is a one to one correspondence between modular geometric lattices and projective spaces, many properties, including Desargues' theorem, can be formulated in the language of lattices, see, e.g., Grätzer [42, Section V.5]. Von Neumann's work exemplifies that lattice theory can be helpful to handle geometric problems in a more elegant and compact way. Analogous applications of lattice theory were given later by others. For example, Jónsson [60] provided lattice identities that hold in a modular geometric lattice if and only if Desargues' theorem holds in the associated projective space. Note that there exists a similar characterization of Pappus' theorem, see Day [29]. For more examples, see Jónsson [62] or Takách [80].

Although von Neumann considered a *complemented* modular lattice L of length $n \geq 4$, his construction of the coordinate ring (without coordinatization) extends to arbitrary modular lattices without complementation, see Artmann [3] and Freese [34], and even to $n = 3$ if L is Arguesian, see Day and Pickering [30].

A concept equivalent to frames is that of *Huhn diamonds*, see Huhn [54]. Since distributive lattices played a central role already in the beginning of lattice theory, cf. Grätzer [42, p. xix], Huhn's original purpose was to generalize the distributive law. He also wanted to find generalizations for many well known theorems and applications of distributive lattices. His new identity, called *n-distributivity*, proved to be a particularly fruitful generalization of distributivity. While distributive lattices among modular lattices are characterized by excluding M_3's (Birkhoff's criteria), in case of n-distributive lattices, M_3's are replaced by Huhn diamonds, see Remark 1.2 for more details. Huhn diamonds are connected to many interesting theorems, for instance, Huhn proved with them that the automorphism group of a finitely presented modular lattice can be infinite, see [55].

Frames and Huhn diamonds are used in the proof of several deep results showing how complicated modular lattices are, only to mention Freese [34], Huhn [55], and Hutchinson [58]. Frames or Huhn diamonds were also used in the theory of congruence varieties, see Hutchinson and Czédli [59], Czédli [17], and Freese, Herrmann and Huhn [36]; and in commutator theory, see Freese and McKenzie [38, Chapter XIII].

Dealing with quasi-fractal generated non-distributive modular lattice varieties, Czédli [18] introduced the concept of *product frames*. This chapter is based on a joint work with Czédli [27]. We show that product frames are closely related to matrices. Namely, the coordinate ring of the so-called *outer frame* of a product frame is a matrix ring over the coordinate ring of the so-called *inner frame* of the product frame, see Theorem 1.7.

Overview of the chapter

Finally, let us give a short overview of this chapter. In Section 1.1 we give an introduction to coordinatization theory. In Section 1.2 we introduce the concept of product frames and prove some lemmas. In Section 1.3 we prove our result on product frames and give some further comments.

Note that in coordinatization theory, the lattice operations join and meet are traditionally denoted by + and $\cdot$ (mostly juxtaposition) such that meets take precedence over joins. In this chapter we follow this tradition. As a general convention for the whole chapter, the indices we use will be positive integers, so $i \le n$ is understood as $1 \le i \le n$.

1.1 Basic definitions and notions

For definition, let $2 \le m$, let L be a nontrivial modular lattice with 0 and 1, and let $\vec{a} = (a_1, \dots, a_m) \in L^m$ and $\vec{c} = (c_{12}, \dots, c_{1m}) \in L^{m-1}$. We say that $(\vec{a}, \vec{c}) = (a_1, \dots, a_m, c_{12}, \dots, c_{1m})$ is a *spanning m-frame* (or a frame of *order* m) of L, if $a_1 \ne a_2$ and the following equations hold for all $j \le m$ and $2 \le k \le m$:

$$\begin{aligned} &\sum_{i \le m} a_i = 1, && a_j \sum_{i \le m,\, i \ne j} a_i = 0, \\ &a_1 + c_{1k} = a_k + c_{1k} = a_1 + a_k, && a_1 c_{1k} = a_k c_{1k} = 0. \end{aligned} \tag{1.1}$$

Notice that if $(\vec{a}, \vec{c})$ is a spanning m-frame, then

$$\text{the } a_i \text{ are the distinct atoms of a Boolean sublattice } \mathbf{2}^m, \tag{1.2}$$

and $\{a_1,\ c_{1k},\ a_k\}$ generates an M_3 with bottom $0 = 0_L$ for $k \in \{2, \dots, m\}$. In particular, a frame of order two is simply an M_3 with $0_{M_3} = 0_L$ and $1_{M_3} = 1_L$.

By the *order* of the frame we mean m. If $(\vec{a}, \vec{c})$ is a spanning m-frame of a principal ideal of L, then we will call it a *frame in L*. Note that von Neumann [69, page 19] calls c_{1k} the axis of perspectivity between the intervals $[0, a_1]$ and $[0, a_k]$, and we will shortly call c_{1k} as the *axis of $\langle a_1, a_k \rangle$-perspectivity.*

Given an m-frame $(\vec{a}, \vec{c})$, we define $c_{k1} = c_{1k}$ for $2 \le k \le n$, and for 1, j, k distinct, let $c_{jk} = (c_{1j} + c_{1k})(a_j + a_k)$. From now on, a frame is always understood in this *extended sense*: $\vec{c}$ includes all the c_{ij}, $i \ne j$, $i, j \le m$. Then, according to Lemma 5.3 in von Neumann [69, page 118] (see also Freese [33]), for i, j, k distinct we have

$$c_{ik} = c_{ki} = (c_{ij} + c_{jk})(a_i + a_k),$$
$$a_i + c_{ij} = a_j + c_{ij} = a_i + a_j, \tag{1.3}$$
$$a_i c_{ij} = a_j c_{ij} = a_i a_j = 0.$$

This means that the index 1 has no longer a special role.

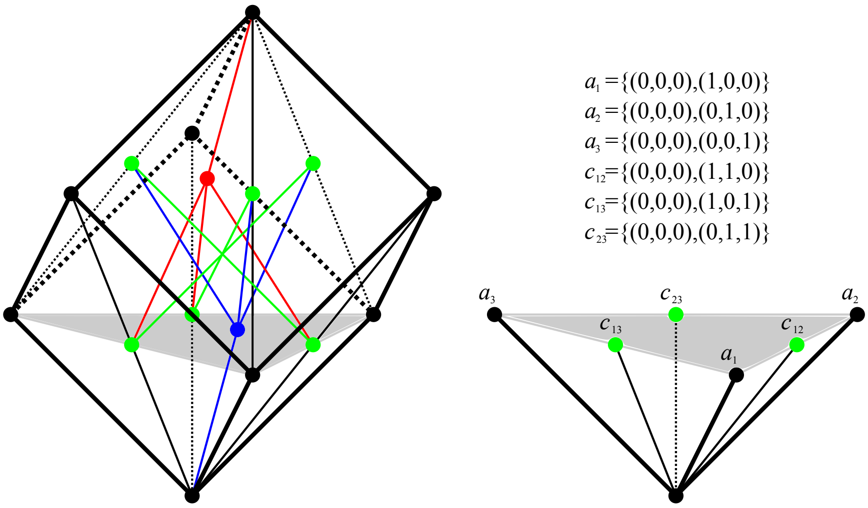

Figure 1.1: $\mathrm{Sub}(\mathbb{Z}_2^3)$ and its canonical 3-frame

Example 1.1 (Canonical m-frame)**.** Let K be a ring with 1. Let v_i denote the vector $(0, \dots, 0, 1, 0, \dots, 0) \in K^m$ (1 at the ith position). Letting $a_i = Kv_i$ and $c_{ij} = K(v_i - v_j)$, we obtain a spanning m-frame of the submodule lattice $\mathrm{Sub}(K^m)$, where K^m is, say, a left module over K in the usual way. This frame is called the *canonical m-frame* of $\mathrm{Sub}(K^m)$. For $m = 3$ and $K = \mathbb{Z}_2$, see Figure 1.1.

This example shows that, sometimes, to unify some definitions or arguments, it is reasonable to allow the formal definition of a *trivial axis* $c_{ii} = 0$, $i \le m$; this convention makes formula (1.3) valid also for $k \in \{i, j\}$. However, according to tradition, the trivial axes do not belong to the frame.

Remark 1.2 (Huhn diamonds)**.** One of many concepts closely related to n-frames is the so-called m-diamond, cf. Herrmann and Huhn [53]. Let $1 \le m$, let L be a nontrivial modular lattice, and let $\vec{a} = (a_1, \ldots, a_{m+1}) \in L^{m+1}$ and $b \in L$. We say that $(\vec{a}, b)$ is an *m-diamond* if the a_i are the distinct atoms of a Boolean sublattice $\mathbf{2}^{m+1}$ and b is a relative complement of each atom in $[a_1a_2, a_1 + \cdots + a_{m+1}]$. This concept was introduced by Huhn [54], cf. also Freese [34].

If L is a nontrivial modular lattice with 0, and $(\vec{a}, b) = (a_1, \ldots, a_m, b)$ is an $(m-1)$-diamond in a modular lattice such that $a_1a_2 = 0$ then $(\vec{a}, \vec{c}) = (\vec{a}, c_{12}, \ldots, c_{1m})$ is an m-frame, where $c_{1j} = (a_1 + a_j)b$. Conversely if $(\vec{a}, \vec{c}) = (a_1, \ldots, a_m, c_{12}, \ldots, c_{1m})$ is an m-frame then $(\vec{a}, b)$ is an $(m-1)$-diamond, where $b = c_{12} + \cdots + c_{1m}$, cf. [53] and [34]. This connection between frames and diamonds allows us to define the canonical m-diamond. For $m = 2$ and $K = \mathbb{Z}_2$, see Figure 1.2.

As a generalization of distributivity, Huhn defined a modular lattice to be *n-distributive* iff it satisfies the following identity:

$$x \vee \bigwedge_{i=0}^{n} y_i = \bigwedge_{j=0}^{n} \Big(x \vee \bigwedge_{\substack{i=0 \\ i \neq j}}^{n} y_i \Big).$$

Since 1-distributivity gives back the distributive law and 1-diamonds generates M_3, Birkhoff's criteria says that a modular lattice is distributive if and only if it does not contain a 1-diamond. Huhn showed that a modular lattice is n-distributive if and only if it does not contain an n-diamond, see [54] for more details.

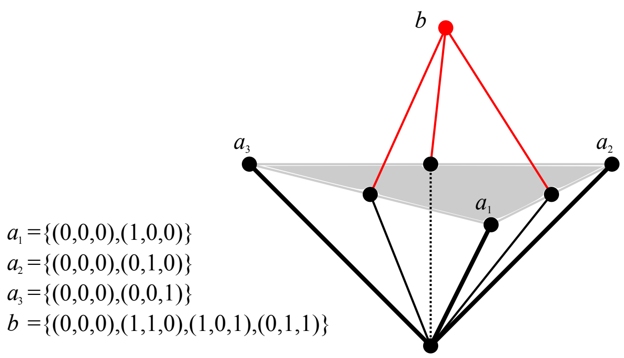

Figure 1.2: The canonical 2-diamond of $\mathrm{Sub}(\mathbb{Z}_2^3)$

In the sequel, assume that L is a modular lattice, and either $m \geq 4$, or $m = 3$ and L is Arguesian. Next, we define the coordinate ring of $(\vec{a}, \vec{c})$ in two, slightly different ways. For $p, q, r \in \{1, \ldots, m\}$ distinct, consider the following projectivities:

$$\begin{aligned} R\binom{p\,q}{r\,q}&\colon [0, a_p + a_q] \to [0, a_r + a_q], \quad x \mapsto (x + c_{pr})(a_r + a_q), \\ R\binom{p\,q}{p\,r}&\colon [0, a_p + a_q] \to [0, a_p + a_r], \quad x \mapsto (x + c_{qr})(a_p + a_r); \end{aligned} \tag{1.4}$$

these are almost the original notations, see von Neumann [69] and Freese [33], the only difference is that we write R rather than P. They are lattice isomorphisms between the indicated principal ideals. For $i, j, k \in \{1, \dots, m\}$ distinct, let

$$\begin{aligned} R\langle i,j\rangle = R\langle a_i, a_j\rangle &= \{x \in L : x + a_j = a_i + a_j,\ \ xa_j = 0\},\\ x \oplus_{ijk} y &= (a_i + a_j)\big((x + a_k)(c_{ik} + a_j) + yR\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big)\big) \text{ and} \\ x \otimes_{ijk} y &= (a_i + a_j)\big(xR\big(\begin{smallmatrix} i & j\\ i & k\end{smallmatrix}\big) + yR\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big)\big) \text{ for } x, y \in R\langle i,j\rangle. \end{aligned} \tag{1.5}$$

Then the operations $\oplus_{ijk}$ and $\otimes_{ijk}$ do not depend on the choice of k, and this definition turns $R\langle i,j\rangle$ into a ring. Moreover, $R\langle i,j\rangle \cong R\langle i',j'\rangle$ for every $i' \neq j'$, see von Neumann [69] or Herrmann [52]. (Notice that von Neumann uses the opposite multiplication.) This $R\langle i,j\rangle$ is called the *coordinate ring* of the frame.

While the above definition seems to be the frequently used one, see Herrmann [52], our needs are better served by von Neumann's original definition, which is more complicated but carries much more information. Following Freese [33], for $i, j, k, h \in \{1, \dots, m\}$ pairwise distinct, let

$$R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big) = R\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big) \circ R\big(\begin{smallmatrix} k & j\\ k & h\end{smallmatrix}\big).$$

We always compose mappings from left to right, that is, $x(R\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big) \circ R\big(\begin{smallmatrix} k & j\\ k & h\end{smallmatrix}\big)) = (xR\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big))R\big(\begin{smallmatrix} k & j\\ k & h\end{smallmatrix}\big)$. Now, the notation $R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big)$ makes sense whenever $i \neq j$ and $k \neq h$; notice that $R\big(\begin{smallmatrix} i & j\\ i & j\end{smallmatrix}\big)$ is the identical mapping.

Next, we consider two small categories. The first one, $\mathcal{C}_1(\vec{a}, \vec{c})$, consists of the pairs (i, j), $i \neq j$ and $i, j \leq m$, as objects, and for any two (not necessarily distinct) objects (i, j) and (k, h), there is exactly one $(i, j) \to (k, h)$ morphism. The second category, $\mathcal{C}_2(\vec{a}, \vec{c})$, consists of the coordinate rings $R\langle i,j\rangle$ of our frame, $i \neq j$, as objects, and all ring isomorphisms among them, as morphism. For a morphism $(i, j) \to (k, h)$ in the first category, let R send this morphism to the mapping $R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big)$. Of course, for an object (i, j) in $\mathcal{C}_1(\vec{a}, \vec{c})$, R sends (i, j) to $R\langle i,j\rangle$. The crucial point is captured in the following lemma.

Lemma 1.3 (von Neumann [69], Day and Pickering [30])**.** *R is a functor from the category $\mathcal{C}_1(\vec{a}, \vec{c})$ to the category $\mathcal{C}_2(\vec{a}, \vec{c})$.*

Proof. The notion of categories came to existence only after von Neumann's fundamental work in lattice theory, recorded later in [69]. Hence it is not useless to give some hints how to extract the above lemma from [69]. If $m \geq 4$, then it follows from pages 119–123 that R is functor, see also Freese [33]. Although von Neumann does not consider $R\langle i,j\rangle$ a ring in itself, it is implicit in [69] that the $R\binom{i\ j}{k\ h}$ are ring isomorphisms. (This becomes a bit more explicit in Freese [33]. With slightly different notation, it is fully explicit in Theorem 2.2 of Herrmann [52].) If $m = 3$, then the lemma follows from Lemma (4.1) of Day and Pickering [30]. □

By an *L-number* (related to the frame $(\vec{a},\vec{c})$) von Neumann means a system $(x^{ij} : i,j \le m,\ i \ne j)$ of elements such that $x^{ij} \in R\langle i,j\rangle$ and $x^{ij}R\binom{i\ j}{k\ h} = x^{kh}$ for all $i \ne j$ and $k \ne h$. (Because there will be lattice entries later, here we use superscripts rather than von Neumann's subscripts.) Clearly, for every (i,j), $i \ne j$, each L-number x is determined by its (i,j)th component x^{ij}. Conversely,

Lemma 1.4 (page 130 of [69], see also Lemma 2.1 in [33])**.** *If $u \in R\langle i,j\rangle$, then there is a unique L-number x such that $x^{ij} = u$.*

Let R^* be the set of L-numbers related to $(\vec{a},\vec{c})$. Von Neumann made R^* into a ring $(R^*, \oplus_{R^*}, \otimes_{R^*})$ such that $R^* \to R\langle i,j\rangle$, $x \mapsto x^{ij}$ is a ring isomorphism for every $i \ne j$. (Of course, von Neumann defined $(R^*, \oplus_{R^*}, \otimes_{R^*})$ first, and later others, including Herrmann [52], transferred the ring structure of R^* to $R\langle i,j\rangle$ by the bijection $R^* \to R\langle i,j\rangle$, $x \mapsto x^{ij}$.)

According to Lemma 1.4 and the previous paragraph, we can perform computations with L-numbers componentwise, and it is sufficient to consider only one component. For $w \in R\langle i,j\rangle$, let $w^* \in R^*$ denote the unique L-number in R^* such that $(w^*)^{ij} = w$. However, we usually make no difference between w and w^*.

To help the reader to understand our calculations in modular lattices while we save a lot of space, the following notations will be in effect. We use

$$=^{i}, \quad =^{\mathrm{f}}, \quad \text{or} \quad =^{\mathrm{L}j}$$

to indicate that formula (i), some basic property of frames, or Lemma j is used, respectively. In many cases, $=^{\mathrm{f}}$ means the same as $=^{1.3}$. When an application of the modular law uses the relation $x \le z$ then, beside using $=^{\mathrm{m}}$, x resp. z will be underlined resp. doubly underlined. For example,

$$(\underline{x} + y)(\underline{\underline{x + z}}) =^{\mathrm{m}} x + y(x + z).$$

The use of the shearing identity (see Grätzer [42, Theorem 347]) is indicated by $=^{\mathrm{s}}$ and underlining the subterm "sheared":

$$x(\underline{y} + z) =^{\mathrm{s}} x\big(y(x + z) + z\big).$$

Even in some other cases, subterms worth noticing are also underlined. If $x_1 \geq x_2 \dots x_k$ for some easy reason, then we write

$$\overline{x_1} x_2 \dots x_k$$

to indicate that this expression is considered as $x_2 \dots x_k$. In other words, overlined meetands will be omitted in the next step. Combining our notations like

$$=^{\text{m,1.2,L3}},$$

we can simultaneously refer to properties like modularity, formulas and lemmas. Formulas, like (1.2), will also be used for the product frame, whose definition comes in the next section.

1.2 The product frame

In this section, we recall the concept of product frame from Czedli [18]. In order to get a detailed picture about product frames, we quote not only the lemmas but also their proofs from [18]. From now on, the general assumption throughout the chapter is that $n \geq 2$, L is a modular lattice, and either $m \geq 4$, or $m = 3$ and L is Arguesian. Let $(\vec{a}, \vec{c})$ be a spanning m-frame of L, and let $(\vec{u}, \vec{v})$ be a spanning n-frame of $[0, a_1] \leq L$. We define a spanning mn frame as follows. For $i, j \leq n$ and $p, q \leq m$, let

$$b_i^p = (u_i + c_{1p})a_p, \quad d_{1j}^{1q} = \big(v_{1j} + c_{1q}(u_j + a_q)\big)(u_1 + b_j^q) \tag{1.6}$$

Let $\vec{b}$ denote the vector of all the b_i^p such that b_1^1 is the first component. Let $\vec{d}$ denote the vector of all the d_{1j}^{1q} such that $(q, j) \neq (1, 1)$.

Lemma 1.5 (Czédli [18, Theorem 1(A)]). *$(\vec{b}, \vec{d})$ is a spanning mn-frame of L, where d_{1j}^{1q} plays the role of the axis of $\langle b_1^1, b_j^q \rangle$-perspectivity.*

We say that $(\vec{b}, \vec{d})$ is the *product frame* of $(\vec{a}, \vec{c})$ and $(\vec{u}, \vec{v})$, while $(\vec{a}, \vec{c})$ resp. $(\vec{u}, \vec{v})$ will be called the *outer* resp. *inner* frame.

Before we prove the previous lemma, we need some preparations. First, let us reformulate (1.6) without relying on trivial axes and providing simpler expressions for some particular values of indices:

$$\begin{aligned} b_i^1 &= u_i \quad \text{for } i \le n, \\ b_i^p &= (u_i + c_{1p})a_p \quad \text{for } i \le n \text{ and } 2 \le p \le m, \\ d_{11}^{1q} &= (u_1 + a_q)c_{1q} \quad \text{for } 2 \le q \le m, \\ d_{1j}^{11} &= v_{1j} \quad \text{for } 2 \le j \le n, \\ d_{1j}^{1q} &= \big(v_{1j} + c_{1q}(u_j + a_q)\big)(u_1 + b_j^q) \quad \text{for } 2 \le j \le n \text{ and } 2 \le q \le m. \end{aligned} \tag{1.7}$$

Second, for $k \le n$, define

$$B_k^p = \sum_{i \le n,\, i \ne k} b_i^p. \tag{1.8}$$

Now, (1.1) together with the isomorphism theorem of modular lattices (cf. Grätzer [42, Theorem 348]) yield that the map $\varphi_p\colon [0, a_1] \to [0, a_p]$, $x \mapsto (x + c_{1p})a_p$ is an isomorphism. Therefore formulas (1.7) together with (1.8) and the definition of the inner frame give

$$a_p = \sum_{i \le n} b_i^p = B_k^p + b_k^p \quad \text{for } k \le n. \tag{1.9}$$

Proof of Lemma 1.5. From (1.9) we conclude

$$\sum_{p \le m} \sum_{i \le n} b_i^p = \sum_{p \le m} a_p =^{\mathrm{f}} 1.$$

Further, for $i \le n$ and $p \le m$,

$$b_i^p \sum_{\substack{j \le n,\, q \le m \\ (q,j) \ne (p,i)}} b_j^q = b_i^p \Big(\sum_{\substack{q \le m \\ q \ne p}} \sum_{j \le n} b_j^q + B_i^p \Big) =^{1.9}$$

$$b_i^p \Big(\underline{\sum_{q \le m,\, q \ne p} a_q} + B_i^p \Big) =^{\mathrm{s},1.9,\mathrm{f}} b_i^p B_i^p = \Big(b_i^1 B_i^1 \Big) \varphi_p =^{\mathrm{f}} 0\varphi_p = 0.$$

This ends the proof of the first two equations of (1.1). Now, we have to show that if $j \le n, q \le m, (q,j) \ne (1,1)$ then $\{b_1^1, d_{1j}^{1q}, b_j^q\}$ generates an M_3 with bottom 0.

If $q = 1$ and $j > 1$, then, by the definition of the inner frame, $\{b_1^1, b_j^1, d_{1j}^{11}\} =^{1.7} \{u_1, v_{1j}, u_j\}$ generates an M_3 with bottom 0. If $q \ge 2$ and $j = 1$ then we have to show that $\{b_1^1, d_{11}^{1q}, b_1^q\} =^{1.7} \{u_1, (u_1 + a_q)c_{1q}, (u_1 + c_{1q})a_q\}$ generates an M_3 with bottom 0. Indeed, we have

$$\underline{u_1} + \underline{\underline{(u_1 + a_q)}}c_{1q} =^{\mathrm{m}} (u_1 + a_q)(u_1 + c_{1q}),$$

$$\underline{u_1} + \underline{\underline{(u_1 + c_{1q})}}a_q =^{\mathrm{m}} (u_1 + c_{1q})(u_1 + a_q),$$

$$\underline{(u_1 + a_q)c_{1q}} + \underline{\underline{(u_1 + c_{1q})}}a_q =^{\mathrm{m}} (u_1 + c_{1q})\Big(\underline{a_q} + \underline{\underline{(u_1 + a_q)}}c_{1q}\Big) =^{\mathrm{m}}$$

$$(u_1 + c_{1q})(u_1 + a_q)(a_q + c_{1q}) =^{\mathrm{f}} (u_1 + c_{1q})(u_1 + a_q)\overline{(a_1 + a_q)},$$

while the meet of any two is 0, since $u_1c_{1q} \le a_1c_{1q} =^{\mathrm{f}} 0$, $u_1a_q \le a_1a_q =^{\mathrm{f}} 0$ and $c_{1q}a_q =^{\mathrm{f}} 0$.

From now on, let $2 \le j \le n$ and $2 \le q \le m$. We have to show that $\{b_1^1, d_{1j}^{1q}, b_j^q\} =^{1.7} \{u_1, \big(v_{1j} + c_{1q}(u_j + a_q)\big)(u_1 + b_j^q), (u_j + c_{1q})a_q\}$ generates an M_3 with bottom 0. The meets are obtained easily:

$$\begin{gathered}
b_1^1 b_j^q =^{1.7} u_1 b_j^q \le^{1.7} a_1 a_q =^{\mathrm{f}} 0,\\
b_1^1 d_{1j}^{1q} =^{1.7} u_1\big(v_{1j} + \underline{c_{1q}(u_j + a_q)}\big)\overline{(u_1 + b_j^q)} =^{\mathrm{sf}} u_1 v_{1j} =^{\mathrm{f}} 0,\\
d_{1j}^{1q} b_j^q =^{1.7} \big(v_{1j} + c_{1q}(u_j + a_q)\big)\overline{(u_1 + b_j^q)} b_j^q =^{1.7} \big(v_{1j} + \underline{c_{1q}(u_j + a_q)}\big)\underline{\underline{(u_j + c_{1q})}} a_q =^{\mathrm{m}}\\
\big(v_{1j}\underline{\underline{(u_j + c_{1q})}} + \underline{c_{1q}(u_j + a_q)}\big) a_q =^{\mathrm{m}}\\
(u_j + c_{1q})\big(v_{1j} + \underline{c_{1q}(u_j + a_q)}\big) a_q =^{\mathrm{sf}} (u_j + c_{1q}) v_{1j} a_q =^{\mathrm{f}} 0.
\end{gathered}$$

The next task is to show that each of the three elements is below the join of the other two. Clearly,

$$d_{1j}^{1q} \le^{1.7} u_1 + b_j^q =^{1.7} b_1^1 + b_j^q.$$

Further,

$$\begin{gathered}
d_{1j}^{1q} + b_j^q =^{1.7} \big(v_{1j} + c_{1q}(u_j + a_q)\big)\underline{\underline{(u_1 + b_j^q)}} + \underline{b_j^q} =^{\mathrm{m}}\\
(u_1 + b_j^q)\big(v_{1j} + b_j^q + c_{1q}(u_j + a_q)\big) =^{1.7}\\
(u_1 + b_j^q)\big(v_{1j} + \underline{(u_j + c_{1q})a_q} + c_{1q}\underline{\underline{(u_j + a_q)}}\big) =^{\mathrm{m}}\\
(u_1 + b_j^q)\Big(v_{1j} + (u_j + a_q)\big(\underline{\underline{(u_j + c_{1q})}} a_q + \underline{c_{1q}}\big)\Big) =^{\mathrm{m}}\\
(u_1 + b_j^q)\big(v_{1j} + (u_j + a_q)(u_j + c_{1q})(a_q + c_{1q})\big) =^{\mathrm{f}}\\
(u_1 + b_j^q)\big(v_{1j} + (u_j + a_q)(u_j + c_{1q})\overline{(a_1 + a_q)}\big) \ge (u_1 + b_j^q)(v_{1j} + u_j) \ge^{\mathrm{f}} u_1 =^{1.7} b_1^1,
\end{gathered}$$

and finally,

$$\begin{gathered}
b_1^1 + d_{1j}^{1q} =^{1.7} \underline{u_1} + \big(v_{1j} + c_{1q}(u_j + a_q)\big)\underline{\underline{(u_1 + b_j^q)}} =^{\mathrm{m}}\\
(u_1 + b_j^q)\big(u_1 + v_{1j} + c_{1q}(u_j + a_q)\big) =^{\mathrm{f}} (u_1 + b_j^q)\big(u_1 + \underline{u_j} + c_{1q}\underline{\underline{(u_j + a_q)}}\big) =^{\mathrm{m}}\\
(u_1 + b_j^q)\big(u_1 + (u_j + c_{1q})(u_j + a_q)\big) \ge b_j^q\big(u_1 + (u_j + c_{1q})(u_j + a_q)\big) =^{1.7} b_j^q. \qquad \square
\end{gathered}$$

Note that one can define d_{ij}^{pq} for "arbitrary" i, j, p, q as we defined c_{ij} for "arbitrary" i, j in Section 1.1. We also mentioned that frames are understood in an extended sense: $\vec{d}$ includes all the d_{ij}^{pq}, $i, j \le n$, $p, q \le m$, $(p, i) \ne (q, j)$. Note that d_{ij}^{pq} are the axis of $\langle b_i^p, b_j^q\rangle$-perspectivity. (To comply with forthcoming notations, we suggest to read the indices of b_i^p downwards, "pi", and column-wise for d_{ij}^{pq}, "$pi\ qj$".)

The following lemma follows easily from the fact that the elements $b_i^1 = u_i$ and $d_{1\,j}^{11} = v_{1j}$ determine both the coordinate ring of the inner frame and the coordinate ring of the product frame, cf. the comments after (1.5).

Lemma 1.6 (Czédli [18]). *If $n \geq 4$, or $n \geq 3$ and L is Arguesian, then $(\vec{b}, \vec{d})$ and $(\vec{u}, \vec{v})$ have isomorphic coordinate rings.*

In sense of the previous lemma, the same notation can be used for the rings associated to the product frame and the inner frame. From now on,

$$\begin{aligned} &S^* = (S^*, \oplus_{S^*}, \otimes_{S^*}) \text{ is the coordinate ring of } (\vec{b}, \vec{d}), \text{ and} \\ &M_n(S^*) = (M_n(S^*), \oplus_{M_n}, \otimes_{M_n}) \text{ is the } n \times n \text{ matrix ring over } S^*. \end{aligned} \tag{1.10}$$

This makes sense, since $mn \geq 4$.

Analogously to Lemma 1.3, the product frame gives rise to a functor and the $S\left\langle \begin{smallmatrix} p & q \\ i & j \end{smallmatrix} \right\rangle = S\langle b_i^p, b_j^q \rangle$ coordinate rings. The previous notations tailored to the product frame are as follows:

$$\begin{aligned} &S\left\langle \begin{smallmatrix} p & q \\ i & j \end{smallmatrix} \right\rangle = \{x \in L : x b_j^q = 0,\ x + b_j^q = b_i^p + b_j^q\}, \\ &S\left(\begin{smallmatrix} pi & qj \\ rk & qj \end{smallmatrix} \right) : S\left\langle \begin{smallmatrix} p & q \\ i & j \end{smallmatrix} \right\rangle \to S\left\langle \begin{smallmatrix} r & q \\ k & j \end{smallmatrix} \right\rangle, \quad x \mapsto (x + d_{i\,k}^{p\,r})(b_k^r + b_j^q), \\ &S\left(\begin{smallmatrix} pi & qj \\ pi & rk \end{smallmatrix} \right) : S\left\langle \begin{smallmatrix} p & q \\ i & j \end{smallmatrix} \right\rangle \to S\left\langle \begin{smallmatrix} p & r \\ i & k \end{smallmatrix} \right\rangle, \quad x \mapsto (x + d_{j\,k}^{q\,r})(b_i^p + b_k^r). \end{aligned} \tag{1.11}$$

(Since we have agreed in reading the indices of, say, $d_{j\,k}^{q\,r}$ column-wise, the space-saving entries qj and rk in $S\left(\begin{smallmatrix} pi & qj \\ pi & rk \end{smallmatrix} \right)$, rather than $\begin{smallmatrix} q \\ j \end{smallmatrix}$ and $\begin{smallmatrix} r \\ k \end{smallmatrix}$, should not be confusing.)

Let us agree that, unless otherwise stated, the superscripts of b and d belong to $\{1, \ldots, m\}$, while all their subscripts to $\{1, \ldots, n\}$. For example, if $d_{i\,j}^{p\,q}$ occurs in a formula, then $p, q \leq m$ and $i, j \leq n$, and also $(p, i) \neq (q, j)$, are automatically stipulated. Similarly, the subscripts of a and c are automatically in $\{1, \ldots, m\}$. This convention allows us, say, to write $\sum_i a_i$ instead of $\sum_{i=1}^m a_i$ without causing any ambiguity. Let us also agree that, unless otherwise stated, we understand our formulas with universally quantified indices, that is, for all meaningful values for the occurring indices.

Finally, we need one more formula:

$$c_{pq} = \sum_i d_{i\,i}^{p\,q}. \tag{1.12}$$

First, we prove

$$c_{1q} = \sum_i d_{i\,i}^{1\,q}. \tag{1.13}$$

As a preparation, we show that

$$\big(v_{1i} + c_{1q}(u_i + a_q)\big)(u_i + a_q) \le c_{1q}. \tag{1.14}$$

Indeed, $\big(v_{1i} + \underline{c_{1q}(u_i + a_q)}\big)\underline{\underline{(u_i + a_q)}} =^{\mathrm{m}} v_{1i}(u_i + \underline{a_q}) + c_{1q}(u_i + a_q) =^{\mathrm{sf}} c_{1q}(u_i + a_q)$, which gives formula (1.14). For $\overline{2 \le i \le n}$, we have

$$d^{1q}_{i\,i} = (d^{11}_{1\,i} + d^{1q}_{1\,i})(b^1_i + b^q_i) =^{1.7} \Big(\underline{v_{1i}} + \underline{\underline{\big(v_{1i} + c_{1q}(u_i + a_q)\big)}}(u_1 + b^q_i)\Big)(u_i + b^q_i) =^{\mathrm{m}}$$
$$\big(v_{1i} + c_{1q}(u_i + a_q)\big)(v_{1i} + u_1 + b^q_i)(u_i + b^q_i) =^{\mathrm{f}}$$
$$\big(v_{1i} + c_{1q}(u_i + a_q)\big)\overline{(u_1 + u_i + b^q_i)}(u_i + b^q_i) =^{1.7}$$
$$\big(v_{1i} + c_{1q}(u_i + a_q)\big)\big(\underline{u_i} + \underline{\underline{(u_i + c_{1q})}}a_q\big) =^{\mathrm{m}}$$
$$\big(v_{1i} + c_{1q}(u_i + a_q)\big)(u_i + c_{1q})(u_i + a_q).$$

Hence

$$\sum_{i \le n} d^{1q}_{i\,i} = \sum_{2 \le i \le n} (d^{1q}_{11} + d^{1q}_{i\,i}) =^{1.7}$$
$$\sum_{2 \le i \le n} \Big(\underline{(u_1 + a_q)c_{1q}} + \big(v_{1i} + c_{1q}(u_i + a_q)\big)\underline{\underline{(u_i + c_{1q})}}(u_i + a_q)\Big) =^{\mathrm{m}}$$
$$\sum_{2 \le i \le n} (u_i + c_{1q})\Big((u_1 + a_q)\underline{\underline{c_{1q}}} + \underline{\big(v_{1i} + c_{1q}(u_i + a_q)\big)(u_i + a_q)}\Big) =^{\mathrm{m}1.14}$$
$$\sum_{2 \le i \le n} \overline{(u_i + c_{1q})}c_{1q}\Big(u_1 + \underline{a_q} + \big(v_{1i} + c_{1q}(u_i + a_q)\big)\underline{\underline{(u_i + a_q)}}\Big) =^{\mathrm{m}}$$
$$\sum_{2 \le i \le n} c_{1q}\Big(u_1 + (u_i + a_q)\big(\underline{a_q} + v_{1i} + c_{1q}\underline{\underline{(u_i + a_q)}}\big)\Big) =^{\mathrm{m}}$$
$$\sum_{2 \le i \le n} c_{1q}\Big(u_1 + (u_i + a_q)\big(v_{1i} + (a_q + c_{1q})(u_i + a_q)\big)\Big) =^{\mathrm{f}}$$
$$\sum_{2 \le i \le n} c_{1q}\Big(u_1 + (u_i + a_q)\overline{\big(v_{1i} + \overline{(a_1 + a_q)}(u_i + a_q)\big)}\Big) =^{\mathrm{f}}$$
$$\sum_{2 \le i \le n} c_{1q}(u_1 + u_i + a_q) = \underline{\underline{c_{1q}}}(u_1 + u_2 + a_q) + \underline{\sum_{3 \le i \le n} c_{1q}(u_1 + u_i + a_q)} =^{\mathrm{m}}$$
$$c_{1q}\Big(u_1 + u_2 + a_q + \sum_{3 \le i \le n} c_{1q}(u_1 + u_i + a_q)\Big) =$$
$$c_{1q}\Big(u_1 + u_2 + \sum_{3 \le i \le n} \big(\underline{a_q} + c_{1q}\underline{\underline{(u_1 + u_i + a_q)}}\big)\Big) =^{\mathrm{m}}$$
$$c_{1q}\Big(u_1 + u_2 + \sum_{3 \le i \le n} (a_q + c_{1q})(u_1 + u_i + a_q)\Big) =^{\mathrm{f}}$$
$$c_{1q}\Big(u_1 + u_2 + \sum_{3 \le i \le n} \overline{(a_1 + a_q)}(u_1 + u_i + a_q)\Big) =$$
$$c_{1q}(u_1 + u_2 + \cdots + u_n + a_q) =^{\mathrm{f}} c_{1q}(a_1 + a_q) = c_{1q}.$$

This proves (1.13). Now, for $p, q \neq 1$, we have

$$d^{pq}_{i\,i} =^{\mathrm{f}} (d^{1p}_{i\,i} + d^{1q}_{i\,i})(b^p_i + b^q_i) \leq$$
$$\Big(\sum_j d^{1p}_{j\,j} + \sum_j d^{1q}_{j\,j}\Big)\Big(\sum_j b^p_j + \sum_j b^q_j\Big) =^{1.9,1.13} (c_{1p} + c_{1q})(a_p + a_q) =^{\mathrm{f}} c_{pq}.$$

This and formula (1.13) give that

$$d^{pq}_{i\,i} \leq c_{pq}. \tag{1.15}$$

Hence

$$a_q + \sum_i d^{pq}_{i\,i} =^{1.9} \sum_i b^q_i + \sum_i d^{pq}_{i\,i} = \sum_i (b^q_i + d^{pq}_{i\,i}) =^{\mathrm{f}}$$
$$\sum_i (b^p_i + b^q_i) = \sum_i b^p_i + \sum_i b^q_i =^{1.9} a_p + a_q =^{\mathrm{f}} a_q + c_{pq}.$$

This together with $a_q c_{pq} = 0$ and formula (1.15) show that $\sum_i d^{pq}_{i\,i}$ and c_{pq} are comparable complements of a_q in $[0, a_p + a_q]$, whence modularity yields (1.12).

1.3 The ring of an outer von Neumann frame

In this section we prove the following theorem.

Theorem 1.7.

(a) *Let L be a lattice with $0, 1 \in L$, and let $m, n \in \mathbb{N}$ with $n \geq 2$. Assume that*

$$L \text{ is modular and } m \geq 4. \tag{1.16}$$

Let $(a_1, \ldots, a_m, c_{12}, \ldots, c_{1m})$ be a spanning von Neumann m-frame of L and $(u_1, \ldots, u_n, v_{12}, \ldots, v_{1n})$ be a spanning von Neumann n-frame of the interval $[0, a_1]$. Let R^* denote the coordinate ring of $(a_1, \ldots, a_m, c_{12}, \ldots, c_{1m})$. Then there is a ring S^* such that R^* is isomorphic to the ring of all $n \times n$ matrices over S^*. If

$$n \geq 4, \tag{1.17}$$

then we can choose S^* as the coordinate ring of $(u_1, \ldots, u_n, v_{12}, \ldots, v_{1n})$.

(b) *The previous part of the theorem remains valid if* (1.16) *and* (1.17) *are replaced by*

$$L \text{ is Arguesian and } m \ge 3 \tag{1.18}$$

and

$$n \ge 3, \tag{1.19}$$

respectively.

Notice that Arguesian lattices are necessarily modular. If $m = 2$, then $R^* = R\langle 1, 2\rangle = \{x \in L : xa_2 = 0 \text{ and } x + a_2 = a_1 + a_2\}$, see (1.5) and Subsection 1.3.1, is just a set, not a ring. If L is not Arguesian and $m = 3$, then R^* is not necessarily a ring. Hence the theorem does not make sense if $m = 2$, or $m = 3$ and L is not Arguesian. Nevertheless, the forthcoming proof still shows that

Remark 1.8. Lemma 1.10 holds even for $m = 2, 3$, provided L is modular.

Next, we give an example to enlighten Theorem 1.7; for $n \ge 4$, the details can be checked based on Theorems II.4.2 and II.14.1 of von Neumann [69].

Example 1.9. Let R be the ring of all $n \times n$ matrices over a field S. Consider the canonical m-frame, with R instead of K, defined in Example 1.1. The coordinate ring R^* of this m-frame is isomorphic to R. Remember from Example 1.1 that $a_1 = R(E, 0, \dots, 0) \in \mathrm{Sub}(R^m)$, where E is the unit matrix in R. Hence the interval $[0, a_1]$ in $\mathrm{Sub}(R^m)$ is isomorphic to the lattice of all left ideals of R. The lattice of these left ideals is known to be isomorphic to the subspace lattice $\mathrm{Sub}(S^n)$ of the vector space S^n. Fix an appropriate isomorphism; it sends the canonical n-frame of $\mathrm{Sub}(S^n)$ to a spanning n-frame $(u_1, \dots, u_n, v_{12}, \dots, v_{1n})$ of $[0, a_1]$. Clearly, the coordinate ring S^* of this n-frame is isomorphic to S. Hence R^* is isomorphic to the ring of all $n \times n$ matrices over S^*.

While $\mathrm{Sub}(R^m)$ is coordinatizable by its construction in Example 1.9, it is worth pointing out that L in Theorem 1.7 is *not coordinatizable* in general. Although some ideas of the proof have been extracted from Example 1.9, Linear Algebra in itself seems to be inadequate to prove Theorem 1.7. (Even if it was an adequate tool, modular lattice theory would probably offer a more elegant treatment, see the last paragraph of Section 2 in [18].) Notice that Herrmann [52] reduces many problems of frame generated modular lattices to Linear Algebra, but our L is not frame-generated in general by evident cardinality reasons.

1.3.1 A pair of reciprocal mappings

For $i,j \le n$, we define a mapping $\varphi_{ij}\colon R^* \to S^*$ as follows. We identify R^* with $R\langle 1,2\rangle = R\langle a_1,a_2\rangle$. So we define $x\varphi_{ij}$ for $x \in R\langle 1,2\rangle$, and, without over-complicating our formulas with writing x^*, we understand $x^*\varphi_{ij}$ as $x\varphi_{ij}$. Similarly, we define the value $x\varphi_{ij}$ in $S\left\langle\begin{smallmatrix}1&1\\i&j\end{smallmatrix}\right\rangle$ but we understand it as $(x\varphi_{ij})^* \in S^*$ without making a notational distinction between $x\varphi_{ij}$ and $(x\varphi_{ij})^*$. Finally, we will put these φ_{ij} together in the natural way to obtain a mapping $\varphi\colon R^* \to M_n(S^*)$: the (i,j)th entry of the matrix $x\varphi$ is defined as $x\varphi_{ij}$. So, the definition of φ is completed by

$$\varphi_{ij}\colon R\langle 1,2\rangle \to S\left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle, \quad x \mapsto x_{ij} = (x + B_j^2)(b_i^1 + b_j^2). \tag{1.20}$$

(We will prove soon that φ_{ij} maps $R\langle 1,2\rangle$ into $S\left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle$.)

In the reverse direction, we will rely on the possibility offered by L-numbers even more: distinct entries of a matrix in $M_n(S^*)$ will be represented with their components of different positions. Let $(e_{ij} : i,j \le n)$ be a matrix over S^*, that is, an element of $M_n(S^*)$. The truth is that e_{ij} belongs to S^*. However, we identify e_{ij} with its component belonging to $S\left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle$, and, again, we do this without notational difference between e_{ij} and its corresponding component in $S\left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle$. Introduce the notation

$$E_{*k} = \sum_i e_{ik}.$$

With this convention, we define

$$\psi\colon M_n(S^*) \to R^*, \quad (e_{ij} : i,j \le n) \mapsto \prod_k (E_{*k} + B_k^2). \tag{1.21}$$

We will prove soon that $\prod_k (E_{*k} + B_k^2)$ belongs to $R\langle 1,2\rangle$, which is identified with R^*.

Next, we formulate an evident consequence of modularity:

$$R\langle i,j\rangle \text{ and } S\left\langle\begin{smallmatrix}p&q\\i&j\end{smallmatrix}\right\rangle \text{ are antichains in } L. \tag{1.22}$$

Indeed, if, say, we had $x < y$ and $x,y \in R\langle i,j\rangle$, then x and y would be comparable complements of a_j, a contradiction. We will often have to prove that two elements of $R\langle i,j\rangle$ or $S\left\langle\begin{smallmatrix}p&q\\i&j\end{smallmatrix}\right\rangle$ are equal; then (1.22) reduces this task to showing that the two elements are comparable.

The rest of this subsection is devoted to the following lemma.

Lemma 1.10. *φ and ψ are bijections, and they are inverse mappings of each other.*

Before proving this lemma, two preliminary statements are necessary.

Lemma 1.11. *Let $j \le m$, and suppose that, for all $i \in \{1, \dots, m\} \setminus \{j\}$, $w_i \in R\langle i, j\rangle$. Then $a_j \sum_{i \ne j} w_i = 0$.*

Proof. Let I_k denote the induction hypothesis "if $|\{i : w_i \ne a_i\}| \le k$, then $a_j \sum_{i \ne j} w_i = 0$". Then I_0 clearly holds by (1.2), and I_{m-1} is our target.

Assume I_{k-1} for an arbitrary $k < m$. We will refer to it with the notation $=^{\mathrm{ih}}$. We want to show I_k. By symmetry, we can assume that $j = m$ and $w_i \ne a_i$ holds only for $i \le k$. Then

$$\begin{aligned}
a_j \sum_{i \ne j} w_i &= a_m(\underline{w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1}} + w_k) \\
&=^{\mathrm{s}} a_m\big((w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1})(w_k + a_m) + w_k\big) \\
&=^{1.5} a_m\big((w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1})(a_k + \underline{a_m}) + w_k\big) \\
&=^{\mathrm{s,ih}} a_m\big((w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1})a_k + w_k\big) \\
&\le^{1.5} a_m\big((a_1 + a_m + \dots + a_{k-1} + a_m + a_{k+1} + \dots + a_{m-1})a_k + w_k\big) \\
&=^{1.2} a_m w_k =^{1.5} 0.
\end{aligned}$$

□

The following easy statement on elements of a *modular* lattice belongs to the folklore; it also occurs as (1) in Huhn [56].

Lemma 1.12. *If $f_i \le g_j$ for all $i \ne j$, $i, j \le k$, then*

$$\prod_{i \le k} g_i + \sum_{i \le k} f_i = \prod_{i \le k} (g_i + f_i).$$

Proof of Lemma 1.10. Let $\prod_k (E_{*k} + B_k^2)$ from (1.21) be denoted by e, and remember that $e_{ij} \in S\left\langle\begin{smallmatrix}1 & 2\\ i & j\end{smallmatrix}\right\rangle$. We have to show that $e \in R\langle 1, 2\rangle$. Let us compute:

$$\begin{aligned}
a_2 e &= \prod_k \big(\underline{\underline{a_2}}(E_{*k} + \underline{B_k^2})\big) =^{\mathrm{m}} \prod_k (a_2 E_{*k} + B_k^2) =^{1.9} \prod_k \big((\underline{b_k^2} + B_k^2)E_{*k} + B_k^2\big) \\
&=^{\mathrm{s}} \prod_k \big((\underline{b_k^2(E_{*k} + B_k^2)} + B_k^2)E_{*k} + B_k^2\big).
\end{aligned}$$

Focusing on the last underlined subterm, observe that the summands e_{ik} of E_{*k} belong to $S\left\langle\begin{smallmatrix}1 & 2\\ i & k\end{smallmatrix}\right\rangle$, and the summands b_j^2, $j \ne k$, of B_k^2 belong to $S\left\langle\begin{smallmatrix}2 & 2\\ j & k\end{smallmatrix}\right\rangle$. Hence, applying Lemma 1.11 to the product frame, we conclude that $b_k^2(E_{*k} + B_k^2) = 0$. Therefore,

$$a_2 e = \prod_k (B_k^2 E_{*k} + B_k^2) = \prod_k B_k^2 =^{1.2} 0. \tag{1.23}$$

Next, we compute

$$\begin{aligned}
a_2 + e &=^{1.9} \sum_k b_k^2 + \prod_k (E_{*k} + B_k^2) =^{\mathrm{L1.12}} \prod_k (E_{*k} + B_k^2 + b_k^2) \\
&=^{1.9} \prod_k \sum_j (e_{jk} + b_k^2 + a_2) =^{1.11} \prod_k \sum_j (b_j^1 + b_k^2 + a_k) \\
&=^{1.9} \prod_k (a_1 + a_2) = a_1 + a_2.
\end{aligned}$$

This and (1.23) imply $e \in R\langle 1,2\rangle$. Hence ψ maps into R^*, as desired.

Next, let $x \in R\langle 1,2\rangle$. To show that φ maps into $M_n(S^*)$, we have to show that $x_{ij} = x\varphi_{ij} = (x + B_j^2)(b_i^1 + b_j^2)$ belongs to $S\left\langle\begin{smallmatrix}1&2\\ i&j\end{smallmatrix}\right\rangle$. This follows easily, since

$$\begin{aligned}
x_{ij} b_j^2 &= (\underline{x} + B_j^2)\overline{(b_i^1 + b_j^2)} b_j^2 =^{\mathrm{s}} \big(x(b_j^2 + B_j^2) + B_j^2\big) b_j^2 \\
&=^{1.9} (x a_2 + B_j^2) b_j^2 =^{1.5} B_j^2 b_j^2 =^{\mathrm{f}} 0, \quad \text{and} \\
x_{ij} + b_j^2 &= (x + B_j^2)(\underline{\underline{b_i^1 + b_j^2}}) + \underline{b_j^2} =^{\mathrm{m}} (b_i^1 + b_j^2)(x + B_j^2 + b_j^2) \\
&=^{1.9} (b_i^1 + b_j^2)(x + a_2) =^{1.5} (b_i^1 + b_j^2)(a_1 + a_2) =^{1.9} b_i^1 + b_j^2.
\end{aligned}$$

Next, we show that $\varphi \circ \psi$ is the identical mapping. Let $x \in R\langle 1,2\rangle$. Then

$$\begin{aligned}
x(\varphi \circ \psi) &= (x\varphi)\psi = (x\varphi_{ij} : i, j \le n)\psi \\
&=^{1.20} \big((x + B_j^2)(b_i^1 + b_j^2) : i, j \le n\big)\psi \\
&=^{1.21} \prod_k y_k, \quad \text{where } y_k = B_k^2 + \sum_i (x + B_k^2)(b_i^1 + b_k^2).
\end{aligned}$$

Observe that it suffices to show that $x \le y_k$ for all $k \le n$, since then (1.22) implies $x = y$. Let us compute:

$$\begin{aligned}
y_k &= \sum_i \big(\underline{B_k^2} + (\underline{\underline{x + B_k^2}})(b_i^1 + b_k^2)\big) =^{\mathrm{m}} \sum_i (x + B_k^2)(b_i^1 + b_k^2 + B_k^2) \\
&\ge^{1.9} \sum_i x(b_i^1 + a_2) = \underline{\underline{x}}(b_1^1 + a_2) + \underline{\sum_{2\le i} x(b_i^1 + a_2)} \\
&=^{\mathrm{m}} x\Big(b_1^1 + a_2 + \sum_{2\le i} x(b_i^1 + a_2)\Big) = x\Big(b_1^1 + \sum_{2\le i}\big(\underline{a_2} + x(\underline{\underline{b_i^1 + a_2}})\big)\Big) \\
&=^{\mathrm{m}} x\Big(b_1^1 + \sum_{2\le i}(a_2 + x)(b_i^1 + a_2)\Big) =^{1.5} x\Big(b_1^1 + \sum_{2\le i}(a_2 + a_1)(b_i^1 + a_2)\Big) \\
&=^{1.9} x\Big(b_1^1 + \sum_{2\le i}(b_i^1 + a_2)\Big) = x\Big(a_2 + \sum_i b_i^1\Big) =^{1.9} x(a_2 + a_1) =^{1.5} x.
\end{aligned}$$

Hence $x \le y_k$, as requested, and $\varphi \circ \psi$ is the identical mapping.

Next, to show that $\psi \circ \varphi$ is the identical mapping, let $e_{ij} \in S\left\langle\begin{smallmatrix}1&2\\ i&j\end{smallmatrix}\right\rangle$ for $i, j \le n$, and denote $(e_{ij} : i, j \le n)\psi = \prod_k (E_{*k} + B_k^2)$ by e. We have already shown that $e \in R\langle 1,2\rangle$, see (1.21), and $e\varphi_{ij} = (e + B_j^2)(b_i^1 + b_j^2) \in S\langle{}^{1\,2}_{i\,j}\rangle$, see (1.20). Since $e_{ij} \le b_i^1 + b_j^2$ by (1.11), $e_{ij} \le e + B_j^2$ would imply $e_{ij} \le e\varphi_{ij}$, and we could derive $e_{ij} = e\varphi_{ij}$ by (1.22). So, it suffices to show that $e_{ij} \le e + B_j^2$. Let us compute:

$$\begin{aligned}
e + B_j^2 &= B_j^2 + \prod_k (E_{*k} + B_k^2) = \underline{B_j^2} + \underline{\underline{(E_{*j} + B_j^2)}} \prod_{k\ne j}(E_{*k} + B_k^2) \\
&=^{\mathrm{m}} (E_{*j} + B_j^2)\Big(\sum_{k\ne j} b_k^2 + \prod_{k\ne j}(E_{*k} + B_k^2)\Big) \\
&=^{\mathrm{L1.12}} (E_{*j} + B_j^2)\prod_{k\ne j}(E_{*k} + B_k^2 + b_k^2) =^{1.9} (E_{*j} + B_j^2)\prod_{k\ne j}\sum_h (e_{hk} + b_k^2 + a_2) \\
&=^{1.11} (E_{*j} + B_j^2)\prod_{k\ne j}\sum_h (b_h^1 + b_k^2 + a_2) =^{1.9} (E_{*j} + B_j^2)\prod_{k\ne j}(a_1 + a_2).
\end{aligned}$$

Since $a_1 + a_2 \ge^{1.9} b_i^1 + b_j^2 \ge^{1.11} e_{ij}$ and $E_{*j} \ge e_{ij}$, the above calculation shows that $e_{ij} \le e + B_j^2$. This completes the proof of Lemma 1.10. □

1.3.2 Addition and further lemmas

Lemma 1.13. *φ and, therefore, ψ are additive.*

Proof. Let $x, y \in R\langle 1,2\rangle$, $z = x \oplus_{123} y$, $x' = x\varphi_{ij} = (x + B_j^2)(b_i^1 + b_j^2)$, $y' = y\varphi_{ij} = (y + B_j^2)(b_i^1 + b_j^2)$ and $z' = z\varphi_{ij} = (z + B_j^2)(b_i^1 + b_j^2)$. It suffices to show that, in $S\langle{}^{1\,2}_{i\,j}\rangle$, we have $x' \oplus^{123}_{i\,j\,i} y' = z'$. Let us compute:

$$\begin{aligned}
x' \oplus^{123}_{i\,j\,i} y' &= (b_i^1 + b_j^2)\Big((x' + b_i^3)(d^{13}_{i\,i} + b_j^2) + y' S\big({}^{1i\;2j}_{3i\;2j}\big)\Big) \\
&= (b_i^1 + b_j^2)\Big((x' + b_i^3)(d^{13}_{i\,i} + b_j^2) + (y' + d^{13}_{i\,i})(b_i^3 + b_j^2)\Big). \qquad (1.24)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
z' &= (z + B_j^2)(b_i^1 + b_j^2) \\
&= (b_i^1 + b_j^2)\Big((a_1 + a_2)\big((x + a_3)(c_{13} + a_2) + yR\big({}^{1\,2}_{3\,2}\big)\big) + B_j^2\Big) \\
&= (b_i^1 + b_j^2)\Big(\underline{\underline{(a_1 + a_2)}}\big((x + a_3)(c_{13} + a_2) + (y + c_{13})(a_3 + a_2)\big) + \underline{B_j^2}\Big) \\
&=^{\mathrm{m}} (b_i^1 + b_j^2)\overline{(a_1 + a_2)}\big((x + a_3)(c_{13} + a_2) + (y + c_{13})(a_3 + a_2) + B_j^2\big) \\
&= (b_i^1 + b_j^2)\big((x + a_3)(\underline{\underline{c_{13} + a_2}}) + \underline{B_j^2} + (y + c_{13})(\underline{\underline{a_3 + a_2}}) + \underline{B_j^2}\big) \\
&=^{\mathrm{m}} (b_i^1 + b_j^2)\big((\underline{x + B_j^2} + a_3)(c_{13} + a_2) + (\underline{y + B_j^2} + c_{13})(a_3 + a_2)\big). \qquad (1.25)
\end{aligned}$$

Now, we can see that the subterms obtained in (1.24) are less than or equal to the corresponding subterms obtained in (1.25). Indeed, $x' \le x + B_j^2$ and $y' \le y + B_j^2$ by definitions, and $b_i^3 \le a_3$, $b_j^2 \le a_2$ and $d^{13}_{i\,i} \le c_{13}$ by (1.15). Hence (1.22) yields $x' \oplus^{123}_{i\,j\,i} y' = z'$. □

Lemma 1.14. *$b^i_j + c_{ik} = b^k_j + c_{ik}$ and $B^i_j + c_{ik} = B^k_j + c_{ik}$.*

Proof. It suffices to deal only with the first equation: $b^i_j + c_{ik} =^{1.15} b^i_j + d^{\,i\,k}_{\,j\,j} + c_{ik} =^{\mathrm{f}} b^k_j + d^{\,i\,k}_{\,j\,j} + c_{ik} =^{1.15} b^i_k + c_{ik}$. □

Lemma 1.15. *Assume that $x, y \in S\left\langle\begin{smallmatrix}1&2\\u&v\end{smallmatrix}\right\rangle$. Then $xR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) = xS\left(\begin{smallmatrix}1u&2v\\1u&3v\end{smallmatrix}\right)$ and, similarly, $yR\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right) = yS\left(\begin{smallmatrix}1u&2v\\3u&2v\end{smallmatrix}\right)$.*

Proof. If $i, j, k \le m$ are pairwise distinct, then we have

$$\begin{aligned} c_{jk}\big(\underline{a_i} + a_k\big) &=^{\mathrm{s}} c_{jk}\big(a_i(c_{jk} + a_k) + a_k\big) \\ &=^{1.3} c_{jk}\big(a_i(a_j + a_k) + a_k\big) =^{1.2} c_{jk}a_k =^{1.3} 0. \end{aligned} \tag{1.26}$$

The outer projectivities $R\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right)$ and $R\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right)$ are lattice isomorphisms that send the interval $[0, a_1 + a_2]$ onto $[0, a_1 + a_3]$ and $[0, a_3 + a_2]$, respectively. Since $S\left\langle\begin{smallmatrix}1&2\\u&v\end{smallmatrix}\right\rangle \subseteq [0, a_1 + a_2]$ is defined in the terminology of lattices and

$$\begin{aligned} b^1_u R\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) &= \big(\underline{b^1_u} + c_{23}\big)\big(\underline{\underline{a_1 + a_3}}\big) =^{\mathrm{m}} b^1_u + c_{23}(a_1 + a_3) =^{1.26} b^1_u, \\ b^2_v R\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) &= \big(b^2_v + c_{23}\big)(a_1 + a_3) =^{\mathrm{L}1.14} \big(\underline{b^3_v} + c_{23}\big)\big(\underline{\underline{a_1 + a_3}}\big) =^{\mathrm{m},1.26} b^3_v, \\ b^1_u R\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right) &= \big(b^1_u + c_{13}\big)(a_3 + a_2) =^{\mathrm{L}1.14} \big(\underline{b^3_u} + c_{13}\big)\big(\underline{\underline{a_3 + a_2}}\big) =^{\mathrm{m},1.26} b^3_u, \\ b^2_v R\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right) &= \big(\underline{b^2_v} + c_{13}\big)\big(\underline{\underline{a_3 + a_2}}\big) =^{\mathrm{m},1.26} b^2_v, \end{aligned}$$

we conclude that these outer projectivities send (the support set of) $S\left\langle\begin{smallmatrix}1&2\\u&v\end{smallmatrix}\right\rangle$ onto $S\left\langle\begin{smallmatrix}1&3\\u&v\end{smallmatrix}\right\rangle$ and $S\left\langle\begin{smallmatrix}3&2\\u&v\end{smallmatrix}\right\rangle$, respectively. Lattice terms are monotone, so we obtain

$$xS\left(\begin{smallmatrix}1u&2v\\1u&3v\end{smallmatrix}\right) = \big(x + d^{\,2\,3}_{\,v\,v}\big)\big(b^1_u + b^3_v\big) \le^{1.15} (x + c_{23})(a_1 + a_3) = xR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right). \tag{1.27}$$

We have seen that both sides of (1.27) belong to $S\left\langle\begin{smallmatrix}1&3\\u&v\end{smallmatrix}\right\rangle$, whence they are equal in virtue of (1.22). The other equation of the lemma follows the same way. □

1.3.3 Multiplication

By an *almost zero matrix* we mean a matrix in which all but possibly one entries are zero. We say that ψ, defined in (1.21), *preserves the multiplication of almost zero matrices*, if $(E \otimes_{M_n} F)\psi = (E\psi) \otimes_{R^*} (F\psi)$ holds for all almost zero matrices $E, F \in M_n(S^*)$.

Lemma 1.16. *If ψ is additive and preserves the multiplication of almost zero matrices, then it is a ring homomorphism.*

Proof. Since each matrix in $M_n(S^*)$ is a sum of almost zero matrices, the lemma follows trivially by ring distributivity. □

Next, we introduce some notations, which will be permanent in the rest of the chapter. Let $E = (e_{ij} : i,j < n) \in M_n(S^*)$ and $F = (f_{ij} : i,j < n) \in M_n(S^*)$ be two almost zero matrices. According to the earlier convention and keeping in mind that b_i^1 is the zero of the ring $S\left\langle{}^{1\,2}_{i\,j}\right\rangle$, this means that there are indices p,q,r,s, fixed from now on, such that

$$\begin{aligned} x &:= e_{pq} \in S\left\langle{}^{1\,2}_{p\,q}\right\rangle, & e_{ij} &= b_i^1 \in S\left\langle{}^{1\,2}_{i\,j}\right\rangle \text{ for } (i,j) \neq (p,q),\\ y &:= f_{rs} \in S\left\langle{}^{1\,2}_{r\,s}\right\rangle, & f_{kh} &= b_k^1 \in S\left\langle{}^{1\,2}_{k\,h}\right\rangle \text{ for } (k,h) \neq (r,s). \end{aligned} \tag{1.28}$$

Let $G = (g_{ij} : i,j < n) = E \otimes_{M_n} F$. By definitions, including the everyday's definition of a product matrix, we have

$$g_{ij} = b_i^1, \text{ the zero of } S\left\langle{}^{1\,2}_{i\,j}\right\rangle, \qquad \text{if } q \neq r \text{ or } (i,j) \neq (p,s); \tag{1.29}$$

$$g_{ps} = xS\left({}^{1p\ 2r}_{1p\ 2s}\right) \otimes^{12\alpha}_{ps\beta} yS\left({}^{1r\ 2s}_{1p\ 2s}\right), \qquad \text{if } q = r; \tag{1.30}$$

where α and β are arbitrary, provided $(1,p) \neq (\alpha,\beta) \neq (2,s)$. We also define

$$e := E\psi, \quad f := F\psi, \quad \text{and, differently,} \quad g := e \otimes_{123} f.$$

The plan is to show that $g\varphi = G$, that is, $g\varphi_{ij} = g_{ij}$ for all $i,j \le n$, since this is equivalent to $G\psi = g$. To prepare a formula for the $g\varphi_{ij}$, we need the following technical lemma.

Lemma 1.17. *For all $j \le n$, we have*

$$B_j^2 + (y + B_r^1 + B_s^2)\prod_{k\neq s}(a_1 + B_k^2) = y + B_r^1 + B_j^2, \tag{1.31}$$

$$B_r^2 + (x + B_p^1 + B_q^2)\prod_{k\neq q}(a_1 + B_k^2) = x + B_p^1 + B_r^2. \tag{1.32}$$

Proof. It suffices to show (1.31), since it implies (1.32) by replacing (y,r,s,j) with (x,p,q,r). Let u denote the left hand side of (1.31). If $j = s$, then

$$\begin{aligned} u &= \underline{B_j^2} + (\underline{\underline{y + B_r^1 + B_j^2}})\prod_{k\neq j}(a_1 + B_k^2) =^{\mathrm{m}} (y + B_r^1 + B_j^2)\Big(B_j^2 + \prod_{k\neq j}(a_1 + B_k^2)\Big)\\ &=^{\mathrm{L1.12}} (y + B_r^1 + B_j^2)\prod_{k\neq j}(a_1 + B_k^2 + b_k^2) =^{1.9} (y + B_r^1 + B_j^2)\prod_{k\neq j}(a_1 + a_2)\\ &=^{1.9,1.5} y + B_r^1 + B_j^2. \end{aligned}$$

If $j \neq s$, then

$$\begin{aligned}
u &= \underline{B_j^2} + (y + B_r^1 + B_s^2)(\underline{\underline{a_1 + B_j^2}}) \prod_{k\neq j,s} (a_1 + B_k^2) \\
&=^{\mathrm{m}} (a_1 + B_j^2)\Big(B_j^2 + (y + B_r^1 + B_s^2) \prod_{k\neq j,s} (a_1 + B_k^2)\Big) \\
&= (a_1 + B_j^2)\Big(b_s^2 + \sum_{k\neq j,s} b_k^2 + (y + B_r^1 + B_s^2) \prod_{k\neq j,s} (a_1 + B_k^2)\Big) \\
&=^{\mathrm{L1.12}} (a_1 + B_j^2)(y + B_r^1 + B_s^2 + b_s^2) \prod_{k\neq j,s} (a_1 + B_k^2 + b_k^2) \\
&=^{1.9} (a_1 + B_j^2)(\underline{y} + B_r^1 + a_2 + \underline{b_s^2})\overline{\prod_{k\neq j,s} (a_1 + a_2)} \\
&=^{1.11,1.9} (a_1 + B_j^2)(\overline{a_1 + a_2}) =^{1.9} B_r^1 + b_r^1 + b_s^2 + B_j^2 \\
&=^{1.11} B_r^1 + y + b_s^2 + B_j^2 =^{1.9} y + B_r^1 + B_j^2
\end{aligned}$$

□

Lemma 1.18. *For every $i, j \leq n$, we have*

$$g\varphi_{ij} = (b_i^1 + b_j^2)\Big(B_p^1 + B_j^2 + B_r^3 + xS\left(\begin{smallmatrix} 1p & 2q \\ 1p & 3q \end{smallmatrix}\right) + yS\left(\begin{smallmatrix} 1r & 2s \\ 3r & 2s \end{smallmatrix}\right)\Big).$$

Proof. Firstly, we express e and, to obtain f, we replace (x, p, q) with (y, r, s):

$$\begin{aligned}
e = E\psi &=^{1.21} \prod_k (E_{*k} + B_k^2) = (E_{*q} + B_q^2) \prod_{k\neq q} (E_{*k} + B_k^2) \\
&=^{1.28,1.9} (x + B_p^1 + B_q^2) \prod_{k\neq q} (a_1 + B_k^2); \qquad (1.33) \\
f &= (y + B_r^1 + B_s^2) \prod_{k\neq s} (a_1 + B_k^2). \qquad (1.34)
\end{aligned}$$

We need some auxiliary equations:

$$\begin{aligned}
B_j^2 + fR\left(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\right) &=^{1.4} \underline{B_j^2} + (f + c_{13})(\underline{\underline{a_3 + a_2}}) =^{\mathrm{m}} (a_3 + a_2)(B_j^2 + f + c_{13}) \\
&=^{1.34} (a_3 + a_2)\Big(c_{13} + B_j^2 + (y + B_r^1 + B_s^2) \prod_{k\neq s} (a_1 + B_k^2)\Big) \\
&=^{1.31} (a_3 + a_2)(c_{13} + y + B_r^1 + B_j^2) \\
&=^{\mathrm{L1.14}} (\underline{\underline{a_3 + a_2}})(c_{13} + y + \underline{B_r^3} + B_j^2) \\
&=^{\mathrm{m}} B_r^3 + (a_3 + a_2)(c_{13} + y + B_j^2), \quad \text{and} \qquad (1.35)
\end{aligned}$$

$$
\begin{aligned}
B_r^3 + eR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) &=^{1.4} \underline{B_r^3} + (e + c_{23})(\underline{\underline{a_1 + a_3}}) =^{\mathrm{m}} (a_1 + a_3)(B_r^3 + e + c_{23}) \\
&=^{\mathrm{L1.14}} (a_1 + a_3)(c_{23} + B_r^2 + e) \\
&=^{1.33} (a_1 + a_3)\Big(c_{23} + B_r^2 + (x + B_p^1 + B_q^2)\prod_{k \neq q}(a_1 + B_k^2)\Big) \\
&=^{1.32} (\underline{\underline{a_1 + a_3}})(c_{23} + x + \underline{B_p^1} + B_r^2) \\
&=^{\mathrm{m}} B_p^1 + (a_1 + a_3)(c_{23} + x + B_r^2).
\end{aligned}
\tag{1.36}
$$

Armed with the previous equations, we obtain

$$
\begin{aligned}
g\varphi_{ij} &=^{1.20} (b_i^1 + b_j^2)(g + B_j^2) \\
&=^{1.5} (b_i^1 + b_j^2)\Big(\underline{B_j^2} + (\underline{\underline{a_1 + a_2}})\big(eR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) + fR\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right)\big)\Big) \\
&=^{\mathrm{m}} (b_i^1 + b_j^2)\overline{(a_1 + a_2)}\big(B_j^2 + eR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) + fR\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right)\big) \\
&=^{1.35} (b_i^1 + b_j^2)\big(eR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) + B_r^3 + (a_3 + a_2)(c_{13} + y + B_j^2)\big) \\
&=^{1.36} (b_i^1 + b_j^2)\big(B_p^1 + (a_1 + a_3)(c_{23} + x + \underline{B_r^2}) + (a_3 + a_2)(c_{13} + y + B_j^2)\big) \\
&=^{\mathrm{L1.14}} (b_i^1 + b_j^2)\big(B_p^1 + (\underline{\underline{a_1 + a_3}})(c_{23} + x + \underline{B_r^3}) + (\underline{\underline{a_3 + a_2}})(c_{13} + y + \underline{B_j^2})\big) \\
&=^{\mathrm{m}} (b_i^1 + b_j^2)\big(B_p^1 + B_r^3 + (a_1 + a_3)(c_{23} + x) + B_j^2 + (a_3 + a_2)(c_{13} + y)\big) \\
&=^{1.4} (b_i^1 + b_j^2)\Big(B_p^1 + B_j^2 + B_r^3 + xR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) + yR\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right)\Big),
\end{aligned}
$$

whence Lemma 1.18 follows by Lemma 1.15. □

Lemma 1.19. *ψ preserves the multiplication of almost zero matrices.*

Proof. Keep the previous notations, and let

$$
x' := xS\left(\begin{smallmatrix}1p&2q\\1p&3q\end{smallmatrix}\right) \in S\left\langle\begin{smallmatrix}1&3\\p&q\end{smallmatrix}\right\rangle, \qquad y' := yS\left(\begin{smallmatrix}1r&2s\\3r&2s\end{smallmatrix}\right) \in S\left\langle\begin{smallmatrix}3&2\\r&s\end{smallmatrix}\right\rangle. \tag{1.37}
$$

We know from Lemma 1.18 that

$$
g\varphi_{ij} = (b_i^1 + b_j^2)(B_p^1 + B_j^2 + B_r^3 + x' + y'). \tag{1.38}
$$

According to (1.29), our first goal is to show that $g\varphi_{ij} = b_i^1$ whenever $q \neq r$ or $(i,j) \neq (p,s)$. Notice that if

$$
b_i^1 \leq B_p^1 + B_j^2 + B_r^3 + x' + y', \tag{1.39}
$$

then $g\varphi_{ij} = b_i^1$ follows from (1.22), so we can aim at (1.39). Since $x' \in S\left\langle\begin{smallmatrix}1&3\\p&q\end{smallmatrix}\right\rangle$ and $y' \in S\left\langle\begin{smallmatrix}3&2\\r&s\end{smallmatrix}\right\rangle$, (1.3) and (1.11) provide us with the following computation rules:

$$
\alpha \neq q \Longrightarrow B_\alpha^3 + x' \geq b_p^1, \tag{1.40}
$$

$$
\beta \neq s \Longrightarrow B_\beta^2 + y' \geq b_r^3. \tag{1.41}
$$

We can assume that $i = p$, since otherwise $b_i^1 \le B_p^1$ gives (1.39). If $r \ne q$, then $B_r^3 + x' \ge^{1.40} b_p^1$ yields (1.39) again. Hence we assume that $q = r$. If $j \ne s$, then $B_j^2 + y' \ge^{1.41} b_r^3$ together with $b_r^3 + x' = b_q^3 + x' \ge^{1.11} b_p^1$ yields (1.39) once more. Therefore, we can assume that $j = s$.

Now, our task is restricted to the case $i = p$, $q = r$, $j = s$. Substituting these indices into (1.38) and computing:

$$\begin{aligned} g\varphi_{ps} &= (b_p^1 + b_s^2)(B_p^1 + B_s^2 + B_r^3 + x' + y') \\ &\ge (b_p^1 + b_s^2)(x' + y') =^{1.37} (b_p^1 + b_s^2)\left(xS\begin{pmatrix} 1p & 2r \\ 1p & 3r \end{pmatrix} + yS\begin{pmatrix} 1r & 2s \\ 3r & 2s \end{pmatrix}\right) \\ &=^{\mathrm{L}1.3} (b_p^1 + b_s^2)\left(xS\begin{pmatrix} 1p & 2r \\ 1p & 2s \end{pmatrix}S\begin{pmatrix} 1p & 2s \\ 1p & 3r \end{pmatrix} + yS\begin{pmatrix} 1r & 2s \\ 1p & 2s \end{pmatrix}S\begin{pmatrix} 1p & 2s \\ 3r & 2s \end{pmatrix}\right) \\ &=^{1.5} xS\begin{pmatrix} 1p & 2r \\ 1p & 2s \end{pmatrix} \otimes_{psr}^{123} yS\begin{pmatrix} 1r & 2s \\ 1p & 2s \end{pmatrix} =^{1.30} g_{ps}. \end{aligned}$$

Hence (1.22) yields that $g\varphi_{ps} = g_{ps}$, indeed. □

Proof of Theorem 1.7. Lemmas 1.10, 1.13, 1.16 and 1.19. □

Chapter 2

Isometrical embeddings

In this chapter we focus on lattice embeddings. They have been heavily studied since the beginning of lattice theory. The first important result was published by Birkhoff [6] in 1935. He proved that *every partition lattice is embeddable into the lattice of subgroups of some group*. Later, in 1946, Whitman [87] showed that *every lattice is embeddable into a partition lattice*. These two results together imply that *every lattice is embeddable into the lattice of subgroups of some group*. These embeddings have considerable consequences; for example, there is no nontrivial lattice identity that holds in all partition lattices or in all subgroup lattices.

Perhaps the best-known proof for Whitman's theorem is due to Jónsson [60]. However, both in Whitman's and Jónsson's proofs, the constructed partition lattices are much bigger than the original ones, for instance, they are infinite even for finite lattices. The question whether a finite lattice is embeddable into a finite partition lattice arose already in Whitman [87]. He conjectured that this question had a positive answer.

Partition lattices belong to a larger class of lattices; they are geometric lattices. A *finite geometric* lattice is an atomistic *semimodular* lattice. The first step towards Whitman's conjecture was a result of Finkbeiner [32]. He proved that *every finite lattice can be embedded into a finite semimodular lattice*. His construction is based on two steps. On the one hand, he showed that every finite lattice that has a so-called *pseudo rank function* can be embedded into a finite semimodular lattice. On the other hand, he pointed out that every finite lattice has a pseudo rank function. His embedding "preserves" the pseudorank function; that is, if L is embedded into S, say $L \leq S$, and p denotes the pseudorank function of L, and h denotes the height function of S then p and h coincide on L. Note that Finkbeiner credits his proof as an unpublished result of Dilworth. The second step towards Whitman's conjecture was a result of Dilworth, which was published later in Crawley and Dilworth [13]. He showed that *every finite lattice can be embedded into a finite geometric lattice*. The last step was made by Pudlák and Tůma [74, 75], who showed in 1977 that Whitman's conjecture is true.

Although Finkbeiner did not manage to prove Whitman's conjecture, his proof drew attention to embeddings that preserve pseudo rank functions. Such embeddings are called *isometrical*. In 1986, blending the results of Finkbeiner and Dilworth, Grätzer and Kiss [43] showed that *every finite lattice with a pseudorank function has an isometrical embedding into a finite geometric lattice*, see also Theorem 2.3. The question whether a finite lattice with a pseudorank function has an isometrical embedding into a partition lattice is still open. Grätzer and Kiss' theorem has a straightforward corollary for semimodular lattices. Given a finite semimodular lattice, its height function is a pseudorank function, and an isometrical embedding (with respect to the height function) is an embedding that preserves the height of each element. It is equivalent to the condition that the embedding preserves the covering relation. Such embeddings are called *cover-preserving*. Now, Grätzer and Kiss' theorem implies that *every finite semimodular lattice has a cover-preserving embedding into a finite geometric lattice*, see also Corollary 2.4. Note that this corollary together with Finkbeiner's result imply Grätzer and Kiss' theorem.

Finkbeiner, Grätzer and Kiss focused on finite lattices. The question arises naturally whether their results can be generalized for infinite lattices. Czédli and Schmidt [23] proved that the corollary of Grätzer and Kiss' theorem can be extended for semimodular lattices of finite length. In [78] we managed to show that Grätzer and Kiss' theorem can also be extended for lattices of finite length, moreover, it can be extended for a larger class of lattices that we called *finite height generated lattices*.

Overview of the chapter

In Section 2.1, as a motivation, we recall some basic examples and prove the corollary of Grätzer and Kiss' theorem, see Corollary 2.4. The construction is due to Wild [89], who noticed that the technique used by Finkbeiner and Dilworth is actually matroid theory. He also noticed that their construction gives a different proof for the corollary of Grätzer and Kiss' theorem. In Section 2.2, we introduce the notion of *finite height* *generated lattices* and prove that they can be embedded isometrically into geometric lattices, see Theorem 2.8.

Notation for the chapter

Given a set S, we define a collection $\mathcal{L}$ of subsets of S to be a *complete lattice of subsets* of S if $\varnothing, S \in \mathcal{L}$ and $\mathcal{L}$ is closed under arbitrary intersection, cf. Crawley and Dilworth [13, Chapter 14]. That is, $\mathcal{L}$ is a complete meet-subsemilattice of the powersetlattice $(2^S; \bigcap, \bigcup)$. Note that a collection of subsets of S is closed under arbitrary intersection iff it is the lattice of closed sets of S with respect to an appropriate closure operator, see , e.g., Burris and Sankappanavar [9]. We will use this concept if we want to emphasize that the lattice $\mathcal{L}$ comes from a closure operator. If S is finite, we usually drop the adjective "complete" and say $\mathcal{L}$ is a lattice of subsets.

We will use $\cap$ resp. $\cup$ for set theoretical intersection resp. union and $\wedge, \vee$ for lattice operations. Sometimes, for example, if we have a complete lattice of subsets, $\wedge$ will coincide with $\cap$. In these cases, we will usually use $\cap$ in order to emphasize this coincidence. Let $(a]$ resp. $[a)$ denote the principal ideal resp. filter generated by a. For the sake of simplicity, sometimes we will write x instead of $\{x\}$, e.g., $X \cup x$ instead of $X \cup \{x\}$, if it is clear that X denotes a set and x denotes an element. $X - Y$ will denote the set theoretical difference of X and Y.

2.1 Motivation: the finite case

Throughout this section let L denote a *finite* lattice. A map $p: L \to \mathbb{N} = \{0, 1, \dots\}$ is called a *pseudorank function* if

(i) it preserves 0, i.e., $p(0) = 0$,
(ii) it is strictly monotone, i.e., $a < b$ implies $p(a) < p(b)$, and
(iii) it is *submodular*, i.e., $p(a \wedge b) + p(a \vee b) \le p(a) + p(b)$.

Example 2.1. Define $h: L \to \mathbb{N}$ to be the height function on L, that is, for $a \in L$, $h(a)$ is the *maximum* of lengths of chains in $[0, a]$. Now, if L is semimodular, that is $a \prec b$ implies $a \vee c \preceq b \vee c$ for all $a, b, c \in L$, then h is a pseudorank function on L, see, e.g., Grätzer [42, Theorem 375].

Example 2.2. Every finite lattice has a pseudorank function, cf. Crawley and Dilworth [13, Lemma 14.1.A] and Finkbeiner [32]. Let us recall Finkbeiner's example. Let p be the map $p: L \to \mathbb{N}$, $a \mapsto 2^{h(1)} - 2^{h(1)-h(a)}$, where h denotes the height function of L. Then p is a pseudorank function. Indeed, $h(0) = 0$ implies that $p(0) = 2^{h(1)} - 2^{h(1)-h(0)} = 2^{h(1)} - 2^{h(1)} = 0$, which shows that p preserves 0. If $a < b$ then $h(a) < h(b)$ implies $p(a) = 2^{h(1)} - 2^{h(1)-h(a)} < 2^{h(1)} - 2^{h(1)-h(b)}$, which shows that p is strictly monotone. Now, to prove the submodularity, let $a, b \in L$. If $a \leq b$ or $a \geq b$ then $p(a \wedge b) + p(a \vee b) = p(a) + p(b)$ trivially holds. Assume that $a \| b$. Then $a, b < a \vee b$, hence $2^{h(1)-h(a)}, 2^{h(1)-h(b)} \leq 2^{h(1)-h(a \vee b)}/2$. This implies $2^{h(1)-h(a)} + 2^{h(1)-h(b)} - 2^{h(1)-h(a \vee b)} \leq 0 \leq 2^{h(1)-h(a \wedge b)}$. Therefore

$$2^{h(1)-h(a)} + 2^{h(1)-h(b)} \leq 2^{h(1)-h(a \wedge b)} + 2^{h(1)-h(a \vee b)}.$$

The required $p(a \wedge b) + p(a \vee b) \leq p(a) + p(b)$ follows immediately, which shows that p is submodular.

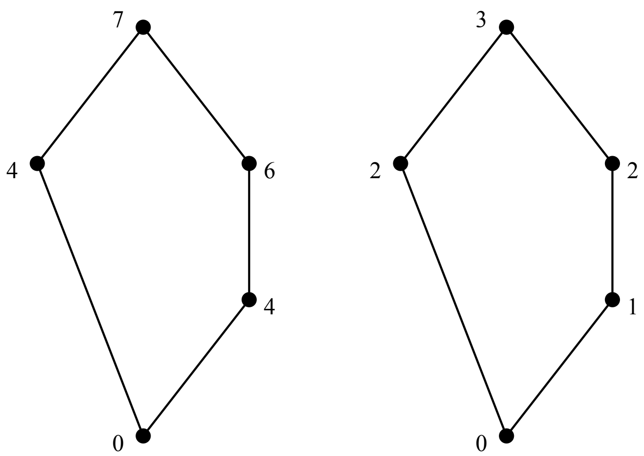

Figure 2.1: Finkbeiner's (left) and another pseudorank function on N_5

Given a finite lattice L with a pseudorank function p, an embedding φ of L into a finite semimodular lattice S is called *isometrical*, if $p = h \circ \varphi$, where h denotes the height function of S, cf. Grätzer and Kiss [43]. Since h is a pseudorank function of S, one can think of an isometrical embedding as "a lattice embedding that preserves the pseudorank function". Observe that if L is semimodular and p is the height function of L then the embedding φ of L into S is isometrical if and only if it preserves the covering relation, i.e., $a \prec b$ implies $\varphi(a) \prec \varphi(b)$ for all $a, b \in L$. Indeed, if L is semimodular then L satisfies the Jordan-Hölder Chain Condition, see, e.g., Grätzer [42, Theorem 374] or Stern [79, Theorem 1.9.1], hence maximal chains of intervals have the same length. It implies that, for any $a \leq b$, $a \prec b$ iff $p(b) = p(a) + 1$. Since S is semimodular, the same holds for S and h. Now, assume that φ is isometrical and let $a \prec b$. Then $h(\varphi(b)) = p(b) = p(a) + 1 = h(\varphi(a)) + 1$, hence $\varphi(a) \prec \varphi(b)$, which implies that φ is cover-preserving. On the other hand, if φ is cover-preserving then it is isometrical, since the covering relation determines the height function, and the Jordan-Hölder Chain Condition holds in S.

Recall that a *finite* geometric lattice is an atomistic, semimodular lattice, see, e.g., Grätzer [42]. In this section, we focus on the following two results.

Theorem 2.3 (Grätzer and Kiss [43, Theorem 3]). *Every finite lattice with a pseudorank function can be embedded isometrically into a geometric lattice.*

Corollary 2.4 (Grätzer and Kiss [43, Lemma 17]). *Every finite semimodular lattice has a cover-preserving embedding into a geometric lattice.*

We will show Wild's construction [89, Theorem 4], which proves Corollary 2.4 and helps better understand the proof of Theorem 2.8. We do not prove Theorem 2.3, since its proof can be obtained from Wild's construction, but it is more complicated, and the main goal of this section is to give some motivation for the next one. On the other hand, both Theorem 2.3 and Corollary 2.4 are special cases of Theorem 2.8 and Corollary 2.9. Before the proof, we need some elementary matroid theory.

2.1.1 Matroids

Matroids are finite structures. This concept is closely related to both linear algebra and graph theory. For example, given a finite subset S of vectors in a vector space, the linearly independent subsets of S form the "independent sets" of a matroid. On the other hand, given the edge set E of a finite graph, the subsets of E that are circuits in the graph form the "circuits" (or "minimal dependent sets") of a matroid. These two examples show that one can think of matroids as a generalization of vector spaces and graphs.

A characteristic feature of matroids is that they can be defined in many different ways: via independent sets, circuits, rank functions, and closure operators. To a certain extent, this property is responsible for the fact that matroid theory can be applied in many different ways. We need only the definition via rank functions and closure operators. For a detailed introduction to matroid theory, see, e.g., Oxley [71].

Given a finite set S, the map $r\colon 2^S \to \mathbb{N}$ is defined to be the *rank function of a matroid* if for any $A, B \subseteq S$,

(R1) $0 \le r(A) \le |A|$;

(R2) $A \subseteq B$ implies $r(A) \le r(B)$;

(R3) $r(A \cup B) + r(A \cap B) \le r(A) + r(B)$.

In this case we say that the pair (S, r) forms a matroid. If S is a finite subset of vectors and $A \subseteq S$ then the linear algebraic rank of A (the maximal number of linearly independent vectors in A) defines a rank function of a matroid. If S is the edge set of a finite graph then the map, which maps to each subset $A \subseteq S$ the maximal number of edges in A whose set contains no circuits, is a rank function of a matroid.

Given a finite set S, the map $\mathrm{cl}\colon 2^S \to 2^S$ is a *closure operator of a matroid* if for any $A, B \subseteq S$ and $a, b \in S$,

(CL1) $A \subseteq \mathrm{cl}(A)$;

(CL2) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

(CL3) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;

(CL4) $b \in \mathrm{cl}(A \cup a) - \mathrm{cl}(A)$ implies $a \in \mathrm{cl}(A \cup b)$.

In this case we say that the pair (S, cl) forms a matroid. A map is called extensive, monotone and idempotent, if it satisfies the first, second, and the third identity, respectively. The first three properties ensure that cl is a closure operator. The last property is the so-called *Exchange Property.*

Lemma 2.5.

(i) *If* (S, r) *forms a matroid then the map* $\mathrm{CL}(r)\colon 2^S \to 2^S$, $A \mapsto \{b \in S : r(A) = r(A \cup b)\}$ *is a closure operator of a matroid.*

(ii) *If* (S, cl) *forms a matroid then the set of closed sets forms a geometric lattice* $\mathcal{L}_{\mathrm{cl}}$. *Moreover, if* h *denotes the height function of* L *then the map* $\mathrm{R}(\mathrm{cl})\colon 2^S \to \mathbb{N}$, $A \mapsto h(\mathrm{cl}(A))$ *is the rank function of a matroid.*

(iii) $\mathrm{R}(\mathrm{CL}(r)) = r$ *and* $\mathrm{CL}(\mathrm{R}(\mathrm{cl})) = \mathrm{cl}$.

Proof. (i) Assume that (S, r) forms a matroid. The fact that cl is extensive follows from the definition. To prove that cl is monotone and idempotent, we need the following property. For any $A \subseteq B \subseteq S$ and $a \in S$,

$$r(A \cup a) = r(A) \text{ implies } r(B \cup a) = r(B). \tag{2.1}$$

If $a \in B$ then $r(B \cup a) = r(B)$ obviously holds. If $a \notin B$ then (R2) and (R3) implies $r(B) \le r(B \cup a) = r(A \cup B \cup a) \le r(A \cup a) + r(B) - r((A \cup a) \cap B) = r(A) + r(B) - r(A) =$

$r(B)$. Notice that (2.1) implies immediately that cl is monotone. To show that cl is idempotent, observe that, for any $A \subseteq S$, $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$, since cl is extensive. To establish the reverse direction, we need the following property. For any $A \subseteq S$ and $a_1, \dots, a_k \in \mathrm{cl}(A)$,

$$r(A \cup a_1 \cup \dots \cup a_k) = r(A). \tag{2.2}$$

We argue by induction on k. If $k = 1$ then (2.2) holds by the definition of cl. Assume that (2.2) holds for $1 \le k \le n$ and let $k = n+1$. Then, using the induction assumption and (2.1) for A, $B = A \cup \{a_1, \dots, a_n\}$ and $a = a_{n+1}$, we obtain (2.2) for $k = n+1$. Thus, by induction, (2.2) holds. Since S is finite, it follows immediately from (2.2) that

$$r(\mathrm{cl}(A)) = r(A). \tag{2.3}$$

Hence for any $a \in \mathrm{cl}(\mathrm{cl}(A))$, we have $r(A \cup a) \le r(\mathrm{cl}(A) \cup a) = r(\mathrm{cl}(A)) = r(A) \le r(A \cup a)$. Therefore $a \in \mathrm{cl}(A)$, which implies that $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$.

We saw that cl is a closure operator. To prove that cl satisfies the Exchange Property, we need the following fact. For any $A \subseteq S$ and $a \in S$,

$$r(A) \le r(A \cup a) \le r(A) + r(a) - r(A \cap a) \le r(A) + 1. \tag{2.4}$$

Now, suppose that $b \in \mathrm{cl}(A \cup a) - \mathrm{cl}(A)$ for some $A \subseteq S$ and $a, b \in S$. Then $r(A \cup a \cup b) = r(A \cup a)$ and $r(A \cup b) \ne r(A)$. From the last inequality and (2.4), we deduce that $r(A \cup b) = r(A) + 1$. Thus $r(A \cup a \cup b) = r(A \cup a) = r(A) + 1 = r(A \cup b)$, hence $a \in \mathrm{cl}(A \cup b)$.

(ii) Assume that (S, cl) forms a matroid. The closed sets trivially form a lattice: if $A, B \in \mathcal{L}_{\mathrm{cl}}$ then $A \wedge B = A \cap B \in \mathcal{L}_{\mathrm{cl}}$ and $A \vee B = \mathrm{cl}(A \cup B) = \bigcap(C \in \mathcal{L}_{\mathrm{cl}} : A \cup B \subseteq C)$. Since the sets $\mathrm{cl}(a)$ $(a \in S)$ are atoms in $\mathcal{L}_{\mathrm{cl}}$ and, for any $A \in \mathcal{L}_{\mathrm{cl}}$, $A = \bigcup(\mathrm{cl}(a) : a \in A)$, the lattice $\mathcal{L}_{\mathrm{cl}}$ is atomistic. To show that it is semimodular, let $A, B, C \in \mathcal{L}_{\mathrm{cl}}$, $A \prec B$. We have to show that $A \vee C \preceq B \vee C$. Picking any $b \in B - A$, $A \prec B$ implies $\mathrm{cl}(A \cup b) = B$. This yields $B \subseteq \mathrm{cl}(A \cup b \cup C)$. Thus we have $A \vee C = \mathrm{cl}(A \cup C) \subseteq \mathrm{cl}(A \cup b \cup C) = \mathrm{cl}(B \cup C) = B \vee C$. Now, the Exchange Property implies that there is not any closed sets between $\mathrm{cl}(A \cup C)$ and $\mathrm{cl}(A \cup b \cup C)$, that is, $\mathrm{cl}(A \cup C) \preceq \mathrm{cl}(A \cup b \cup C)$.

Let r be the map defined in (ii). Then (R1) and (R2) hold by definition. To prove (R3), let $A, B \subseteq S$. Then the definition of r and the fact that h is the height function of a geometric lattice imply $r(A \cup B) + r(A \cap B) = h(\mathrm{cl}(A \cup B)) + h(\mathrm{cl}(A \cap B)) \le h(\mathrm{cl}(A) \vee \mathrm{cl}(B)) + h(\mathrm{cl}(A) \cap \mathrm{cl}(B)) \le h(\mathrm{cl}(A)) + h(\mathrm{cl}(B)) = r(A) + r(B)$.

(iii) If (S, r) is a matroid then $\mathcal{L}_{\mathrm{CL}(r)}$ is a geometric lattice and r and the height function of $\mathcal{L}_{\mathrm{CL}(r)}$ coincide on the closed sets. This together with (2.3) yield

$\mathrm{R}(\mathrm{CL}(r)) = r$. Let (S, cl) be a matroid and $A \subseteq S$. Then $A \in \mathcal{L}_{\mathrm{cl}}$ if and only if for any $a \in S - A$, $\mathrm{R}(\mathrm{cl})(A \cup a) = h(\mathrm{cl}(A \cup a)) > h(\mathrm{cl}(A))$, where h denotes the height function of $\mathcal{L}_{\mathrm{cl}}$. This implies that the closed sets with respect to cl and the closed sets with respect to $\mathrm{CL}(\mathrm{R}(\mathrm{cl}))$ are the same. Thus $\mathrm{cl} = \mathrm{CL}(\mathrm{R}(\mathrm{cl}))$. □

2.1.2 Embeddings with matroids

Now, we are in position to show Wild's embedding with matroids [89, Theorem 4], which proves Corollary 2.4.

Recall that every finite lattice is isomorphic to a lattice of subsets. Moreover, the base set can be chosen to be the set of nonzero join-irreducible elements of L. An element $a \in L$ is *join irreducible* if for all $x, y \in L$, $a = x \vee y$ implies $a = x$ or $a = y$, see, e.g., Grätzer [42, Section I.6]. Let $J(L)$ denote the set of nonzero join-irreducible elements of L. For any element $a \in J(L)$, let a_0 denote its unique lower cover. For $x \in L$, let $\downarrow x = (x] \cap J(L)$. Then the set $\{\downarrow x : x \in L\}$ forms a closure system and the corresponding lattice of subsets $\mathcal{L}$ is isomorphic to L. Indeed, $L \to \mathcal{L}$, $x \mapsto \downarrow x$ defines an isomorphism. Assume that $L \cong \mathcal{L}$ is semimodular and let h denote the height function of L.

Lemma 2.6 (Wild [89, Lemma 3] and Welsh [84, Theorem 2 of Chapter 8])**.** *The map $r \colon 2^{J(L)} \to \mathbb{N}$, $A \mapsto \min\{h(\downarrow x) + |A - \downarrow x| : x \in L\}$ defines a rank function of a matroid.*

Proof. It is straightforward that r satisfies (R1) and (R2). To check (R3), observe that for any $A, B, X, Y \subseteq J(L)$,

$$|A - X| + |B - Y| \geq |(A \cup B) - (X \cup Y)| + |(A \cap B) - (X \cap Y)|,$$

hence

$$\begin{aligned} r(A) + r(B) &= \min_{x,y \in L} \{h(\downarrow x) + |A - \downarrow x| + h(\downarrow y) + |B - \downarrow y|\} \\ &\geq \min_{x,y \in L} \{h(\downarrow x \vee \downarrow y) + \\ &\quad + h(\downarrow x \cap \downarrow y) + |(A \cup B) - (\downarrow x \cup \downarrow y)| + |(A \cap B) - (\downarrow x \cap \downarrow y)|\} \\ &\geq \min_{x,y \in L} \{h(\downarrow x \vee \downarrow y) + \\ &\quad + h(\downarrow x \cap \downarrow y) + |(A \cup B) - (\downarrow x \vee \downarrow y)| + |(A \cap B) - (\downarrow x \cap \downarrow y)|\} \\ &\geq r(A \cup B) + r(A \cap B). \end{aligned}$$

□

Notice that the last proof uses the fact that L is semimodular, since h must be a pseudorank function. Otherwise the map r would not be necessarily a rank function of a matroid.

Lemma 2.7. *Let r be the rank function of the previous lemma and let* $\mathrm{cl} = \mathrm{CL}(r)$. *Then, for any $x \in L$, $r(\wr x) = h(\wr x)$ and $\mathrm{cl}(\wr x) = \wr x$.*

Proof. To show the first part of the lemma, observe that for any $x, y \in L$

$$|\wr x - \wr y| = |\wr x - (\wr x \cap \wr y)| \geq h(\wr x) - h(\wr x \cap \wr y) \geq h(\wr x) - h(\wr y), \tag{2.5}$$

hence

$$\begin{aligned} h(\wr x) &= h(\wr x) + |\wr x - \wr x| \\ &\geq r(\wr x) = \min_{y \in L}\{h(\wr y) + |\wr x - \wr y|\} \geq h(\wr y) + h(\wr x) - h(\wr y) = h(\wr x). \end{aligned}$$

To show the second part of the lemma, it is enough to prove that for any $x \in L$ and $a \in J(L) - \wr x$, $r(\wr x \cup a) > r(\wr x)$. Let $x \in L$ and $a \in J(L) - \wr x$. Observe that for any $y \in L$,

$$h(\wr x \vee \wr y) - h(\wr y) \leq |\wr x - \wr y|,$$

hence

$$\begin{aligned} r(\wr x) < h(\wr x) + 1 &\leq \min_{y \in L}\{h(\wr x \vee \wr y) + |(\wr x \cup a) - (\wr x \vee \wr y)|\} \\ &= \min_{y \in L}\{h(\wr y) + (h(\wr x \vee \wr y) - h(\wr y)) + |(\wr x \cup a) - (\wr x \vee \wr y)|\} \\ &\leq \min_{y \in L}\{h(\wr y) + |\wr x - \wr y| + |(\wr x \cup a) - (\wr x \vee \wr y)|\} \\ &\leq \min_{y \in L}\{h(\wr y) + |(\wr x \cup a) - \wr y|\} = r(\wr x \cup a). \end{aligned}$$

□

Proof of Corollary 2.4. Assume that L is a finite semimodular lattice. Let us identify L with the lattice $\mathcal{L}$ defined above, and let h be its height function. By Lemma 2.6 and 2.5(ii), we can define r, cl, and $\mathcal{L}_{\mathrm{cl}}$. Then $\mathcal{L}_{\mathrm{cl}}$ is a geometric lattice, whose height function is the restriction of r to it. We show that $\mathcal{L}$ is a cover-preserving sublattice of $\mathcal{L}_{\mathrm{cl}}$, that is, $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\mathrm{cl}}$ and two elements cover each other in $\mathcal{L}$ iff they cover each other in $\mathcal{L}_{\mathrm{cl}}$. By Lemma 2.7, $\mathcal{L} \subseteq \mathcal{L}_{\mathrm{cl}}$, and for any $x \in L$, $r(\wr x) = h(\wr x)$, which yields that $\mathcal{L}$ is a meet-subsemilattice of $\mathcal{L}_{\mathrm{cl}}$ and two elements cover each other in $\mathcal{L}$ iff they cover each other in $\mathcal{L}_{\mathrm{cl}}$.

To show that $\mathcal{L}$ is also a join-subsemilattice of $\mathcal{L}_{\mathrm{cl}}$, let $\wr a \vee \wr b$ denote the join of two elements in $\mathcal{L}_{\mathrm{cl}}$. It means that $\wr a \vee \wr b = \mathrm{cl}(\wr a \cup \wr b)$. We show

$$\wr a \vee \wr b = \wr(a \vee b) \tag{2.6}$$

by induction on $h(a)+h(b)$. If $h(a)+h(b) \le 1$ then (2.6) holds trivially. Assume that $k > 1$ and (2.6) holds if $h(a)+h(b) < k$. Let $a, b \in L$ such that $h(a)+h(b) = k$. We may assume that $h(a) > 0$ and a has a lower cover $c \prec a$. Then, by the semimodularity, $c \vee b \preceq a \vee b$. If $c \vee b = a \vee b$ then, by the induction hypothesis, $\wr a \vee \wr b \le \wr(a \vee b) = \wr(c \vee b) = \wr c \vee \wr b \le \wr a \vee \wr b$. If $c \vee b \prec a \vee b$ then, by the induction hypothesis, $\wr a \vee \wr b = \wr a \vee (\wr c \vee \wr b) = \wr a \vee \wr(c \vee b)$. Observe that $\wr a \subsetneq \wr(c \vee b)$, and $\wr(c \vee b) \prec \wr(a \vee b)$, since $\mathcal{L}$ is a meet-subsemilattice of $\mathcal{L}_{\mathrm{cl}}$ and $r(\wr(a \vee b)) = h(a \vee b) = h(c \vee b) + 1 = r(\wr(c \vee b)) + 1$. Thus $\wr a \vee \wr b = \wr a \vee \wr(c \vee b) = \wr(a \vee b)$. □

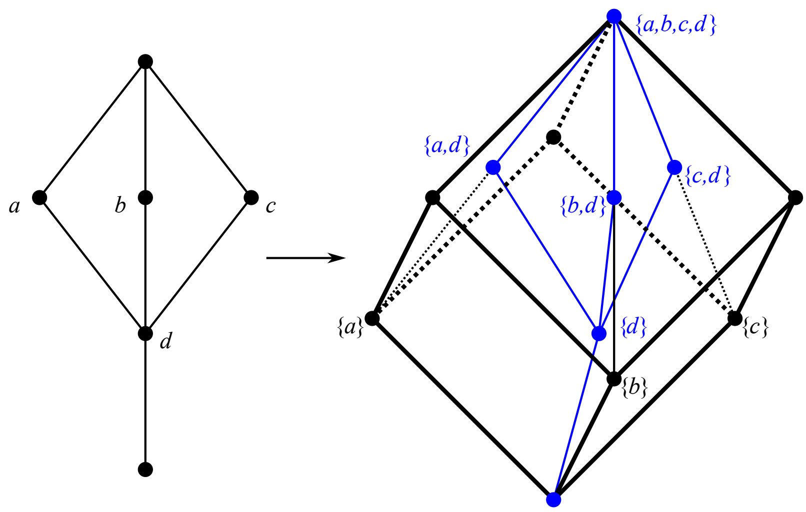

Figure 2.2: The cover-preserving embedding of L (left) into G (right)

Before we turn to the general case, let us show an example. Let L be the six element lattice of Figure 2.2. It is a modular lattice, thus it is also semimodular. We want to use Wild's construction to find a cover-preserving embedding of L into a geometric lattice. Now, $J(L) = \{a, b, c, d\}$, and the corresponding lattice of subsets is $\mathcal{L} = \{\varnothing, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}\}$. One can easily check that the rank function r, which is defined in Lemma 2.6, is:

$r(A)$	A
0	$\varnothing$
1	$\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$
2	$\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$
3	$\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{a,b,c,d\}$

Using Lemma 2.5, one can determine the closure operator $\mathrm{cl} = \mathrm{CL}(r)$. Indeed, every subset is closed except the three-element subsets. The corresponding geometric lattice is

$$G = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c,d\}\},$$

see Figure 2.2.

2.2 The general case

The matroid theoretical approach of Section 2.1 helps us understand the general proof. However, instead of using the toolkit of (infinite) matroid theory, we generalize the concepts of rank function and closure operator of a matroid. Thus we do not mention any possible concept of an infinite matroid. It seems less complicated, since the theory of infinite matroids is much more difficult than the theory of finite ones. Indeed, even the definition of infinite matroids is not clear, since there are various reasonable ways to define them, see Oxley [70, 72].

First, we need to generalize the concept of pseudorank function. For any semimodular lattice, we want the height function to be a pseudorank function, cf. Example 2.1. Given a lattice L with a lower bound 0, the height of an element $a \in L$ is defined to be the *supremum* of lengths of chains in $[0,a]$. Let a function $p\colon L \to \mathbb{N}_\infty = \{0,1,\dots,\infty\}$ be called a *pseudorank function* if it has the following properties:

(i) $p(0) = 0$;

(ii) $a \le b$ implies $p(a) \le p(b)$ for all $a,b \in L$;

(iii) $a < b$ implies $p(a) < p(b)$ for all $a,b \in L$ of finite height;

(iv) $p(a \wedge b) + p(a \vee b) \le p(a) + p(b)$ for all $a,b \in L$;

(v) $p(a) < \infty$ iff a is of finite height.

Note that if L is finite, the new definition of pseudorank function coincides the old one. It is an easy consequence of the Jordan-Hölder Chain Condition, see, e.g., Stern [79, Theorem 1.9.1], that in any semimodular lattice, the elements of finite height form a sublattice. Indeed, let S be a semimodular lattice with 0 and let $x,y \in S$ be elements of finite height, say $h(x), h(y) < \infty$. Then any maximal chain of $[0, x\wedge y]$ can be extended to a maximal chain of $[0,x]$, whose lengths are at most $h(x)$, hence $h(x \wedge y) < \infty$. On the other hand, if $0 \prec x_1 \prec \cdots \prec x_k = x$ is a maximal chain of $[0,x]$ then, by the semimodularity, $y \preceq x_1 \vee y \preceq \cdots \preceq x_k \vee y = x \vee y$ is a maximal chain of $[y, x \vee y]$. Together with a maximal chain of $[0, y]$, it shows that $[0, x \vee y]$ contains a finite maximal chain. Now, the Jordan-Hölder Chain Condition implies that all maximal chains of $[0, x \vee y]$ are finite, moreover, they have the same length, hence $h(x \vee y) < \infty$. This shows that in any semimodular lattice, the elements of finite height form a sublattice. Hence the height function of an *arbitrary* semimodular lattice is a pseudorank function.

Again, for a lattice L with a pseudorank function p, an embedding φ of L into a semimodular lattice S is called *isometrical*, if $p = h \circ \varphi$, where h denotes the height function of S. Recall that a geometric lattice is an atomistic, semimodular *algebraic* lattice, see, e.g., Grätzer [42].

To formulate the general statements corresponding to Theorem 2.3 and Corollary 2.4, we need the concept of a finite height generated lattice. A lattice is said to be *finite height generated* if it is complete and every element is the join of some elements of finite height. Note that lattices of finite length are finite height generated. To show a finite height generated lattice that is not of finite length, consider, for instance, $\mathbb{N}_\infty$ with the usual ordering.

Theorem 2.8. *Every finite height generated algebraic lattice with a pseudorank function can be embedded isometrically into a geometric lattice.*

Corollary 2.9. *Every finite height generated semimodular algebraic lattice has a cover-preserving embedding into a geometric lattice.*

2.2.1 Basic concepts and lemmas

We say that a pseudorank function r on a complete lattice $\mathcal{L}$ of subsets of S is a *rank function* if

$$r(A) - r(B) \leq |A - B| \quad \text{for all } A, B \in \mathcal{L} \text{ of finite height.} \tag{2.7}$$

Note that this concept differs from that of rank function of a matroid. However, they are very close to each other. Notice that if $B = \varnothing$ then (2.7) and (R1) are similar. Also note that if S is finite, our rank function on a complete lattice of subsets of S is a strictly increasing rank function in sense of P. Crawley and R.P. Dilworth [13].

For every finite height generated lattice with a pseudorank function, we are going to construct a complete lattice of subsets, which is isomorphic to the lattice and on which the pseudorank function becomes a rank function. Note that our construction is an extension of that of P. Crawley and R.P. Dilworth [13, Lemma 14.1.B].

For the rest of this subsection, let us fix a finite height generated lattice L with a pseudorank function p. Note that join-irreducible elements are defined the same way in general as we did in the finite case, cf. Grätzer [42, Section I.6]. Let $J \subseteq J(L)$ denote the set of nonzero join-irreducible elements of finite height. Recall that, for $f \in J$, f_0 denotes the unique lower cover of f. The set J is a poset with respect to the restriction of the partial ordering of L. Since L is finite height generated, the set $\{(a] \cap J : a \in L\}$ of order-ideals of J forms a complete lattice of subsets that is isomorphic to L. However, (2.7) does not necessarily hold. To avoid this problem, we need sufficiently many elements in the ground set of the required complete lattice of subsets.

Let $\{X_f : f \in J\}$ be a collection of pairwise disjoint sets such that $|X_f| = p(f) - p(f_0)$. Set $S = \bigcup(X_f : f \in J)$. For each $a \in L$, define $\wr a = \bigcup(X_f : f \in J, f \leq a)$. Then $\wr 0 = \varnothing$, $\wr 1 = S$ and $\wr(\bigwedge A) = \bigcap(\wr a : a \in A)$ for all $A \subseteq L$. Consequently, the collection $\mathcal{L} = \{\wr a : a \in L\}$ forms a complete lattice of subsets of S. Since L is finite height generated, the map $\varphi : L \to \mathcal{L}, a \mapsto \wr a$ is an isomorphism. For each $a \in L$, define $r(\wr a) = p(a)$.

Lemma 2.10. *The above defined r is a rank function on $\mathcal{L}$ with $p = r \circ \varphi$.*

Proof. Certainly, r is a pseudorank function. This fact will be used hereafter without further reference. To prove (2.7), observe that it suffices to show, that

$$r(\wr a) - r(\wr b) \leq |\wr a - \wr b| \quad \text{for all } a, b \in L, a \geq b \text{ of finite height.} \tag{2.8}$$

Indeed, if (2.8) holds then for any $c, d \in L$ of finite height, we have $r(\wr c) - r(\wr d) \leq r(\wr c) - r(\wr c \cap \wr d) \leq |\wr c - (\wr c \cap \wr d)|$ and $\wr c - (\wr c \cap \wr d) = \wr c - \wr d$.

We prove (2.8) by induction on $h(a)$, where h denotes the height function of L. Let $a, b \in L, a \geq b$ be arbitrary elements of finite height. The case $h(a) = 0$ is trivial. Suppose that $h(a) > 0$.

If $a = b$ then (2.8) holds trivially. If $a > b$ and a is join-irreducible then $b = a_0$ and (2.8) holds by definition. If $a > b$ and a is not join-irreducible then there exists an element $f \in J$ such that $a = b \vee f$ and $f < a$. Using the induction hypothesis and the submodularity of r, we obtain $r(\wr a) - r(\wr b) \leq r(\wr f) - r(\wr b \cap \wr f) \leq |\wr f - (\wr b \cap \wr f)| \leq |\wr a - \wr b|$.

If $a > b$ and $a \not\succ b$ then there is an element $c \in L$ such that $b < c \prec a$. Hence the induction hypothesis and the previous paragraph yields $r(\wr a) - r(\wr b) = r(\wr a) - r(\wr c) + r(\wr c) - r(\wr b) \leq |\wr a - \wr c| + |\wr c - \wr b| = |\wr a - \wr b|$. ☐

Observe that if L is semimodular and p is the height function of L then $|X_f| = 1$ for all $f \in J$. Thus X_f can be chosen to be $\{f\}$ and $S = J$. Besides, if L is also finite then $\wr a$ denotes exactly the same set here and in Section 2.1. Consequently, the corresponding (complete) lattices of subsets coincide as well.

Let $F \subseteq L$ denote the set of elements of finite height. Since L has a pseudorank function, F is a sublattice. Notice that

$$\text{for any finite } A \subseteq S \text{ there is } x \in F \text{ such that } A \subseteq \wr x. \tag{2.9}$$

For each $x \in F$, we define r_x to be the map

$$r_x: 2^S \to \mathbb{N} = \{0, 1, \dots\}, \quad A \mapsto \min\{r(\wr y) + |(A \cap \wr x) - \wr y| : y \in F\}.$$

Observe that the above definition is an extension of the one in Lemma 2.6. Given a set $A \subseteq S$ and an element $x \in F$, we say that $y \in F$ *represents* $r_x(A)$ if $r_x(A) = r(\wr y) + |(A \cap \wr x) - \wr y|$. Some important properties of r_x can be found in the following statements. Some of them might be familiar from Lemmas 2.6 and 2.7.

Lemma 2.11.

(i) $r_x(A) = \min\{r(\wr y) + |(A \cap \wr x) - \wr y| : y \in [0, x]\}$ *for all* $A \subseteq S$.

(ii) $0 \le r_x(A) = r_x(A \cap \wr x) \le \min\{|A \cap \wr x|, r(\wr x)\}$ *for all* $A \subseteq S$.

(iii) $A \subseteq B$ *implies* $r_x(A) \le r_x(B)$ *for all* $A, B \subseteq S$.

(iv) $r_x(A) = r_y(A)$ *for all* $A \subseteq S$ *and* $x, y \in F$ *satisfying* $A \subseteq \wr x \cap \wr y$.

(v) *If* $x \ge y$ *then* $r_x(\wr y) = r(\wr y)$ *for all* $x, y \in F$.

Proof. The first four statements follow easily from the definition. To prove (v), let $x \ge y$ be elements of F. By (iv) and (ii), we have $r_x(\wr y) = r_y(\wr y) \le r(\wr y)$. To prove the opposite direction, let $u \in F$ represent $r_y(\wr y)$. Then by (2.7), we obtain $r_y(\wr y) = r(\wr u) + |\wr y - \wr u| \ge r(\wr u) + r(\wr y) - r(\wr u) = r(\wr y)$. □

Lemma 2.12. *Let* $A \subseteq B \subseteq S$ *and* $x \in F$. *Suppose that* $u \in F$ *represents* $r_x(A)$ *and* $v \in F$ *represents* $r_x(B)$. *Then* $u \wedge v$ *represents* $r_x(A)$ *and* $u \vee v$ *represents* $r_x(B)$, *that is*

$$r_x(A) = r(\wr u \cap \wr v) + |(A \cap \wr x) - (\wr u \cap \wr v)| \quad \textit{and} \tag{2.10}$$

$$r_x(B) = r(\wr u \vee \wr v) + |(B \cap \wr x) - (\wr u \vee \wr v)|. \tag{2.11}$$

Proof. First, we need some elementary calculations.

$$\begin{aligned}|(A\cap \wr x)-(\wr u\cap \wr v)| = |(A\cap \wr x)-\wr u| + |(A\cap \wr x\cap \wr u)-\wr v| \leq\\ \leq |(A\cap \wr x)-\wr u| + |(B\cap \wr x\cap \wr u)-\wr v| =\\ = |(A\cap \wr x)-\wr u| + |(B\cap \wr x)-\wr v| - |(B\cap \wr x)-(\wr u\cup \wr v)| \leq\\ \leq |(A\cap \wr x)-\wr u| + |(B\cap \wr x)-\wr v| - |(B\cap \wr x)-(\wr u\vee \wr v)|.\end{aligned}$$

Now, using the definition of r_x, the submodularity of r and the above calculations, we obtain the following inequalities

$$\begin{aligned}r_x(A) &\leq r(\wr u\cap \wr v) + |(A\cap \wr x)-(\wr u\cap \wr v)| \leq\\ &\leq r(\wr u)+r(\wr v)-r(\wr u\vee \wr v) + |(A\cap \wr x)-(\wr u\cap \wr v)| \leq\\ &\leq r(\wr u)+r(\wr v)-r(\wr u\vee \wr v)+\\ &+|(A\cap \wr x)-\wr u| + |(B\cap \wr x)-\wr v| - |(B\cap \wr x)-(\wr u\vee \wr v)| =\\ &= r_x(A) + \underline{r_x(B) - r(\wr u\vee \wr v) - |(B\cap \wr x)-(\wr u\vee \wr v)|} \leq r_x(A).\end{aligned}$$

Therefore the above inequalities are equalities. Thus the underlined part is zero, which gives (2.11), while (2.10) is the first inequality. □

Corollary 2.13. *For any $A\subseteq S$ and $x\in F$, there exists a smallest and a largest element in $[0,x]$ that represents $r_x(A)$.*

Lemma 2.14. *Let $A\subseteq S$ and $a\in S$. If $r_x(A\cup a) = r_x(A)$ and $y\in F$ represents $r_x(A\cup a)$ then the following hold:*

(i) *y also represents $r_x(A)$ and*

(ii) *$a\notin \wr x$ or $a\in \wr y$.*

Proof. $r_x(A\cup a) = r(\wr y) + |((A\cup a)\cap \wr x)-\wr y| \geq r(\wr y) + |(A\cap \wr x)-\wr y| \geq r_x(A) = r_x(A\cup a)$ implies that y also represents $r_x(A)$ and $|((A\cup a)\cap \wr x)-\wr y| = |(A\cap \wr x)-\wr y|$. Hence $a\notin \wr x$ or $a\in \wr y$. □

Lemma 2.15. *Let $A\subseteq S$ and $B=\{b\in S: r_x(A\cup b) = r_x(A)\}$. Then $r_x(B) = r_x(A)$.*

Proof. By the monotonicity of r_x, that is Lemma 2.11(iii), we know that $r_x(A)\leq r_x(B)$. By Corollary 2.13, there exists a largest element $y\in[0,x]$ that represents $r_x(A)$. Let $b\in B-A$ be an arbitrary element. Let $z\in[0,x]$ represent $r_x(A\cup b)$. Then Lemma 2.14(i) implies that z represents $r_x(A)$, and Lemma 2.14(ii) implies that $b\notin \wr x$ or $b\in \wr z$. We also have $\wr z\subseteq \wr y$, because y is the largest element that represents $r_x(A)$. Hence $b \notin \lightning x$ or $b \in \lightning y$. Consequently, $(B - A) \cap \lightning x \subseteq \lightning y$, which yields that

$$r_x(A) = r(\lightning y) + |(A \cap \lightning x) - \lightning y| = r(\lightning y) + |(B \cap \lightning x) - \lightning y| \geq r_x(B). \qquad \square$$

We saw in Section 2.1 that finite geometric lattices and closure operators of matroids are closely related. In general, let $\mathrm{cl}: S \to S$ be a closure operator. We say that cl is *algebraic*, if for any set $A \subseteq S$ and any element $a \in \mathrm{cl}(A)$ there is a finite subset $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0)$. The Exchange Property is defined the same way in general as we did in the finite case. Now, for an algebraic closure operator that satisfies the Exchange Property, the lattice of closed sets forms a geometric lattice. Indeed, it is algebraic, since the closure operator is algebraic. It is atomistic, since the Exchange Property ensures that closures of one element sets are atoms, and every closed set is the join of the closures of its one element subsets. Finally, the semimodularity follows from the Exchange Property.

Although we do not use it, let us mention the fact that every geometric lattice can be obtained from an appropriate algebraic closure operator that satisfies the Exchange Property, see, e.g., Grätzer [42, Section V.3].

Using r_x, we define two kinds of closure operators on S: cl_x for each $x \in F$ and cl. Namely, for any $A \subseteq F$,

$$\begin{aligned} \mathrm{cl}_x(A) &= \{a \in S : r_y(A \cup a) = r_y(A) \text{ for all } y \in F \cap [x)\}, \\ \mathrm{cl}(A) &= \bigcup \{\mathrm{cl}_y(A) : y \in F\}. \end{aligned}$$

Notice that

$$\mathrm{cl}_x(A) \subseteq \mathrm{cl}_y(A) \text{ if } x \leq y. \tag{2.12}$$

Lemma 2.16. *The functions* $\mathrm{cl}_x: 2^S \to 2^S, A \mapsto \mathrm{cl}_x(A)$ *and* $\mathrm{cl}: 2^S \to 2^S, A \mapsto \mathrm{cl}(A)$ *are algebraic closure operators. Moreover,* cl *satisfies the Exchange Property.*

Proof. The extensivity of cl_x is immediate from the definition. To prove the monotonicity, let $A \subseteq B \subseteq S$. By the definition of cl_x, it is enough to prove that $r_y(B \cup a) = r_y(B)$ for all $a \in \mathrm{cl}_x(A)$ and all $y \in F \cap [x)$. Suppose indirectly that there are elements $a \in \mathrm{cl}_x(A)$ and $y \in F \cap [x)$ such that $r_y(B \cup a) > r_y(B)$. Then $r_y(B \cup a) = r_y(B) + 1$. By Corollary 2.13, there exists a smallest element $u \in [0, y]$ that represents $r_y(A \cup a)$. Let $v \in [0, y]$ represent $r_y(B)$. Then v also represents $r_y(B \cup a)$, thus $a \in \lightning y$ and $a \notin \lightning v$ must hold. Using Lemma 2.12 for $A \cup a \subseteq B \cup a$, we obtain that $u \wedge v$ represents $r_y(A \cup a)$. Then $u \leq u \wedge v$ gives $\lightning u \subseteq \lightning v$. By

Lemma 2.14(ii) for $r_y(A \cup a)$ and u, we have that $a \notin \between y$ or $a \in \between u \subseteq \between v$, which contradicts the fact that $a \in \between y$ and $a \notin \between v$. Consequently, cl_x is monotone.

To prove that cl_x is idempotent, let $A \subseteq S$. For any $y \in F \cap [x)$, we have $A \subseteq \mathrm{cl}_x(A) \subseteq B_y = \{b \in S : r_y(A \cup b) = r_y(A)\}$. By Lemma 2.15 and the monotonicity of r_y, we also have $r_y(A) = r_y(\mathrm{cl}_x(A)) = r_y(B_y)$. Now, for any $a \in \mathrm{cl}_x(\mathrm{cl}_x(A))$ and any $y \in F \cap [x)$,

$$r_y(A) \le r_y(A \cup a) \le r_y(\mathrm{cl}_x(A) \cup a) = r_y(\mathrm{cl}_x(A)) = r_y(A).$$

Hence $a \in \mathrm{cl}_x(A)$ and $\mathrm{cl}_x(\mathrm{cl}_x(A)) \subseteq \mathrm{cl}_x(A)$. The other direction follows immediately from the extensivity of cl_x. Consequently, cl_x is idempotent. We conclude that cl_x is a closure operator.

To prove that cl_x is algebraic, let $A \subseteq S$ and $a \in \mathrm{cl}_x(A)$. Let $y = \bigwedge\{z \in F \cap [x) : a \in \between z\}$. By (2.9), we obtain $y \in F$. First, let $A_0 \subseteq A \cap \between y$ be a finite subset such that $r_y(A_0)$ is maximal. Then $r_y(A_0) = r_y(A)$. Indeed, the maximality of $r_y(A_0)$ implies that $r_y(A_0 \cup b) = r_y(A_0)$ for all $b \in A$. Using Lemma 2.15 for $B = \{b \in S : r_y(A_0 \cup b) = r_y(A_0)\}$, we obtain $r_y(A_0) = r_y(B)$. Therefore $r_y(A_0) = r_y(A) = r_y(B)$ by the monotonicity of r_y. Now, we have $r_y(A_0) \le r_y(A_0 \cup a) \le r_y(A \cup a) = r_y(A) = r_y(A_0)$, hence $r_y(A_0 \cup a) = r_y(A_0)$. Finally, let $z \in F \cap [x)$. If $a \notin \between z$ then $r_z(A_0) = r_z(A_0 \cup a)$ trivially holds. If $a \in \between z$ then $y \le z$ by the definition of y. Using Lemma 2.11(iv) for $A_0 \cup a \subseteq \between y \subseteq \between z$, we obtain $r_z(A_0) = r_y(A_0) = r_y(A_0 \cup a) = r_z(A_0 \cup a)$. Hence $a \in \mathrm{cl}_x(A_0)$. We conclude that cl_x is an algebraic closure operator.

The extensivity and monotonicity of cl follow immediately from those of cl_x. To prove the idempotency of cl, let $A \subseteq S$ and suppose that $a \in \mathrm{cl}(\mathrm{cl}(A))$. By definition, $a \in \mathrm{cl}_x(\mathrm{cl}(A))$ for some $x \in F$. Since cl_x is algebraic, there is a finite subset $A_0 \subseteq \mathrm{cl}(A)$ such that $a \in \mathrm{cl}_x(A_0)$. By (2.12) and the definition of cl, $A_0 \subseteq \mathrm{cl}_y(A)$ for some $y \in F$. By (2.12) and the monotonicity and idempotency of cl_x, we have $a \in \mathrm{cl}_x(\mathrm{cl}_y(A)) \subseteq \mathrm{cl}_{x\vee y}(\mathrm{cl}_{x\vee y}(A)) = \mathrm{cl}_{x\vee y}(A) \subseteq \mathrm{cl}(A)$. Hence $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$. The other direction follows immediately from the extensivity of cl. Consequently, cl is a closure operator. It is algebraic since cl_x is algebraic for all $x \in F$.

To prove that cl satisfies the Exchange Property, let $A \subseteq S$ and $a, b \in S$ such that $a \in \mathrm{cl}(A \cup b) - \mathrm{cl}(A)$. Since cl is algebraic, there is a finite subset $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0 \cup b) - \mathrm{cl}(A_0)$. Hence $a \in \mathrm{cl}_x(A_0 \cup b) - \mathrm{cl}_x(A_0)$ for some $x \in F$. By (2.9) and (2.12), we can assume that $A_0 \cup \{a, b\} \subseteq \between x$. By the definition of cl_x, there are $u, v \in F \cap [x)$ such that $r_u(A_0 \cup a) = r_u(A_0) + 1$ and $r_v(A_0 \cup b) = r_v(A_0) + 1$, since $a, b \notin \mathrm{cl}_x(A_0)$. Using this and Lemma 2.11(iv) for $A_0 \cup \{a, b\} \subseteq \between x$, we obtain that $r_y(A_0 \cup a) = r_y(A_0 \cup b) = r_y(A_0) + 1$ for all $y \in F \cap [x)$. The assumption $a \in \mathrm{cl}_x(A_0 \cup b)$ implies $r_y(A_0 \cup \{a, b\}) = r_y(A_0 \cup b) = r_y(A_0 \cup a)$ for all $y \in F \cap [x)$. Now, $b \in \mathrm{cl}_x(A_0 \cup a) \subseteq \mathrm{cl}(A \cup a)$ follows immediately. Hence cl satisfies the Exchange Property. □

2.2.2 The main proofs

Before the proof of Theorem 2.8, we need a short technical lemma about finite height generated *algebraic* lattices.

Lemma 2.17. *If L is a finite height generated algebraic lattice and the elements of finite height form a sublattice then its elements of finite height are exactly its compact elements.*

Proof. Suppose that $a \in L$ is compact. Then $a = \bigvee B$ for some elements $B \subseteq L$ of finite height, because L is finite height generated. Since a is compact, there is a finite $B_0 \subseteq B$ with $a = \bigvee B_0$. Hence a is of finite height. Now, suppose that $b \in L$ is of finite height. Then $b = \bigvee A$ for some compact elements $A \subseteq L$, because L is algebraic. Since b is of finite height, there is a finite $A_0 \subseteq A$ with $b = \bigvee A_0$. Hence b is compact. □

Proof of Theorem 2.8. Given a finite height generated algebraic lattice L with a pseudorank function p, define $\mathcal{L}$ and r as we did in Subsection 2.2.1. We will also use S for the ground set of $\mathcal{L}$ and $F \subseteq L$ for the set of elements of finite height. Recall that F is a sublattice, since L has a pseudorank function. Denote $\mathcal{L}_{\mathrm{cl}}$ the complete lattice of subsets that corresponds to the closure operator cl. By Lemma 2.16, $\mathcal{L}_{\mathrm{cl}}$ is a geometric lattice. It is enough to prove that $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\mathrm{cl}}$ such that r and the height function of $\mathcal{L}_{\mathrm{cl}}$ coincide on $\mathcal{L}$. Then $\overline{\varphi}\colon L \to \mathcal{L}_{\mathrm{cl}}, x \mapsto \downarrow x$ is an isometrical embedding.

First, we show that $\mathcal{L} \subseteq \mathcal{L}_{\mathrm{cl}}$. Let $x \in L$ and $a \in S - \downarrow x$. Suppose, for a contradiction, that $a \in \mathrm{cl}(\downarrow x)$. Then, by definition, there is a $y \in F$ with $a \in \mathrm{cl}_y(\downarrow x)$. By (2.9) and (2.12), we can assume that $a \in \downarrow y$. Notice that $a \in \mathrm{cl}_y(\downarrow x)$ implies that $r_y(\downarrow x \cup a) = r_y(\downarrow x)$. Let $z \in F$ represent $r_y(\downarrow x \cup a)$. Then Lemma 2.14(ii) and $a \in \downarrow y$ implies that $a \in \downarrow z$. Since $a \in \downarrow z - \downarrow x$, we have $\downarrow z \neq \downarrow x \cap \downarrow z = \downarrow(x \wedge z)$, hence $x \wedge z < z$. However, $(\downarrow x \cap \downarrow y) - \downarrow(x \wedge z) = (\downarrow x \cap \downarrow y) - (\downarrow x \cap \downarrow z) = (\downarrow x \cap \downarrow y) - \downarrow z = ((\downarrow x \cup a) \cap \downarrow y) - \downarrow z$. Since r is strictly monotone for elements of finite height, we obtain that $r_y(\downarrow x) \leq r(\downarrow(x \wedge z)) + |(\downarrow x \cap \downarrow y) - \downarrow(x \wedge z)| < r(\downarrow z) + |(\downarrow x \cap \downarrow y) - \downarrow(x \wedge z)| =$

$r(\downarrow z) + |((\downarrow x \cup a) \cap \downarrow y) - \downarrow z| = r_y(\downarrow x \cup a)$, which contradicts $r_y(\downarrow x) = r_y(\downarrow x \cup a)$. This proves that $\downarrow x \in \mathcal{L}_{\text{cl}}$ and $\mathcal{L} \subseteq \mathcal{L}_{\text{cl}}$. Moreover, the meet operation both on $\mathcal{L}$ and $\mathcal{L}_{\text{cl}}$ is the intersection, therefore $\mathcal{L}$ is a meet-subsemilattice of $\mathcal{L}_{\text{cl}}$.

To prove that $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\text{cl}}$, observe that the join of two elements $\downarrow x, \downarrow y \in \mathcal{L}$ in the larger lattice $\mathcal{L}_{\text{cl}}$ is $\text{cl}(\downarrow x \cup \downarrow y)$. Since $\text{cl}(\downarrow x \cup \downarrow y) \subseteq \downarrow(x \vee y)$, it is enough to show that $\text{cl}(\downarrow x \cup \downarrow y) \supseteq \downarrow(x \vee y)$. Let $a \in \downarrow(x \vee y)$. By definition, it means that $a \in X_b$ for some $b \in J \cap (x \vee y]$. Since L is finite height generated, $x = \bigvee X$ and $y = \bigvee Y$ for some $X, Y \subseteq F$. By Lemma 2.17, b is compact, hence $b \le \bigvee X_0 \vee \bigvee Y_0$ for some finite $X_0 \subseteq X$, $Y_0 \subseteq Y$. Let $x_0 = \bigvee X_0$ and $y_0 = \bigvee Y_0$. By Lemma 2.17, $x_0, y_0 \in F$. Now, $b \le x_0 \vee y_0$ implies $a \in \downarrow(x_0 \vee y_0)$. In order to prove that $a \in \text{cl}(\downarrow x \cup \downarrow y)$, it is enough to show that $a \in \text{cl}(\downarrow x_0 \cup \downarrow y_0)$. We prove that $a \in \text{cl}_{x_0 \vee y_0}(\downarrow x_0 \cup \downarrow y_0)$, which yields $a \in \text{cl}(\downarrow x_0 \cup \downarrow y_0)$. As a preparation for this, we show that

$$r_z(\downarrow x_0 \cup \downarrow y_0) = r_z(\downarrow(x_0 \vee y_0)) \text{ for all } z \in F \cap [x_0 \vee y_0). \tag{2.13}$$

Since $r_z(\downarrow x_0) = r(\downarrow x_0)$ by Lemma 2.11(v), x_0 represents $r_z(\downarrow x_0)$. Similarly, y_0 represents $r_z(\downarrow y_0)$. Assume that z_0 represents $r_z(\downarrow x_0 \cup \downarrow y_0)$. Using Lemma 2.12 twice for $\downarrow x_0 \subseteq \downarrow x_0 \cup \downarrow y_0$ and $\downarrow y_0 \subseteq \downarrow x_0 \cup \downarrow y_0$, we obtain that $x_0 \vee y_0 \vee z_0$ represents $r_z(\downarrow x_0 \cup \downarrow y_0)$. However, $\downarrow x_0 \cup \downarrow y_0 \subseteq \downarrow(x_0 \vee y_0 \vee z_0)$, hence $r_z(\downarrow x_0 \cup \downarrow y_0) = r(\downarrow(x_0 \vee y_0 \vee z_0)) + |((\downarrow x_0 \cup \downarrow y_0) \cap \downarrow z) - \downarrow(x_0 \vee y_0 \vee z_0)| = r(\downarrow(x_0 \vee y_0 \vee z_0))$. On the other hand, $\downarrow x_0 \cup \downarrow y_0 \subseteq \downarrow(x_0 \vee y_0)$, which implies $r_z(\downarrow x_0 \cup \downarrow y_0) \le r(\downarrow(x_0 \vee y_0)) + |((\downarrow x_0 \cup \downarrow y_0) \cap \downarrow z) - \downarrow(x_0 \vee y_0)| = r(\downarrow(x_0 \vee y_0)) \le r(\downarrow(x_0 \vee y_0 \vee z_0))$. Together with $r(\downarrow(x_0 \vee y_0 \vee z_0)) = r_z(\downarrow x_0 \cup \downarrow y_0)$, we obtain that $r_z(\downarrow x_0 \cup \downarrow y_0) = r(\downarrow(x_0 \vee y_0))$. Lemma 2.11(v) yields that $r(\downarrow(x_0 \vee y_0)) = r_z(\downarrow(x_0 \vee y_0))$, which finishes the proof of (2.13). Now, $\downarrow x_0 \cup \downarrow y_0 \subseteq \downarrow x_0 \cup \downarrow y_0 \cup a \subseteq \downarrow(x_0 \vee y_0)$, the monotonicity of r_z and (2.13) implies that $r_z(\downarrow x_0 \cup \downarrow y_0 \cup a) = r_z(\downarrow x_0 \cup \downarrow y_0)$ for all $z \in F \cap [x_0 \vee y_0)$. Hence $a \in \text{cl}_{x_0 \vee y_0}(\downarrow x_0 \cup \downarrow y_0)$. We conclude that $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\text{cl}}$.

To prove that the embedding is isometrical, we have to show that $r(\downarrow x) = h(\downarrow x)$ for all $x \in L$, where h denotes the height function of $\mathcal{L}_{\text{cl}}$. By the definition of finite height generated lattices, it suffices to prove that $r(\downarrow x) = h(\downarrow x)$ for all $x \in F$. We use induction on the height of $x \in F$. If $x = 0$ then $r(\downarrow 0) = 0 = h(\downarrow 0)$.

Suppose that $0 \le x \prec y$ and $r(\downarrow x) = h(\downarrow x)$. Since r is a rank function on $\mathcal{L}$, we have a set $A = \{a_1, \dots, a_{k-1}, a_k\} \subseteq \downarrow y - \downarrow x$ with $k = r(\downarrow y) - r(\downarrow x)$ distinct elements. Let $A_0 = \downarrow x$ and $A_i = \downarrow x \cup a_1 \cup \dots \cup a_i$ for all $i \in \{1, \dots, k\}$. Assume that $z \in F \cap [y)$. Clearly, $r_z(A_i) \le r(\downarrow x) + i$. By Lemma 2.11(v), $r_z(\downarrow x) = r(\downarrow x)$. Hence $r_z(A_i) \le r_z(\downarrow x) + i$. The opposite direction is also true. Suppose, for a contradiction, that $r_z(A_i) < r(\downarrow x) + i$ for some $i \in \{1, \ldots, k\}$. Let i be minimal for this property, that is $r_z(A_i) < r(\downarrow x) + i$ and $r_z(A_{i-1}) = r(\downarrow x) + i - 1$. Note that such i exists, since $r_z(A_0) = r(\downarrow x)$. Then, by the monotonicity of r_z, $r_z(A_i) = r(\downarrow x) + i - 1 = r_z(A_{i-1})$. Let u_i represent $r_z(A_i)$. Applying Lemma 2.14(ii) for $r_z(A_i)$ and u_i, we obtain that $a_i \in \downarrow u_i$, since $a_i \in \downarrow y \subseteq \downarrow z$. Notice that x represents $r_z(\downarrow x)$ by Lemma 2.11(v). Using Lemma 2.12 for $\downarrow x \subseteq A_i$, we obtain that $x \vee u_i$ represents $r_z(A_i)$. We conclude from $\downarrow x \subset A_i \subseteq \downarrow y, x \prec y, a_i \in \downarrow u_i$ and the construction of S and $\mathcal{L}$ that $x \vee u_i \geq y$. Now, we have

$$r(\downarrow x) + i > r_z(A_i) = r(\downarrow (x \vee u_i)) + |(A_i \cap \downarrow z) - \downarrow (x \vee u_i)| =$$
$$= r(\downarrow (x \vee u_i)) \geq r(\downarrow y) = r(\downarrow x) + k \geq r(\downarrow x) + i,$$

which is a contradiction. Therefore

$$r_z(A_i) = r_z(\downarrow x) + i \text{ for all } i \in \{1, \ldots, k\} \text{ and all } z \in F \cap [y). \tag{2.14}$$

Hence for every $f \in F$, we have that $r_z(A_{i-1}) \neq r_z(A_i)$ for $z = f \vee y$, which shows that $a_i \notin \mathrm{cl}_f(A_{i-1})$ for all $f \in F$, that is $a_i \notin \mathrm{cl}(A_{i-1})$. This gives that $\mathrm{cl}(A_{i-1}) \neq \mathrm{cl}(A_i)$. Clearly, $\mathrm{cl}(A_{i-1}) \leq \mathrm{cl}(A_{i-1} \cup a_i) = \mathrm{cl}(A_i)$. Thus

$$\downarrow x = \mathrm{cl}(A_0) \prec \mathrm{cl}(A_1) \prec \cdots \prec \mathrm{cl}(A_k). \tag{2.15}$$

We know from (2.14) and the definition of k that $r_z(A_k) = r_z(\downarrow y)$ for all $z \in F \cap [y)$. Since r_z is monotone, $r_z(A_k) = r_z(A_k \cup b)$ for all $b \in \downarrow y - A_k$. Hence $b \in \mathrm{cl}_y(A_k) \subseteq \mathrm{cl}(A_k)$ for all $b \in \downarrow y - A_k$, and we obtain that $\downarrow y \subseteq \mathrm{cl}(A_k)$. This, together with $A_k \subseteq \downarrow y$, yields that $\mathrm{cl}(A_k) = \downarrow y$. Consequently, we conclude from (2.15), the semimodularity of $\mathcal{L}_{\mathrm{cl}}$ and the induction hypothesis that $h(\downarrow y) = h(\downarrow x) + k = r(\downarrow x) + k = r(\downarrow y)$. □

Proof of Corollary 2.9. Let L be a finite height generated semimodular algebraic lattice. Consider the height function $h_L : L \to \mathbb{N}_\infty$. We conclude from Theorem 2.8 that L has an isometrical embedding ψ into a geometric lattice G with respect to h_L. Assume that $x \prec y$ in L and choose a minimal element f of finite height in $(y] - (x]$. Let g be a lower cover of f. Then $x = x \vee g$ and $y = x \vee f$. Now, $\psi(f)$ covers $\psi(g)$, since $h_G(\psi(f)) - h_G(\psi(g)) = h_L(f) - h_L(g) = 1$, where h_G denotes the height function of G. Hence the semimodularity of G implies that $\psi(y) = \psi(x) \vee \psi(f)$ covers $\psi(x) = \psi(x) \vee \psi(g)$. Therefore ψ is cover-preserving. □

Remark 2.18. If L is of finite length, the construction of $\mathcal{L}_{\mathrm{cl}}$ becomes more simple: we need only r_1 since $\mathrm{cl} = \mathrm{cl}_1$. Note that in this case r_1 is a rank function on $\mathcal{L}_{\mathrm{cl}}$.

2.2.3 Examples

Let L_1 be the five element lattice of Figure 2.3. It is not semimodular. Although Example 2.2 gives a pseudorank function $L_1 \to \mathbb{N}$, $0 \mapsto 0$, $r \mapsto 4$, $s \mapsto 6$, $t \mapsto 4$ and $1 \mapsto 7$, one can easily find a "nicer" one, which has smaller values. Let p be the map $p: L \to \mathbb{N}$, $p(0) = 0$, $p(r) = 2$, $p(s) = 2$, $p(t) = 1$ and $p(1) = 3$. It is a pseudorank function. See also Figure 2.1.

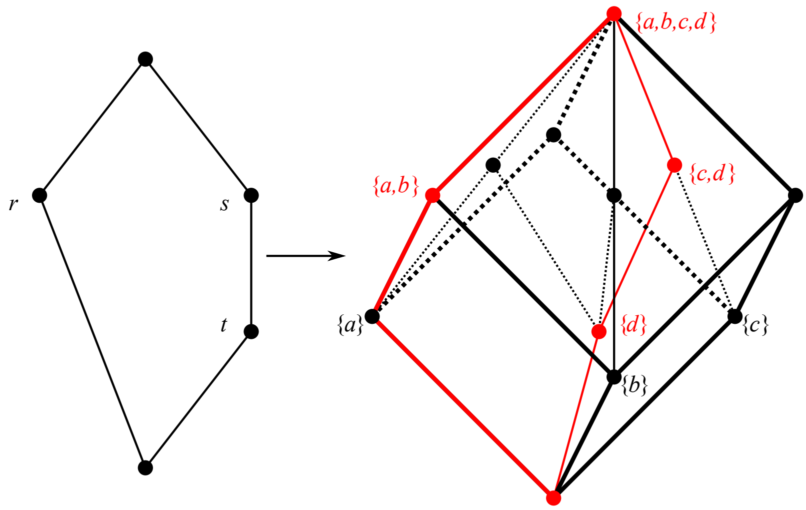

Figure 2.3: The isometrical embedding of L_1 (left) into G_1 (right)

We want to embed L_1 isometrically into a geometric lattice. Notice that the nonzero join-irreducible elements are r, s and t. Since $p(r) - p(r_0) = 2$ and $p(s) - p(s_0) = p(t) - p(t_0) = 1$, $|X_r| = 2$ and $|X_s| = |X_t| = 1$. Let $X_r = \{a, b\}$, $X_s = \{c\}$ and $X_t = \{d\}$. Thus $S = \{a, b, c, d\}$. Considering Remark 2.18, it is enough to calculate r_1:

$r_1(A)$	A
0	$\varnothing$
1	$\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$
2	$\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$
3	$\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{a,b,c,d\}$

Now, one can easily determine $\mathrm{cl} = \mathrm{cl}_1$. Indeed, every subset is closed, except the three element subsets. The corresponding geometric lattice is

$$G_1 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c,d\}\},$$

see in Figure 2.3.

Let $L_2 = \mathbb{N}$ with the usual ordering, see Figure 2.4. We want to embed L_2 into a geometric lattice. Although it is not a finite height generated lattice, it can be extended to a finite height generated lattice $\mathbb{N} = [0,\infty) \leq \mathbb{N}_\infty$, which is also an algebraic lattice. Indeed, the elements of $\mathbb{N}$ are exactly the compact elements of $\mathbb{N}_\infty$ and $\infty = \bigvee \mathbb{N}$.

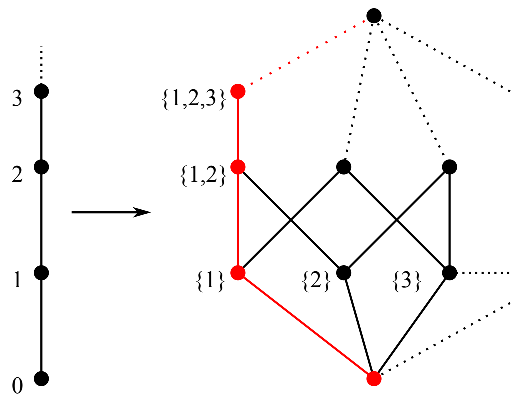

Figure 2.4: The isometrical embedding of L_2 (left) into G_2 (right)

Let p denote the height function of $\mathbb{N}_\infty$, which is a pseudorank function, since $\mathbb{N}_\infty$ is semimodular. Notice that the nonzero join irreducible elements of finite height are $\mathbb{N} - \{0\}$. Since p is the height function, it is also a rank function, cf. (2.7). Thus S can be chosen to be $\mathbb{N} - \{0\}$. Observe that for any $x \in \mathbb{N}$, $\downarrow x = (0, x]$, and any $A \subseteq S$, $r_x(A) = |A \cap \downarrow x|$. For any $a \in S - A$, if $x \geq a$ then $r_x(A \cup a) = |(A \cup a) \cap \downarrow x| = |(A \cap \downarrow x) \cup a| > |A \cap \downarrow x| = r_x(A)$. Hence, $\mathrm{cl}_x(A) = A$. This implies that $\mathrm{cl}(A) = A$ for every subset $A \subseteq S$. The corresponding geometric lattice G_2 is the Boolean lattice of all subsets of S, see Figure 2.4.

Chapter 3

Mal'cev conditions

The classic theorem of Mal'cev [67] states that the congruences of any algebra of a variety $\mathcal{V}$ permute if and only if there is a ternary term p such that $\mathcal{V}$ satisfies the following identities:

$$p(x, y, y) = x \text{ and } p(x, x, y) = y.$$

Jónsson [63] and Day [28] proved similar results for distributivity and modularity. These results led to the concept of Mal'cev(-type) conditions, see Grätzer [40]. Using Grätzer's concept, Jónsson's resp. Day's result says that the class of congruence distributive resp. congruence modular varieties can be defined by a Mal'cev condition, cf. Theorem 3.1 and 3.2. Later, beside the concept of Mal'cev condition, two similar concepts appeared, the strong and weak Mal'cev conditions, cf. Taylor [81].

After Mal'cev's, Jónsson's and Day's results, many classes of varieties have proved to be definable by (strong/weak) Mal'cev conditions. Both permutability and distributivity have some generalizations, the so-called n-permutability and n-distributivity. Hagemann and Mitschke [50] characterized n-permutability ($n \geq 2$) by a strong Mal'cev condition. On the other hand, n-distributivity, which was introduced by Huhn [54], turned out to be equivalent with distributivity in congruence varieties, cf. Nation [68]. Thus Jónsson's result [63] also characterizes congruence n-distributivity by a Mal'cev condition. Let us mention here that distributivity and n-distributivity are not equivalent in general. Distributivity implies n-distributivity, but, e.g., M_3 is an n-distributive lattice that is not distributive (if $n > 2$), cf. Remark 1.2.

As for congruence modularity, Gumm [49] improved Day's result and found a Mal'cev condition for congruence modularity that contains ternary terms, see also Lakser, Taylor and Tschantz [65]. Then Czédli and Horváth [21] proved that every lattice identity that implies modularity in congruence varieties can be characterized by a Mal'cev condition. Their proof is heavily based on one of their former paper with Radeleczki [22]. Note that it is still an open problem whether all congruence lattice identities can be characterized by a Mal'cev condition. On the other hand, Wille [90] and Pixley [73] showed that every congruence lattice identity can be characterized by a weak Mal'cev condition.

In connection with Mal'cev conditions, we consider important to mention that Csákány was the first person from Szeged, who dealt with Mal'cev condition, for example, one of his results is the characterization of regular varieties by a Mal'cev condition [14]. He also wrote his thesis for the doctor of science degree about Mal'cev conditions and their applications [15].

Nowadays, Mal'cev conditions, especially Jónsson's, Day's and Gumm's terms, are frequently used in universal algebra and related areas such as CSP, cf., e.g., Barto and Kozik [4, 5].

Observe that, in case of groups, rings and modules, congruences are determined by normal subgroups, ideals and submodules. Although one congruence class does not usually determine the whole congruence, these examples show that given an algebra with a constant operation symbol c, the congruence class that contains c can play a special role. To recall a related concept from Chajda [11], let $\lambda : p(x_1, \ldots, x_n) \le q(x_1, \ldots, x_n)$ be a lattice identity, and let $\mathcal{V}$ be a variety with a constant operation symbol 0 in its type. We say that λ holds for the congruences of $\mathcal{V}$ at 0 if for every $\mathbf{A} \in \mathcal{V}$ and for all congruences $\alpha_1, \ldots, \alpha_n$ of $\mathbf{A}$, we have $[0]p(\alpha_1, \ldots, \alpha_n) \subseteq [0]q(\alpha_1, \ldots, \alpha_n)$. In particular, if λ is $\alpha_1 \wedge (\alpha_2 \vee \alpha_3) \le (\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3)$ resp. $(\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_3) \le \alpha_1 \vee (\alpha_2 \wedge (\alpha_1 \vee \alpha_3))$, then we say that $\mathcal{V}$ is *congruence distributive* resp. *congruence modular at* 0.

This concept is not as trivial as it may seem. For example, while the variety $\mathcal{S}$ of meet semilattices with 0 is congruence distributive at 0, the dual of the distributive law does not hold for congruences of $\mathcal{S}$ at 0, see Example 3.9.

Returning to Mal'cev conditions, Chajda [11] has given a Mal'cev condition characterizing congruence distributivity at 0, and Czédli [16] has pointed out that the satisfaction of λ for congruences at 0 can always be characterized by a *weak* Mal'cev condition. (This is particularly useful when each congruence α is determined by $[0]\alpha$, see the comment following Prop. 2 in Czédli [16].) Later, Chajda and Halaš [12] took some steps towards characterizing congruence modularity at 0. Then we gave a Mal'cev condition in [77] that characterizes congruence modularity at

0. Note that Jónsson's and Day's characterization of congruence distributivity and congruence modularity follows from the characterization of congruence distributivity and congruence modularity at 0, cf. Remark 3.6.

Overview of the chapter

In Section 3.1 we recall the precise definition of Mal'cev conditions, and we formulate Jónsson's and Day's results. In Section 3.2 we show Chajda's characterization of congruence distributivity at 0 and our characterization of congruence modularity at 0. We close this section with some examples and concluding remarks.

Notation for the chapter

Throughout this chapter, algebras are typeset in bold capital letters, e.g., $\mathbf{A}$, their underlying sets are typeset in capital letters, e.g., A, and varieties are typeset in calligraphy letters, e.g., $\mathcal{V}$. For a given algebra $\mathbf{A} \in \mathcal{V}$ and elements $a, b \in A$, $\Theta(a,b)$ denotes the smallest congruence of $\mathbf{A}$ that contains (a,b).

3.1 Definition of a Mal'cev condition

A class K of varieties is defined by a *strong Mal'cev condition* iff there exist polynomial symbols $p_1, \dots, p_k$ and a finite set Σ of equations in $p_1, \dots, p_k$ such that a variety $\mathcal{V}$ of type τ belongs to K if and only if each polynomial symbol can be associated with a term of type τ such that the equations of Σ become identities that hold in $\mathcal{V}$. The classical result of Mal'cev says that the class of congruence permutable varieties is definable by a strong Mal'cev condition.

A class K of varieties is defined by a *Mal'cev condition* iff there exists a sequence K_i ($i \in \mathbb{N}$) of classes such that each K_i is defined by a strong Mal'cev condition, $K_i \subseteq K_{i+1}$ for all $i \in \mathbb{N}$, and $K = \bigcup_{i=0}^{\infty} K_i$. The following two theorems show the two most known examples of classes defined by a Mal'cev condition: the class of congruence distributive resp. congruence modular varieties.

Theorem 3.1 (Jónsson [63, Theorem 2.1]). *For a variety $\mathcal{V}$ of algebras, the following conditions are equivalent:*

(i) *$\mathcal{V}$ is congruence distributive, that is* $\operatorname{Con}\mathbf{A}$ *is distributive for all* $\mathbf{A} \in \mathcal{V}$*;*

(ii) *there is a natural number n and a sequence of terms $D_0, D_1, \ldots, D_n$ in three variables such that $\mathcal{V}$ satisfies the following identities:*

$$D_0(x,y,z) = x \text{ and } D_n(x,y,z) = z; \tag{D1}$$

$$D_i(x,y,x) = x \qquad \text{for all } i; \tag{D2}$$

$$D_i(x,z,z) = D_{i+1}(x,z,z) \qquad \text{for } i \text{ odd;} \tag{D3}$$

$$D_i(x,x,z) = D_{i+1}(x,x,z) \qquad \text{for } i \text{ even.} \tag{D4}$$

Theorem 3.2 (Day [28, Theorem 1])**.** *For a variety $\mathcal{V}$ of algebras, the following conditions are equivalent:*

(i) *$\mathcal{V}$ is congruence modular, that is* $\operatorname{Con}\mathbf{A}$ *is modular for all $\mathbf{A} \in \mathcal{V}$;*

(ii) *there is a natural number n and a sequence of terms $M_0, M_1, \ldots, M_n$ in four variables such that $\mathcal{V}$ satisfies the following identities*

$$M_0(x,y,z,w) = x \text{ and } m_n(x,y,z,w) = w; \tag{M1}$$

$$M_i(x,y,y,x) = x \qquad \text{for all } i; \tag{M2}$$

$$M_i(x,y,y,w) = M_{i+1}(x,y,y,w) \qquad \text{for } i \text{ odd;} \tag{M3}$$

$$M_i(x,x,w,w) = M_{i+1}(x,x,w,w) \qquad \text{for } i \text{ even.} \tag{M4}$$

Note that the terms D_i resp. M_i are usually called Jónsson terms resp. Day terms.

Finally, a class K of varieties is defined by a *weak Mal'cev condition* iff there exists a sequence K_i $(i \in \mathbb{N})$ of classes such that each K_i is defined by a Mal'cev condition and $K = \bigcap_{i=0}^{\infty} K_i$.

3.2 Congruences of algebras with constants

Recall that congruences of a given variety $\mathcal{V}$ with a constant satisfy the identity $\lambda : p(x_1, \ldots, x_n) \leq q(x_1, \ldots, x_n)$ at 0 iff for every $\mathbf{A} \in \mathcal{V}$ and for all congruences $\alpha_1, \ldots, \alpha_n$ of $\mathbf{A}$, we have $[0]p(\alpha_1, \ldots, \alpha_n) \subseteq [0]q(\alpha_1, \ldots, \alpha_n)$. Notice that if the congruences of $\mathcal{V}$ satisfy λ then they satisfy λ at 0, too. Chajda [11] pointed out that some slight modification of Jónsson's proof of Theorem 3.1 gives a Mal'cev condition for congruence distributivity at 0.

Theorem 3.3 (Chajda [11, Theorem 1])**.** *For a variety $\mathcal{V}$ of algebras with a constant 0, the following conditions are equivalent:*

(i) Con $\mathbf{A}$ *is distributive at* 0 *for all* $\mathbf{A} \in \mathcal{V}$*;*

(ii) *there is a natural number* n *and there are binary terms* d_i $(i = 0, \dots, n)$ *such that* $\mathcal{V}$ *satisfies the following identities:*

$$d_0(x,y) = 0 \text{ and } d_n(x,y) = y; \tag{d1}$$

$$d_i(x,0) = 0 \qquad \text{for all } i; \tag{d2}$$

$$d_i(x,x) = d_{i+1}(x,x) \qquad \text{for } i \text{ odd;} \tag{d3}$$

$$d_i(0,y) = d_{i+1}(0,y) \qquad \text{for } i \text{ even.} \tag{d4}$$

Instead of a proof, we only note that the terms $d_i(x,y)$ are obtained from the Jónsson terms: $d_i(x,y) = D_i(0,x,y)$. Chajda and Halaš [12] observed that in varieties that are congruence modular at 0 the terms $m_i(x,y,z) = M_i(0,x,y,z)$ obtained from the Day terms must hold. However, they did not manage to prove that these terms also characterize congruence modularity at 0. In the next theorem we show that an appropriate modification of Day's proof of Theorem 3.2 works.

Theorem 3.4. *For a variety* $\mathcal{V}$ *of algebras with a constant* 0*, the following conditions are equivalent:*

(i) Con $\mathbf{A}$ *is modular at* 0 *for all* $\mathbf{A} \in \mathcal{V}$*;*

(ii) *there is a natural number* n *and there are ternary terms* m_i $(i = 0, \dots, n)$ *such that* $\mathcal{V}$ *satisfies the following identities:*

$$m_0(x,y,z) = 0 \text{ and } m_n(x,y,z) = z; \tag{m1}$$

$$m_i(x,x,0) = 0 \qquad \text{for all } i; \tag{m2}$$

$$m_i(x,x,z) = m_{i+1}(x,x,z) \qquad \text{for } i \text{ odd;} \tag{m3}$$

$$m_i(0,z,z) = m_{i+1}(0,z,z) \qquad \text{for } i \text{ even.} \tag{m4}$$

For a fixed algebra $\mathbf{A}$, congruences $\beta, \gamma \in \operatorname{Con} A$ and integer $k \geq 0$, let $\Delta_k = \Delta_k(\mathbf{A}, \beta, \gamma)$ denote the relation $\beta \circ \gamma \circ \dots \circ \gamma \circ \beta$ with $2k+1$ factors. Notice that Δ_k is reflexive, symmetric and it is compatible with the operations of $\mathbf{A}$. Such relations are called *tolerances*, cf. Chajda [10]. Before the proof we need the following lemma.

Lemma 3.5. *Suppose that we have the ternary terms* m_i *given above. Let us fix an algebra* $\mathbf{A} \in \mathcal{V}$*, congruences* $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$ *and elements* $a, d \in A$*. If* $\alpha \geq \gamma$*,* $(0,a) \in (\alpha \wedge \beta) \vee \gamma$ *and for some integer* $k \geq 0$*,* $(a,d) \in \alpha \cap \Delta_k$ *then* $(0,d) \in (\alpha \wedge \beta) \vee \gamma$*.*

Proof. We prove the lemma by induction over k. The lemma is trivially true for $k = 0$. Suppose that the lemma is true for some integer $k \geq 0$ and let $(0,a) \in (\alpha \wedge \beta) \vee \gamma$

and $(a,d) \in \alpha \cap \Delta_{k+1}$. We have to prove that $(0,d) \in (\alpha \wedge \beta) \vee \gamma$. As $(a,d) \in \alpha \cap \Delta_{k+1} = \alpha \cap (\Delta_k \circ \gamma \circ \beta)$, there exist elements $b, c \in A$ such that

$$a\, \Delta_k\, b, \quad b\, \gamma\, c, \quad c\, \beta\, d.$$

Define $e_i = m_i(b,c,d)$ for $i \leq n$. We show (by induction over i) that $(0,e_i) \in (\alpha \wedge \beta) \vee \gamma$ for $i \leq n$. Then by (m1), we have $(0,d) = (0,e_n) \in (\alpha \wedge \beta) \vee \gamma$. By (m1), we have $e_0 = 0$, hence $(0,e_0) \in (\alpha \wedge \beta) \vee \gamma$ is obvious. Suppose that for some $i < n$, we have $(0,e_i) \in (\alpha \wedge \beta) \vee \gamma$. We show that $(0,e_{i+1}) \in (\alpha \wedge \beta) \vee \gamma$.

For arbitrary $j \leq n$, by (m2), we have

$$\begin{aligned} e_j &= m_j(b,c,d)\ \Delta_k\ m_j(a,d,d); \\ e_j &= m_j(b,c,d)\ \gamma\ m_j(b,b,a)\ (\alpha \wedge \beta) \vee \gamma\ m_j(b,b,0) = \\ &= m_j(0,0,0)\ (\alpha \wedge \beta) \vee \gamma\ m_j(a,a,a)\ \alpha\ m_j(a,d,d). \end{aligned}$$

Since $\gamma \leq (\alpha \wedge \beta) \vee \gamma \leq \alpha$, we have

$$e_j\ \alpha \cap \Delta_k\ m_j(a,d,d). \tag{3.1}$$

For i even, by (m4), we have

$$m_i(a,d,d)\ (\alpha \wedge \beta) \vee \gamma\ m_i(0,d,d) = \\ m_{i+1}(0,d,d)\ (\alpha \wedge \beta) \vee \gamma\ m_{i+1}(a,d,d). \tag{3.2}$$

By the induction hypothesis over i, we have $(0,e_i) \in (\alpha \wedge \beta) \vee \gamma$. Using the induction hypothesis over k for (3.1) with $j = i$, we obtain $(0, m_i(a,d,d)) \in (\alpha \wedge \beta) \vee \gamma$. Then by (3.2), we have $(0, m_{i+1}(a,d,d)) \in (\alpha \wedge \beta) \vee \gamma$. Using the induction hypothesis over k for (3.1) with $j = i+1$, we obtain $(0,e_{i+1}) \in (\alpha \wedge \beta) \vee \gamma$.

For i odd, by (m3), we have

$$e_i = m_i(b,c,d)\ \gamma\ m_i(b,b,d) = m_{i+1}(b,b,d)\ \gamma\ m_{i+1}(b,c,d) = e_{i+1}$$

By the induction hypothesis over i, we have $(0,e_i) \in (\alpha \wedge \beta) \vee \gamma$, hence we obtain $(0,e_{i+1}) \in (\alpha \wedge \beta) \vee \gamma$. $\square$

Proof of Theorem 3.4. (i)⇒(ii). Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y,z)$ denote the $\mathcal{V}$-free algebra over $\{x,y,z\}$. The variety $\mathcal{V}$ is closed under forming subalgebras and direct products, therefore $\mathcal{V}$ contains $\mathbf{F}$. We define congruence relations on $\mathbf{F}$ by

$$\alpha = \Theta(x,y) \vee \Theta(0,z), \quad \beta = \Theta(0,x) \vee \Theta(y,z), \quad \gamma = \Theta(x,y).$$

By (i), we have $z \in [0]\alpha \wedge (\beta \vee \gamma) = [0](\alpha \wedge \beta) \vee \gamma$, which means $(0, z) \in (\alpha \wedge \beta) \vee \gamma$. It follows that there is a natural number n and there are ternary terms m_i ($i = 0, \dots, n$) such that

$$m_0(x, y, z) = 0 \text{ and } m_n(x, y, z) = z; \tag{3.3}$$

$$m_i(x, y, z)\ \alpha \wedge \beta\ m_{i+1}(x, y, z) \qquad \text{for } i \text{ even;} \tag{3.4}$$

$$m_i(x, y, z)\ \gamma\ m_{i+1}(x, y, z) \qquad \text{for } i \text{ odd.} \tag{3.5}$$

As $\mathbf{F}$ is a $\mathcal{V}$-free algebra with free generators x, y and z, Equation (3.3) proves (m1). Equation (m2), (m3) and (m4) also follow from (3.3), (3.4) and (3.5). For example, to prove (m2), let us consider the homomorphism φ defined by $x\varphi = x$, $y\varphi = x$ and $z\varphi = 0$. Both $\alpha \wedge \beta$ and γ are contained by $\ker \varphi$, hence (3.4) and (3.5) imply

$$\begin{aligned} m_i(x, x, 0) &= m_i(x\varphi, y\varphi, z\varphi) = m_i(x, y, z)\varphi \\ &= m_{i+1}(x, y, z)\varphi = m_{i+1}(x\varphi, y\varphi, z\varphi) \\ &= m_{i+1}(x, x, 0) \end{aligned}$$

for all i. Using (m1), this proves (m2). Similar arguments prove (m3) and (m4). The details are left to the reader.

(ii)⇒(i). For any $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\alpha \wedge (\beta \vee \gamma) = \bigcup_{k=0}^{\infty} \alpha \cap \Delta_k$. Hence, using Lemma 3.5 for $a = 0$, we obtain (i). □

Remark 3.6. Since both Chajda's resp. our proof is heavily based on Jónsson's resp. Day's original ones, it is not surprising that Theorem 3.3 resp. Theorem 3.4 imply the nontrivial direction (ii)⇒(i) of Theorem 3.1 resp. Theorem 3.2. Indeed, let $\mathcal{V}$ be a variety of algebras of type τ. Let us extend τ with a (new) constant operator 0 and let τ_0 denote the extended type. For any algebra $\mathbf{A} \in \mathcal{V}$ and for any element $a \in A$, let $\mathbf{A}_a$ denote the algebra of type τ_0, where $0_{\mathbf{A}} = a$. Let $\mathcal{V}_0$ be the variety of algebras of type τ_0 generated by all algebras of the form $\mathbf{A}_a$. If $T_0, \dots, T_n$ are Jónsson resp. Day terms over $\mathcal{V}$ then $T_0, \dots, T_n$ are Jónsson resp. Day terms over $\mathcal{V}_0$, too. Now, substitute 0 for the first variable of T_i to obtain t_i. Then we can apply either Theorem 3.3 or Theorem 3.4 to the terms $t_0, \dots, t_n$, which implies that the original variety $\mathcal{V}$ is congruence distributive resp. congruence modular.

Remark 3.7. Congruence distributivity at 0 obviously implies congruence modularity at 0. Note that the converse is not true: congruence modularity at 0 is in fact a weaker concept than congruence distributivity at 0. Indeed, in the variety of groups, congruences are determined by normal subgroups, hence congruences of groups satisfy any identity λ iff they satisfy λ at 0. Therefore the variety of groups is congruence modular at 0 but it is not congruence distributive at 0.

Example 3.8. Let $\mathcal{G}_0$ denote the variety of idempotent groupoids with zero (idempotent groupoids that have a constant operation symbol 0 satisfying $0x = x0 = 0$). Then

- $\mathcal{G}_0$ is both congruence distributive and modular at 0, but
- $\mathcal{G}_0$ is neither congruence distributive nor modular in the usual sense.

To show that $\mathcal{G}_0$ is congruence distributive at 0 use Theorem 3.3 with $n = 2$ and

$$d_0(x,y) = 0, \quad d_1(x,y) = xy, \quad d_2(x,y) = y.$$

Congruence modularity at 0 follows from congruence distributivity at 0, but for the sake of completeness, note that it also follows from Theorem 3.4 with $n = 3$ and

$$\begin{aligned} m_0(x,y,z) &= 0, & m_1(x,y,z) &= xz \\ m_2(x,y,z) &= yz, & m_3(x,y,z) &= z. \end{aligned}$$

To verify the second part, observe that the variety $\mathcal{S}$ of meet semilattices with 0 is a subvariety of $\mathcal{G}_0$, and recall from Freese and Nation [39] that $\mathcal{S}$ satisfies no nontrivial congruence lattice identity.

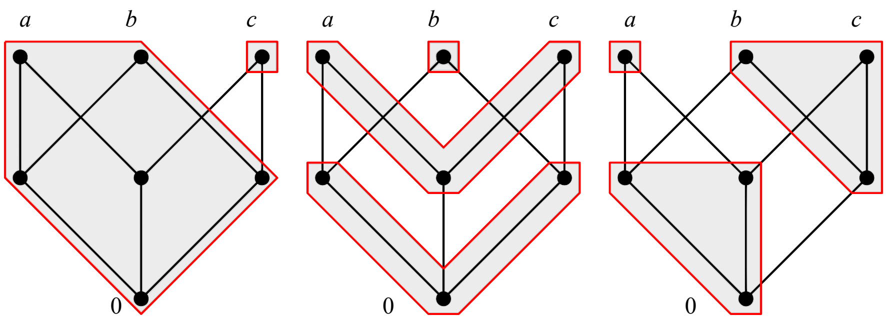

Figure 3.1: The meet semilattice $\mathbf{S} \in \mathcal{S}$ and its congruences

Example 3.9 (Czédli [16]). We have just seen that $\mathcal{S}$ is congruence distributive at 0. The following semilattice shows that the dual of the distributive law does not hold for congruences of $\mathcal{S}$ at 0. Consider the seven element semilattice $\mathbf{S}$ depicted in Figure 3.1 and its congruences α, β and γ corresponding to the following partitions.

$$\alpha\colon \big\{\{a,b,a\wedge b,a\wedge c,b\wedge c,0\},\{c\}\big\},$$
$$\beta\colon \big\{\{a,c,a\wedge c\},\{b\},\{a\wedge b,b\wedge c,0\}\big\},$$
$$\gamma\colon \big\{\{a\},\{a\wedge c,a\wedge b,0\},\{b,c,b\wedge c\}\big\}.$$

Then $[0](\alpha\vee\beta)\wedge(\alpha\vee\gamma) = S \supsetneq S\setminus\{c\} = [0]\alpha\vee(\beta\wedge\gamma)$, which shows that the dual of the distributive law fails for congruences of $\mathbf{S}$ at 0.

The core of this counterexample is the fact that $[0](\Phi\vee\Psi) = [0]\Phi\cup[0]\Psi$ need not hold for all congruences Φ and Ψ. On the other hand, $[0](\Phi\wedge\Psi) = [0]\Phi\cap[0]\Psi$ holds for all congruences Φ and Ψ. This yields that the dual of the distributive law implies distributivity for congruences at 0. Indeed, assume that the dual of the distributive law holds for congruences of a variety $\mathcal{V}$, and let $\mathbf{A}\in\mathcal{V}$ and $\alpha,\beta,\gamma\in\operatorname{Con}\mathbf{A}$. Then

$$\begin{aligned}[0](\alpha\wedge\beta)\vee(\alpha\wedge\gamma) &= [0]\big((\alpha\wedge\beta)\vee\alpha\big)\wedge\big((\alpha\wedge\beta)\vee\gamma\big)\\ &= [0]\alpha\wedge\big((\alpha\wedge\beta)\vee\gamma\big) = [0]\alpha\cap[0]\big((\alpha\wedge\beta)\vee\gamma\big)\\ &= [0]\alpha\cap[0]\big((\alpha\vee\gamma)\wedge(\beta\vee\gamma)\big) = [0]\alpha\wedge\big((\alpha\vee\gamma)\wedge(\beta\vee\gamma)\big)\\ &= [0]\alpha\wedge(\beta\vee\gamma).\end{aligned}$$

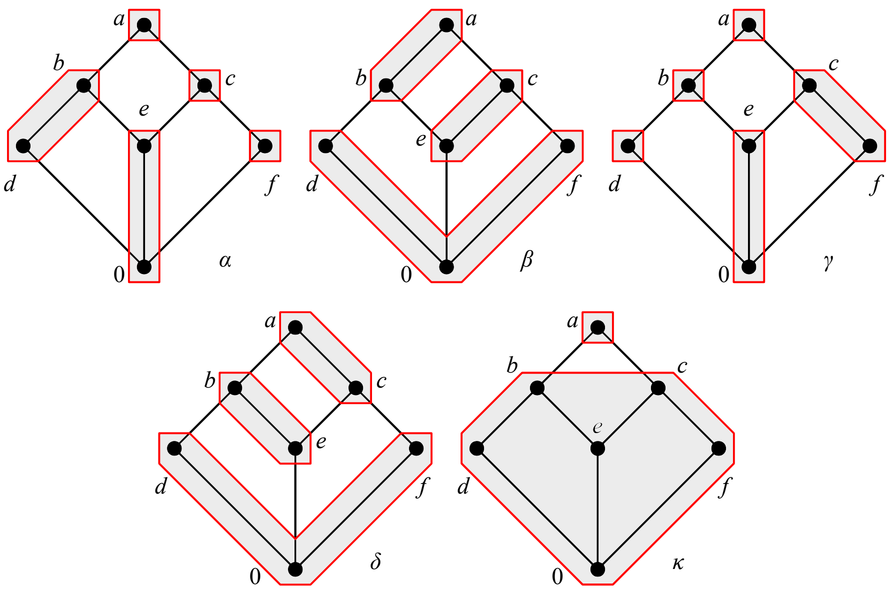

Figure 3.2: The meet semilattice $\mathbf{T}\in\mathcal{S}$ and its congruences

Remark 3.10. One may feel that there is a hope to combine Theorem 3.4 with the result of Czédli and Horváth [21]. However, this seems difficult, since [21] is based on the following fact, see Czédli and Horváth [20] and [21]: for any two tolerances in a congruence modular variety, the transitive closure commutes with the intersection; in their notation $\Phi^* \cap \Psi^* = (\Phi \cap \Psi)^*$. For varieties that are congruence modular at 0, the analogous statement, $[0]\Phi^* \cap \Psi^* = [0](\Phi \cap \Psi)^*$, is not true. Indeed, consider the seven element meet semilattice $\mathbf{T} \in \mathcal{S}$ depicted in Figure 3.2, and the congruences α, β, γ, δ and κ represented by the following partitions.

$$\alpha\colon \big\{\{a\},\{c\},\{f\},\{0,e\},\{b,d\}\big\}, \qquad \beta\colon \big\{\{0,d,f\},\{a,b\},\{c,e\}\big\},$$
$$\gamma\colon \big\{\{a\},\{b\},\{d\},\{0,e\},\{c,f\}\big\}, \qquad \delta\colon \big\{\{0,d,f\},\{b,e\},\{a,c\}\big\},$$
$$\kappa\colon \big\{\{a\},T \setminus \{a\}\big\}.$$

Then $\Phi = \alpha \circ \beta \circ \alpha$ and $\Psi = \gamma \circ \delta \circ \gamma$ are tolerances of $\mathbf{T}$. Since $0 \mathrel{\alpha} 0 \mathrel{\beta} d \mathrel{\alpha} b \mathrel{\beta} a$ and $0 \mathrel{\gamma} 0 \mathrel{\delta} f \mathrel{\gamma} c \mathrel{\delta} a$, we have $(0,a) \in \Phi^* \cap \Psi^*$. On the other hand, $\Phi \cap \Psi \subseteq \kappa$ yields that $(0,a) \notin (\Phi \cap \Psi)^*$.

Summary

In my doctoral dissertation, three problems of modular and semimodular lattices are studied. Modularity and semimodularity are two closely related concepts of lattice theory. Indeed, the concept of semimodularity is proved to be the most useful generalization of modularity. The class of semimodular lattices contains properly the class (variety) of modular lattices. However, if a lattice is of finite length and both itself and its dual are semimodular then it is also modular. The three chapters of my dissertation are based on the papers [27, 78] and [77].

In the first chapter, we are dealing with a problem of coordinatization theory, one of the oldest and deepest part of lattice theory. In the first section, we introduce the concept of a von Neumann frame and mention a related concept called Huhn diamond. Without any proof, we recall some basic results of coordinatization theory, which are used later in the chapter. In the second section, we define the concept of a product frame and some related concepts: the outer and inner frames. These concepts are due to Gábor Czédli. In the third section, we prove a joint result with Gábor Czédli, which says that the coordinate ring associated to the outer frame is the matrix ring of the coordinate ring associated to the product frame, see [27].

In the second chapter, we study isometrical embeddings of lattices with pseudorank functions into geometric lattices. This problem has a close connection to semimodular lattices. First of all, geometric lattices form the best known class of semimodular lattices. On the other hand, if L is a semimodular lattice then its height function is a pseudorank function, and the isometrical embedding of L preserves the height of each element, moreover it also preserves the covering relation under some necessary conditions. In the first section, we recall a proof of Marcel Wild [89], which shows that every finite semimodular lattice has a cover-preserving embedding into a geometric lattice. This argument is a motivation for the second section, where we prove a generalization of an embedding result of George Grätzer and Emil W. Kiss [43], see [78].

In the third chapter, we are dealing with Mal'cev conditions, which play a central role in universal algebra. We characterize a generalization of congruence modularity by a Mal'cev condition. Assume that the type of an algebra **A** has a constant operation symbol 0. Then those classes of congruences of **A** that contain 0 form a lattice with respect to set inclusion. In contrast to, e.g., groups or rings, this lattice differs from the congruence lattice in general. Similarly to congruence modularity, we call **A** congruence modular at 0, if the above defined lattice is modular. Proving the conjecture of Ivan Chajda, we show that congruence modularity at 0 can be characterized by a Mal'cev condition, see [77].

To understand the dissertation, we assume basic knowledge of lattice theory and universal algebra, but the reader is also directed to Grätzer [42] and Burris and Sankappanavar [9]. We define any deeper concept that occurs in the dissertation, but the reader also can find some references to them.

Now, we recall the major results of the dissertation chapter by chapter.

Von Neumann frames

For definition, let $2 \le m$, let L be a nontrivial modular lattice with 0 and 1, and let $\vec{a} = (a_1, \dots, a_m) \in L^m$ and $\vec{c} = (c_{12}, \dots, c_{1m}) \in L^{m-1}$. We say that $(\vec{a}, \vec{c}) = (a_1, \dots, a_m, c_{12}, \dots, c_{1m})$ is a *spanning m-frame* (or a frame of *order m*) of L, if $a_1 \ne a_2$ and the following equations hold for all $j \le m$ and $2 \le k \le m$:

$$\sum_{i \le m} a_i = 1, \qquad a_j \sum_{i \le m,\, i \ne j} a_i = 0,$$

$$a_1 + c_{1k} = a_k + c_{1k} = a_1 + a_k, \qquad a_1 c_{1k} = a_k c_{1k} = 0.$$

Let us mention here that in coordinatization theory, the lattice operations join and meet are traditionally denoted by + and · (mostly juxtaposition) such that meets take precedence over joins.

To understand the concept of von Neumann frames better, let us consider the following example. Let K be a ring with 1. Let v_i denote the vector $(0, \dots, 0, 1, 0, \dots, 0) \in K^m$ (1 at the ith position). Letting $a_i = K v_i$ and $c_{1j} = K(v_1 - v_j)$, we obtain a spanning m-frame of the submodule lattice $\mathrm{Sub}(K^m)$, where K^m is, say, a left module over K in the usual way. This frame is called the *canonical m-frame* of $\mathrm{Sub}(K^m)$.

We also need the concept of a coordinate ring. If $m \ge 4$ and $(\vec{a}, \vec{c}) = (a_1, \dots, a_m, c_{12}, \dots, c_{1m})$ is a spanning m-frame of L then one can define addition and multiplication on the set $R\langle 1,2 \rangle = \{x \in L : x + a_2 = a_1 + a_2,\ x a_2 = 0\}$ such that $R\langle 1,2 \rangle$ forms a ring with a unit. This ring is called the *coordinate ring* of $(\vec{a}, \vec{c})$. Note that the ring construction also works if $m = 3$ and L is Arguesian.

Now, we are in position to formulate the main result of the first chapter.

Theorem ([27, Theorem 1.1])**.**

(a) *Let L be a lattice with $0, 1 \in L$, and let $m, n \in \mathbb{N}$ with $n \geq 2$. Assume that*

$$L \text{ is modular and } m \geq 4. \tag{a1}$$

Let $(\vec{a}, \vec{c}) = (a_1, \ldots, a_m, c_{12}, \ldots, c_{1m})$ be a spanning von Neumann m-frame of L and $(\vec{u}, \vec{v}) = (u_1, \ldots, u_n, v_{12}, \ldots, v_{1n})$ be a spanning von Neumann n-frame of the interval $[0, a_1]$. Let R^* denote the coordinate ring of $(\vec{a}, \vec{c})$. Then there is a ring S^* such that R^* is isomorphic to the ring of all $n \times n$ matrices over S^*. If

$$n \geq 4, \tag{a2}$$

then we can choose S^* as the coordinate ring of $(\vec{u}, \vec{v})$.

(b) *The previous part of the theorem remains valid if* (a1) *and* (a2) *are replaced by*

$$L \text{ is Arguesian and } m \geq 3 \tag{b1}$$

and

$$n \geq 3, \tag{b2}$$

respectively.

We could formulate the theorem without recalling the concepts of a product frame and the corresponding outer and inner frames. However, it is worth mentioning here that S^* is the coordinate ring associated to the product frame that occurs in the proof of the theorem. While $(\vec{a}, \vec{c})$ and $(\vec{u}, \vec{v})$ are the corresponding outer and inner frames, respectively.

Isometrical embeddings

Given a lattice L with a lower bound 0, a function $p: L \to \mathbb{N}_\infty = \{0, 1, \ldots, \infty\}$ is called a *pseudorank function* if it has the following properties:

(i) $p(0) = 0$;

(ii) $a \leq b$ implies $p(a) \leq p(b)$ for all $a, b \in L$;

(iii) $a < b$ implies $p(a) < p(b)$ for all $a, b \in L$ of finite height;
(iv) $p(a \wedge b) + p(a \vee b) \leq p(a) + p(b)$ for all $a, b \in L$;
(v) $p(a) < \infty$ iff a is of finite height.

In case of finite lattices, this definition coincide that of Finkbeiner [32] and Stern [79]. It is an easy consequence of the Jordan-Hölder Chain Condition that the height function of any semimodular lattice is a pseudorank function.

Consider a lattice L with a lower bound 0, a pseudorank function $p: L \to \mathbb{N}_\infty$ and a geometric lattice G whose height function is denoted by h. Then L is embeddable *isometrically* into G iff there is a lattice embedding $\varphi: L \to G$ such that $p = h \circ \varphi$, cf. Grätzer and Kiss [43].

We need one more concept in order to formulate the main result of this chapter, which generalizes a result of Grätzer and Kiss [43]. A lattice is said to be *finite height generated* iff it is complete and every element is the join of some elements of finite height. Note that lattices of finite length are finite height generated. To show a finite height generated lattice that is not of finite length, consider, for instance, $\mathbb{N}_\infty$ with the usual ordering.

Theorem ([78, Theorem 1]). *Every finite height generated algebraic lattice with a pseudorank function can be embedded isometrically into a geometric lattice.*

This theorem has a straidforward corollary for semimodular lattices. A lattice embedding is said to be *cover-preserving* iff it preserves the covering relation.

Corollary ([78, Corollary 2]). *Every finite height generated semimodular algebraic lattice has a cover-preserving embedding into a geometric lattice.*

Mal'cev conditions

Let $\mathcal{V}$ be a variety that has a constant operation symbol 0 in its type. We say that $\mathcal{V}$ is congruence modular at 0 iff for every algebra $\mathbf{A} \in \mathcal{V}$ and for all congruences α, β and γ of $\mathbf{A}$, we have $[0]\alpha \vee (\beta \wedge (\alpha \vee \gamma)) = [0](\alpha \vee \beta) \wedge (\alpha \vee \gamma)$, cf. Chajda [11] and Chajda and Halaš [12]. Notice that congruence modularity implies congruence modularity at 0, for instance, any group or ring variety is congruence modular at 0, since it is congruence modular. However, the converse is not true.

The main result of the third chapter characterizes congruence modularity at 0 by a Mal'cev condition. A similar result for congruence modularity was published by Day [28]. Note that our proof is heavily based on that of Day.

Theorem ([77, Theorem 1]). *For a variety $\mathcal{V}$ of algebras with a constant 0, the following conditions are equivalent:*

(i) $\operatorname{Con}\mathbf{A}$ *is modular at* 0 *for all* $\mathbf{A} \in \mathcal{V}$*;*

(ii) *there is a natural number* n *and there are ternary terms* m_i $(i = 0, \ldots, n)$ *such that* $\mathcal{V}$ *satisfies the following identities:*

$$m_0(x, y, z) = 0 \text{ and } m_n(x, y, z) = z; \tag{m1}$$

$$m_i(x, x, 0) = 0 \quad \text{for all } i; \tag{m2}$$

$$m_i(x, x, z) = m_{i+1}(x, x, z) \quad \text{for } i \text{ odd}; \tag{m3}$$

$$m_i(0, z, z) = m_{i+1}(0, z, z) \quad \text{for } i \text{ even}. \tag{m4}$$

Összefoglaló

Doktori értekezésem a moduláris és féligmoduláris hálók témakörének egy-egy problémájával foglalkozik. Már a neveik alapján is sejthető, hogy a két említett hálótulajdonság szoros kapcsolatban áll egymással. A féligmodularitás a modularitásnak az egyik legismertebb általánosítása. Moduláris hálók mindig féligmodulárisak, valamint – mivel a modularitás önduális tulajdonság – moduláris hálók duálisa is féligmoduláris. Érdemes megjegyezni, hogy véges magasságú hálók esetén a fenti észrevétel megfordítható: ha egy véges magasságú háló és duálisa is féligmoduláris, akkor moduláris. Az értekezés három fejezete rendre az [27, 78] és [77] dolgozatok eredményein alapul.

Az első fejezetben a moduláris hálók egyik legrégebbi és legmélyebb témakörével, a Neumann-féle koordinátázáselmélettel foglalkozom. A fejezet első részében bevezetem a Neumann-féle keret fogalmát és röviden kitérek az ezzel ekvivalens Huhn-gyémánt fogalmára. Bizonyítás nélkül hivatkozom a témakör azon eredményeire, amelyekre a fejezetben később szükségem lesz. A fejezet második részében a Czédli Gábor által definiált szorzatkeret, valamint a hozzá tartozó külső és belső keret fogalmát ismertetem. A fejezet harmadik részében a Czédli Gáborral közös eredményünket bizonyítom, mely szerint a külső kerethez tartozó koordinátagyűrű a szorzatkerethez tartozó koordinátagyűrű feletti mátrixgyűrű [27].

A második fejezetben pszeudorang függvénnyel rendelkező hálók geometriai hálókba történő izometrikus beágyazásával foglalkozom. Ez több ponton is szervesen kötődik a féligmoduláris hálók témaköréhez. Egyrészt a geometriai hálók a féligmoduláris hálók egyik legismertebb részosztálya. Másrészt ha a fent említett beágyazás során tetszőleges háló helyett féligmoduláris hálót veszünk, valamint a pszeudorang függvényt a (féligmoduláris) háló magasságfüggvényének választjuk, akkor az erre vonatkozó izometrikus beágyazás olyan hálóbeágyazás, ami megőrzi a magasságfüggvényt, sőt, bizonyos feltételek mellett a fedés relációt is. A fejezetben Grätzer György és Kiss Emil [43] véges hálókra vonatkozó izometrikus beágyazását általánosítom algebrai hálók egy „szép” osztályára. A fejezet első részében Marcel Wild [89] matroidokkal történő fedésőrző beágyazása található véges féligmoduláris hálókra, ami motivációt ad az általános eset bizonyításához, amit a fejezet második részében közlök [78].

A harmadik fejezetben Mal’cev feltételekkel foglalkozom. Algebrák kongruenciahálóinak számos tulajdonságára született Mal’cev feltétel. A fejezetben a kongruencia-modularitás egy általánosítására mutatok Mal’cev feltételt. Olyan algebrák esetén, amiknek a típusában a csoportokhoz vagy a gyűrűkhöz hasonlóan szerepel konstans műveleti jel, az adott konstanst tartalmazó kongruenciaosztályok hálót alkotnak. Ellentétben a csoportokkal és gyűrűkkel, általános esetben a konstanst tartalmazó kongruenciaosztály nem feltétlenül határozza meg a teljes kongruenciát, és a konstanst tartalmazó kongruenciaosztályok hálója nem feltétlenül egyezik meg a kongruenciahálóval. A kongruencia-modularitáshoz hasonló fogalom definiálható ebben az esetben is, amit 0-nál vett kongruencia-modularitásnak hívunk. Ivan Chajda sejtését igazolva megmutatom, hogy a 0-nál vett kongruencia-modularitás jellemezhető Mal’cev feltétellel [77].

Az értekezés megértéséhez elegendőek a hálóelmélet és az univerzális algebra alapfogalmai, amelyek mindegyike előfordul az egyetemi tanulmányok során, de megtalálható Grätzer [42], valamint Burris és Sankappanavar [9] könyveiben is. Minden egyéb fogalmat, melynek ismeretét előre nem feltételeztem, az értekezésben külön definiáltam és hivatkozással láttam el.

A következőkben fejezetenként röviden ismertetem az értekezésben található eredményeimet.

Neumann-féle keretek

Rögzítsünk egy L korlátos moduláris hálót és egy $m \geq 2$ egész számot, továbbá legyen $\vec{a} = (a_1, \dots, a_m) \in L^m$ és $\vec{c} = (c_{12}, \dots, c_{1m}) \in L^{m-1}$. Azt mondjuk, hogy $(\vec{a}, \vec{c}) = (a_1, \dots, a_m,\ c_{12}, \dots, c_{1m})$ az L háló *feszítő m-kerete*, ha $a_1 \neq a_2$ és minden $j \leq m$ és $2 \leq k \leq m$ indexre teljesülnek az alábbi összefüggések:

$$\sum_{i \leq m} a_i = 1, \qquad a_j \sum_{i \leq m,\, i \neq j} a_i = 0,$$

$$a_1 + c_{1k} = a_k + c_{1k} = a_1 + a_k, \qquad a_1 c_{1k} = a_k c_{1k} = 0.$$

Ezen a ponton érdemes megjegyezni, hogy a koordinátázáselméletben a hálóműveleteket ($\vee$ és $\wedge$) hagyományosan rendre összeadás ($+$) és szorzás ($\cdot$) jelöli.

Ahhoz, hogy a Neumann-féle keretek fogalmát jobban megértsük, tekintsük a következő példát. Legyen K egységelemes gyűrű. Ekkor rögzített $m \geq 2$ egész számra K^m tekinthető K feletti baloldali modulusnak. Jelölje v_i a $(0,\dots,0,1,0,\dots,0) \in K^m$ vektort, ahol az 1 az i-edik koordinátában szerepel. Könnyen ellenőrizhető, hogy a K^m (baloldali) részmodulusai által alkotott (korlátos, moduláris) hálóban az $a_i = Kv_i$ és $c_{1j} = K(v_1 - v_j)$ elemek feszítő m-keretet alkotnak. Ezt a keretet nevezik *kanonikus m-keretnek.*

A későbbiekben szükségünk lesz még a koordinátagyűrű fogalmára. Ha $m \geq 4$ és $(\vec{a},\vec{c}) = (a_1,\dots,a_m,\ c_{12},\dots,c_{1m})$ az L moduláris háló feszítő m-kerete, akkor az $R\langle 1,2\rangle = \{x \in L : x + a_2 = a_1 + a_2,\ xa_2 = 0\}$ halmazon definiálható egy összeadás és egy szorzás művelet, melyre nézve $R\langle 1,2\rangle$ egységelemes gyűrűt alkot. Ezt nevezzük az $(\vec{a},\vec{c})$ keret *koordinátagyűrűjének.* Mindez akkor is érvényben marad, ha $m = 3$ és L Désargues-féle.

A fent felsorolt fogalmak segítségével már megfogalmazható az értekezés első fejezetének fő eredménye.

Tétel ([27, Theorem 1.1])**.**

(a) *Legyen L korlátos háló, és legyenek $m,n \geq 2$ egész számok. Tegyük fel, hogy*

$$L \text{ moduláris és } m \geq 4. \tag{a1}$$

Legyen $(\vec{a},\vec{c}) = (a_1,\dots,a_m,c_{12},\dots,c_{1m})$ az L háló feszítő m-kerete és $(\vec{u},\vec{v}) = (u_1,\ \dots,\ u_n,v_{12},\dots,v_{1n})$ a $[0,a_1]$ intervallum feszítő n-kerete. Jelölje R^* az $(\vec{a},\vec{c})$ kerethez tartozó koordinátagyűrűt. Ekkor létezik olyan S^* gyűrű, amire R^* izomorf az S^* feletti $(n\times n)$-es mátrixok gyűrűjével. Ha

$$n \geq 4, \tag{a2}$$

akkor S^* választható az $(\vec{u},\vec{v})$ kerethez tartozó koordinátagyűrűnek.

(b) *A tétel előző része érvényben marad akkor is, ha az* (a1) *és* (a2) *feltételeket rendre a következőkre cseréljük*

$$L \text{ Désargues-féle és } m \geq 3, \tag{b1}$$

valamint

$$n \geq 3. \tag{b2}$$

Bár a tétel megfogalmazásához nem szükséges ismerni a már említett szorzatkeret valamint a külső és belső keret fogalmát, jegyezzük meg, hogy a tételben szereplő $(\vec{a}, \vec{c})$ m-keretet nevezzük *külső*, az $(\vec{u}, \vec{v})$ n-keretet pedig *belső* keretnek. Érdemes azt is megemlíteni a szorzatkeret tényleges definíciója nélkül, hogy az S^* gyűrű lényegében a szorzatkerethez tartozó koordinátagyűrűt jelöli.

Izometrikus beágyazások

Adott L alulról korlátos háló esetén a $p: L \to \mathbb{N}_\infty = \{0, 1, \ldots, \infty\}$ függvényt *pszeudorang függvénynek* nevezzük, ha teljesülnek rá az alábbi feltételek:

(i) $p(0) = 0$;

(ii) minden $a \leq b$ elemre $p(a) \leq p(b)$;

(iii) minden $a < b$ véges magasságú elemre $p(a) < p(b)$;

(iv) $p(a \wedge b) + p(a \vee b) \leq p(a) + p(b)$ minden a, b elemre és

(v) $p(a) < \infty$ pontosan akkor teljesül, ha a véges magasságú elem.

A fenti definíció véges hálók esetén megegyezik Finkbeiner [32] valamint Stern [79] definíciójával. Vegyük észre, hogy ha L féligmoduláris, akkor a Jordan–Hölder-láncfeltétel közvetlen következménye, hogy a magasságfüggvény teljesíti a fenti feltételeket, ezért pszeudorang függvény.

Legyen adott egy (alulról korlátos) L háló, egy $p: L \to \mathbb{N}_\infty$ pszeudorang függvény és egy G geometriai háló, melynek magasságfüggvényét jelölje h. Azt mondjuk, hogy L *izometrikusan beágyazható* G-be, ha létezik olyan $\varphi: L \to G$ beágyazás, amire $p = h \circ \varphi$ teljesül, vö. Grätzer és Kiss [43].

Ahhoz, hogy megfogalmazzuk az értekezés második fejezetének fő eredményét, amely Grätzer és Kiss [43] véges hálókra vonatkozó hasonló eredményét általánosítja, szükségünk van még egy fogalomra. Egy teljes hálót nevezzünk *majdnem alacsonynak*, ha minden eleme előáll véges magasságú elemek egyesítéseként. Például $\mathbb{N}_\infty$ a szokásos rendezésre nézve majdnem alacsony.

Tétel ([78, Theorem 1]). *Minden majdnem alacsony pszeudorang függvénnyel rendelkező algebrai háló beágyazható izometrikusan egy geometriai hálóba.*

Az előző tételnek megfogalmazható féligmoduláris hálókra egy közvetlen következménye. Nevezzünk egy hálóbeágyazást *fedésőrzőnek*, ha megőrzi a fedés relációt.

Következmény ([78, Corollary 2]). *Minden majdnem alacsony féligmodulárs algebrai hálónak létezik fedésőrző beágyazása egy geometriai hálóba.*

Mal'cev feltételek

Legyen $\mathcal{V}$ olyan varietás, aminek a típusában szerepel a 0 konstans műveleti jel. Ekkor azt mondjuk, hogy $\mathcal{V}$ *kongruencia-moduláris a 0-nál*, ha tetszőleges $\mathbf{A} \in \mathcal{V}$ algebra bármely α, β és γ kongruenciájára teljesül a következő összefüggés: $[0]\alpha \vee (\beta \wedge (\alpha \vee \gamma)) = [0](\alpha \vee \beta) \wedge (\alpha \vee \gamma)$, vö. Chajda [11] valamint Chajda és Halaš [12]. Jegyezzük meg, hogy a kongruencia-modularitásból következik a kongruencia-modularitás a 0-nál, például bármilyen csoport- vagy gyűrűvarietás mindig kongruencia-moduláris a 0-nál, hiszen kongruencia-moduláris. Ezzel szemben a kongruencia-modularitás nem feltétlenül következik a 0-nál vett kongruencia-modularitásból.

Az értekezés harmadik fejezetének fő eredménye Day [28] kongruencia-modularitásra vonatkozó eredményének megfelelőjeként a 0-nál vett kongruencia-modularitást jellemzi Mal'cev feltétellel.

Tétel ([77, Theorem 1])**.** *Legyen $\mathcal{V}$ olyan varietás, aminek a típusában szerepel a* 0 *konstans műveleti jel. Ekkor az alábbi állítások ekvivalensek:*

(i) *$\mathcal{V}$ kongruencia-moduláris a* 0*-nál;*

(ii) *létezik n természetes szám és léteznek m_i $(i = 0, \ldots, n)$ háromváltozós kifejezések úgy, hogy $\mathcal{V}$-ben teljesülnek az alábbi azonosságok:*

$$m_0(x,y,z) = 0 \text{ és } m_n(x,y,z) = z; \tag{m1}$$

$$m_i(x,x,0) = 0 \quad \text{minden } i \text{ indexre;} \tag{m2}$$

$$m_i(x,x,z) = m_{i+1}(x,x,z) \quad \text{minden páratlan } i \text{ indexre;} \tag{m3}$$

$$m_i(0,z,z) = m_{i+1}(0,z,z) \quad \text{minden páros } i \text{ indexre.} \tag{m4}$$

Bibliography

[1] Adaricheva, K., Czédli, G.: *Notes on the description of join-distributive lattices by permutations.* arXiv:1210.3376v1 [math.RA] 11 Oct 2012

[2] Adaricheva, K., Gorbunov, V.A., Tumanov, V.I.: *Join-semidistributive lattices and convex geometries.* Advances in Mathematics **173**, 1–49 (2003)

[3] Artmann, B.: *On coordinates in modular lattices with a homogeneous basis.* Illinois J. Math. **12**, 626–648 (1968)

[4] Barto, L., Kozik, M.: *New conditions for Taylor varieties and CSP.* 25th Annual IEEE Symposium on Logic in Computer Science LICS 2010, 100–109, IEEE Computer Soc., Los Alamitos, CA (2010)

[5] Barto, L., Kozik, M.: *Absorbing subalgebras, cyclic terms and the constraint satisfaction problem.* Log. Methods in Comput. Sci. **8**, 1:07, 1–26 (2012)

[6] Birkhoff, G.: *On the structure of abstract algebras.* Proc. Cam. Phil. Soc. **31**, 433–454 (1935)

[7] Birkhoff, G.: *Von Neumann and Lattice Theory.* Bull. Amer. Math. Soc. **64**, 50–56 (1958)

[8] Birkhoff, G.: *Lattice Theory.* Third edition. American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I. (1967)

[9] Burris, S.N., Sankappanavar, H.P.: *A Course in Universal Algebra.* Springer Verlag, Berlin, New York (1981)

[10] Chajda, I.: *Algebraic Theory of Tolerance Relations.* Palacky University Olomouc (1991)

[11] Chajda, I.: *Congruence distributivity in varieties with constants.* Arch. Math. (Brno) **22**, 121–124 (1986)

[12] Chajda, I., Halaš, R.: *Congruence modularity at* 0. Discuss. Math. Algebra and Stochastic Methods **17**, no. 1, 57–65 (1997)

[13] Crawley, P., Dilworth, R.P.: *Algebraic theory of lattices.* Prentice-Hall, Englewood Cliffs, NJ (1973)

[14] Csákány, B.: *Characterizations of regular varieties.* Acta Sci. Math. (Szeged) **31**, 187–189 (1970)

[15] Csákány, B.: *Mal'cev típusú tulajdonságok és alkalmazásaik.* thesis for the doctor of science degree, Szeged (1974) (Hungarian)

[16] Czédli, G.: *Notes on congruence implication.* Archivum Math. Brno **27**, 149–153 (1991)

[17] Czédli, G.: *How are diamond identities implied in congruence varieties.* Algebra Universalis **30**, 291–293 (1993)

[18] Czédli, G.: *The product of von Neumann n-frames, its characteristic, and modular fractal lattices.* Algebra Universalis **60**, 217–230 (2009)

[19] Czédli, G.: *Coordinatization of join-distributive lattices.* arXiv:1208.3517v3 [math.RA] 12 Oct 2012

[20] Czédli, G., Horváth, E.K.: *Congruence distributivity and modularity permit tolerances.* Acta Univ. Palacki. Olomuc., Fac. Rerum Natur. Math. **41**, 39–42 (2002)

[21] Czédli, G., Horváth, E.K.: *All congruence lattice identities implying modularity have Mal'tsev conditions.* Algebra Universalis **50**, 69–74 (2003)

[22] Czédli, G., Horváth, E.K., Radeleczki, S.: *On tolerance lattices of algebras in congruence modular varieties.* Acta Math. Hungar. **100**, 9–17 (2003)

[23] Czédli, G., Schmidt, E.T.: *A cover-preserving embedding of semimodular lattices into geometric lattices.* Advances in Mathematics **225**, 2455–2463 (2010)

[24] Czédli, G., Schmidt, T.: *The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices.* Algebra Universalis **66**, 69–79 (2011)

[25] Czédli, G., Schmidt, E.T.: *Slim semimodular lattices. I. A visual approach.* Order 29, 481–497 (2012)

[26] Czédli, G., Schmidt, E.T.: *Slim semimodular lattices. II. A description by patchwork systems.* Order, published online August 29, 2012

[27] Czédli, G., Skublics, B.: *The ring of an outer von Neumann frame in modular lattices.* Algebra Universalis **64**, 187–202 (2010)

[28] Day, A.: *A characterization of modularity for congruence lattices of algebras.* Canad. Math. Bull. **12**, 167–173 (1969)

[29] Day, A.: *In search of a Pappian lattice identity.* Canad. Math. Bull. **24**, 187–198 (1981)

[30] Day, A., Pickering, D.: *The coordinatization of Arguesian lattices.* Trans. Amer. Math. Soc. **278**, 507-522 (1983)

[31] Dedekind, R.: *Über die von drei Moduln erzeugte Dualgruppe.* Math. Ann. **53**, 371–403 (1900) (German)

[32] Finkbeiner, D.T.: *A semimodular imbedding of lattices.* Canad. J. Math. **12**, 582–591 (1960)

[33] Freese, R.: *Projective geometries as projective modular lattices.* Trans. Amer. Math. Soc. **251**, 329–342 (1979)

[34] Freese, R.: *The variety of modular lattices is not generated by its finite members.* Trans. Amer. Math. Soc. **255**, 277–300 (1979)

[35] Freese, R.: *Free modular lattices.* Trans. Amer. Math. Soc. **261**, 81–91 (1980)

[36] Freese, R., Herrmann, C., Huhn, A.P.: *On some identities valid in modular congruence varieties.* Algebra Universalis **12**, 322–334 (1981)

[37] Freese, R., Ježek, J., Nation, J.B.: *Free Lattices.* Mathematical Surveys and Monographs, vol. 42, Amer. Math. Soc., Providence, RI (1995)

[38] Freese, R., McKenzie, R.: *Commutator theory for congruence modular varieties.* London Mathematical Society Lecture Notes Series **125**, Cambridge University Press, Cambridge (1987)

[39] Freese, R., Nation, J.B.: *Congruence lattices of semilattices.* Pacific J. Math. **49**, 51–58 (1973)

[40] Grätzer, G.: *Two Mal'cev-type theorems in universal algebra.* J. Combinatorial Theory **8**, 334–342 (1970)

[41] Grätzer, G.: *General Lattice Theory.* Birkhäuser Verlag, Basel-Stuttgart (1978); Second edition: Birkhäuser Verlag (1998)

[42] Grätzer, G.: *Lattice Theory: foundation.* Birkhäuser Verlag, Basel (2011)

[43] Grätzer, G., Kiss, E.W.: *A construction of semimodular lattices.* Order **2**, 351–365 (1986)

[44] Grätzer, G., Knapp, E.: *Notes on planar semimodular lattices I. Construction.* Acta Sci. Math. (Szeged) **73**, 445–462 (2007)

[45] Grätzer, G., Knapp, E.: *Notes on planar semimodular lattices II. Congruences.* Acta Sci. Math. (Szeged) **74**, 37–47 (2008)

[46] Grätzer, G., Knapp, E.: *Notes on planar semimodular lattices III. Rectangular lattices.* Acta Sci. Math. (Szeged) **75**, 29–48 (2009)

[47] Grätzer, G., Nation, J.B.: *A new look at the Jordan-Hölder theorem for semimodular lattices.* Algebra Universalis **64**, 309–311 (2010)

[48] Grätzer, G., Schmidt, E.T.: *Characterizations of congruence lattices of abstract algebras.* Acta. Sci. Math. (Szeged) **24**, 34–59 (1963)

[49] Gumm, H.P.: *Congruence modularity is permutability composed with distributivity.* Arch. Math. (Basel) **36**, 569-576 (1981)

[50] Hagemann, J., Mitschke, A.: *On n-permutable congruences.* Algebra Universalis **3**, 8–12 (1973)

[51] Herrmann, C.: *On the word problem for the modular lattice with four generators.* Math. Ann. **265**, 513–527 (1983)

[52] Herrmann, C.: *On the arithmetic of projective coordinate systems.* Trans. Amer. Math. Soc. **284**, 759–785 (1984)

[53] Herrmann, C., Huhn, A.P.: *Lattices of normal subgroups which are generated by frames.* Lattice Theory (Proc. Colloq., Szeged, 1974), pp. 97–136. Colloq. Math. Soc. János Bolyai, vol. 14, North Holland, Amsterdam (1976)

[54] Huhn, A.P.: *Schwach distributive Verbände. I.* Acta Sci. Math. (Szeged) **33**, 297–305 (1972)

[55] Huhn, A.P.: *On G. Grätzer's problem concerning automorphisms of a finitely presented lattice.* Algebra Universalis **5**, 65–71 (1975)

[56] Huhn, A.P.: *On n-distributive systems of elements of a modular lattice.* Publ. Math. Debrecen **27**, 107–115 (1980)

[57] Hutchinson, G.: *Recursively unsolvable word problems for modular lattices and diagram-chasing.* J. Algebra **26**, 385–399 (1973)

[58] Hutchinson, G.: *Embedding and unsolvability theorems for modular lattices.* Algebra Universalis **7**, 47–84 (1977)

[59] Hutchinson, G., Czédli, G.: *A test for identities satisfied in lattices of submodules.* Algebra Universalis **8**, 269–309 (1978)

[60] Jónsson, B.: *On the representation of lattices.* Math. Scand. **1**, 193–206 (1953)

[61] Jónsson, B.: *Modular lattices and Desargues' theorem.* Math. Scand. **2**, 295–314 (1954)

[62] Jónsson, B.: *Lattice theoretic approach to projective and affine geometry.* 1959 The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957-Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski) pp. 188–203 Studies in Logic and the Foundations of Mathematics North-Holland Publishing Co., Amsterdam

[63] Jónsson, B.: *Algebras whose congruence lattices are distributive.* Math. Scand. **21**, 110–121 (1967)

[64] Jónsson, B.: *Congruence varieties.* Algebra Universalis **10**, 355–394 (1980)

[65] Lakser, H., Taylor, W., Tschantz, S.T.: *A new proof of Gumm's Theorem.* Algebra Universalis **20**, 115–122 (1985)

[66] Lipshitz, L.: *The undecidability of the word problems for projective geometries and modular lattices.* Trans. Amer. Math. Soc. **193**, 171–180 (1974)

[67] Mal'cev, A.I.: *On the General Theory of Algebraic Systems.* Mat. Sb. N.S. **35(77)**, 3–20 (1954) (Russian)

[68] Nation, J.B.: *Varieties whose congruences satisfy certain lattice identities.* Algebra Universalis **4**, 78–88 (1974)

[69] Neumann, J. von: *Continuous Geometry.* (Foreword by Israel Halperin), Princeton University Press, Princeton (1960, 1998)

[70] Oxley, J.G.: *Infinite matroids.* Proc. London Math. Soc. **37**, 259–272 (1978)

[71] Oxley, J.G.: *Matroid Theory.* Oxford University Press, Oxford, New York, Toronto (1992); Second edition: Oxford University Press (2011)

[72] Oxley, J.G.: *Infinite matroids.* In: White, N. (ed.) Matroid application. Encyclopedia of Mathematics and its Applications, vol. 40, pp. 73–90. Cambridge University Press (1992)

[73] Pixley, A.F.: *Local Mal'cev conditions.* Canad. Math. Bull. **15**, 559–568 (1972)

[74] Pudlák, P., Tůma, J.: *Every finite lattice can be embedded in the lattice of all equivalences over a finite set.* Comment. Math. Carolin. **18**, 409–414 (1977)

[75] Pudlák, P., Tůma, J.: *Every lattice can be embedded in a finite partition lattice.* Algebra Universalis **10**, 74–95 (1980)

[76] Rotman, J.J.: *An Introduction to the Theory of Groups.* 4th edition: Graduate Texts in Mathematics, vol. 148, Springer (1995)

[77] Skublics, B.: *Congruence modularity at 0.* Algebra Universalis **66**, 63–67 (2011)

[78] Skublics, B.: *Isometrical embeddings of lattices into geometric lattices.* Order, published online November 9, 2012

[79] Stern, M.: *Semimodular Lattices: Theory and Applications.* Encyclopedia of Mathematics and its Applications, vol. 73, Cambridge University Press (1999)

[80] Takách, G.: *Pappian implies Arguesian for modular lattices.* Algebra Universalis **42**, 51–52 (1999)

[81] Taylor, W.: *Characterizing Mal'cev conditions.* Algebra Universalis **3**, 351–395 (1973)

[82] Veblen, O., Young, J.W.: *A Set of Assumptions for Projective Geometry.* American Journal of Mathematics **30**, 347-–380 (1908)

[83] Veblen, O., Young, J.W.: *Projective Geometry Volume I and II.* Ginn and Co., Boston (1910, 1917)

[84] Welsh, D.: *Matroid Theory.* Academic Press, London, New York, San Francisco (1976)

[85] Whitman, P.M.: *Free lattices.* Ann. of Math. (2) **42**, 325–330 (1941)

[86] Whitman, P.M.: *Free lattices II.* Ann. of Math. (2) **43**, 104–115 (1942)

[87] Whitman, P.M.: *Lattices, equivalence relations, and subgroups.* Bull. Amer. Math. Soc. **52**, 507–522 (1946)

[88] Wielandt, H.: *Eine Verallgemeinerung der invarianten Untergruppen.* Math. Zeit. **45**, 209–244 (1939) (German)

[89] Wild, M.: *Cover preserving embedding of modular lattices into partition lattices.* Discrete Math. **112**, 207–244 (2002)

[90] Wille, R.: *Kongruenzklassengeometrien.* Lecture Notes in Mathematics, vol. 113, Springer-Verlag, Berlin-New York (1970) (German)