On subuniverses of lattices and semilattices

Abstract of Ph.D. thesis

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Introduction

A lattice is an abstract structure in mathematics. According to Roman [12], the beginnings of lattice theory can be dated to the early 1890s, when the concept was developed by Richard Dedekind, during investigating subgroups of abelian groups. By Grätzer [10], he carried out related research on ideals of algebraic numbers and he introduced the concept of modularity as well. According to Grätzer [10], George Boole's propositional logic independently led to the concept of Boolean algebras in the first half of the nineteenth century. This was followed at the end of the nineteenth century by Charles S. Pierce and Ernst Schröder's investigation on the axiomatics of Boolean algebras, when they introduced the lattice concept.

The concept was developed further by Garrett Birkhoff in the mid-thirties of the last century in a brilliant series of papers, in which by Grätzer [10] he demonstrated the importance of lattice theory. Birkhoff monograph [5] turned lattice theory into a major branch of abstract algebra. With the papers mentioned above and work done by Valère Glivenko, Karl Menger, John von Neumann and Oystein Ore, lattice theory has become a standard branch of modern algebra. For more details, see Grätzer [10], Roman [12] and Rota [13].

The role of von Neumann deserves a separate mention. From `https://en.wikipedia.org/wiki/John_von_Neumann` we see that John von Neumann (1903-1957) was a Hungarian-American mathematician, physicist, computer scientist, engineer and polymath. Von Neumann is generally regarded as the foremost mathematician of his time and is said to be the last representative of the great mathematicians (see [11]). He integrated pure and applied sciences. Notions like von Neumann algebra, prizes and https://njszt.hu/hu are named after him. His excellence also manifested itself in lattice theory, and his work substantially contributed to the fact that lattice theory eventually became a separate branch of mathematics. The founder of lattice theory and Universal Algebra, Garrett Birkhoff himself wrote in [6] that

"‘John von Neumann's brilliant mind blazed over lattice theory like a meteor, during a brief period centering around 1935-1937."’

and

"‘One wonders what would have been the effect on lattice theory, if von Neumann's intense two-year preoccupation with lattice theory had continued for twenty years!"’

Another milestone in the history of lattice theory was the year 1971, when the first journal devoted to lattices was founded. This journal called Algebra Universalis is still going strong. Well, it is also devoted to universal algebra not just lattice theory, but these two branches have a lot in common at the topical level and personal level. The founder, George Grätzer, is famous for producing results, papers and monographs on lattice theory; his 61-times coauthor, Elégius Tamás Schmidt (1936-2016) also deserves a mention.

Within a Ph.D. thesis, we cannot hope to give a reasonable survey of what transpired in lattice theory after the progress made by Birkhoff and John von Neumann. Instead of doing so, the reader is referred to the paper Rota [13], and to introductory sections of the monographs Grätzer [10] and Roman [12].

In addition to the above-mentioned lattice theorist Grätzer (Hungarian-Canadian) and Schmidt (Hungarian), it is worth mentioning that Hungarian lattice theorists Gábor Szász, András Huhn, Gábor Czédli and Sándor Radeleczki have made substantial contributions and have had a huge impact on this branch of mathematics with their researches.

The topic of the dissertation is restricted to finite lattices and semilattices. Next, we briefly summarise the main results upon which the dissertation is built on.

Some large numbers of subuniverses of finite lattices

In Chapter 2 (which is based on a joint paper with Horváth [2]), we proved that the fourth largest number of subuniverses of an n-element lattice is $21.5 \cdot 2^{n-5}$ for $n \geq 6$, and the fifth largest number of subuniverses of an n-element lattice is $21.25 \cdot 2^{n-5}$ for $n \geq 7$. Also, we described the n-element lattices with exactly $21.5 \cdot 2^{n-5}$ (for $n \geq 6$) and $21.25 \cdot 2^{n-5}$ (for $n \geq 7$) subuniverses. For a lattice L, $\mathrm{Sub}(L)$ denote its *sublattice lattice.*

Lemma 1. *If $|L| = n$ for the lattice L, and S is a partial sublattice* *of L with $|S| = k$ and with $|\operatorname{Sub}(S)| = m$, then $|\operatorname{Sub}(L)| \leq m \cdot 2^{n-k}$.*

The following lemma can be proved with a computer program. The program for counting subuniverses is available on the webpage of G. Czédli: http://www.math.u-szeged.hu/~czedli/, (subsize, a program for counting subuniverses 2019). The dissertation contains the standard proof for each case.

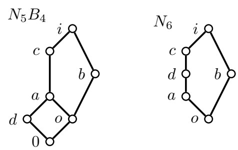

Figure 0.1: Lattices N_5B_4 and N_6

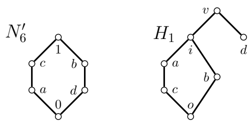

Figure 0.2: Lattice N_6' and partial lattice H_1

Lemma 2. *For the lattices and a partial lattice given in figures 0.1 to 0.3, the following five assertions hold.*

(i) $|\operatorname{Sub}(N_6)| = 43 = 21.5 \cdot 2^{6-5}$,

(ii) $|\operatorname{Sub}(N_5B_4)| = 69 = 17.25 \cdot 2^{7-5}$,

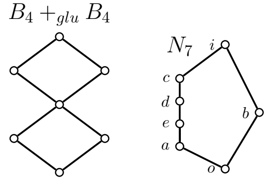

Figure 0.3: Lattices $B_4 +_{\mathrm{glu}} B_4$ and N_7

(iii) $|\operatorname{Sub}(N_6')| = 37 = 18.5 \cdot 2^{6-5}$,

(iv) $|\operatorname{Sub}(H_1)| = 79 = 19.75 \cdot 2^{7-5}$,

(v) $|\operatorname{Sub}(N_7)| = 83 = 20.75 \cdot 2^{7-5}$,

The following theorem states the main result of the chapter:

For a natural number $n \in \mathbb{N}^+$, let

$$\mathrm{NS}(n) := \{|\operatorname{Sub}(L)| : L \text{ is a lattice of size } |L| = n\}.$$

Theorem 3. *The following two assertions hold.*

(i) The fourth largest number in $\mathrm{NS}(n)$ *is* $21.5 \cdot 2^{n-5}$ *for* $n \geq 6$*. Furthermore, for* $n \geq 6$*, an* n*-element lattice* L *has exactly* $21.5 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} N_6 +_{\mathrm{glu}} C_1$*, where* C_0 *and* C_1 *are chains.*

(ii) The fifth largest number in $\mathrm{NS}(n)$ *is* $21.25 \cdot 2^{n-5}$ *for* $n \geq 7$*. Furthermore, for* $n \geq 7$*, an* n*-element lattice* L *has exactly* $21.25 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C_1$*, where* C_0 *and* C_1 *are chains.*

Several large numbers of subuniverses of finite semilattices

In Chapter 3 (which is based on a joint paper with Horváth [3]), motivated by the results of Chapter 2, we proved that the first largest number of subuniverses of an n-element semilattice is $2^n = 32 \cdot 2^{n-5}$, the second largest number is $28 \cdot 2^{n-5}$ and the third one is $26 \cdot 2^{n-5}$, where $n \geq 5$. Also, we described the n-element semilattices with exactly $32 \cdot 2^{n-5}$, $28 \cdot 2^{n-5}$, or $26 \cdot 2^{n-5}$ subuniverses. Following Czédli [7] and [8], we define the *relative number of subuniverses* of A as follows:

$$\sigma_k(A) := |\operatorname{Sub}(A)| \cdot 2^{k-n}.$$

Similarly, if $\mathcal{B} = (B, F_B)$, then

$$\sigma_k(\mathcal{B}) := |\operatorname{Sub}(\mathcal{B})| \cdot 2^{k-n}.$$

An element u of a semilattice L is called a narrow element, or *narrows* , if $u \neq 1_L$ and $L = \uparrow u \cup \downarrow u$. That is, if $u \neq 1_L$ and $x \| u$ holds for no $x \in L$.

Lemma 4. *If $(K, \vee)$ is a subsemilattice and H is a subset of a finite semilattice $(L, \vee)$, then the following three assertions hold.*

(i) With the notation $t := |H \cap S : S \in Sub(L, \vee)|$, we have that

$$\sigma_k(L, \vee) \leq t \cdot 2^{k-|H|}.$$

(ii) $\sigma_k(L, \vee) \leq \sigma_k(K, \vee)$.

(iii) Assume, in addition, that $(K, \vee)$ has no narrows. Then $\sigma_k(L, \vee) = \sigma_k(K, \vee)$ if and only if $(L, \vee)$ is (isomorphic to) $C_0 +_{\mathrm{ord}} (K, \vee) +_{\mathrm{glu}} C_1$, where C_1 is a chain, and C_0 is a chain or the emptyset.

The following lemma can be proved using a computer program. The program for counting subuniverses is available on the webpage of G. Czédli: http://www.math.u-szeged.hu/ czedli/ (subsize, a program for counting subuniverses 2019). The dissertation also contains the standard proof for each case.

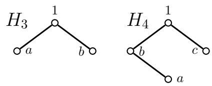

Figure 0.4: Partial lattices H_3 and H_4

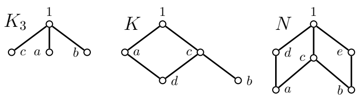

Figure 0.5: Partial lattices K_3, K and N

Lemma 5. *For the join-semilattices given in figures 0.4 to 0.6, the following seven assertions hold.*

(i) $\sigma_5(H_3) = 28$,

(ii) $\sigma_5(H_4) = 26$,

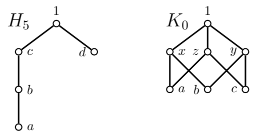

Figure 0.6: Partial lattices H_5 and K_0

(iii) $\sigma_5(H_5) = 25$,

(iv) $\sigma_5(K_3) = 24$,

(v) $\sigma_5(K) = 23$,

(vi) $\sigma_5(N) = 19.5$,

(vii) $\sigma_5(K_0) = 15.25$.

The following theorem summarizes the main result of the chapter:

Theorem 6. *If $5 \le n \in \mathbb{N}^+$, then the following three assertions hold.*

(i) The first largest number in $\mathrm{NS}(n)$ is $2^n = 32 \cdot 2^{n-5}$. Furthermore, an n-element semilattice $(L, \vee)$ has exactly 2^n subuniverses if and only if $(L, \vee)$ is a chain.

(ii) The second largest number in $\mathrm{NS}(n)$ is $28 \cdot 2^{n-5}$. Furthermore, an n-element semilattice $(L, \vee)$ has exactly $28 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_3 +_{\mathrm{glu}} C_1$ or $(L, \vee) \cong C_0 +_{\mathrm{ord}} H_3 +_{\mathrm{glu}} C_1$, where C_0 and C_1 are finite chains.

(iii) *The third largest number in* $\mathrm{NS}(n)$ *is* $26 \cdot 2^{n-5}$. *Furthermore, an* n*-element semilattice* $(L, \vee)$ *has exactly* $26 \cdot 2^{n-5}$ *subuniverses if and only if* $(L, \vee) \cong H_4 +_{\mathrm{glu}} C_1$ *or* $(L, \vee) \cong C_0 +_{\mathrm{ord}} H_4 +_{\mathrm{glu}} C_1$, *where* C_0 *and* C_1 *are finite chains.*

The number of subuniverses, congruences, weak congruences of semilattices defined by trees

In Chapter 4 (which is based on joint manuscript with Horváth and Németh [4]), first we determined the number of subuniverses of semilattices defined by arbitrary and special kinds of trees via combinatorial considerations, as follows:

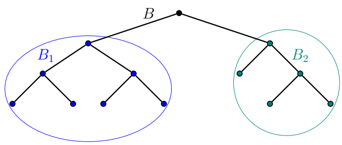

Figure 0.7: The Left and right maximal subtrees

Lemma 7. *If* $(T, \vee)$ *is a semilattice defined by a tree* T, *then*

$$|\mathrm{Sub}(T, \vee)| = \prod_{i=1}^{n} (|\mathrm{Sub}(T_i, \vee)|) + \sum_{i=1}^{n} (|\mathrm{Sub}(T_i, \vee)|) - (n-1),$$

where $T_1, \dots, T_n$ *is a repetition free list of maximal subtrees of the tree* T.

Corollary 7.1. *If* $(B, \vee)$ *is a semilattice defined by a binary tree* B*, then*

$$|\mathrm{Sub}(B, \vee)| = |\mathrm{Sub}(B_1, \vee)| \cdot |\mathrm{Sub}(B_2, \vee)| + (|\mathrm{Sub}(B_1, \vee)| + |\mathrm{Sub}(B_1, \vee)|) - 1,$$

where B_1*,* B_2 *are the left and right maximal subtrees of the tree, respectively.*

Corollary 7.2. *If* $(B, \vee)$ *is a semilattice defined by a prickly-snake* B *of height* h*, then*

$$|\mathrm{Sub}(B, \vee)| = 3\,|\mathrm{Sub}(B_1, \vee)| + 1 = \frac{5 \cdot 3^h - 1}{2},$$

where B_1 *is the left maximal subtree of the tree.*

Second, using a result of Freese and Nation [9], we gave a formula for the number of congruences of semilattices defined by arbitrary and special kinds of trees, as follows:

Lemma 8. *If* $(T, \vee)$ *is a semilattice defined by a tree* T*, then*

$$|\mathrm{Con}(T, \vee)| = 2^{|T|-1} = 2^{\sum_{i=1}^{n} |T_i|} = 2^n \cdot \prod_{i=1}^{n} |\mathrm{Con}(T_i, \vee)|\,,$$

where $T_1, \ldots, T_n$ *is a repetition free list of maximal subtrees of the tree* T*.*

Corollary 8.1. *If* $(B, \vee)$ *is a semilattice defined by a binary tree* B*, then*

$$|\mathrm{Con}(B, \vee)| = 2^{|B_1|+|B_2|} = 4 \cdot |\mathrm{Con}(B_1, \vee)| \cdot |\mathrm{Con}(B_2, \vee)|\,,$$

where B_1, B_2 are the left and right maximal subtrees of the tree, respectively.

Corollary 8.2. *If $(B, \vee)$ is a semilattice defined by a prickly-snake B of height h, then*

$$|\mathrm{Con}(B, \vee)| = 4 \cdot |\mathrm{Con}(B_1, \vee)| = 4^h,$$

where B_1 is the left maximal subtree of the tree.

Corollary 8.3. *If $(B, \vee)$ is a semilattice defined by a perfect binary tree B of height h, then*

$$|\mathrm{Con}(B, \vee)| = 4 \cdot |\mathrm{Con}(B_1, \vee)|^2 = 2^{2^{h+1}-2},$$

where B_1 is the left maximal subtree of the tree.

Third, using both results, we proved a formula for the number of weak congruences of semilattices defined by a binary tree, like so:

Lemma 9. *If $(B, \vee)$ is a semilattice defined by a binary tree B and $1' \notin B$, then*

$$\left|\mathrm{Cw}\Big((B, \vee) +_{\mathrm{ord}} \{1'\}\Big)\right| = 3 \cdot |\mathrm{Cw}(B, \vee)| - 1.$$

Theorem 10. *If $(B, \vee)$ is a semilattice defined by a binary tree B, then*

$$|\mathrm{Cw}(B, \vee)| = 4(|\mathrm{Cw}(B_1, \vee)| \cdot |\mathrm{Cw}(B_2, \vee)|) - (|\mathrm{Cw}(B_1, \vee)| + |\mathrm{Cw}(B_2, \vee)|),$$

where B_1, B_2 are the left and right maximal subtrees of the tree, respectively.

Corollary 10.1. *If* $(B, \vee)$ *is a semilattice defined by a prickly-snake* B *of height* h*, then*

$$|\mathrm{Cw}(B, \vee)| = 7 \cdot |\mathrm{Cw}(B_1, \vee)| - 2 = \frac{5 \cdot 7^h + 1}{3},$$

where B_1 *is the left maximal subtree of the tree.*

Finally, we solved two related nontrivial recurrences by applying the method of Aho and Sloane, as in the following theorems:

Theorem 11. *If* $(B, \vee)$ *is a semilattice defined by a perfect binary tree* B *of height* h*, then*

$$|\mathrm{Sub}(B, \vee)| = |\mathrm{Sub}(B_1, \vee)|^2 + 2\,|\mathrm{Sub}(B_1, \vee)| - 1,$$

where B_1 *is the left maximal subtree of the tree.*

Moreover,

$$|\mathrm{Sub}(B, \vee)| = \left\lceil C^{2^{h+1}} \right\rceil - 1, \quad C = 1.6784589651254\ldots$$

where $\lceil x \rceil$ *denotes the least integer greater than or equal to* x*.*

Theorem 12. *If* $(B, \vee)$ *is a semilattice defined by a perfect binary tree* B *of height* h*, then*

$$|\mathrm{Cw}(B, \vee)| = 4 \cdot |\mathrm{Cw}(B_1, \vee)|^2 - 2 \cdot |\mathrm{Cw}(B_1, \vee)|\,,$$

where B_1 *is the left maximal subtree of the tree.*

Moreover,

$$|\mathrm{Cw}(B, \vee)| = \left\lceil \tfrac{1}{4}\, C^{2^{h+1}} \right\rceil, \quad C = 2.61803398874989\ldots$$

where $\lceil x \rceil$ *denotes the least integer greater than or equal to* x*.*

$(1+1+2)$-generated lattices of quasiorders

In Chapter 5 (which is based on a joint paper with Czédli [1]), we proved that the lattice $\mathrm{Quo}(n)$ of all quasiorders (also known as preorders) of an n-element set is $(1+1+2)$-generated for $n = 3$, $n = 6$ (when $\mathrm{Quo}(6)$ consists of 209 527 elements), as follows:

Let $A = \{a, b, c, d, f, g\}$. We define the following quasiorders of A:

$$\begin{aligned} \alpha &:= e(d, f) \vee e(f, g), & \beta &:= \alpha \vee e(b, c) \vee q(b, a) \\ \gamma &:= e(a, b) \vee e(a, d) \vee e(c, f), & \delta &:= e(b, c) \vee e(c, g) \vee e(a, f). \end{aligned} \tag{0.1}$$

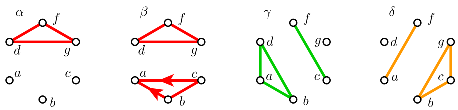

Figure 0.8: α, β, γ, and δ

Theorem 13. *With the quasiorders defined in* (0.1), $\{\alpha, \beta, \gamma, \delta\}$ *is a* $(1+1+2)$*-generating set of the quasiorder lattice* $\mathrm{Quo}(6) = \mathrm{Quo}(\{a, b, c, d, f, g\})$. *Hence,* $\mathrm{Quo}(6)$ *is* $(1+1+2)$*-generated.*

Corollary 13.1. $\mathrm{Quo}(3)$ *is* $(1+1+2)$*-generated.*

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