# On subuniverses of lattices and semilattices

Ph.D. thesis

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# 1

# Introduction

A lattice is an abstract structure in mathematics. According to Roman [44], the beginnings of lattice theory can be dated to the early 1890s, when the concept was developed by Richard Dedekind, during investigating subgroups of abelian groups. By Grätzer [33], he carried out related research on ideals of algebraic numbers and he introduced the concept of modularity as well. According to Grätzer [33], George Boole's propositional logic independently led to the concept of Boolean algebras in the first half of the nineteenth century. This was followed at the end of the nineteenth century by Charles S. Pierce and Ernst Schröder's investigation on the axiomatics of Boolean algebras, when they introduced the lattice concept.

The concept was developed further by Garrett Birkhoff in the mid-thirties of the last century in a brilliant series of papers, in which by Grätzer [33] he demonstrated the importance of lattice theory. Birkhoff monograph [8] turned lattice theory into a major branch of abstract algebra. With the papers mentioned above and work done by Valère Glivenko, Karl Menger, John von Neumann and Oystein Ore, lattice theory has become a standard branch of modern algebra. For more details, see Grätzer [33], Roman [44] and Rota [45].

The role of von Neumann deserves a separate mention. From `https://en.wikipedia.org/wiki/John_von_Neumann` we see that John von Neumann (1903-1957) was a Hungarian-American mathematician, physicist, computer scientist, engineer and polymath. Von Neumann is generally regarded as the foremost mathematician of his time and is said to be the last representative of the great mathematicians (see [39]). He integrated pure and applied sciences. Notions like von Neumann algebra, prizes and https://njszt.hu/hu are named after him. His excellence also manifested itself in lattice theory, and his work substantially contributed to the fact that lattice theory eventually became a separate branch of mathematics. The founder of lattice theory and Universal Algebra, Garrett Birkhoff himself wrote in [9] that

"‘John von Neumann’s brilliant mind blazed over lattice theory like a meteor, during a brief period centering around 1935-1937."’

and

"‘One wonders what would have been the effect on lattice theory, if von Neumann’s intense two-year preoccupation with lattice theory had continued for twenty years!"’

Another milestone in the history of lattice theory was the year 1971, when the first journal devoted to lattices was founded. This journal called Algebra Universalis is still going strong. Well, it is also devoted to universal algebra not just lattice theory, but these two branches have a lot in common at the topical level and personal level. The founder, George Grätzer, is famous for producing results, papers and monographs on lattice theory; his 61-times coauthor, Elégius Tamás Schmidt (1936-2016) also deserves a mention.

Within a Ph.D. thesis, we cannot hope to give a reasonable survey of what transpired in lattice theory after the progress made by Birkhoff and John von Neumann. Instead of doing so, the reader is referred to the paper Rota [45], and to introductory sections of the monographs Grätzer [33] and Roman [44].

In addition to the above-mentioned lattice theorist Grätzer (Hungarian-Canadian) and Schmidt (Hungarian), it is worth mentioning that Hungarian lattice theorists Gábor Szász, András Huhn, Gábor Czédli and Sándor Radeleczki have made substantial contributions and have had a huge impact on this branch of mathematics with their researches.

In the rest of our introduction, we restrict our attention to those results that directly motivated our work and are closely connected to it.

In recent years, intensive research has been carried out on finite lattices.

Freese [31] was able to prove that an $n$-element lattice has at most $2^{n-1}$ congruences. Inspired by this result, Czédli [19] showed that if $L$ has fewer than $2^{n-1}$ congruences, then it has at most $2^{n-2}$ congruences. He also described the $n$-element lattices with exactly $2^{n-2}$ congruences. Continuing the work of Freese and Czédli, Kulin and Mureşan [36] studied the smallest and the largest numbers of congruences of $n$-element lattices. They proved that the third, fourth and fifth largest numbers of congruences of an $n$–element lattice are $5 \cdot 2^{n-5}$ when $n \geq 5$, $2^{n-3}$ and $7 \cdot 2^{n-6}$ when $n \geq 6$, respectively. They also determined the structures of $n$–element lattices having these numbers. The above papers, as well as the paper of Czédli and Horváth [25], in which the authors found the first three largest numbers of subuniverses of finite lattices, motivated us to investigate the large numbers of subuniverses of lattices further and find the fourth and fifth largest numbers of subuniverses of n-element lattices. The results were published in one of our joint papers with Horváth and Chapter 2 is based on this paper, [3].

Chapter 3 is based on other joint paper with Horváth, [4], the aim of which was to determine the first three largest numbers of subuniverses of $n$-element semilattices. The first largest number is $2^n$, the second largest number is $28 \cdot 2^{n-5}$ and the third largest number is $26 \cdot 2^{n-5}$. We also described the $n$-element semilattices with exactly $2^n$, $28 \cdot 2^{n-5}$ or $26 \cdot 2^{n-5}$ subuniverses. We were inspired in writing the above paper by similar or analogous papers by various lattice theorists, for instance Adaricheva [1], Czédli [21] and [23].

Chapter 4 is based on joint manuscript with Horváth and Németh, [5], and the main aim here was to determine the number of subuniverses, congruences and weak congruences of semilattices defined by rooted binary trees. The concept of weak congruences is a tool for studying congruences and subalgebras of the same algebra together. The first researcher who studied the compatible symmetric and transitive relations on an algebra was F. Šik, together with his Ph.D. student T.D. Mai (see [55]). Šešelja and Tepavčević wrote a book [55] on weak congruences. The purpose of their book was to present the basic properties of weak congruences, in particular, their lattices, and to show how they can be applied in universal algebra. The book was published after several years of systematic studies by the authors, and it contains the bibliography on these and related topics up to 2001. Some later results on weak congruences can be found in, for example, Czédli et al. [28] and Šešelja and Tepavčević [56].

Chapter 5 is based on a joint paper with Czédli, [2], whose main goal was to prove that the 209 527-element lattice Quo(6) is $(1+1+2)$-generated. The idea that large lattices can have small generating sets goes back to 1976 when Poguntke and Rival [41] proved that each lattice can be embedded into a four-generated finite simple lattice. It turned out much later that three generators are sufficient if we drop the simplicity assumption; see Czédli [17]. Partition lattices, which are the same as equivalence lattices up to an isomorphism, are known to be simple. Hence, Pudlák and Tůma's result that every finite lattice is embeddable into a finite partition lattice (see [42]), superseded Poguntke and Rival's result in 1980. However, Poguntke and Rival's result still served as the motivation for Strietz [50] and [51] to prove that *equivalence lattice* Equ($n$) over $n$-element set is four-generated for $3 \leq n \in \mathbb{N}^+$ and it is $(1 + 1 + 2)$-generated for $10 \leq n \in \mathbb{N}^+$. In 1983, Zádori [57] furnished an entirely new method for finding four-element generating sets of Equ($n$) and extended Strietz's result by proving that Equ($n$) is $(1+1+2)$-generated even for $7 \leq n \in \mathbb{N}^+$. He left open the problem of whether Equ($n$) for $n \in \{5, 6\}$ is $(1+1+2)$-generated. Then 37 years later, Czédli and Oluoch [27] published a solution to Zádori's problem on Equ(5) and Equ(6). The existence of a four-element generating set is basically about sublattices. Indeed, the meaning of "$\{a, b, c, d\}$ generates L" is that no proper sublattice of $L$ includes $\{a, b, c, d\}$ as a subset. Their method for Equ(6) encouraged us to focus on the analogous problem for Quo(6). Now we can see more clearly the relationship between these ideas and the general aim of the dissertation.

# 2

# Some large numbers of subuniverses of finite lattices

This chapter is based on a joint paper with Horváth [3]. By a subuniverse, we mean a sublattice or the emptyset. We prove that the fourth largest number of subuniverses of an $n$-element lattice is $21.5 \cdot 2^{n-5}$ for $n \geq 6$, and the fifth largest number of subuniverses of an $n$-element lattice is $21.25 \cdot 2^{n-5}$ for $n \geq 7$. Also, we describe the $n$-element lattices with exactly $21.5 \cdot 2^{n-5}$ (for $n \geq 6$) and $21.25 \cdot 2^{n-5}$ (for $n \geq 7$) subuniverses.

## 2.1 Notations used in the chapter

All the lattices in this chapter will be assumed to be *finite*. For a lattice $L$, let $\mathrm{Sub}(L)$ denote its *sublattice lattice*; it consists of all the *subuniverses* of $L$. A subset $X$ of $L$ is in $\mathrm{Sub}(L)$ if and only if $X$ is closed with respect to join and meet. In particular $\emptyset \in \mathrm{Sub}(L)$. Note that, for $X \in \mathrm{Sub}(L)$, $X$ is a sublattice of $L$ if and only if $X$ is nonempty.

For a natural number $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, let

$$\mathrm{NS}(n) := \{|\mathrm{Sub}(L)| : L \text{ is a lattice of size } |L| = n\}.$$

That is, $k \in \mathrm{NS}(n)$ if and only if some $n$-element lattice has exactly $k$ subuniverses. The acronym NS means Number of Sublattices, and $L$ has only

$|\operatorname{Sub}(L)| - 1$ *sublattices.*

Let $P$ and $Q$ be posets with disjoint underlying sets. Then the *ordinal sum* $P +_{\text{ord}} Q$ is the poset on $P \cup Q$ with $s \leq t$ iff either $s, t \in P$ and $s \leq t$; or $s, t \in Q$ and $s \leq t$; or $s \in P$ and $t \in Q$. In other words, every element of $P$ is less than every element of $Q$, and the relations in $P$ and $Q$ remain the same. To draw the Hasse diagram of $P +_{\text{ord}} Q$, we place the Hasse diagram of $Q$ above that of $P$ and then connect every minimal element of $Q$ with all maximal elements of $P$ (see Figure 2.1). If $K$ with 1 and $L$ with 0 are finite posets, then their glued sum $K +_{\text{glu}} L$ is the ordinal sum of the posets $K \setminus \{1_K\}$, the singleton poset, and $L \setminus \{0_L\}$, in this order (see Figure 2.2). Note that $+_{\text{glu}}$ is an associative but not a commutative operation.

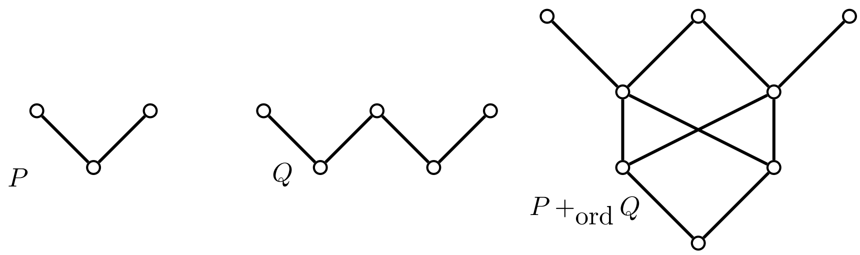


Figure 2.1: The ordinal sum $P +_{\text{ord}} Q$ of $P$ and $Q$

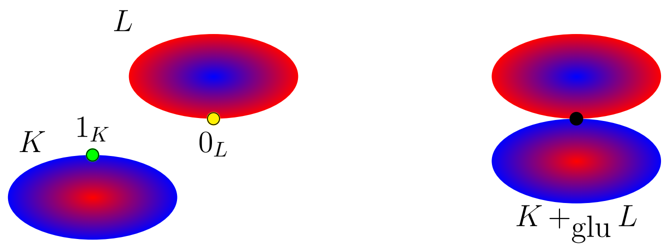


Figure 2.2: The glued sum $K +_{\text{glu}} L$ of $K$ and $L$

For elements $u, v \in L$, the *interval* $[u, v] := \{x \in L : u \leq x \leq v\}$ is defined only if $u \leq v$. For $u$ in $L$, the principal ideal generated by $u$ is $\downarrow u := \{x \in L : x \leq u\}$, while the principal filter is $\uparrow u := \{x \in L : u \leq x\}$. We can also write $\downarrow_L u$ and $\uparrow_L u$ to specify the lattice $L$. For $u, v$ in $L$, we say that $u$ and $v$ are *incomparable* if $u \nleq v$ and $v \nleq u$, we write $u \parallel v$ to denote that $u$ and $v$ are incomparable. We say that $u$ is *join-irreducible* if $u$ has at most one lower cover; and *meet-irreducible* is defined dually. If an element is both *join-irreducible* and *meet-irreducible*, then it is called *doubly irreducible*. An $u$ in $L$ is *isolated* if $u$ is *doubly irreducible* and $L = \downarrow u \cup \uparrow u$. That is, $u$ is *doubly irreducible* and $x \parallel u$ holds for no $x$ in $L$. By an *isolated edge* we mean a prime interval $[u, v]$ such that $u \prec v$ (i.e. $v$ covers $u$, if and only if $u < v$ and there is no $x$ such that $u < x < v$ ) and $L = \downarrow u \cup \uparrow v$.

Let $F$ be a set of binary operation symbols. By a *binary partial algebra* $\mathcal{A}$ of type $F$ we mean a structure $\mathcal{A} = (A; F_A)$ such that $A$ is a nonempty set, $F_A = \{f_A : f \in F\}$, and for each $f \in F$, $f_A$ is a map from a subset $\mathrm{Dom}(f_A)$ of $A^2$ to $A$. If $\mathrm{Dom}(f_A) = A^2$ for all $f \in F$, then $\mathcal{A}$ is a *binary algebra* (without the adjective "partial"). In particular, every lattice is a binary algebra; note that we write $\vee$ and $\wedge$ instead of $\vee_A$ and $\wedge_A$ when the meaning is clear from the context. A *subuniverse* of $\mathcal{A}$ is a subset $X$ of $A$ such that $X$ is closed with respect to all partial operations; that is, whenever $x, y \in X$, $f \in F$ and $(x, y) \in \mathrm{Dom}(f_A)$, then $f_A(x, y) \in X$. The set of subuniverses of $\mathcal{A}$ will be denoted by $\mathrm{Sub}(\mathcal{A})$. For example: for a lattice $(L, \vee, \wedge)$, assume that $S \subseteq L$; for $a, b, c \in S$, if $a \vee b = c$, then we say that $a \vee_S b$ is defined in $S$ and equal to $c$. While if $a \vee b$ is not in $S$, then we say that $a \vee_S b$ is not defined in $S$; $a \wedge_S b$ should be considered dually.

**Example 2.1.1.** *As an example of a partial algebra, we define the partial lattice $H_1$ as follows: for $x \parallel y$, $(x, y) \in \mathrm{Dom}(\wedge)$ if and only if $\{x, y\} \subseteq \{o, i, a, b, c\}$ and $(x, y) \in \mathrm{Dom}(\vee)$ if and only if $\{x, y\} \subseteq \{o, i, a, b, c\}$ or $\{x, y\} = \{d, i\}$; whenever $x \wedge y$ or $x \vee y$ are defined, they are $\inf\{x, y\}$ and $\sup\{x, y\}$, respectively. For the Hasse diagram of $H_1$; see Figure 2.8.*

We use the term partial lattice here to mean that $S$ is a partial algebra with two binary operations denoted by $\vee$ and $\wedge$. A *subuniverse* of $S$ is a subset $Y$ of $S$ such that if $a, b \in Y$ and $a \vee_S b$ is defined in $S$, then $a \vee_S b \in Y$; and the same is true for $\wedge_S$. We say that the partial lattice $S$ is a partial sublattice of the lattice $\mathcal{L} = (L; \vee_L, \wedge_L)$ if $S$ is a subposet of $L$ and for all $a, b \in S$, such that $a \parallel b$ and $(a, b) \in \mathrm{Dom}(\vee_S)$ we have that $a \vee_S b = a \vee_L b$. Mostly we denoted lattices by $L$ and partial lattices by $S$.

## 2.2 Preliminaries

Czédli and Horváth [25] proved that the first three largest numbers in $\mathrm{NS}(n)$ are $32 \cdot 2^{n-5}, 26 \cdot 2^{n-5}$, and $23 \cdot 2^{n-5}$, where $5 \leq n \in \mathbb{N}^{+}$. Moreover, an $n$-element lattice $L$ has exactly $2^n$ subuniverses if and only if $L$ is a chain. Also, they described the $n$-element lattices with exactly $26 \cdot 2^{n-5}$, and $23 \cdot 2^{n-5}$ subuniverses, as in the following theorem:

**Theorem 2.2.1** (Czédli and Horváth [25]). *If $5 \leq n \in \mathbb{N}^{+}$, then the following two assertions hold.*

*(i) The second largest number in $\mathrm{NS}(n)$ is $26 \cdot 2^{n-5}$. Furthermore, an $n$-element lattice $L$ has exactly $26 \cdot 2^{n-5}$ subuniverses if and only if $L \cong C_0 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C_1$, where $C_0$ and $C_1$ are chains while $B_4$ denotes the four-element Boolean lattice.*

*(ii) The third largest number in $\mathrm{NS}(n)$ is $23 \cdot 2^{n-5}$. Furthermore, an $n$-element lattice $L$ has exactly $23 \cdot 2^{n-5}$ subuniverses if and only if $L \cong C_0 +_{\mathrm{glu}} N_5 +_{\mathrm{glu}} C_1$, where $C_0$ and $C_1$ are chains while $N_5$ is the five-element nonmodular lattice.*

Next, we recall two lemmas, they will be used when proving Theorem 2.4.1.

**Lemma 2.2.2** (Czédli and Horváth [25]). *If $K$ is a sublattice and $H$ is a subset of a finite lattice $L$, then the following three assertions hold.*

*(i) With the notation $t := |\{H \cap G : G \in \mathrm{Sub}(L)\}|$, we have that $|\mathrm{Sub}(L)| \leq t \cdot 2^{|L|-|H|}$.*

*(ii) $|\mathrm{Sub}(L)| \leq |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$.*

*(iii) Assume, in addition, that $K$ has neither an isolated element, nor an isolated edge. Then $|\mathrm{Sub}(L)| = |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$ if and only if $L$ is (isomorphic to) $C_0 +_{\mathrm{glu}} K +_{\mathrm{glu}} C_1$ for some chains $C_0$ and $C_1$.*

The $n$-element chain and $n$-element Boolean lattice are denoted by $C^{(n)}$ and $B_n$, respectively.

**Lemma 2.2.3** (Czédli and Horváth [25]). *For the lattices given in Figures 2.3 to 2.5, the following seven assertions hold.*

*(i)* $|\mathrm{Sub}(B_4)| = 13 = 26 \cdot 2^{4-5}$,

*(ii)* $|\mathrm{Sub}(N_5)| = 23 = 23 \cdot 2^{5-5}$,

*(iii)* $|\mathrm{Sub}(B_4 +_{\mathrm{glu}} B_4)| = 85 = 21.25 \cdot 2^{7-5}$,

*(iv)* $|\mathrm{Sub}(B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} B_4)| = 169 = 21.125 \cdot 2^{8-5}$,

*(v)* $|\mathrm{Sub}(C^{(2)} \times C^{(3)})| = 38 = 19 \cdot 2^{6-5}$,

*(vi)* $|\mathrm{Sub}(M_3)| = 20 = 20 \cdot 2^{5-5}$,

*(vii)* $|\mathrm{Sub}(B_8)| = 74 = 9.25 \cdot 2^{8-5}$.

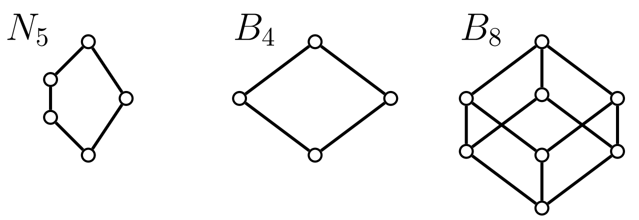


Figure 2.3: Lattices $N_5, B_4$ and $B_8$

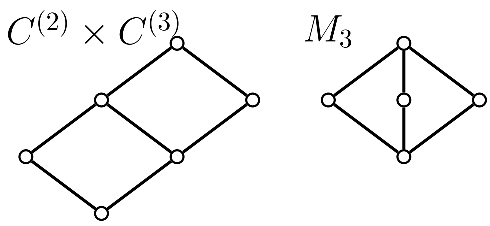


Figure 2.4: Lattices $C^{(2)} \times C^{(3)}$ and $M_3$

To shorten the proof of the Theorem 2.2.1, the authors of [25] proved Lemma 2.2.4. Note that a $k$-element antichain will be called a *$k$-antichain.*

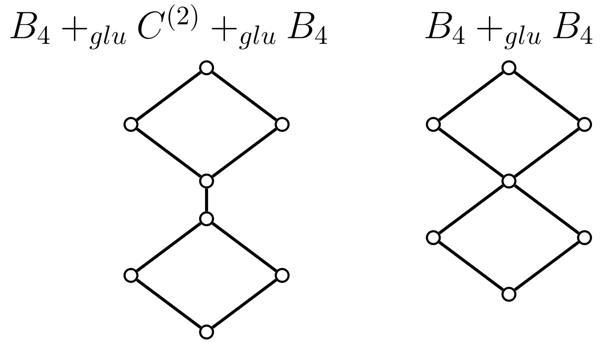


Figure 2.5: Lattices $B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} B_4$ and $B_4 +_{\mathrm{glu}} B_4$

**Lemma 2.2.4** (Czédli and Horváth [25]). *If an $n$-element lattice $L$ has a 3-antichain, then we have that* $|\operatorname{Sub}(L)| \le 20 \cdot 2^{n-5}$.

In order to prove the Lemma 2.2.4, the authors of [25] used two well-known facts, Lemmas 2.2.5 and 2.2.6. Implicitly, they often used the well-known Homomorphism Theorem 2.2.7.

**Lemma 2.2.5** (Rival and Wille [43]). *For every lattice $K$ generated by $\{a, b, c\}$ such that $a < c$, there is a unique surjective homomorphism $\varphi$ from the (so-called* finitely presented*) lattice $F_{\mathrm{lat}}(\tilde{a}, \tilde{b}, \tilde{c})$, given in Figure 2.6, onto $K$ such that $\varphi(\tilde{a}) = a$, $\varphi(\tilde{b}) = b$, and $\varphi(\tilde{c}) = c$.*

**Lemma 2.2.6** (Folklore). *For every join-semilattice $S$ generated by $\{a, b, c\}$, there is a unique surjective homomorphism $\varphi$ from the free join-semilattice $F_{\mathrm{jsl}}(\tilde{a}, \tilde{b}, \tilde{c})$, given in Figure 2.6, onto $S$ such that $\varphi(\tilde{a}) = a$, $\varphi(\tilde{b}) = b$, and $\varphi(\tilde{c}) = c$.*

Before recalling the well-known theorem, note that we compose homomorphism from right to left, that is $(f \circ g)(x) = f(g(x))$.

**Theorem 2.2.7** (Sankappanavar and Burris [46]). *Suppose $\alpha : A \to B$ is homomorphism onto $B$. Then there is an isomorphism $\beta$ from $A/ker(\alpha)$ to $B$ defined by $\alpha = \beta \circ \nu$, where $\nu$ is the natural homomorphism from $A$ to $A/ker(\alpha)$.*

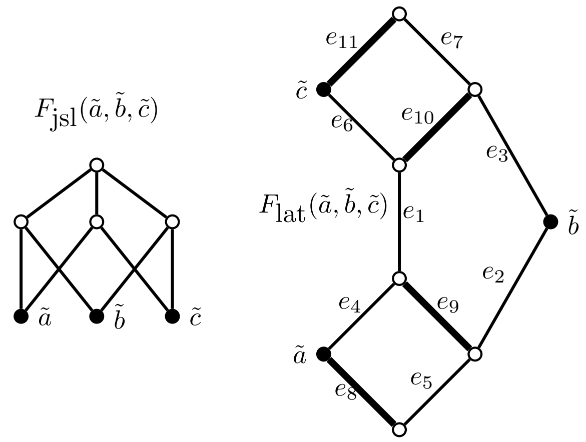


Figure 2.6: $F_{\mathrm{jsl}}(\tilde a,\tilde b,\tilde c)$ and $F_{\mathrm{lat}}(\tilde a,\tilde b,\tilde c)$

## 2.3 Auxiliary results

Below, we recall and prove only a special case of Lemma 2.3 from Czédli [20].

**Lemma 2.3.1.** *If $|L| = n$ for the lattice $L$, and $S$ is a partial sublattice of $L$ with $|S| = k$ and with $|\operatorname{Sub}(S)| = m$, then $|\operatorname{Sub}(L)| \le m \cdot 2^{n-k}$.*

*Proof.* First, we show that any subuniverse of $L$ is an extension of a subuniverse of $S$. Let $X \in \operatorname{Sub}(L)$, and let the restriction of $X$ to $S$ be $Y := X \cap S$. If $a, b \in Y$ and $a \vee b$ is defined in $S$, then $a \vee_S b = a \vee_L b \in X$ because $a, b \in Y \subseteq X$ and $X$ is closed under $\vee_L$. However $a \vee_S b \in S$, so $a \vee_S b \in S \cap X = Y$. We obtained that $Y$ is closed under $\vee_S$. Similarly, $Y$ is closed under $\wedge_S$. So, $Y$ is a subuniverse of $S$, and $X$ is an extension of $Y$. Clearly, any $Y \in \operatorname{Sub}(S)$ has $2^{n-k}$ extensions for a subset of $L$, and the number of subuniverses cannot be more than this. Since we have $m$ choices for $Y$, we see that $|\operatorname{Sub}(L)| \le m \cdot 2^{n-k}$. □

The following lemma can be proved with a computer program (see Appendix A). The program for counting subuniverses is available on the webpage of G. Czédli: http://www.math.u-szeged.hu/~czedli/, (subsize, a program for counting subuniverses 2019). Note that several of our computational results have also been verified by another program called sublatts, available at the same webpage.

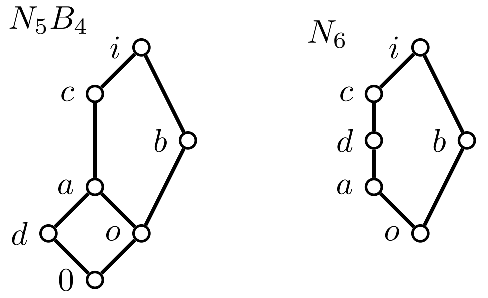


Figure 2.7: Lattices $N_5B_4$ and $N_6$

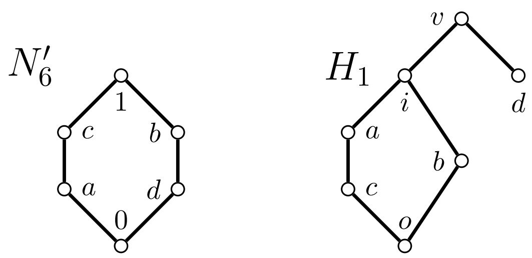


Figure 2.8: The lattice $N_6'$ and the partial lattice $H_1$ defined in Example 2.1.1

**Lemma 2.3.2.** *For the lattices and a partial lattice given by Figures 2.7–2.9 and, in case of $H_1$, by Example 2.1.1, the following five assertions hold.*

*(i)* $|\operatorname{Sub}(N_6)| = 43 = 21.5 \cdot 2^{6-5}$,

*(ii)* $|\operatorname{Sub}(N_5B_4)| = 69 = 17.25 \cdot 2^{7-5}$,

*(iii)* $|\operatorname{Sub}(N_6')| = 37 = 18.5 \cdot 2^{6-5}$,

*(iv)* $|\operatorname{Sub}(H_1)| = 79 = 19.75 \cdot 2^{7-5}$,

*(v)* $|\operatorname{Sub}(N_7)| = 83 = 20.75 \cdot 2^{7-5}$,

*Proof.* The notation given by Figure 2.7 to 2.9, will be used.

And for later reference, note that

$$\text{if } L \text{ is a chain, then } |\operatorname{Sub}(L)| = 2^{|L|}. \tag{2.1}$$

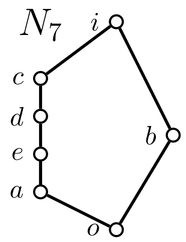


Figure 2.9: Lattice $N_7$

For (i), observe that

$$|\{S \in \mathrm{Sub}(N_6) : b \notin S\}| = 32, \qquad \text{by 2.1,}$$

$$|\{S \in \mathrm{Sub}(N_6) : b \in S, \{a, d, c\} \cap S = \emptyset\}| = 4, \text{ and}$$

$$|\{S \in \mathrm{Sub}(N_6) : b \in S, \{a, d, c\} \cap S \neq \emptyset\}| = 7,$$

whereby $|\,\mathrm{Sub}(N_6)| = 32 + 4 + 7 = 43$, and this proves (i).

The proof using the above mentioned computer program is:

**Input:**

```
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

    \beginjob
\name
N_6
\size
6
\elements
oabcid
\P edges
\P oa ad dc ci ob bi
\constraints
a+b=i b+d=i c+b=i , a*b=o c*b=o b*d=o
```

\endofjob
\enddata

**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 14:16:45) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2019 ]
|A|=6, A(without commas)={oabcid}. Constraints:
edges
oa ad dc ci ob bi
a+b=i b+d=i c+b=i a*b=o c*b=o b*d=o
Result for A=N_6: |Sub(A)| = 43, whence
sigma(A) = |Sub(A)|*2 ^ (5-|A|) = 21.5000000000000000 .

For (ii), observe that

$$|\{S \in \mathrm{Sub}(N_5B_4) : d \notin S\}| = 46, \qquad \text{by Lemma 2.2.2(iii) and Lemma 2.2.3 (ii),}$$
$$|\{S \in \mathrm{Sub}(N_5B_4) : d \in S, b \notin S\}| = 20, \text{ and}$$
$$|\{S \in \mathrm{Sub}(N_5B_4) : d \in S, b \in S\}| = 3,$$

whereby $|\,\mathrm{Sub}(N_5B_4)| = 46 + 20 + 3 = 69$, and this proves (ii). For the program calculation, see Appendix A.1.

For (iii), observe that

$$|\{S \in \mathrm{Sub}(N_6') : b \notin S\}| = 23, \qquad \text{by Lemma 2.2.2(iii) and Lemma 2.2.3(ii)}$$
$$|\{S \in \mathrm{Sub}(N_6') : d \in S, \{a, c\} \cap\ S \neq \emptyset\}| = 6, \text{ and}$$
$$|\{S \in \mathrm{Sub}(N_6') : \{a, c\} \cap\ S = \emptyset\}| = 8,$$

whereby $|\,\mathrm{Sub}(N_6')| = 23 + 6 + 8 = 37$, and this proves (iii). For the program calculation, see Appendix A.2.

For (iv), notice that $H_1$ is a partial lattice (see 2.1.1), but not a lattice, so subuniverses are those subsets that are closed with respect to all partial operations (see [20]). Observe that

$|\{S \in \mathrm{Sub}(H_1) : d \notin S\}| = 46,$ by Lemma 2.2.2(iii) and Lemma 2.2.3 (ii) ,

$|\{S \in \mathrm{Sub}(H_1) : \{d, v\} \subseteq S\}| = 23,$ and

the remaining subuniverses are the following: $\{b, d\}$, $\{o, b, d\}$, and all

the elements of $P(\{o, a, c\})$ with $d$,

whereby $|\,\mathrm{Sub}(H_1)| = 46 + 23 + 2 + 8 = 79$, and this proves (iv). For the program calculation, see Appendix A.3.

For (v), observe that

$$|\{S \in \mathrm{Sub}(N_7) : b \notin S\}| = 64, \qquad \text{by 2.1},$$

$$|\{S \in \mathrm{Sub}(N_7) : b \in S, \{a, e, d, c\} \cap S = \emptyset\}| = 4, \text{ and}$$

$$|\{S \in \mathrm{Sub}(N_7) : b \in S, \{a, e, d, c\} \cap S \neq \emptyset\}| = 15,$$

whereby $|\,\mathrm{Sub}(N_7)| = 64 + 4 + 15 = 83$, and this proves (v). For the program calculation, see Appendix A.4.

□

## 2.4 The fourth and fifth largest numbers of subuniverses of finite lattices

**Theorem 2.4.1.** *The following two assertions hold.*

*(i) The fourth largest number in* $\mathrm{NS}(n)$ *is* $21.5 \cdot 2^{n-5}$ *for* $n \geq 6$. *Furthermore, for* $n \geq 6$, *an* $n$*-element lattice* $L$ *has exactly* $21.5 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} N_6 +_{\mathrm{glu}} C_1$, *where* $C_0$ *and* $C_1$ *are chains.*

*(ii) The fifth largest number in* $\mathrm{NS}(n)$ *is* $21.25 \cdot 2^{n-5}$ *for* $n \geq 7$. *Furthermore, for* $n \geq 7$, *an* $n$*-element lattice* $L$ *has exactly* $21.25 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C_1$, *where* $C_0$ *and* $C_1$ *are chains.*

*Proof.* We shall prove part (i).

Let $L$ be an $n$-element lattice. We obtain from Lemma 2.2.2 (iii) and from 2.3.2(i) that if

$$L \cong C_0 +_{\mathrm{glu}} N_6 +_{\mathrm{glu}} C_1 \text{ for finite chains } C_0 \text{ and } C_1, \tag{2.2}$$

then $|\operatorname{Sub}(L)| = 21.5 \cdot 2^{n-5}$.

We know from Theorem 2.2.1(ii) that the third largest number in $\mathrm{NS}(n)$ is $23 \cdot 2^{n-5}$. Hence, in order to complete the proof of Theorem 2.4.1 (i), it suffices to exclude the existence of a lattice $L$ such that

$$\begin{aligned} &|L| = n,\ 21.5 \cdot 2^{n-5} \le |\operatorname{Sub}(L)| < 23 \cdot 2^{n-5}, \\ &\text{but } L \text{ is not of the form given in (2.2).} \end{aligned} \tag{2.3}$$

Suppose, for a contradiction, that $L$ is a lattice satisfying (2.3). If $L$ had a 3-antichain, then we would have by Lemma 2.2.4 that $|\operatorname{Sub}(L)| < 20 \cdot 2^{n-5}$, contradicting the first inequality in (2.3). If $L$ had at most one 2-antichain, then we would have that $|\operatorname{Sub}(L)| \ge 26 \cdot 2^{n-5}$ by Theorem 2.2.1, contradicting the second inequality in (2.3). Hence

$$L \text{ has at least two 2-antichains but it has no 3-antichain.} \tag{2.4}$$

We will prove that

$$L \text{ cannot have two } \textit{non-disjoint} \text{ 2-antichains.} \tag{2.5}$$

Suppose on the contrary that $\{a, b\}$ and $\{c, b\}$ are two distinct 2-antichains in $L$. Since there is no 3-antichain in $L$, we can assume that $a < c$. With $K := [\{a, b, c\}]$, let $\varphi\colon F_{\mathrm{lat}}(\tilde a, \tilde b, \tilde c) \to K$ be the unique lattice homomorphism from Lemma 2.2.5, and let $\Theta$ be the kernel of $\varphi$. We follow the notations of Figure 2.6. There are three cases.

**Case 1**: $\Theta$ does not collapse $e_1$ and it does not collapse at least one of $e_4$ and $e_6$. By duality, we can assume that $e_4$ is not collapsed. Since $e_1$ generates the monolith congruence, i.e. the smallest nontrivial congruence of the $N_5$ sublattice of $F_{\mathrm{lat}}(\tilde a, \tilde b, \tilde c)$, no other edge of the $N_5$ sublattice is collapsed. Now, $e_4$ is perspective to $e_5$, $e_9$ is perspective to $e_8$. Hence, $N_5B_4$ is a sublattice of $L$ and we conclude that $|\operatorname{Sub}(L)| \le 17.25 \cdot 2^{n-5}$ by Lemma 2.2.2 (ii) and by Lemma 2.3.2(ii). Therefore, Case 1 is excluded by (2.3).

**Case 2**: $\Theta$ does not collapse $e_1$ but it collapses both $e_4$ and $e_6$. Since $e_1$ generates the monolith congruence of the $N_5$ sublattice of $F_{\rm lat}(\tilde{a},\tilde{b},\tilde{c})$, no other edge of this $N_5$ sublattice is collapsed. Hence, as $N_5$ is a sublattice of $L$, clearly, $\{a,b,c\}$ generates a pentagon $N_5$. We know from Lemma 2.2.3(ii) that $|\operatorname{Sub}(N_5)| = 23$, and we also have assumed in 2.3 that $|\operatorname{Sub}(L)| < 23 \cdot 2^{n-5}$. Applying Lemma 2.2.2 (iii) for $K := N_5$, we see that

$$\left.\begin{array}{l} L \text{ cannot be of the form } L \cong C_0 +_{\rm glu} N_5 +_{\rm glu} C_1 \text{ for} \\ \text{finite chains } C_0 \text{ and } C_1. \end{array}\right\} \tag{2.6}$$

Let $o$ and $i$ stand for the least and the largest elements of the abovementioned $N_5$ sublattice, respectively. By rewording (2.6),

$$\text{we exclude the possibility that } {\downarrow}o \text{ is a chain, } {\uparrow}i \text{ is a chain, and } [o,i] = N_5. \tag{2.7}$$

Thus, at least one of the three parts of (2.7) fails. If ${\downarrow}o$ is *not* a chain, then we have a sublattice of the form either $B_4 +_{\rm glu} B_4$ or $B_4 +_{\rm glu} C^{(2)} +_{\rm glu} B_4$; but then the number of sublattices could be at most $21.25 \cdot 2^{n-5}$ by Lemma 2.2.3 (iii) and (iv) and by Lemma 2.2.2 (ii). Hence, ${\downarrow}o$ is a chain. By duality, ${\uparrow}i$ is a chain, too.

Next, we show that $x \nparallel o$ and $x \nparallel i$ for every $x \in L$. To do so, it is sufficient to deal with the relation between $x$ and $i$. Observe that if $x$ was incomparable with $i$ then we would have a copy of $H_1$ and $|\operatorname{Sub}(L)|$ would be at most $19.75 \cdot 2^{n-5}$ by Lemma 2.3.2(iv) and Lemma 2.3.1. Hence, $x \nparallel i$ and $x \nparallel o$, as required. For later reference, let us summarize:

$$\left.\begin{array}{l} {\downarrow}o \text{ and } {\uparrow}i \text{ are chains,} \\ \text{and for all } x \in L \setminus ({\downarrow}o \cup {\uparrow}i \cup N_5)\text{, we have that} \\ x \in [o,i]\text{; in particular, } L = {\downarrow}o \cup {\uparrow}i \cup [o,i]. \end{array}\right\} \tag{2.8}$$

Let us remind that $N_5$ above denotes the sublattice $\{o,i,a,b,c\}$.

Next, (2.6) and (2.8) yield an element $d \in [o,i] \setminus N_5$. The absence of 3-antichains, see (2.4), together with (2.8) imply that either $d$ is comparable with $b$ or it is comparable with both $a$ and $c$. According to these two possibilities, the argument for Case 2 splits into two subcases.

**Case 2a**: $d$ is comparable with $b$. By duality, we can assume that $d < b$. If $a \vee d = i$, then $\{o,i,a,b,c,d\} \cong N_6'$ would easily lead to $|\operatorname{Sub} L| \leq 18.5 \cdot 2^{6-5}$, by Lemma 2.2.2(ii) and Lemma 2.3.2(iii). Hence, $a \vee d = i$ is excluded, that is, $a \vee d \neq i$. So $v := a \vee d < i$. Observe that $v \not\leq b$ since otherwise $a \leq b$ would be a contradiction. We also have that $v \not\geq b$ since otherwise $v \geq a$ and $v \geq b$ would lead to $v \geq a \vee b = i$, contradicting that $v < i$. So $v \parallel b$. But $\{c, v, b\}$ is not an antichain, so $v \parallel b$ and $c \parallel b$ implies that $v > c$ or $v \leq c$. However, if $v \leq c$, then $d \leq v \leq c$ and $d \leq b$ would lead to $d \leq c \wedge b = o$, a contradiction. Hence, $v > c$. Note that $c \vee d = (c \vee a) \vee d = c \vee (a \vee d) = c \vee v = v$. So we have that

$$a \vee b = i,\ a \wedge b = o,\ c \vee b = i,\ c \wedge b = o,\ a \vee d = v, \text{ and } v \vee b = i. \tag{2.9}$$

Let $G_7$ denote the seven-element partial lattice $\{o, i, a, b, c, d, v\}$ defined by the equalities listed in (2.9) (see Figure 2.10). We have that it has exactly $19.5 \cdot 2^{7-5}$ subuniverses (for the calculation, see Appendix A.5), whence $|\operatorname{Sub}(L)| \leq 19.5 \cdot 2^{n-5}$ by Lemma 2.3.1. Since this estimate violates (2.3) Case 2a cannot occur.

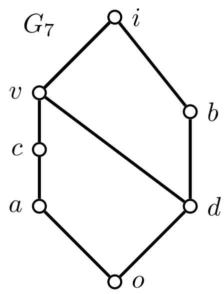


Figure 2.10: Partial lattice $G_7$

For the sake of possible future applications, note that the equalities

$$a \wedge d = o,\ c \wedge d = o,\ c \vee d = v \tag{2.10}$$

also hold. If $G_7^-$ denotes the partial lattice on the base set $\{o, i, a, b, c, d, v\}$ defined jointly by (2.9) and (2.10), then we have that $|\operatorname{Sub}(G_7^-)| = 16.75 \cdot 2^{7-5}$ (see Appendix A .6), and this would lead to $|\operatorname{Sub}(L)| \leq 16.75 \cdot 2^{n-5}$.

For later reference, note that since $d \in [o, i] \setminus N_5$ was arbitrary,

$$\text{for all } x \in [o, i] \setminus N_5 = [o, i] \setminus \{o, i, a, b, c\}, \text{ we have that } x \parallel b. \tag{2.11}$$

**Case 2b**: $d$ is comparable with $a$ and $c$. Now $\{a, c, d\}$ is a chain. Since $b, d \in [o, i]$, $o \leq b \wedge d$ and $b \vee d \leq i$. But $d \parallel b$ by (2.11), so $b \wedge d < b$ and $b < b \vee d$, and we have that $o \leq b \wedge d < b < b \vee d \leq i$. If $o < b \wedge d$, then $o < b \wedge d < b$ shows that $b \wedge d \in [o, i] \setminus N_5$, so the comparability $b \wedge d < b$ would violate (2.11). Hence $b \wedge d = o$. Similarly, $b \vee d = i$. So $\{a, c, d\}$ is a chain and for all $x \in \{a, c, d\}$ we have that $b \wedge x = o$ and $b \vee x = i$. This allows us to assume that $a < d < c$ since otherwise we can relabel the elements of the chain $\{a, c, d\}$. Also, since $b$ is a complement of every element of this chain in the interval $[o, i]$, the elements $o, i, a, b, c, d$ form a sublattice isomorphic to $N_6$. Since we know from (2.3) that $L$ is not of the form (2.2), the third line of (2.8) implies that there is an element $e \in [o, i] \setminus N_6$. By (2.11), $e \parallel b$. Since there is no 3-antichain, $\{a, c, d, e\}$ is a chain. Related to $\{o, i, a, b, c\}$, $e$ plays the same role as $d$. Hence $b \wedge e = o$ and $b \vee e = i$, and we have that $\{o, i, a, b, c, d, e\} \cong N_7$. Armed with this $N_7$, Lemmas 2.2.2(ii) and 2.3.2 (v) give that $|\operatorname{Sub}(L)| \leq 20.75 \cdot 2^{n-5}$, which contradicts (2.3). Therefore, Case 2b is excluded, too. Consequently, Case 2 cannot occur.

**Case 3**: $\Theta$ collapses $e_1$. Since $a \parallel b$ and $c \parallel b$, none of the thick edges $e_8, \ldots, e_{11}$ is collapsed by $\Theta$. Observe that at least one of $e_4$ and $e_6$ is *not* collapsed by $\Theta$, since otherwise $\langle \tilde{a}, \tilde{c} \rangle$ would belong to $\Theta = \ker(\varphi)$ by transitivity and so we would have that $a = c$, which would be a contradiction. By duality, we can assume that $e_4$ is not collapsed by $\Theta$. Since $e_2$, $e_3$, and $e_5$ are perspective to $e_{10}$, $e_9$, and $e_4$, respectively, these edges are not collapsed either. So, with the exception of $e_1$, no edge among the elements of $\downarrow(\tilde{a} \vee \tilde{b})$ in Figure 2.6 is collapsed. Thus, the $\varphi(\downarrow(\tilde{a} \vee \tilde{b}))$ is a sublattice (isomorphic to) $C^{(2)} \times C^{(3)}$ in $L$. Hence, $|\operatorname{Sub}(L)| \leq 19 \cdot 2^{n-5}$ by Lemma 2.2.2 (ii) and Lemma 2.2.3(v), which contradicts our assumption that $L$ satisfies (2.3) and shows that Case 3 cannot occur.

Now that all the three cases are excluded, we have proved (2.5).

To provide a convenient tool to exploit (2.4) and (2.5), we claim that

$$\begin{aligned} &\text{if } x, y, z \in L \text{ such that } |\{x, y, z\}| = 3 \text{ and } x \parallel y, \\ &\text{then either } \{x, y\} \subseteq \downarrow z, \text{ or } \{x, y\} \subseteq \uparrow z. \end{aligned} \tag{2.12}$$

To see this, assume the premise. Since $L$ has no 3-antichain, $z$ is comparable to one of $x$ and $y$. By duality and symmetry, we can assume that $x < z$.

Since $z \leq y$ would imply $x \leq y$, we have that $z \nleq y$. Also, we have that $z \nparallel y$ since otherwise the presence of the 2-antichains $\{x, y\}$ and $\{z, y\}$ would contradict (2.5). Hence, $y < z$, and we have proved (2.12).

Resuming our considerations based on the indirect assumption that $L$ violates (2.3), observe that (2.4) and (2.5) give us a four-element subset $\{a, b, c, d\}$ of $L$ such that $a \parallel b$ and $c \parallel d$. By duality and (2.12), we can assume that $\{a, b\} \subseteq {\downarrow}c$. Applying (2.12) to $\{a, b, d\}$ as well, we obtain that $\{a, b\}$ is included either in ${\uparrow}d$, or in ${\downarrow}d$. Since the first alternative would lead to $d \leq a \leq c$ and this would contradict $c \parallel d$, we have that $\{a, b\} \subseteq {\downarrow}d$. Thus, $\{a, b\} \subseteq {\downarrow}c \cap {\downarrow}d = {\downarrow}(c \wedge d)$, and we obtain that $u := a \vee b \leq c \wedge d =: v$. Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, $S$ is a sublattice isomorphic to $B_4 +_{\mathrm{glu}} B_4$ or $B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} B_4$. Using Lemma 2.2.2(ii) together with (iii) and (iv) of Lemma 2.2.3, we find that $|\operatorname{Sub}(L)| \leq 21.25 \cdot 2^{n-5}$. This inequality contradicts (2.3) and completes the proof of part (i) of Theorem 2.4.1.

We will now prove part (ii). Let $L$ be an $n$-element lattice. We obtain from Lemma 2.2.2 (iii) and Lemma 2.2.3 (iii) that if

$$L \cong C_0 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C_1 \text{ for finite chains } C_0 \text{ and } C_1, \tag{2.13}$$

then $|\operatorname{Sub}(L)| = 21.25 \cdot 2^{n-5}$. In order to complete the proof of part (ii) of Theorem 2.4.1, it suffices to exclude the existence of a lattice $L$ such that

$$\begin{aligned} &|L| = n,\ 21.25 \cdot 2^{n-5} \leq |\operatorname{Sub}(L)| < 21.5 \cdot 2^{n-5}, \\ &\text{but } L \text{ is not of the form given in (2.13).} \end{aligned} \tag{2.14}$$

Suppose for a contradiction that $L$ is a lattice satisfying (2.14).

We claim that

$$L \text{ cannot have two } \textit{non-disjoint} \text{ 2-antichains.} \tag{2.15}$$

Suppose on the contrary that $\{a, b\}$ and $\{c, b\}$ are two distinct 2-antichains in $L$. By Lemma 2.2.4, there is no 3-antichain in $L$. Hence we can assume that $a < c$.

With $K := [\{a, b, c\}]$, let $\varphi\colon F_{\mathrm{lat}}(\tilde{a}, \tilde{b}, \tilde{c}) \to K$ be the unique lattice homomorphism from Lemma 2.2.5, and let $\Theta$ be the kernel of $\varphi$. We will use the notation of Figure 2.6, and we will borrow a lot from the proof of part (i) of Theorem 2.4.1. Again, there are three cases.

**Case A**: $\Theta$ does not collapse $e_1$ and it does not collapse at least one of $e_4$ and $e_6$. By duality, we can assume that $e_4$ is not collapsed. Since $e_1$ generates the smallest nontrivial congruence of the $N_5$ sublattice of $F_{\rm lat}(\tilde a,\tilde b,\tilde c)$, no other edge of the $N_5$ sublattice is collapsed. Now, $e_4$ is perspective to $e_5$, $e_9$ is perspective to $e_8$. Hence, $N_5B_4$ is a sublattice of $L$ and we conclude that $|\operatorname{Sub}(L)|\leq 17.25\cdot 2^{n-5}$ by Lemma 2.2.2 (ii) and by Lemma 2.3.2(ii). Thus, Case A is excluded by (2.14).

**Case B**: $\Theta$ does not collapse $e_1$ but it collapses both $e_4$ and $e_6$. Since $e_1$ generates the monolith congruence of the $N_5$ sublattice of $F_{\rm lat}(\tilde a,\tilde b,\tilde c)$, no other edge of this $N_5$ is collapsed. Hence, $N_5$ is a sublattice of $L$ and, clearly, it is generated by $\{a,b,c\}$. If $L$ was of the form $C_0+_{\rm glu}N_5+_{\rm glu}C_1$ with chains $C_0$ and $C_1$, then Lemma 2.2.2(iii) and Lemma 2.2.3 (ii) would give that $|\operatorname{Sub}(L)|=23\cdot 2^{n-5}$, violating (2.14). Hence,

$$L \text{ is not of the form } C_0+_{\rm glu}N_5+_{\rm glu}C_1 \text{ with chains } C_0 \text{ and } C_1. \tag{2.16}$$

Let $o$ and $i$ stand for the least and the largest elements of the above-mentioned $N_5$ sublattice, respectively.

In order to prove by contradiction that $\downarrow o$ is chain, suppose the contrary. Then $B_4+_{\rm glu}B_4$ or $B_4+_{\rm glu}C^{(2)}+_{\rm glu}B_4$ is a sublattice of $L$.
If $B_4+_{\rm glu}C^{(2)}+_{\rm glu}B_4$ is a sublattice, then Lemma 2.2.2 (ii) and Lemma 2.2.3 (iv) give that $|\operatorname{Sub}(L)|\leq 21.125\cdot 2^{n-5}$, contradicting (2.14). However, if $B_4+_{\rm glu}B_4$ is a sublattice, then we have to argue a bit more that in the proof of part (i) of Theorem 2.4.1. Indeed, then Lemma 2.2.2 (ii) and Lemma 2.2.3 (iii) only give that $|\operatorname{Sub}(L)|\leq 21.25\cdot 2^{n-5}$, which together with (2.14) yield that $|\operatorname{Sub}(L)|=21.25\cdot 2^{n-5}$. However, then Lemma 2.2.2 (iii) and Lemma 2.2.3 (iii) give that $L$ is of the form $C_0+_{\rm glu}B_4+_{\rm glu}B_4+_{\rm glu}C_1$ with chains $C_0$ and $C_1$, which contradicts the presence of $N_5$ in $L$. This proves that $\downarrow o$ is a chain. Since the dual argument also works, we obtain that

$$\downarrow o \text{ and } \uparrow i \text{ are chains.} \tag{2.17}$$

As in the proof of the first part of Theorem 2.4.1, $x \parallel i$ would make Lemma 2.3.2(iv) and Lemma 2.3.1 applicable to a copy of $H_1$ and

$|\operatorname{Sub}(L)| \le 19.75 \cdot 2^{n-5}$ would contradict (2.14). Hence, taking the Duality Principle into account,

$$\text{every element of } L \text{ is comparable to } o \text{ and } i. \tag{2.18}$$

It follows from (2.17) and (2.18) that

$$L = {\downarrow}o \cup {\uparrow}i \cup [o,i], \text{ and both } {\downarrow}o \text{ and } {\uparrow}i \text{ are chains.} \tag{2.19}$$

By Lemma 2.2.4,

$$\text{for every } x \in [o,i] \setminus \{o,i,a,b,c\} = [o,i] \setminus N_5, \text{ neither } \{a,b,x\} \text{ nor } \{c,b,x\} \text{ is a 3-antichain.} \tag{2.20}$$

We assert that

$$\text{if } x \in [o,i] \setminus N_5, \text{ then } x \parallel b. \tag{2.21}$$

When proving (2.21) below, we write $d$ instead of $x$; so $d \in [o,i] \setminus N_5$. For the sake of contradiction, suppose that $d \nparallel b$. By duality, we can assume that $d < b$. Consider the element $v := a \vee d$. If we had $v = i$, then $\{o,i,a,b,c,d\} \cong N_6'$ would easily lead to $|\operatorname{Sub}(L)| \le 18.5 \cdot 2^{n-5}$ via lemmas 2.2.2(ii) and 2.3.2(iii). Hence $v < i$. We have that $v \not\le b$, because otherwise we would obtain that $a \le b$. Since $v \ge b$ would lead to $v = v \vee b \ge a \vee b = i$, it follows that $v \parallel b$. Hence, since none of $\{a,v,b\}$ and $\{c,v,b\}$ is a 3-antichain by Lemma 2.2.4, it follows that $v \nparallel a$ and $v \nparallel c$. So the three-element set $\{a,c,v\}$ is a chain in $L$. Since $v \le c$ would give that $d \le (a \vee d) \wedge b = v \wedge b \le c \wedge b = o$, we obtain from $v \nparallel c$ that $c < v$. Note that $c \vee d = (c \vee a) \vee d = c \vee (a \vee d) = c \vee v = v$. Now $\{o,i,a,b,c,d,v\}$ becomes a partial $G_7$-sublattice (see Figure 2.10) of $L$; see (2.9). Since $|\operatorname{Sub}(G_7)| = 19.5 \cdot 2^{7-5}$ by Appendix A.5 and so Lemma 2.3.1 yields that $|\operatorname{Sub}(L)| \le 19.5 \cdot 2^{n-5}$, contradicting (2.14). This proves (2.21).

Next, observe that if $[o,i] = N_5$ held, then (2.19) and Theorem 2.2.1(ii) would give that $|\operatorname{Sub}(L)| = 23 \cdot 2^{n-5}$, contradicting (2.14). Hence $[o,i] \ne N_5$ and we can pick an element

$$d \in [o,i] \setminus N_5.$$

By (2.21), $d \parallel b$. Since $a \parallel b$ but $\{a,b,d\}$ is not a 3-antichain by (2.20), we conclude that $d \nparallel a$. Similarly, by (2.21), $d \parallel b$. Since $c \parallel b$ but $\{c,b,d\}$ is not a 3-antichain by (2.20), we have that $d \nparallel c$. We have seen that

$$\{a, c, d\} \text{ is a chain.} \tag{2.22}$$

There are two subcases depending on $d \in [a, c]$ or $d \notin [a, c]$.

**Case B1**: $d \in [a, c]$, Then $a < d < c$. Clearly, $\{o, i, a, b, c, d\}$ forms a sublattice isomorphic to $N_6$. For simplicity, we will write $N_6 = \{o, i, a, b, c, d\}$. Using (2.14), the equality $|\operatorname{Sub}(N_6)| = 21.5 \cdot 2^{6-5}$ from Lemma 2.3.2(i), and Lemma 2.2.2 (iii), we obtained that $L$ is *not* of the form $C_0 +_{\text{glu}} N_6 +_{\text{glu}} C_1$ with $C_0$ and $C_1$ being chains. Hence, using (2.19), we can pick an element

$$e \in [o, i] \setminus N_6. \tag{2.23}$$

Combining (2.21) and (2.23), we obtain that $e \parallel b$. We already know that $y \parallel b$ for all $y \in \{a, d, c\}$. Since there is no 3-antichain by Lemma 2.2.4, $e \parallel b$ and $y \parallel b$ give that $e \nparallel y$ for all $y \in \{a, d, c\}$. Hence,

$$\{a, d, c, e\} \text{ is a chain, whence so is } \{o, i, a, d, c, e\}. \tag{2.24}$$

Now if $e$ belonged to $[a, c]$, then $\{o, i, a, b, c, d, e\}$ would be a sublattice isomorphic to $N_7$ and so Lemma 2.2.2(ii) and Lemma 2.3.2(v) together would contradict (2.14). Hence, $e \notin [a, c]$. By (2.24), this means that either $o < e < a$, or $c < e < i$. By duality, we can assume the second alternative, that is, $c < e < i$. Since $e \wedge b = o$ would clearly imply that $\{o, i, a, b, c, d, e\}$ is a sublattice isomorphic to $N_7$, which has just been excluded, we can assume that $f := e \wedge b > o$. We turn the 8-element set $\{o, i, a, b, c, d, e, f\}$ into a partial lattice $Q_8$ as follows. To do so, we use the notation $S = Q_8 \setminus \{e, f\}$; note that $S$ is a sublattice of $L$ and $S$ is isomorphic to $N_6$. For $x, y \in Q_8$, such that $x \parallel y$, we let

$$x \wedge y := \begin{cases} x \wedge_S y, & \text{if } x, y \in S, \\ f, & \text{if } \{x, y\} = \{e, b\}, \\ o, & \text{if } x \in \{a, d, c\} \text{ and } y = f, \\ o, & \text{if } y \in \{a, d, c\} \text{ and } x = f \end{cases}$$

and

$$x \vee y := \begin{cases} x \vee_S y, & \text{if } x, y \in S, \\ i, & \text{if } \{x, y\} = \{e, b\}, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

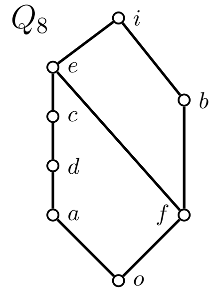


Figure 2.11: Partial sublattice $Q_8$

It is straightforward to compute that $|\operatorname{Sub}(Q_8)| = 16.375 \cdot 2^{8-5}$ (see Figure 2.11, for the calculation see Appendix A.7). Hence, by Lemma 2.3.1, $|\operatorname{Sub}(L)| \leq 16.375 \cdot 2^{n-5}$, contradicting (2.14). We conclude that Case B1 cannot occur.

**Case B2**: $d \notin [a, c]$. Based on (2.22), duality allows us to assume that $c < d$. Then $o < a < c < d < i$. If we had that $d \wedge b = o$, then $\{o, i, a, b, c, d\}$ would form a sublattice isomorphic to $N_6$ and, apart from interchanging the role of $c$ and that of $d$, we would have Case B1. But this would be a contradiction since we have already seen that Case B1 cannot occur. Hence, $x := d \wedge b \neq o$. Using that $d \parallel b$ by (2.21) (applied to $d$), we obtain that $o < x < b$. Since $b$ covers $o$, understood in $N_5$, it follows that $x \notin N_5$. However, then $x \in [o, i] \setminus N_5$ and $x < b$ contradicts (2.21). This shows that Case B2 cannot occur.

Now, after that both Case B1 and Case B2 have been excluded, it follows that Case B cannot occur.

**Case C**: $\Theta$ collapses $e_1$. Then, exactly in the same way as in Case 1 (used in the proof of part (i) of Theorem 2.4.1), duality allows us to assume that $\Theta$ does not collapse $e_4$ and $\varphi(\downarrow(\tilde{a} \vee \tilde{b}))$ is a sublattice (isomorphic to)

$C^{(2)} \times C^{(3)}$ in $L$. Hence, $|\operatorname{Sub}(L)| \le 19 \cdot 2^{n-5}$ by Lemma 2.2.2 (ii) and Lemma 2.2.3(v), contradicting (2.14). This shows that Case C cannot occur.

After excluding each of Cases A, B, and C, we have shown the validity of (2.15).

Next, observe that $|\operatorname{Sub}(L)| < 20 \cdot 2^{n-5}$ and $|\operatorname{Sub}(L)| \ge 26 \cdot 2^{n-5}$ would contradict the first inequality and the second inequality of (2.14), respectively. Hence, the argument between (2.3) and (2.4) remains valid for the current situation, and $L$ still satisfies (2.4). Since (2.5) is the same as (2.15), we know that $L$ satisfies (2.5). Hence the sentence preceding (2.12) applies, and we obtain that $L$ satisfies (2.12).

Based on (2.4), we can pick two 2-antichains, $\{a, b\}$ and $\{c, d\}$, in $L$. They are disjoint by (2.15), so $|\{a, b, c, d\}| = 4$. Since $a \parallel b$, (2.12) applies to $\{a, b, c\}$, and we obtain that $\{a, b\} \subseteq {\downarrow}c$ or $\{a, b\} \subseteq {\uparrow}c$. By duality, we can assume that $\{a, b\} \subseteq {\downarrow}c$. Similarly, (2.12) to applies to the set $\{a, b, d\}$ as well, and we get that $\{a, b\}$ is included either in ${\uparrow}d$, or in ${\downarrow}d$. Since the first alternative would lead to $d < a < c$ and so it would contradict that $c \parallel d$, we have that $\{a, b\} \subseteq {\downarrow}d$. Thus, $\{a, b\} \subseteq {\downarrow}c \cap {\downarrow}d = {\downarrow}(c \wedge d)$, and we obtain that $u := a \vee b \le c \wedge d =: v$. Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, $S$ is a sublattice isomorphic to $B_4 +_{\text{glu}} B_4$ or $B_4 +_{\text{glu}} C^{(2)} +_{\text{glu}} B_4$. However, if $S$ was isomorphic to $B_4 +_{\text{glu}} C^{(2)} +_{\text{glu}} B_4$, then Lemma 2.2.2(ii) together with Lemma 2.2.3(iv) would give that $|\operatorname{Sub}(L)| \le 21.125 \cdot 2^{n-5}$, contradicting (2.14).

Hence $S$ is isomorphic to $B_4 +_{\text{glu}} B_4$. Lemma 2.2.2(ii) together with Lemma 2.2.3(iii) imply that $|\operatorname{Sub}(L)| \le 21.25 \cdot 2^{n-5}$. On the other hand, $|\operatorname{Sub}(L)| \ge 21.25 \cdot 2^{n-5}$ holds by (2.14). So we have that $|\operatorname{Sub}(L)| = 21.25 \cdot 2^{n-5}$. Hence, Lemma 2.2.2(iii) together with Lemma 2.2.3(iii) give that $L$ is of the form (2.13). This indicates that $L$ violates (2.14), contradicting the initial assumption that $L$ satisfies (2.14). This completes the proof of part (ii) of Theorem 2.4.1 and that of the whole Theorem 2.4.1. $\square$

## 2.5 Some related results

Freese [31] was able to prove that an $n$-element lattice $L$ has at most $2^{n-1}$ congruences. Motivated with this result, Czédli [19] proved that if $L$ has fewer than $2^{n-1}$ congruences, then it has at most $2^{n-2}$ congruences. Also, he described the $n$-element lattices with exactly $2^{n-2}$ congruences, as it is in the following theorem:

**Theorem 2.5.1** (Czédli [19]). *If $L$ is a finite lattice of size $n = |L|$ and $|\operatorname{Con}(L)|$ is size of the congruence lattice $L$, then the following hold.*

*(i) $L$ has at most $2^{n-1}$ congruences. Furthermore, $|\operatorname{Con}(L)| = 2^{n-1}$ if and only if $L$ is a chain.*

*(ii) If $L$ has less than $2^{n-1}$ congruences, then it has at most $2^{n-1}/2 = 2^{n-2}$ congruences.*

*(iii) $|\operatorname{Con}(L)| = 2^{n-2}$ if and only if $L$ is of the form $C_1 +_{\text{glu}} B_4 +_{\text{glu}} C_2$ such that $C_1$ and $C_2$ are chains and $B_4$ is the four-element Boolean lattice, see figure 2.3.*

Continuing the work of Freese and Czédli, Kulin and Mureşan studied in [36] the smallest and the largest numbers of congruences of finite lattices of $n$ elements. They proved that the third, fourth, and fifth largest numbers of congruences of an $n$–element lattice are $5 \cdot 2^{n-5}$ if $n \geq 5$, $2^{n-3}$ and $7 \cdot 2^{n-6}$ if $n \geq 6$, respectively. They also determine the structures of the $n$–element lattices having those numbers.

A finite lattice is said to be planar if it has a Hasse diagram that can be drawn in the plane with nonintersecting lines. In [20], Czédli proved that if an $n$-element finite lattice $L$ has at least $83 \cdot 2^{n-8}$ sublattices, then it is a planar lattice. Czédli in [22] showed that if $L$ has more than $2^{n-5}$ congruences, then $L$ is planar. In both cases, the result is sharp for large $n$; see Remark 1.3 in Czédli [20] and Remark 1.2 in Czédli [22].

# 3

# Several large numbers of subuniverses of finite semilattices

This chapter is based on a joint paper with Horváth [4]. Let $(L, \vee)$ be a finite $n$-element semilattice. We prove that the first largest number of subuniverses of an $n$-element semilattice is $2^n = 32 \cdot 2^{n-5}$, the second largest number is $28 \cdot 2^{n-5}$ and the third one is $26 \cdot 2^{n-5}$, where $n \geq 5$. Also, we describe the $n$-element semilattices having exactly $32 \cdot 2^{n-5}$, $28 \cdot 2^{n-5}$, or $26 \cdot 2^{n-5}$ subuniverses.

## 3.1 Notations used in this chapter

All the semilattices in this chapter will be assumed to be finite. Our notation and terminology is standard, see, for example, Chajda et al. [12]. However, we recall some notions and introduce some auxiliary concepts.

On a semilattice $(L, \vee)$, we have a natural partial ordering defined by

$$x \leq y \iff x \vee y = y.$$

Conversely, if $(L, \leq)$ is a partial order in which any two elements $x, y$ have a least upper bound $x \vee y$, then $(L, \vee)$ is a semilattice. For any $x, y$ in a join-semilattice, $x \wedge y$ is defined by their infimum provided it exists; and if this infimum does not exist, then $x \wedge y$ is undefined. For the definition of ordinal sum and glued sum of posets, we direct the reader to Section 2.1.

Now let us define the semilattices $H_3$ and $H_4$, which will be used later (see Figure 3.1). The three-element semilattice $\{a, b, 1\}$ defined by

$$a \| b \text{ and } a \vee b = 1,$$

is called $H_3$, while $H_4$ is a four-element semilattice $\{a, b, c, 1\}$ with $a < b$, defined as follows:

$$a \| c,\ b \| c \text{ and } a \vee c = b \vee c = 1.$$

For general results on semillatices we direct the reader to the book of Chajda et al. [12]. An element $u$ of a semilattice $L$ is called a narrow element, or *narrows* for short, if $u \neq 1_L$ and $L = {\uparrow}u \cup {\downarrow}u$. That is, if $u \neq 1_L$ and $x \| u$ holds for no $x \in L$.

The concept of a partial algebra was defined earlier in Section 2.1. Let us agree that whenever we say that $S = (\{a_1, \ldots, a_n\}, \vee)$ is a partial semilattice with

$$x_1 \vee y_1 = z_1, \ldots, x_m \vee y_m = z_m, \tag{3.1}$$

then the Dom($\vee$)=$\{(x_i, y_i) : 1 \leq i \leq m\} \cup \{(y_i, x_i) : 1 \leq i \leq m\}$ and for $(u, v) \in$ Dom($\vee$), $u \vee v$ is defined by 3.1 together with commutativity.

Now we have a structure $(A, \vee)$ such that $A$ is a non-empty set with a partial operation $\vee$, which is a map from Dom($\vee$) $\subseteq A^2$ to $A$. A *subuniverse* of $A$ is a subset $X$ of $A$ such that $X$ is closed with respect to this partial operation; i.e, if $x, y \in X$ and $(x, y) \in$ Dom($\vee$), then $x \vee y \in X$. The set of subuniverses of $A$ will be denoted by Sub($A$). In particular $\emptyset \in$ Sub($A$). Following Czédli [20] and [23], we define the *relative number of subuniverses* of $A$ as follows:

$$\sigma_k(A) := |\operatorname{Sub}(A)| \cdot 2^{k-n}.$$

Similarly, if $\mathcal{B} = (B, F_B)$, then

$$\sigma_k(\mathcal{B}) := |\operatorname{Sub}(\mathcal{B})| \cdot 2^{k-n}.$$

In his papers Czédli set $k = 8$, while here we will set $k = 5$. This notion could have been introduced in the previous chapter as well, but we did not want to deviate from the published paper.

## 3.2 Preliminaries

We were inspired by similar or analogous results concerning lattices and semilattices. Recall that Czédli and Horváth [25] proved that the first three largest numbers of subuniverses of an $n$-element lattices are $32 \cdot 2^{n-5}, 26 \cdot 2^{n-5}$, and $23 \cdot 2^{n-5}$, where $5 \le n \in \mathbb{N}^+$. In the joint paper with Horváth [3], we showed that the fourth and fifth largest numbers are $21.5 \cdot 2^{n-5}$ (for $n \ge 6$) and $21.25 \cdot 2^{n-5}$ (for $n \ge 7$), respectively. Also we described the $n$-element lattices producing these numbers in Theorem 2.4.1.

The following lemma is from Czédli [20], and it demonstrates the importance of the concept of *relative number* of subuniverses. Also, it will be used later in the proof of Lemma 3.3.1 and Theorem 3.4.1.

**Lemma 3.2.1** ( Czédli [20], Lemma 2.3)**.** *If $\mathcal{B} = (B, F_B)$ is a weak partial subalgebra of a finite partial algebra $\mathcal{A} = (A, F_A)$, then $\sigma_k(\mathcal{A}) \le \sigma_k(\mathcal{B})$, for any $k$.*

## 3.3 Auxiliary results

**Lemma 3.3.1.** *If $(K, \vee)$ is a subsemilattice and $H$ is a subset of a finite semilattice $(L, \vee)$, then the following three assertions hold.*

*(i) With the notation $t := |H \cap S : S \in \mathrm{Sub}(L, \vee)|$, we have that*

$$\sigma_k(L, \vee) \le t \cdot 2^{k-|H|}.$$

*(ii) $\sigma_k(L, \vee) \le \sigma_k(K, \vee)$.*

*(iii) Assume, in addition for the previous assumptions, that $(K, \vee)$ has no narrows. Then $\sigma_k(L, \vee) = \sigma_k(K, \vee)$ if and only if $(L, \vee)$ is (isomorphic to) $C_0 +_{\mathrm{ord}} (K, \vee) +_{\mathrm{glu}} C_1$, where $C_1$ is a chain, and $C_0$ is a chain or the empty set.*

*Proof.* It is a routine to derive part (i) and (ii) from the proof of Lemma 3.2.1. The argument actually provides a bit more than stated in (i) and (ii); namely, for later reference, note the following:

$$\text{If } \sigma_k(L,\vee) = \sigma_k(K,\vee) \text{, then for every } S \in \mathrm{Sub}(K,\vee) \text{ and every subset } X \text{ of } L \setminus K \text{ we have that } S \cup X \in \mathrm{Sub}(L,\vee). \tag{3.2}$$

Next, to prove part (iii), let $n := |(L,\vee)|$ and $m := |(K,\vee)|$. Let $k = 5$, the case of any other $k$ being analogous. Now assume that $(K,\vee)$ has no narrows. First, let $(L,\vee) = C_0 +_{\mathrm{ord}} (K,\vee) +_{\mathrm{glu}} C_1$. It is obvious that whenever $X \subseteq L \setminus K$ and $S \in \mathrm{Sub}(K,\vee)$, $S \cup X \in \mathrm{Sub}(L,\vee)$. Since $L \setminus K$ has $2^{|L|-|K|}$ subsets, $|\mathrm{Sub}(L,\vee)| \geq |\mathrm{Sub}(K,\vee)| \cdot 2^{|L|-|K|}$. Dividing this inequality by $2^{n-5} = 2^{m-5} \cdot 2^{|L|-|K|}$, we get the required equality, which is the converse inequality stated in part (ii).

Conversely, assume the equality stated in (iii). We claim that

$$\text{for all } y \in K \text{ and for all } x \in L \setminus K,\ y \not\parallel x. \tag{3.3}$$

Suppose we have the contrary. If $y \in K$, then $\{y\} \in \mathrm{Sub}(K)$. If $x \in L \setminus K$ and $y || x$, then $\{y, x\}$ is not a subuniverse of $L$, which contradicts (3.2). Next, we claim that

$$\text{for all } x \in L \setminus K,\ x \not> 1_K \text{ implies that for all } y \in K,\ x < y. \tag{3.4}$$

Suppose the contrary is valid and let us pick an $x$ in $L \setminus K$ and a $y \in K$ such that $x \not> 1_K$ and $x \not< y$. Using (3.3) and $x \neq y$, we have that $y < x < 1_K$. Let $p := \bigvee\{s \in K : s < x\}$, which exists by finiteness and $y \leq p \leq x$. In fact, $p \in K$ as $K$ is a subsemilattice of $L$ but $x \notin K$, so $y \leq p < x$. Now assume that $u \in K$ such that $u \not\leq p$. We know from (3.3) that $u \not\parallel x$. If we had $u \leq x$ (actually, $u < x$ since $x \notin K$), then $u$ would be one of the joinands defining $p$ and so $u \leq p$ would be a contradiction. Hence $x < u$, and so $p < x < u$ implies $p < u$. We have seen that, for any $u \in K$, $u \not\leq p$ implies $p < u$. In other words, $K = {\uparrow_K} p \cup {\downarrow_K} p$, which means $p$ is a narrows, contradicting our assumption about $K$. Thus, (3.4) holds. Lastly, we show that $L \setminus K$ is a chain. Indeed, if $L \setminus K$ is not a chain, say $a||b$, $a \in L \setminus K$ and

$b \in L \setminus K$, then $\emptyset \in \mathrm{Sub}(K)$ extended by $\{a, b\} \notin \mathrm{Sub}(L)$ would contradict (3.2). Define $C_1 = \{x \in L \setminus K : x \geq 1_K\}$, which is a chain (a subchain of $L \setminus K$). Let $C_0 = (L \setminus K) \setminus C_1$, which is either a chain or empty. If $C_0$ is empty, then $L$ is $K +_{glu} C_1$, as required. If $C_0$ is nonempty, then its elements are less than any element of $K$ by (3.4), and so $L = C_0 +_{\mathrm{ord}} K +_{glu} C_1$, as required. $\square$

The following lemma can be proved using a computer program (see Appendix B). The program for counting subuniverses is available on the webpage of G. Czédli: http://www.math.u-szeged.hu/ czedli/ (subsize, a program for counting subuniverses 2019).

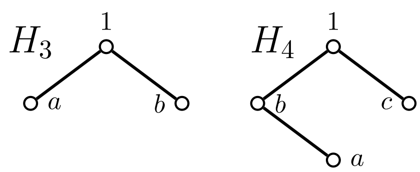


Figure 3.1: Partial lattices $H_3$ and $H_4$

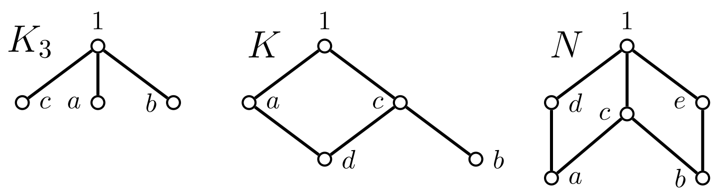


Figure 3.2: Partial lattices $K_3$, $K$ and $N$

**Lemma 3.3.2.** *For the join-semilattices given in figures 3.1 to 3.3, the following seven assertions hold.*

*(i)* $\sigma_5(H_3) = 28$,

*(ii)* $\sigma_5(H_4) = 26$,

*(iii)* $\sigma_5(H_5) = 25$,

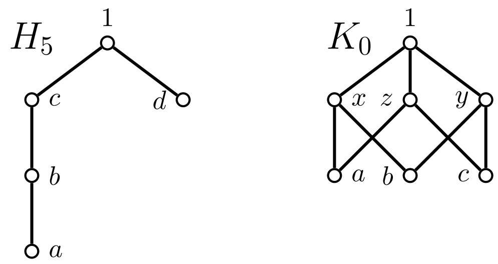


Figure 3.3: Partial lattices $H_5$ and $K_0$

*(iv)* $\sigma_5(K_3) = 24$,

*(v)* $\sigma_5(K) = 23$,

*(vi)* $\sigma_5(N) = 19.5$,

*(vii)* $\sigma_5(K_0) = 15.25$.

*Proof.* The same notations used in figures 3.1 to 3.3 will be applied. For later reference, note that if $(L, \vee)$ is a chain, then $|Sub(L, \vee)| = 2^{|(L,\vee)|}$.

For (i), notice that

$$|\{S \in \mathrm{Sub}(H_3, \vee) : a \notin S\}| = 4, \qquad \text{(S is chain)},$$

$$|\{S \in \mathrm{Sub}(H_3, \vee) : a \in S, \{b\} \cap \ S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \mathrm{Sub}(H_3, \vee) : a \in S, \{b\} \cap \ S \neq \emptyset\}| = 1,$$

whereby $|\,\mathrm{Sub}(H_3, \vee)| = 4+2+1 = 7 = 28 \cdot 2^{3-5}$, which means that $\sigma_5(H_3) = 28$; and this proves case (i).

The proof using the above mentioned computer program is:

**Input:**

\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*
\beginjob
\name

```
H_3
\size
3
\elements
ab1
\P edges
\P a1 b1
\constraints
a+b=1 ,
\endofjob
\enddata
```

**Output:**

```
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 18:56:26) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

    |A|=3, A(without commas)={ab1}. Constraints:
edges
a1 b1
a+b=1
Result for A=H_3: |Sub(A)| = 7, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 28.0000000000000000 .

    The computation took 8/1000 seconds.
```

For (ii), let us compute

$$|\{S \in \mathrm{Sub}(H_4, \vee) : a \notin S\}| = 7, \qquad \text{by (i)} ,$$

$$|\{S \in \mathrm{Sub}(H_4, \vee) : a \in S, \{b, c\} \cap\ S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \mathrm{Sub}(H_4, \vee) : a \in S, \{b, c\} \cap\ S \neq \emptyset\}| = 4.$$

Hence, $|\operatorname{Sub}(H_4, \vee)| = 7+2+4 = 13 = 26 \cdot 2^{4-5}$, which means that $\sigma_5(H_4) = 26$; and this proves case (ii). For the program calculation, see Appendix B.8.

For (iii), let us compute

$$|\{S \in \operatorname{Sub}(H_5, \vee) : d \notin S\}| = 16, \qquad \text{(S is chain)},$$

$$|\{S \in \operatorname{Sub}(H_5, \vee) : d \in S, \{a, b, c\} \cap\ S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \operatorname{Sub}(H_5, \vee) : d \in S, \{a, b, c\} \cap\ S \neq \emptyset\}| = 7.$$

Hence, $|\operatorname{Sub}(H_5, \vee)| = 16 + 2 + 7 = 25 = 25 \cdot 2^{5-5}$, which means that $\sigma_5(H_5) = 25$; and this proves case (iii). For the program calculation, see Appendix B.9.

For (iv), let us compute

$$|\{S \in \operatorname{Sub}(K_3, \vee) : a \notin S\}| = 7, \qquad \text{(by (i))},$$

$$|\{S \in \operatorname{Sub}(K_3, \vee) : a \in S, \{b, c\} \cap\ S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \operatorname{Sub}(K_3, \vee) : a \in S, \{b, c\} \cap\ S \neq \emptyset\}| = 3.$$

Hence, $|\operatorname{Sub}(K_3, \vee)| = 7+2+3 = 12 = 24 \cdot 2^{4-5}$, which means that $\sigma_5(K_3) = 24$; and this proves case (iv). For the program calculation, see Appendix B.10.

In order to prove (v), note that $(B_4, \vee)$ has $|\operatorname{Sub}(B_4)| + 1$, see Appendix B.11. Now, let us compute

$$|\{S \in \operatorname{Sub}(K, \vee) : b \notin S\}| = 14, \qquad (S \text{ is } B_4),$$

$$|\{S \in \operatorname{Sub}(K, \vee) : b \in S, \{a, c, d, \} \cap S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \operatorname{Sub}(K, \vee) : b \in S, \{a, c, d, \} \cap S \neq \emptyset\}| = 7,$$

whereby $|\operatorname{Sub}(K, \vee)| = 14 + 2 + 7 = 23 = 23 \cdot 2^{5-5}$, which means that $\sigma_5(K, \vee) = 23$; and this proves case (v). For the program calculation, see Appendix B.12.

For (vi), let us compute

$$|\{S \in \operatorname{Sub}(N, \vee) : d \notin S\}| = 23, \qquad \text{(by (v))},$$

$$|\{S \in \operatorname{Sub}(N, \vee) : d \in S, \{a, b, c, e\} \cap S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \operatorname{Sub}(N, \vee) : d \in S, \{a, b, c, e\} \cap S \neq \emptyset\}| = 14.$$

Hence, $|\operatorname{Sub}(N, \vee)| = 23 + 2 + 14 = 39 = 19.5 \cdot 2^{6-5}$, which means that $\sigma_5(N, \vee) = 19.5$; and this proves case (vi). For the program calculation, see Appendix B.13.

For (vii), let us compute

$$|\{S \in \operatorname{Sub}(K_0, \vee) : b \notin S\}| = 39, \qquad \text{(by (vi))},$$

$$|\{S \in \operatorname{Sub}(K_0, \vee) : b \in S, \{a, x, c, y, z\} \cap S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \operatorname{Sub}(K_0, \vee) : b \in S, \{a, x, c, y, z\} \cap S \neq \emptyset\}| = 20.$$

Hence, $|\operatorname{Sub}(K_0, \vee)| = 39 + 2 + 20 = 61 = 15.25 \cdot 2^{7-5}$, which means that $\sigma_5(K_0, \vee) = 15.25$; and this proves case (vii). For the program calculation, see Appendix B.14. $\square$

## 3.4 The first three largest numbers of the subuniverses of finite semilattices

For a natural number $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, let

$$\operatorname{NS}(n) := \{|\operatorname{Sub}(L)| : L \text{ is a semilattice of size } |L| = n\}.$$

**Theorem 3.4.1.** *If $5 \le n \in \mathbb{N}^+$, then the following three assertions hold.*

*(i) The first largest number in $\operatorname{NS}(n)$ is $2^n = 32 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L, \vee)$ has exactly $2^n$ subuniverses if and only if $(L, \vee)$ is a chain.*

*(ii) The second largest number in $\operatorname{NS}(n)$ is $28 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L, \vee)$ has exactly $28 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_3 +_{\text{glu}} C_1$ or $(L, \vee) \cong C_0 +_{\text{ord}} H_3 +_{\text{glu}} C_1$, where $C_0$ and $C_1$ are finite chains.*

*(iii) The third largest number in $\operatorname{NS}(n)$ is $26 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L, \vee)$ has exactly $26 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_4 +_{\text{glu}} C_1$ or $(L, \vee) \cong C_0 +_{\text{ord}} H_4 +_{\text{glu}} C_1$, where $C_0$ and $C_1$ are finite chains.*

*Proof.* Part (i) is trivial. For part (ii), let $(L, \vee)$ be an $n$-element semilattice. We know from Lemma 3.3.1(iii) and 3.3.2(i) that if

$$\begin{gathered}(L, \vee) \cong H_3 +_{\text{glu}} C_1 \text{ or } (L, \vee) \cong C_0 +_{\text{ord}} H_3 +_{\text{glu}} C_1, \\ \text{where } C_0 \text{ and } C_1 \text{ are chains,}\end{gathered} \tag{3.5}$$

then $\sigma_5(L, \vee) = \sigma_5(H_3) = 28$, indeed. Conversely, assume that $\sigma_5(L, \vee) = 28$. Then it follows from part (i) that $(L, \vee)$ is not a chain. So $(L, \vee)$ has two incomparable elements, $a$ and $b$. Clearly, $\{a, b, a \vee b\}$ is a join-subsemilattice isomorphic to $H_3$. But $\sigma_5(H_3)$ is also 28 by Lemma 3.3.2(i). Thus, Lemma 3.3.1(iii) immediately tell us that $(L, \vee)$ is of the desired form. With this, we have completed the proof of part (ii) of Theorem 3.4.1.

We will now prove part (iii).

Assume that $(L, \vee)$ is of the given form. Then $\sigma_5(L, \vee) = 26$ follows from Lemma 3.3.1(iii) and Lemma 3.3.2(ii). In order to prove the converse, namely the nontrivial implication, assume that $\sigma_5(L, \vee) = 26$. By Theorem 3.4.1(i), $(L, \vee)$ has two incomparable elements, $a$ and $b$. By Theorem 3.4.1(ii), $\{a, b\}$ is not the only 2-element antichain in $(L, \vee)$ since otherwise $\sigma_5(L, \vee)$ would be 28. To complete the proof, consider the following cases.

**Case 1:** There is an antichain $\{c, d\}$ disjoint from $\{a, b\}$, where the elements $a$, $b$, $c$, $d$ are distinct. Now let $x := a \vee b$ and $y := c \vee d$. Let $t := |\{a, b, c, d, x, y\}|$. Depending on $t \in \{4, 5, 6\}$, there are three cases. The number of possible cases can be reduced by symmetry: $a$ and $b$ play a symmetric role, so do $c$ and $d$, and so do $\{a, b\}$ and $\{c, d\}$ and hence x and y. We have to consider only three sub-cases. These are:

**Sub-case 1a:** Here $t = 6$. Take the partial algebra $U_1 = \{a, b, c, d, x, y\}$ with $a \vee b = x$ and $c \vee d = y$ (see Figure 3.4). This six-element partial algebra has $\sigma_5(U_1) = 24.5$, which can be verified by the above-mentioned computer program (see Appendix B.15). By Lemma 3.2.1, we find that $\sigma_5(L, \vee) \leq \sigma_5(U_1) \leq 24.5$, contradicting $\sigma_5(L, \vee) = 26$. Thus, this case is ruled out.

**Sub-case 1b:** Here $t = 5$. By symmetry, $y = c \vee d$ is not a new element, so $y = c \vee d$ is either $x$ or $a$. The case $y = b$ need not be considered because $a$ and $b$ are symmetric. Therefore, this sub-case 1b is split into two cases as follows:

First, when $y = x$, this case is captured by taking the partial algebra $U_2 = \{a, b, c, d, x\}$ with $a \vee b = x$, $c \vee d = x$ (see Figure 3.4). It has $\sigma_5(U_2) = 25$ using the computer program (see Appendix B.16). Like the above, this implies that $\sigma_5(L, \vee) \leq \sigma_5(U_2) \leq 25$, which contradicts the assertion $\sigma_5(L, \vee) = 26$.

Second, when $y = a$, this case is covered by taking the partial algebra $U_3 = \{a, b, c, d, x\}$ with $a \vee b = x$, $c \vee d = a$ (see Figure 3.4). This five-element partial algebra has $\sigma_5(U_3) = 24 < 26$ (for the calculation, see Appendix B.17), and we get a contradiction as before.

From the above we can see that Sub-case 1b is excluded since so are both of its subcases.

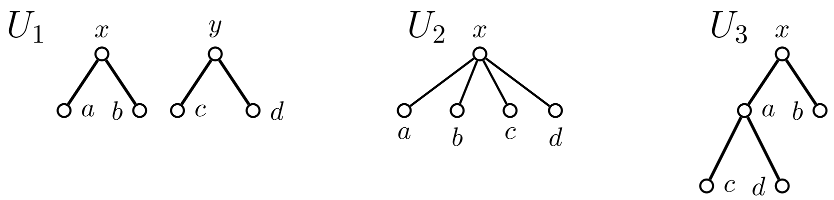


Figure 3.4: Partial lattices $U_1$, $U_2$ and $U_3$

**Sub-case 1c:** Here $t = 4$. Then $x = a \vee b$ is either $c$ or $d$. By symmetry, we can assume that $a \vee b = c$. However, then $a < c < c \vee d$, $b < c < c \vee d$, whereby $y = c \vee d$ is bigger than any of the elements $a, b, c, d$, which contradicts $t = 4$. So this case is excluded.

After having ruled out all of the sub-cases, we find that Case 1 is excluded. That is, no two-element antichain is disjoint from $\{a, b\}$. But recall that there is another two-element antichain, whereby, by $a$-$b$ symmetry, we consider

**Case 2:** there is an element $c$ such that $a$ and $c$ are incomparable. Now, there are two sub-cases according to the position of $b$ and $c$.

**Sub-case 2a:** Here $b$ and $c$ are also incomparable. Now, we need to investigate how many of the elements $a \vee b$, $a \vee c$, and $b \vee c$ are equal to $a \vee b \vee c$. The answer could be $0, 1, 2$ or $3$. Hence using symmetry, it suffices to consider just the following four join-semilattices. The first join-semilattice is $K_0 = \{a, b, c, z, x, y, 1\}$ with edges $ax$, $bx$, $by$, $cy$, $az$, $cz$, $x1$, $y1$, $z1$ and equalities $a \vee b = x$, $b \vee c = y$, $a \vee c = z$, $x \vee z = 1$, $x \vee y = 1$, $z \vee y = 1$, $a \vee y = 1$, $c \vee x = 1$, $b \vee z = 1$ (see Figure 3.3); this gives $\sigma_5(K_0) = 15.25$ by

Lemma 3.3.2(vii). The second join-semilattice is $K_1 = \{a, b, c, x, y, 1\}$ with edges $ax$, $bx$, $by$, $cy$, $x1$, $y1$ and equalities $a \vee b = x$, $b \vee c = y$, $a \vee c = 1$, $x \vee y = 1$, $x \vee c = 1$, $y \vee a = 1$ (see Figure 3.5); this gives $\sigma_5(K_1) = 18.5$ (for the calculation, see Appendix B.18).

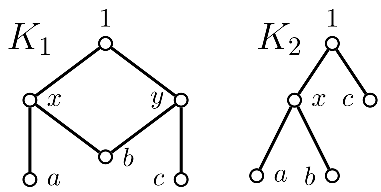


Figure 3.5: Partial lattices $K_1$ and $K_2$

The third is $K_2 = \{a, b, c, x, 1\}$ with edges $ax, bx, x1, c1$ and equalities $a \vee b = x$, $a \vee c = 1$, $b \vee c = 1$, $x \vee c = 1$ (see Figure 3.5); this gives $\sigma_5(K_2) = 22$ (for the calculation, see Appendix B.19). The fourth is $K_3 = \{a, b, c, 1\}$ with edges $a1, b1, c1$, and equalities $a \vee b = 1$, $a \vee c = 1$, $b \vee c = 1$ (see Figure 3.2); this gives $\sigma_5(K_3) = 24$ by Lemma 3.3.2(iv). Since one of $K_0$, $K_1$, $K_2$, and $K_3$ is a subsemilattice of $(L, \vee)$ and all the four $\sigma_5$ values of these join-semilattices are smaller than 26, sub-case 2a is excluded.

**Sub-case 2b:** Here $b \nparallel c$ while $a \| c$ and $a \| b$. Now, $b$ and $c$ have a symmetric role. So we can assume that $b < c$. Suppose, for the sake of contradiction, that $x := a \vee b < a \vee c := 1$. Assuming these comparabilities and incomparabilities, $|\{a, b, c, x, 1\}| = 5$; for example if $x = a \vee b = c$ is impossible since it would mean $a < c$. Apart from the notation used, the join-subsemilattice $\{a, b, c, x, 1\}$ here is the same as $K$ in Figure 3.2. By Lemma 3.3.2 (v), $\sigma_5(K) = 23 < 26$, which leads to a contradiction. So $a \vee b < a \vee c$ fails but $a \vee b \leq a \vee c$ since $b < c$. Therefore, with $1 = a \vee b = a \vee c$, $\{a, b, c, 1\}$ is a subsemilattice (isomorphic to) $H_4$.

Now that all other possibilities have been ruled out, we know that $H_4$ is a join-subsemilattice of $(L, \vee)$. Note that $H_4$ has no narrows. Therefore, by Lemma 3.3.1(iii), $(L, \vee)$ is of the desired form. Then, the proof of Theorem 3.4.1 is complete.

$\square$

## 3.5 Some related results

In [21], Czédli found the four largest numbers of congruences of $n$-element semilattices. He, also described the $n$-element semilattices producing these numbers, as in the following theorem.

**Theorem 3.5.1** (Czédli [21]). *If $\langle S; \wedge\rangle$ is a finite meet-semilattice of size $n = |S| > 1$, then the following hold.*

*(i) $\langle S; \wedge\rangle$ has at most $2^{n-1} = 32 \cdot 2^{n-6}$ congruences. Furthermore, we have that $|\operatorname{Con}(S; \wedge)| = 2^{n-1}$ if and only if $\langle S; \wedge\rangle$ is a tree semilattice.*

*(ii) If $\langle S; \wedge\rangle$ has fewer than $2^{n-1} = 32 \cdot 2^{n-6}$ congruences, then it has at most $28 \cdot 2^{n-6}$ congruences. Furthermore, $|\operatorname{Con}(S; \wedge)| = 28 \cdot 2^{n-6}$ if and only if $\langle S; \wedge\rangle$ is a quasi-tree semilattice and its nucleus is the four-element Boolean lattice; see Figure 2 (Appendix C) for $n = 6$.*

*(iii) If $\langle S; \wedge\rangle$ has fewer than $28 \cdot 2^{n-6}$ congruences, then it has at most $26 \cdot 2^{n-6}$ congruences. Furthermore, $|\operatorname{Con}(S; \wedge)| = 26 \cdot 2^{n-6}$ if and only if $\langle S; \wedge\rangle$ is a quasi-tree semilattice such that its nucleus is the pentagon $N_5$; see Figure 5 (Appendix C) and $S_1, ..., S_3$ in Figure 3 (Appendix C).*

*(iv) If $\langle S; \wedge\rangle$ has fewer than $26 \cdot 2^{n-6}$ congruences, then it has at most $25 \cdot 2^{n-6}$ congruences. Furthermore, $|\operatorname{Con}(S; \wedge)| = 25 \cdot 2^{n-6}$ if and only if $\langle S; \wedge\rangle$ is a quasi-tree semilattice such that its nucleus is either $F$, or $N_6$; see Figure 5 (Appendix C) and $S_4, ..., S_7$ in Figure 4.*

Czédli in [23] was able to prove that an $n$-element finite semilattice with at least $127 \cdot 2^{n-8}$ subsemilattices is planar. More precisely:

**Theorem 3.5.2** (Czédli [23]). *Let $L$ be a finite semilattice, and let $n := |L|$ denote the number of its elements. If $L$ has at least $127 \cdot 2^{n-8}$ subsemilattices, then it is a planar semilattice.*

The theorem above is sharp; see Remark 1.2 in Czédli [23]. Chajda [10] studied join-semilattices and lattices with the greatest element 1 where every interval $[p, 1]$ is a lattice with an antitone involution. He characterized these semilattices by means of an induced binary operation, called sectionally antitone involution. This characterization is performed by means of identities, so the classes of these semilattices or lattices form varieties. He also investigated the congruence properties of these varieties. Libkin and Muchnik in [38] proved that if $S$ is an arbitrary semilattice and $S_1, S_2$ are its disjoint subsemilattices, then $S_1$ and $S_2$ can be separated via a separatory subsemilattice. That is, there exists a separatory subsemilattice $D \subseteq S$ such that $S_1 \subseteq D, S_2 \subseteq S - D$. This result immediately implies that any semilattice with two or more elements has a proper separatory subsemilattice. They also proved that a lattice $L$ satisfies the separation condition iff it is distributive and series-parallel. A lattice $L$ is called series-parallel, if it does not contain a subposet whose diagram looks like the letter $N$.

# 4

# The number of subuniverses, congruences, weak congruences of semilattices defined by trees

This chapter is based on joint manuscript with Horváth and Németh [5]. Here we determine the number of subuniverses of semilattices defined by arbitrary and special kinds of trees using combinatorial considerations. Using a result of Freese and Nation [32], we provide a formula for the number of congruences of semilattices defined by arbitrary and special kinds of trees. Using both results, we prove a formula for the number of weak congruences of semilattices defined by a binary tree and we discuss some special cases. We solve two related nontrivial recurrences by applying the method of Aho and Sloane.

## 4.1 Notations used in this chapter

A rooted tree defines a semilattice in a natural way, where the root is the greatest element. Recall that on a semilattice $(L, \vee)$, we have a natural partial order defined by

$$x \leq y \iff x \vee y = y.$$

Conversely, if $(L, \leq)$ is partial order in which any two elements $x, y$ have a least upper bound $x \vee y$, then $(L, \vee)$ is a semilattice. For any $x, y$ in a join-semilattice, $x \wedge y$ is defined by their infimum provided it exists; if this infimum does not exist, then $x \wedge y$ is undefined.

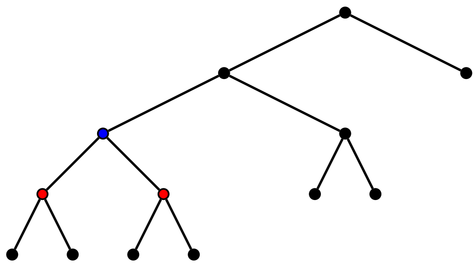

Figure 4.1: Binary tree

A binary tree is a rooted tree in which every node has at most two children (see, e.g., Figure 4.1). A binary tree is called a full binary tree if each of its nodes except the root is either a leaf node or an internal node having one parent and two children; for example, see $B_9$ in Figure 4.2. The binary tree's height is the number of edges of the longest path from the root node to a leaf node in the tree, i.e., the length of the longest path from the root node to any leaf node in the tree. The perfect binary tree is a binary tree in which all interior nodes have two children, and all the leaves have the same path length to the root; for example, see $B_7$ in Figure 4.2.

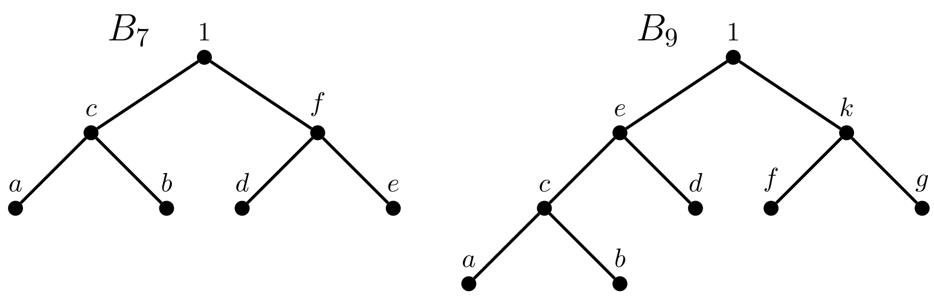


Figure 4.2: Perfect binary tree and full binary tree

If $B$ is a binary tree, then $(B, \vee)$ denotes the semilattice defined naturally on $B$. We will denote the left maximal and right maximal subtrees by $B_1$ and

$B_2$, respectively, whose roots are the left and right coatoms of $B$, respectively (see, e.g., Figure 4.3).

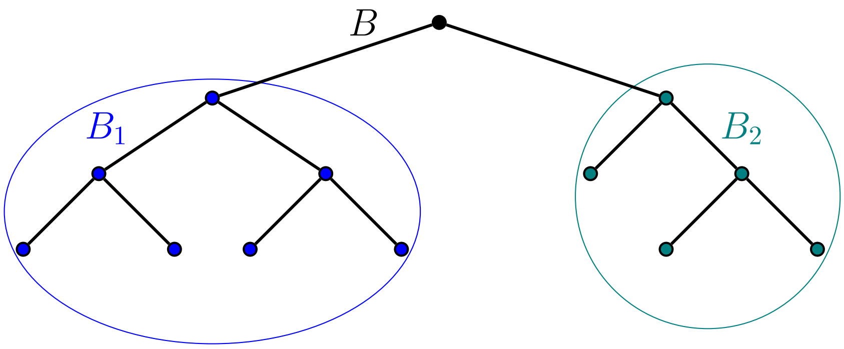


Figure 4.3: The Left and right maximal subtrees

Now we will introduce a new terminology. We are going to call a full binary tree a *prickly-snake*, if in all but the last level, there is exactly one parent (see, e.g., Figure 4.4), because of the shape of the graph. It is not hard to see that the trees in Figure 4.4 are isomorphic. In this chapter we consider the most “left-sided" structure, which is located on the right-hand side of Figure 4.4.

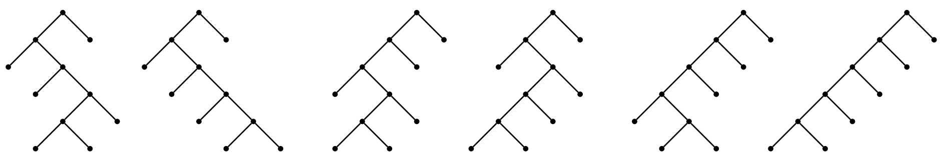

Figure 4.4: Prickly-snake graphs

## 4.2 The cardinality of the subuniverse lattice of a semilattice defined by a tree

**Lemma 4.2.1.** *If* $(T, \vee)$ *is a semilattice defined by a tree* $T$*, then*

$$|\mathrm{Sub}(T, \vee)| = \prod_{i=1}^{n}(|\mathrm{Sub}(T_i, \vee)|) + \sum_{i=1}^{n}(|\mathrm{Sub}(T_i, \vee)|) - (n-1),$$

*where $T_1, \ldots, T_n$ is a repetition free list of maximal subtrees of the tree $T$.*

*Proof.* Let $T_i^*$ be subuniverses of $T_i$ where $i \in \{1, 2, ..., n\}$. Then all $T_i^*$ are subuniverses of $T$, as well. In this way we get

$$|\mathrm{Sub}(T_1, \vee)| + |\mathrm{Sub}(T_2, \vee)| + ... + |\mathrm{Sub}(T_n, \vee)| - (n-1)$$

subuniverses because $|\mathrm{Sub}(T_1, \vee)|, |\mathrm{Sub}(T_2, \vee)|, ..., |\mathrm{Sub}(T_n, \vee)|$ are all count $\emptyset$. Also, $\{1\} \cup \bigcup_{i=1}^n T_i^*$ is a subuniverse, and the number of such subuniverses is $\prod_{i=1}^n (|\mathrm{Sub}(T_i, \vee)|)$. Hence we counted all the subuniverses. □

**Corollary 4.2.1.1.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$, then*

$$|\mathrm{Sub}(B, \vee)| = |\mathrm{Sub}(B_1, \vee)| \cdot |\mathrm{Sub}(B_2, \vee)| + (|\mathrm{Sub}(B_1, \vee)| + |\mathrm{Sub}(B_2, \vee)|) - 1,$$

*where $B_1$, $B_2$ are the left and right maximal subtrees of the tree, respectively.*

**Corollary 4.2.1.2.** *If $(B, \vee)$ is a semilattice defined by a prickly-snake $B$ of height $h$, then*

$$|\mathrm{Sub}(B, \vee)| = 3\,|\mathrm{Sub}(B_1, \vee)| + 1 = \frac{5 \cdot 3^h - 1}{2},$$

*where $B_1$ is the left maximal subtree of the tree.*

*Proof.* Apply Corollary 4.2.1.1 and webpage https://oeis.org/ with integer sequence A060816. □

**Remark 4.2.2.** See also https://erich-friedman.github.io/mathmagic/1000.html

**Theorem 4.2.3.** *If $(B, \vee)$ is a semilattice defined by a perfect binary tree $B$ of height $h$, then*

$$|\mathrm{Sub}(B, \vee)| = |\mathrm{Sub}(B_1, \vee)|^2 + 2\,|\mathrm{Sub}(B_1, \vee)| - 1,$$

*where $B_1$ is the left maximal subtree of the tree.*

*Moreover,*

$$|\mathrm{Sub}(B, \vee)| = \left\lceil C^{2^{h+1}} \right\rceil - 1, \quad C = 1.6784589651254\ldots$$

*where $\lceil x \rceil$ denotes the least integer greater than or equal to $x$.*

*Proof.* To get the recurrence formula, use Corollary 4.2.1.1. Now we solve the recurrence.

For the sake of convenience, let $a_n := |\mathrm{Sub}(B, \vee)|+1$, where $B$ is a perfect binary tree of height $(n-1)$. With this notation, our recurrence reads as

$$a_n = a_{n-1}^2 - 1, \quad a_0 = 2.$$

In the rest of the proof we will apply the method of Aho and Sloane [6, Section 3].

It is obvious that (i) the sequence $(a_n)$ is monotone increasing, and (ii) $a_n \geq 2$ for all $n$.

Denoting

$$x_n := \log a_n\,, \quad y_n := \log\left(1 - \frac{1}{a_n^2}\right), \tag{4.1}$$

the recurrence

$$a_{n+1} = a_n^2 - 1 = a_n^2\left(1 - \frac{1}{a_n^2}\right)$$

can be rewritten as

$$x_{n+1} = 2x_n + y_n.$$

It is clear that

$$\begin{aligned}
x_1 &= 2x_0 + y_0, \\
x_2 &= 4x_0 + 2y_0 + y_1, \\
x_3 &= 8x_0 + 4y_0 + 2y_1 + y_2, \\
&\vdots \\
x_n &= 2^n x_0 + \sum_{k=0}^{n-1} 2^{n-1-k} y_k.
\end{aligned}$$

Now, let

$$A_n := 2^n x_0 + \sum_{k=0}^{\infty} 2^{n-1-k} y_k\,, \quad B_n := \sum_{k=n}^{\infty} 2^{n-1-k} y_k\,.$$

These series are (absolutely) convergent since, from (i) and (ii), $y_n < 0$ and $|y_n| \geq |y_{n+1}|$, hence

$$|2^{n-1-k} y_k| \leq \frac{2^{n-1}|y_0|}{2^k}.$$

Clearly $B_n < 0$ for all $n$ and

$$|B_n| = \frac{1}{2}|y_n| + \frac{1}{4}|y_{n+1}| + \frac{1}{8}|y_{n+2}| + \cdots < |y_n| \,. \tag{4.2}$$

With this notation

$$a_n = e^{x_n} = e^{A_n - B_n} = e^{A_n} e^{-B_n}. \tag{4.3}$$

We will now rewrite the first term as follows:

$$\begin{aligned} e^{A_n} &= \exp\left(2^n x_0 + \sum_{k=0}^{\infty} 2^{n-1-k} y_k\right) \\ &= \exp\left(2^n \left(x_0 + \sum_{k=0}^{\infty} 2^{-1-k} y_k\right)\right) = C^{2^n}, \end{aligned}$$

where

$$C := \exp\left(x_0 + \sum_{k=0}^{\infty} 2^{-1-k} y_k\right) = \exp\left(x_0 + \frac{y_0}{2} + \frac{y_1}{4} + \frac{y_2}{8} + \cdots\right)$$

is a constant (independent of $n$). Using *Mathematica* (see Appendix D.20), we find that

$$C = 1.6784589651254\ldots .$$

Now, consider the second term in (4.3). By (4.2), we have

$$0 < e^{-B_n} = e^{|B_n|} < e^{|y_n|} = \exp\left|\log\left(1 - \frac{1}{a_n^2}\right)\right| = \exp\left(\log\left(\frac{1}{1 - \frac{1}{a_n^2}}\right)\right).$$

Using the elementary inequality $\frac{1}{1-u} \le 1 + 2u$ for $0 \le u \le \frac{1}{2}$ gives that $1 < \frac{1}{1-\frac{1}{a_n^2}} < 1 + \frac{2}{a_n^2}$, and, therefore,

$$e^{-B_n} < 1 + \frac{2}{a_n^2}.$$

Substituting this to (4.3) and using (ii), we have

$$e^{A_n} \le a_n \le e^{A_n}\left(1 + \frac{2}{a_n^2}\right) \le e^{A_n}\left(1 + \frac{2}{(e^{A_n})^2}\right) \le e^{A_n} + \frac{2}{e^{A_n}} < e^{A_n} + 1.$$

Since $a_n$ is an integer, the formula in Theorem 4.2.3 follows. □

## 4.3 The cardinality of the congruence lattice of a semilattice defined by a binary tree

For a finite join-semilattice $S = \langle S; \vee \rangle$, we will use the notation $S^+ := S \setminus \{1\}$. Then $\langle S^+; \wedge \rangle$ is a partial algebra, which we shall call the partial meet-semilattice associated with $S$. Recall that by a partial subalgebra of $\langle S^+; \wedge \rangle$ we mean a subset $X$ of $S^+$ such that whenever $x, y \in X$ and $x \wedge y$ is defined in $\langle S^+; \wedge \rangle$, $x \wedge y \in X$. As for the set inclusion relation $\subseteq$, the set of all partial subalgebras of $\langle S^+; \wedge \rangle$ turns out to be a lattice, which we will denote by $\mathrm{Sub}(\langle S^+; \wedge \rangle)$. For convenience, our convention is that $\emptyset \in \mathrm{Sub}(\langle S^+; \wedge \rangle)$. The dual of the following Lemma 4.3.1 is due to Freese and Nation [32], but we formulate a dual version of Lemma 3.1 stated in Czédli's paper [21].

**Lemma 4.3.1** (Czédli [21])**.** *For every finite join-semilattice $\langle S; \vee \rangle$, the lattice $\mathrm{Con}(S; \vee)$ is dually isomorphic to $\mathrm{Sub}(S^+; \wedge)$. In particular, we have that $|\mathrm{Con}(S; \vee))| = |\mathrm{Sub}(S^+; \wedge)|$. See Figure 4.5 for an illustration.*

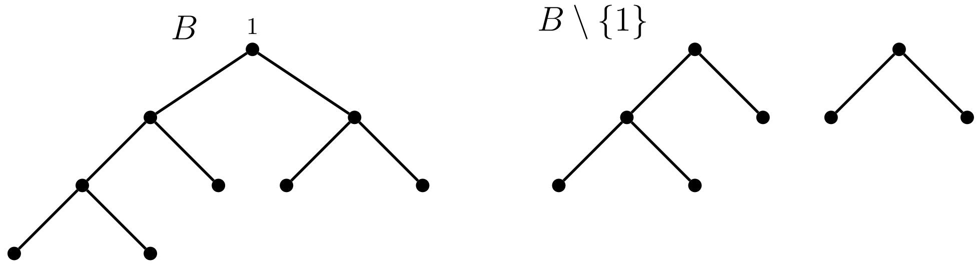


Figure 4.5: Illustration of Lemma 4.3.1.

**Lemma 4.3.2.** *If $(T, \vee)$ is a semilattice defined by a tree $T$, then*

$$|\mathrm{Con}(T, \vee)| = 2^{|T|-1} = 2^{\sum_{i=1}^{n} |T_i|} = 2^n \cdot \prod_{i=1}^{n} |\mathrm{Con}(T_i, \vee)| \,,$$

*where $T_1, \ldots, T_n$ is a repetition free list of maximal subtrees of the tree $T$.*

*Proof.* By Lemma 4.3.1, $|\mathrm{Con}(T, \vee)| = |\mathrm{Sub}(\langle T^+; \wedge \rangle)|$ . Since $(T, \vee)$ is a semilattice defined by a tree $T$, then $x \wedge y$ is defined in $T^+$ only if $x$ and $y$ form a comparable pair in $T^+$ i.e. $x$ and $y$ form a comparable pair in $(T_i, \wedge)$, $i \in \{1, 2, ..., n\}$. Whence $(x \wedge y) \in \{x, y\}$. Hence every subset of $T^+$ belongs to $\mathrm{Sub}(\langle T^+; \wedge \rangle)$. Now $T^+$ has $2^{|T|-1}$ subsets, and $|T| - 1 = \sum_{i=1}^{n} |T_i|$, so

$$\left|\mathrm{Sub}(\langle T^+; \wedge \rangle)\right| = 2^{|T|-1} = 2^{\sum_{i=1}^{n} |T_i|}.$$

Now in the same way $|\mathrm{Con}(T_i, \vee)| = \left|\mathrm{Sub}(\langle T_i^+; \wedge \rangle)\right| = 2^{|T_i|-1}$, so

$$2^{\sum_{i=1}^{n} |T_i|} = 2^n \cdot \prod_{i=1}^{n} (|\mathrm{Con}(T_i, \vee)| \, .$$

□

**Corollary 4.3.2.1.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$, then*

$$|\mathrm{Con}(B, \vee)| = 2^{|B_1|+|B_2|} = 4 \cdot |\mathrm{Con}(B_1, \vee)| \cdot |\mathrm{Con}(B_2, \vee)| \, ,$$

*where $B_1, B_2$ are the left and right maximal subtrees of the tree, respectively.*

**Corollary 4.3.2.2.** *If $(B, \vee)$ is a semilattice defined by a prickly-snake $B$ of height $h$, then*

$$|\mathrm{Con}(B, \vee)| = 4 \cdot |\mathrm{Con}(B_1, \vee)| = 4^h,$$

*where $B_1$ is the left maximal subtree of the tree.*

*Proof.* Just apply Corollary 4.3.2.1, and the fact that it is a geometric sequence. □

**Remark 4.3.3.** We mention here the famous combinatorial identity $\sum_{i+j=n} \binom{2i}{i}\binom{2j}{j} = 4^n$; Paul Erdős found it interesting as well, see Sved [52] and Duarte and Oliveira [30]. Also, the number of congruences in the prickly-snake is equal to the total number of cells in the first $2^n$ rows of the Pascal rhombus (mod 2), as shown in the manuscript of Stockmeyer [49] (end of Chapter 3). Another relevant and interesting identity is in Janjić's paper [34], called Identity 21, and from by Example 22/3, it describes the number of ternary words of length $2n - 2$ containing one subword 22. This number also appears in the paper by Merca and Cuza [40] in connection with the power sums of cosine functions. We also mention here the paper of Barry [7](Chapter 11), concerning generalized Ballot transform pairs.

**Corollary 4.3.3.1.** *If $(B, \vee)$ is a semilattice defined by a perfect binary tree $B$ of height $h$, then*

$$|\mathrm{Con}(B, \vee)| = 4 \cdot |\mathrm{Con}(B_1, \vee)|^2 = 2^{2^{h+1}-2},$$

*where $B_1$ is the left maximal subtree of the tree.*

*Proof.* The first part follows from Corollary 4.3.2.1. By Lemma 4.3.2, $|\mathrm{Con}(B_1, \vee)| = 2^{|B_1|-1}$. Now, using the fact that the number of nodes of the perfect binary tree of height $s$ is $2^{s+1} - 1$, the height of $B_1$ is $h - 1$, and the last part of the statement follows. See also https://oeis.org/ with integer sequence A051191. □

## 4.4 The number of weak congruences of semilattices defined by full binary tree

A weak congruence on an algebra $A$ is a compatible weak equivalence on $A$, i.e., a symmetric and transitive subuniverse of $A^2$. The collection $\mathrm{Cw}(A)$ of weak congruences on an algebra $A$ is an algebraic lattice under inclusion (see [55]). Note that $\mathrm{Con}(A)$, $\mathrm{Sub}(A)$ and, for any subalgebra B of A, $\mathrm{Sub}(B)$ are sublattices of $\mathrm{Cw}(A)$ (see Figure 4.6).

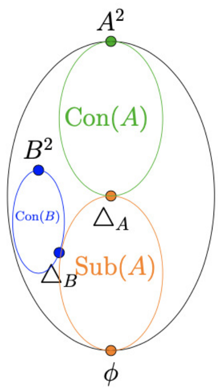


Figure 4.6: $\mathrm{Cw}(A)$

Recall that the subset $K$ of a semilattice $(\mathcal{L}, \vee)$ is called convex if and only if $x, y \in K$, $z \in (\mathcal{L}, \vee)$ and $x \leq z \leq y$ imply $z \in K$.

**Lemma 4.4.1.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$ and $1' \notin B$, then*

$$\Big|\mathrm{Cw}\Big((B,\vee) +_{\mathrm{ord}} \{1'\}\Big)\Big| = 3 \cdot |\mathrm{Cw}(B,\vee)| - 1.$$

*Proof.* By the *support* of a weak congruence $\theta$ we mean the subsemilattice $\{x : (x,x) \in \theta\}$. Denote $(B,\vee) +_{\mathrm{ord}} \{1'\}$ by $(B',\vee)$. We count the weak congruences of $(B',\vee)$ by classifying them according to their restrictions to $(B,\vee)$. Let $\theta$ be a *nonempty* week congruence of $(B,\vee)$. Then, as we are going to show below, there are exactly three weak congruences of $(B',\vee)$ that restrict to $\theta$. The first is $\theta$ itself. The second is $\theta \cup \{(1',1')\}$; it is a weak congruence of $(B',\vee)$ since $1'$ is $\vee$-irreducible. We are left with the third case when at least one pair $(x,1')$ is added to $\theta$ such that $x \neq 1'$. Denote such an extension by $\theta'$; we need to show that $\theta'$ exists and it is uniquely determined. Let $S$ be the support of $\theta$; then, clearly, $S' := S \cup \{1'\}$ is the support of $\theta'$. Note that $\theta'$ belongs to $\mathrm{Con}(S')$. Observe that

$$\text{the } \theta'\text{-classes are convex subsets of } (S',\vee). \tag{4.4}$$

Indeed, if $a,b,c \in S'$, $a \le c \le b$, and $(a,b) \in \theta'$, then $(c,b) = (a \vee c, b \vee c) \in \theta'$, so $c \in a/\theta'$, proving (4.4).

Next, let $i$ denote the greatest element of $S$; it exists since $S$ is finite and $i = \bigvee S$. It follows easily from (4.4) that, for any $y$, $(y,1') \in \theta'$ implies that $y \in i/\theta$. Since there is at least one such $y$ (namely, $y := x$), transitivity yields that

$$\theta' = \theta \cup (i/\theta \times \{1'\}) \cup (\{1'\} \times i/\theta) \cup \{(1',1')\}. \tag{4.5}$$

Hence, we obtain that $\theta'$ is uniquely determined by $\theta$. Since $1'$ is $\vee$-irreducible, it is easy to see that (4.5) defines a weak congruence of $B'$ the restriction of which is $\theta$.

The argument above shows that the any nonempty weak congruence of $(B,\vee)$, there corresponds exactly three weak congruences of $(B',\vee)$; this justifies the coefficient 3 in the lemma. The subtrahend $-1$ is explained by the trivial fact that $i$ and $\theta'$ in (4.5) do not exists when $\theta = \emptyset$. $\square$

**Theorem 4.4.2.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$, then*

$$|\mathrm{Cw}(B, \vee)| = 4(|\mathrm{Cw}(B_1, \vee)| \cdot |\mathrm{Cw}(B_2, \vee)|) - (|\mathrm{Cw}(B_1, \vee)| + |\mathrm{Cw}(B_2, \vee)|),$$

*where $B_1$, $B_2$ are the left and right maximal subtrees of the tree, respectively.*

*Proof.* Both $\mathrm{Cw}(B_1 +_{\mathrm{ord}} \{1'\}, \vee)$ and $\mathrm{Cw}(B_2 +_{\mathrm{ord}} \{1'\}, \vee)$ contain the $\emptyset$ and the only congruence on the singleton $\{1'\}$, is

$$\begin{aligned}
|\mathrm{Cw}(B, \vee)| &= 1 + \sum_{\substack{B^* \in \mathrm{Sub}\, B \\ B^* \neq \emptyset}} |\mathrm{Con}(B^*, \vee)| \\
&= 1 + \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i, \\ B_i^* \neq \emptyset}} 4\, |\mathrm{Con}(B_1^*, \vee)\, \mathrm{Con}(B_2^*, \vee)| + |\mathrm{Cw}(B_1 +_{\mathrm{ord}} \{1'\}, \vee)| \\
&\qquad + |\mathrm{Cw}(B_2 +_{\mathrm{ord}} \{1'\}, \vee)|) - 1 - 2,
\end{aligned}$$

using Lemma 4.4.1,

$$\begin{aligned}
&= \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i, \\ B_i^* \neq \emptyset}} 4\, |\mathrm{Con}(B_1^*, \vee)| \cdot |\mathrm{Con}(B_2^*, \vee)| + (3\, |\mathrm{Cw}(B_1, \vee)| - 1) \\
&\qquad + (3\, |\mathrm{Cw}(B_2, \vee)| - 1) - 2 \\
&= \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i, \\ B_i^* \neq \emptyset}} 4\, |\mathrm{Con}(B_1^*, \vee)| \cdot |\mathrm{Con}(B_2^*, \vee)| + 4\, |\mathrm{Cw}(B_1, \vee)| + 4\, |\mathrm{Cw}(B_2, \vee)| \\
&\qquad - |\mathrm{Cw}(B_1, \vee)| - |\mathrm{Cw}(B_2, \vee)| - 4.
\end{aligned}$$

Now for $B_i$,

$$|\mathrm{Cw}(B_i, \vee)| = 1 + \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i \\ B_i^* \neq \emptyset}} |\mathrm{Con}(B_i^*, \vee)|\,,$$

and let us use

$$|\mathrm{Cw}(B_1,\vee)|\cdot|\mathrm{Cw}(B_2,\vee)| = (1+\sum_{\substack{B_1^*\in \mathrm{Sub}\, B_1\\ B_1^*\neq\emptyset}}|\mathrm{Con}(B_1^*,\vee)|)$$

$$(1+\sum_{\substack{B_2^*\in \mathrm{Sub}\, B_2\\ B_i^*\neq\emptyset}}|\mathrm{Con}(B_2^*,\vee)|)$$

$$=\sum_{\substack{B_i^*\in \mathrm{Sub}\, B_i\\ B_i^*\neq\emptyset}}|\mathrm{Con}(B_1^*,\vee)|\cdot|\mathrm{Con}(B_2^*,\vee)|+|\mathrm{Cw}(B_1,\vee)|+|\mathrm{Cw}(B_2,\vee)|-1.$$

Then we arrive at

$$|\mathrm{Cw}(B,\vee)| = 4\,|\mathrm{Cw}(B_1,\vee)|\cdot|\mathrm{Cw}(B_2,\vee)|-|\mathrm{Cw}(B_1,\vee)|-|\mathrm{Cw}(B_2,\vee)|\,.$$

□

**Corollary 4.4.2.1.** *If $(B,\vee)$ is a semilattice defined by a prickly-snake $B$ of height $h$, then*

$$|\mathrm{Cw}(B,\vee)| = 7\cdot|\mathrm{Cw}(B_1,\vee)|-2=\frac{5\cdot 7^h+1}{3},$$

*where $B_1$ is the left maximal subtree of the tree.*

*Proof.* Apply Theorem 4.4.2 and webpage https://oeis.org/ with integer sequence A199420, to prove this. □

**Theorem 4.4.3.** *If $(B,\vee)$ is a semilattice defined by a perfect binary tree $B$ of height $h$, then*

$$|\mathrm{Cw}(B,\vee)| = 4\cdot|\mathrm{Cw}(B_1,\vee)|^2-2\cdot|\mathrm{Cw}(B_1,\vee)|\,,$$

*where $B_1$ is the left maximal subtree of the tree.*

*Moreover,*

$$|\mathrm{Cw}(B,\vee)| = \left\lceil \tfrac{1}{4}\,C^{2^{h+1}}\right\rceil,\quad C=2.61803398874989\ldots$$

*where $\lceil x\rceil$ denotes the least integer greater than or equal to $x$.*

*Proof.* To get the recurrence formula, use Theorem 4.4.2. Now we solve the recurrence.

For the sake of convenience, let now $a_n := |\mathrm{Cw}(B, \vee)|$, where $B$ is a perfect binary tree of height $(n-1)$. With this notation, our recurrence is

$$a_{n+1} = 4a_n^2 - 2a_n, \quad a_0 = 1.$$

In what follows, we again apply the method of Aho and Sloane [6, Section 3].

It is clear that (i) the sequence $(a_n)$ is monotone increasing; (ii) $a_n \geq 2$ for all $n$; moreover, (iii) $a_n \geq 2^{2n-1}$.

Using the notation

$$x_n := \log a_n\,, \quad y_n := \log\left(1 - \frac{1}{2a_n}\right), \tag{4.1$'$}$$

our recurrence reads as follows:

$$x_{n+1} = \log 4 + 2x_n + y_n.$$

It is clear that

$$x_n = (2^n - 1)\log 4 + 2^n x_0 + \sum_{k=0}^{n-1} 2^{n-1-k} y_k.$$

Now let

$$A_n := (2^n - 1)\log 4 + 2^n x_0 + \sum_{k=0}^{\infty} 2^{n-1-k} y_k\,, \quad B_n := \sum_{k=n}^{\infty} 2^{n-1-k} y_k\,.$$

These series are convergent again, like that in the proof of Theorem 4.2.3. Moreover, it is clear that $B_n < 0$ for all $n$ and

$$|B_n| = \frac{1}{2}|y_n| + \frac{1}{4}|y_{n+1}| + \frac{1}{8}|y_{n+2}| + \cdots < |y_n| \tag{4.2$'$}$$

and

$$a_n = e^{x_n} = e^{A_n - B_n} = e^{A_n} e^{-B_n}. \tag{4.3$'$}$$

The first term is the leading one. We find that

$$\begin{aligned} e^{A_n} &= \exp\left((2^n - 1)\log 4 + 2^n x_0 + \sum_{k=0}^{\infty} 2^{n-1-k} y_k\right) \\ &= \exp\left(-\log 4 + 2^n\left(\log 4 + x_0 + \sum_{k=0}^{\infty} 2^{-1-k} y_k\right)\right) = \frac{1}{4} \cdot C^{2^n}, \end{aligned}$$

where

$$C := \exp\left(\log 4 + x_0 + \sum_{k=0}^{\infty} 2^{-1-k} y_k\right) = 4\exp\left(x_0 + \frac{y_0}{2} + \frac{y_1}{4} + \frac{y_2}{8} + \cdots\right)$$

is a constant (independent of $n$). Using *Mathematica* (see Appendix D.21), we found that

$$C = 2.61803398874989\ldots.$$

Now, consider the second term in (4.3′). By (4.2′) we have

$$0 < e^{-B_n} = e^{|B_n|} < e^{|y_n|} = \exp\left|\log\left(1 - \frac{1}{2a_n}\right)\right| = \exp\left(\log \frac{1}{1 - \frac{1}{2a_n}}\right).$$

With the elementary inequality $\frac{1}{1-u} \leq 1 + 2u$ for $0 \leq u \leq \frac{1}{2}$ means that $1 < \frac{1}{1-\frac{1}{2a_n}} < 1 + \frac{1}{a_n}$, and, therefore, we see that

$$e^{-B_n} < 1 + \frac{1}{a_n}.$$

Substituting this to (4.3′), using (ii) and (iii), we have

$$e^{A_n} \leq a_n \leq e^{A_n}\left(1 + \frac{1}{a_n}\right) \leq e^{A_n}\left(1 + \frac{1}{e^{A_n}}\right) < e^{A_n} + 1.$$

Since $a_n$ is an integer, the assertion follows. □

## 4.5 Some related results

Libkin and Gurvich [37] studied those semilattices whose diagrams are trees. They characterized them as semilattices whose convex subsemilattices form a convex geometry. Moreover, they characterized atomistic semilattices with tree-diagram by lattice theoretic and graph theoretic means.

The concept of weak congruences is a tool for studying congruences and subalgebras of the same algebra together. The first researcher who studied the compatible symmetric and transitive relations on algebra was Šik, together with his Ph.D. student Mai (see [55]). Šešelja and Tepavčević wrote the book [55] on weak congruences. The purpose of their book was to present the basic properties of weak congruences, especially, their lattices, and to show how the results can be applied in universal algebra. The book was published after several years of systematic studies by the authors and it contains a bibliography on these and related topics up to 2001. Some later results on weak congruences can be found, for example in Chajda et al. [13], Czédli et al. [28], Šešelja and Tepavčević [47], Vojvodić and Šešelja [54] and Šešelja et al. [56].

# 5

# $(1+1+2)$-generated lattices of quasiorders

This chapter is based on a joint paper with Czédli [2]. A lattice is $(1+1+2)$-generated if it has a four-element generating set such that exactly two of the four generators are comparable. We prove that the lattice $\mathrm{Quo}(n)$ of all quasiorders (also known as preorders) of an $n$-element set is $(1+1+2)$-generated for $n = 3$ (trivially), $n = 6$ (when $\mathrm{Quo}(6)$ consists of $209\,527$ elements).

## 5.1 Notations used in this chapter

Given a set $A$, a relation $\rho \subseteq A^2$ is a *quasiorder* (also known as a *preorder*) if $\rho$ is reflexive and transitive. With respect to set inclusion, the set of all quasiorders of $A$ form a lattice $\mathrm{Quo}(A) = \langle \mathrm{Quo}(A), \subseteq \rangle$, called the *quasiorder lattice* of $A$. The meet of $\rho, \tau \in \mathrm{Quo}(A)$ is their intersection, and so we can say that $\rho \wedge \tau = \rho \cap \tau$. The join $\rho \vee \tau$ of $\rho$ and $\tau$ is the transitive closure of $\rho \cup \tau$. That is, for $x, y \in A$, we have $\langle x, y \rangle \in \rho \vee \tau$ if and only if there exists an $n \in \mathbb{N}^+ := \{1, 2, 3, 4, \dots\}$ and there are elements $z_0 = x, z_1, z_2, \dots, z_{n-1}, z_n = y$ in $A$ such that $\langle z_{i-1}, z_i \rangle \in \rho \cup \tau$ for all $i \in \{1, \dots, n\}$. Symmetric quasiorders are *equivalences* (also known as *equivalence relations*). The equivalences of

$A$ also form a lattice, called the *equivalence lattice* $\mathrm{Equ}(A)$ of $A$, which is a sublattice of $\mathrm{Quo}(A)$.

Since we are only interested in these lattices up to isomorphism, we will often write $\mathrm{Equ}(|A|)$ and $\mathrm{Quo}(|A|)$ instead of $\mathrm{Equ}(A)$ and $\mathrm{Quo}(A)$, respectively. In particular, $\mathrm{Quo}(6)$, which plays a distinguished role in this chapter, is the quasiorder lattice with a six-element underlying set. Note that $\mathrm{Equ}(A)$ and $\mathrm{Quo}(A)$ are complete lattices but the concept of *complete* lattices occurs only in the introductory section alongside a survey of the literature. Below, we only deal with *finite* lattices, which are complete, of course.

A four-element subset $X$ of a poset (partially ordered set) $Y$ is a (1+1+2)-*subset* of $Y$ if exactly two elements of $X$ are comparable. A subset $X$ of a lattice $L$ is a $(1+1+2)$-*generating set* of $L$ if $X$ is a $(1+1+2)$-subset of $L$ that generates $L$. If a lattice $L$ has a $(1+1+2)$-generating set, then we say that $L$ is $(1+1+2)$-*generated*.

## 5.2 Preliminaries

In 1976, Poguntke and Rival [41] proved that each lattice can be embedded into a four-generated finite simple lattice. (It turned out much later that three generators are sufficient if we drop the simplicity assumption; see Czédli [17].) Partition lattices, which are the same as equivalence lattices up to isomorphism, are well known to be simple. Thus, Pudlák and Tůma's result that every finite lattice is embeddable into a finite partition lattice (see [42]) superseded Poguntke and Rival's result in 1980. However, Poguntke and Rival's result still served well as the motivation for Strietz [50] and [51] to prove that $\mathrm{Equ}(n)$ is four-generated for $3 \le n \in \mathbb{N}^+$ and it is $(1+1+2)$-generated for $10 \le n \in \mathbb{N}^+$.

In 1983, Zádori [57] gave an entirely new method to find four-element generating sets of $\mathrm{Equ}(n)$ and extended Strietz's result by proving that $\mathrm{Equ}(n)$ is $(1+1+2)$-generated even for $7 \le n \in \mathbb{N}^+$. His method was the basis of all the more involved methods that were used to find small generating sets of $\mathrm{Equ}(A)$ and $\mathrm{Quo}(A)$ over the past three and a half decades. During this period, four-element generating sets and even $(1+1+2)$-generating sets (in

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Equ}(n)\rvert$ | 1 | 2 | 5 | 15 | 52 | 203 | 877 |
| $\lvert\mathrm{Quo}(n)\rvert$ | 1 | 4 | 29 | 355 | 6 942 | 209 527 | 9 535 241 |

Table 5.1: $|\mathrm{Equ}(n)|$ and $|\mathrm{Quo}(n)|$ for $n \in \{1, 2, \dots, 7\}$

the sense of complete generation) of $\mathrm{Equ}(A)$ were given for all infinite sets $A$ with "accessible" cardinalities (see Czédli [15], [14], and [16]). Even the lion's share of Czédli and Oluoch [27] is based on Zádori's method. Also, this period witnessed that extensions of his method were used to find small generating sets of $\mathrm{Quo}(A)$ by Chajda and Czédli [11], Czédli [18], Czédli and Kulin [26], and Takách [53]. Even the methods used by Dolgos [29] and Kulin [35] show lots of similarity with Zádori's method. Theorem 5.2.1 of this section will summarize the strongest results on $(1 + 1 + 2)$-generating sets of quasiorder lattices that were proved earlier.

Note that sometimes the structure $\langle \mathrm{Quo}(A), \bigvee, \bigwedge, {}^{-1}\rangle$ with $\rho^{-1} := \{\langle x, y\rangle : \langle y, x\rangle \in \rho\}$ rather than the complete lattice $\langle \mathrm{Quo}(A), \bigvee, \bigwedge\rangle$ was considered. So the title of Takách [53] should not mislead the reader since, after removing the operation $\rho \mapsto \rho^{-1}$, Takách [53] yields a six-element generating set of $\langle \mathrm{Quo}(A), \bigvee, \bigwedge\rangle$.

In 1983, Zádori [57] left the problem whether $\mathrm{Equ}(n)$ is $(1 + 1 + 2)$-generated for $n \in \{5, 6\}$ open. The difficulty of these small values lies in the fact that his method does not work for small $n$. This explains why it took 37 years to solve Zádori's problem on $\mathrm{Equ}(5)$ and $\mathrm{Equ}(6)$; see Czédli and Oluoch [27] for the solution. While [27] contains a traditional proof that $\mathrm{Equ}(6)$ is $(1+1+2)$-generated, computer programs were used to show that $\mathrm{Equ}(5)$ is not. This shows that "small" equivalence lattices create more difficulties than larger ones. The quotient marks indicate that $\mathrm{Equ}(5)$ and $\mathrm{Equ}(6)$ are not so small (see Table 5.1). This table has been taken from Sloane [48]. Note that the $|\mathrm{Equ}(n)|$ row and the first five numbers of the $|\mathrm{Quo}(n)|$ row of Table 5.1 also occur in Chajda and Czédli [11] and were obtained by a straightforward computer program twenty-five years ago.

Our knowledge on small generating sets of quasiorder lattices evolved in parallel to and in interactions with the analogous question about equivalence lattices. Small generating sets of *infinite* (complete) quasiorder lattices were given in Chajda and Czédli [11] even before dealing with infinite equivalence lattices. Surprisingly, it was quasiorders that showed the way how to pass from finite equivalence lattices to infinite ones. Prior to paper [2], our knowledge on small generating sets of $\mathrm{Quo}(A)$ was summarized in the last sentence of Theorem 1.1 in Czédli [18] and in Theorem 3.5 and Lemma 3.3 of Czédli and Kulin [26] as follows.

**Theorem 5.2.1** (Czédli [18] and Czédli and Kulin [26])**.** *If $A$ is a non-singleton set with accessible cardinality, then the following assertions hold.*

*(i) If $|A| \neq 4$, then* $\mathrm{Quo}(A)$ *is four-generated as a complete lattice.*

*(ii) If* $13 \leq |A|$ *is a finite* odd *number, then* $\mathrm{Quo}(A)$ *is* $(1+1+2)$*-generated.*

*(iii) If* $56 \leq |A|$ *is a finite* even *number, then* $\mathrm{Quo}(A)$ *is* $(1+1+2)$*-generated.*

*(iv) If $A$ is infinite, then* $\mathrm{Quo}(A)$ *is* $(1 + 1 + 2)$*-generated as a complete lattice.*

*(v) If $A$ is finite and $|A| \geq 3$, then* $\mathrm{Quo}(A)$ *is not a three-generated lattice.*

Note that ZFC has a model in which all infinite sets are of accessible cardinalities. We do not know whether $\mathrm{Quo}(A)$ has a four-element generating set if $|A| = 4$. (Based on our experience with computer programs for equivalence lattices, see Czédli and Oluoch [27], we guess that this question could be solved by a computer program that would require days of computer time if the same personal computer was used as in case of [27]. Developing such a computer program was not pursued at the time of writing.)

Although the paper [2] which this chapter is based on, presents $(1+1+2)$-generating sets for several new values of $n$, the existence of $(1 + 1 + 2)$-generating sets remains an unsolved problem for a few values. The only value of $n \geq 2$ for which we know that $\mathrm{Quo}(n)$ is *not* $(1+1+2)$-generated is $n = 2$. This follows trivially from $|\mathrm{Quo}(2)| = 4$ since a four-element lattice cannot have a three-element antichain.

Lastly, we note the following about quasiorder lattices. Armed with our tools based on Zádori's method, the large values of $n$ create less difficulty than the small values (apart from very small values where the problem is trivial or easy). In view of the history of equivalence lattices, it is not a surprise that paper [2] extends the scope of (ii) and (iii) of Theorem 5.2.1 by adding some *slightly smaller* numbers $n$. The surprise is that now we also add a significantly smaller number, $n = 6$, where Quo(6) is a huge lattice but Zádori's method cannot be used.

## 5.3 A $(1+1+2)$-generating set of Quo$(6)$

The least quasiorder $\{\langle x, x\rangle : x \in A\}$ of $A$ will be denoted by $\Delta = \Delta_A$. For elements $x$ and $y$ of $A$, the following two members of Quo$(A)$ will play a particularly important role in our proofs:

$$q(x,y) = \{\langle x,y\rangle\} \cup \Delta \quad \text{and} \quad e(x,y) = e(y,x) = \{\langle x,y\rangle, \langle y,x\rangle\} \cup \Delta. \tag{5.1}$$

We allow that $x = y$; however, $q(x,x) = \Delta$ and $e(x,x) = \Delta$ will not play any significant role in our proofs. The atoms of Quo$(A)$ and those of Equ$(A)$ are exactly the $q(x,y)$ and the $e(x,y)$ with $x \neq y \in A$. The importance of $q(x,y)$ and $e(x,y)$ lies in the following well-known and trivial fact: for any non-singleton set $A$ and for every $\rho \in$ Quo$(A)$ and $\theta \in$ Equ$(A)$,

$$\rho = \bigvee\{q(x,y) : \langle x,y\rangle \in \rho\} \qquad \text{and} \qquad \theta = \bigvee\{e(x,y) : \langle x,y\rangle \in \theta\}. \tag{5.2}$$

In other words, Quo$(A)$ and Equ$(A)$ are atomistic.

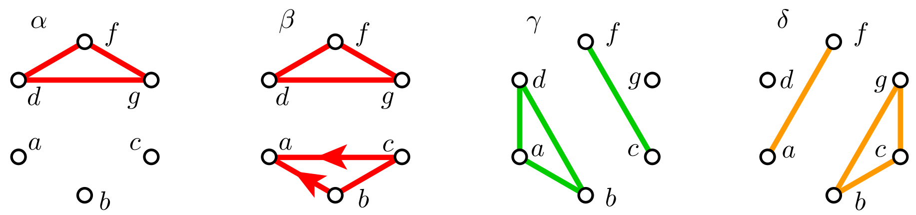


Figure 5.1: $\alpha$, $\beta$, $\gamma$, and $\delta$

Next, let $A = \{a, b, c, d, f, g\}$. We define the following quasiorders of $A$:

$$\begin{aligned} \alpha &:= e(d,f) \vee e(f,g), & \beta &:= \alpha \vee e(b,c) \vee q(b,a) \\ \gamma &:= e(a,b) \vee e(a,d) \vee e(c,f), & \delta &:= e(b,c) \vee e(c,g) \vee e(a,f). \end{aligned} \tag{5.3}$$

These quasiorders are shown in Figure 5.1 using the corresponding directed graphs. (The edges without arrows are directed in both ways.) Namely, for $\rho \in \{\alpha, \beta, \gamma, \delta\}$ and $x, y \in A$, we have that $\langle x, y\rangle \in \rho$ if and only if there is a directed path (possibly of length 0) in the graph corresponding to $\rho$ in Figure 5.1.

For the sake of the following remark, let $\langle u_1, u_2, u_3, u_4, u_5, u_6\rangle := \langle b, a, c, d, f, g\rangle$ and $\beta^\star := \beta \vee q(a, b) = \beta \vee q(u_2, u_1)$.

**Remark 5.3.1.** *Each of $\alpha$, $\beta^\star$, $\gamma$, and $\delta$ is an equivalence, $\alpha < \beta^\star$, and it was proved in Czédli and Oluoch [27] that $\{\alpha, \beta^\star, \gamma, \delta\}$ is a $(1+1+2)$-generating set of* $\mathrm{Equ}(6) = \mathrm{Equ}(\{u_1, u_2, u_3, u_4, u_5, u_6\})$.

We are going to prove the following statement.

**Theorem 5.3.2.** *With the quasiorders defined in* (5.3), *$\{\alpha, \beta, \gamma, \delta\}$ is a $(1+1+2)$-generating set of the quasiorder lattice* $\mathrm{Quo}(6) = \mathrm{Quo}(\{a, b, c, d, f, g\})$. *Hence,* $\mathrm{Quo}(6)$ *is $(1+1+2)$-generated.*

As Remark 5.3.1 indicates, our generating set is only slightly different from the one used in Czédli and Oluoch [27]. However, this little difference results in a substantial change in the complexity of the proofs. Indeed, while only six equations were necessary in [27] to prove that $\{\alpha, \beta^\star, \gamma, \delta\}$ generates $\mathrm{Equ}(6)$, we are going to use twenty-five equations, (5.8)–(5.32), to prove Theorem 5.3.2.

*Proof of Theorem 5.3.2.* First, we fix our notation and describe the corresponding technique. For $\rho \in \mathrm{Quo}(A)$, let $\Theta(\rho) := \rho \cap \rho^{-1} = \{\langle x, y\rangle : \langle x, y\rangle \in \rho \text{ and } \langle y, x\rangle \in \rho\}$, which is the largest equivalence relation of $A$ included in $\rho$. On the quotient set $A/\Theta(\rho)$, we can define a relation $\rho/\Theta(\rho)$ as follows: for $\Theta(\rho)$-blocks $x/\Theta(\rho)$ and $y/\Theta(\rho)$ in $A/\Theta(\rho)$, we let

$$\langle x/\Theta(\rho), y/\Theta(\rho)\rangle \in \rho/\Theta(\rho) \overset{\text{def}}{\iff} \langle x, y\rangle \in \rho.$$

We know from the folklore of algebra that $\rho/\Theta(\rho)$ is well defined and it is a partial order, the so-called *order induced by $\rho$*. Hence, $A/\Theta(\rho) = \langle A/\Theta(\rho), \rho/\Theta(\rho)\rangle$ is a poset. For several choices of $\rho$, we will frequently draw the Hasse diagram of this poset in order to provide a visual description of $\rho$. In such a diagram, the $\Theta(\rho)$-blocks are indicated by rectangles. However, we will adopt the following convention:

$$\begin{array}{l} \text{if } \{x\} \text{ is a singleton block of } \Theta(\rho) \text{ such that} \\ \text{for every } \{x\} \neq Y \in A/\Theta(\rho) \text{ we have that} \\ \langle \{x\}, Y\rangle \notin \rho/\Theta(\rho) \text{ and } \langle Y, \{x\}\rangle \notin \rho/\Theta(\rho), \\ \text{then } \{x\} \text{ is not indicated in the Hasse diagram} \\ \text{of } A/\Theta(\rho). \end{array} \tag{5.4}$$

In other words, if $x \in A$ has the property that

$$(\forall y \in A)\Big(\{\langle x, y\rangle, \langle y, x\rangle\} \cap \rho \neq \emptyset \Longrightarrow x = y\Big),$$

then (the necessary singleton) block $x/\Theta(\rho)$ is not indicated in the Hasse diagram. For example, the quasiorders defined in (5.3) are visualized by diagrams as follows.

$$\alpha : \boxed{d, f, g}, \ \beta : \begin{array}{cc} \boxed{a} & \boxed{d, f, g} \\ | & \\ \boxed{b, c} & \end{array}, \ \gamma : \boxed{a, b, d}\ \boxed{c, f}, \ \delta : \boxed{b, c, g}\ \boxed{a, f}. \tag{5.5}$$

Since we are going to perform a lot of computations with quasiorders, convention (5.4) and the above-mentioned visual approach will be helpful for the reader in the rest of the proof. Note that if a diagram according to our convention is given and $x \neq y \in A$, then we have $\langle x, y\rangle \in \rho$ if and only if both the block $x/\Theta(\rho)$ of $x$ and that of $y$ are drawn in the (Hasse) diagram and $x/\Theta(\rho) \leq y/\Theta(\rho)$ according to the diagram. In particular, if $x$ and $y$ are in the same block, then $\langle x, y\rangle \in \rho$. Note also that our computations in the proof never require dealing with pairs of the form $\langle x, x\rangle$. Although the following observation is quite easy to prove, it will substantially ease our task later on.

**Observation 5.3.3** (Disjoint Paths Principle)**.** *For $k, s \in \mathbb{N}^+$ and a set $B$, let $x, y,\ u_0 = x, u_1, \dots, u_{k-1}, u_k = y,\ v_0 = x, v_1, \dots, v_{s-1}, v_s = y$ be elements of $B$ such that $\{u_1, \dots, u_{k-1}\} \cap \{v_1, \dots, v_{s-1}\} = \emptyset$, $|\{u_1, \dots, u_{k-1}\}| = k - 1$,* *and* $|\{v_1,\dots,v_{s-1}\}| = s-1$. *For* $i \in \{1,\dots,k\}$ *and* $j \in \{1,\dots,s\}$, *let* $p_i \in \{e,q\}$ *and* $r_j \in \{e,q\}$; *see* (5.1) *for the meaning of* $q$ *and* $e$. *Assume that there is an* $i' \in \{1,\dots,k\}$ *such that* $p_{i'} = q$ *or there is a* $j' \in \{1,\dots,s\}$ *such that* $r_{j'} = q$. *Then*

$$q(x,y) = \Big(\bigvee_{i=1}^{k} p(u_{i-1},u_i)\Big) \wedge \Big(\bigvee_{j=1}^{s} r(v_{j-1},v_j)\Big). \tag{5.6}$$

Similar observations (sometimes under the name "Circle Principle") were previously formulated in Czédli [15], [18, Lemma 2.1], Czédli and Kulin [16, Lemma 2.5], Kulin [35, Lemma 2.2], and were used implicitly in Chajda and Czédli [11], Czédli [14] and [16], and Takách [53]. However, Observation 5.3.3 is slightly stronger than its precursors.

Our argument proving Observation 5.3.3 runs as follows. We can assume that $x \neq y$. Let $\rho$ denote the quasiorder given on the right of the equality symbol in (5.6). Since the pair $\langle x,y\rangle$ belongs to both meetands in (5.6) by transitivity, the inequality $q(x,y) \leq \rho$ is clear. Since $\{u_1,\dots,u_{k-1}\} \cap \{v_1,\dots,v_{s-1}\} = \emptyset$, we obtain that

$$\rho \leq \Big(\bigvee_{i=1}^{k} e(u_{i-1},u_i)\Big) \wedge \Big(\bigvee_{j=1}^{s} e(v_{j-1},v_j)\Big) = e(x,y). \tag{5.7}$$

Using the fact that the existence of $i'$ or $j'$ together with $|\{u_1,\dots,u_{k-1}\}| = k-1$, and $|\{v_1,\dots,v_{s-1}\}| = s-1$ easily exclude that $\langle y,x\rangle \in \rho$, (5.7) implies that $\rho \leq q(x,y)$. Combining this with the previously established converse inequality, we conclude (5.6) and the validity of Observation 5.3.3. Here, for the sake of brevity, we will often refer to (5.6) rather than Observation 5.3.3.

Now, resuming the proof of Theorem 5.3.2, let $S$ denote the sublattice generated by $\{\alpha,\beta,\gamma,\delta\}$ in $\mathrm{Quo}(6) = \mathrm{Quo}(\{a,b,c,d,f,g\})$. Since $S$ is closed with respect to $\wedge$ and $\vee$, it will be clear that the quasiorders given on the left of the equality signs below in (5.8)–(5.32) all belong to $S$, provided we use quasiorders already in $S$ on the right of our equality signs. (To see that the quasiorders on the right are in $S$, we will cite the relevant earlier equations except, possibly (5.5).) We also need to check that the equalities we claim below hold in $\mathrm{Quo}(6)$. We are going to check this either with the help of the diagrams given on the right of the equalities in question, or this will prompt follows from (5.6). After these instructions on how to read the equations shown below, we are ready to compute; the details are easy to follow provided (5.5) is in the reader's visual field.

$e(b,c) = \beta \wedge \delta$ by (5.5);
$$\begin{array}{ll} \boxed{a} & \boxed{d,f,g} \\ \mid & \\ \boxed{b,c} & \end{array} \wedge \boxed{b,c,g}\,\boxed{a,f}\,. \tag{5.8}$$

$q(b,a) = \beta \wedge \gamma$ by (5.5);
$$\begin{array}{ll} \boxed{a} & \boxed{d,f,g} \\ \mid & \\ \boxed{b,c} & \end{array} \wedge \boxed{a,b,d}\,\boxed{c,f}\,. \tag{5.9}$$

$e(d,f) = \alpha \wedge (\gamma \vee e(b,c))$ by (5.5) and (5.8);
$$\boxed{d,f,g} \wedge \boxed{a,b,d,c,f}\,. \tag{5.10}$$

$q(g,f) = \alpha \wedge (\delta \vee q(b,a))$ by (5.5) and (5.9);
$$\boxed{d,f,g} \wedge \begin{array}{c} \boxed{a,f} \\ \mid \\ \boxed{b,c,g} \end{array}\,. \tag{5.11}$$

$e(a,d) = \gamma \wedge (e(d,f) \vee \delta)$ by (5.5) and (5.10);
$$\boxed{a,b,d}\,\boxed{c,f} \wedge \boxed{b,c,g}\,\boxed{a,f,d}\,. \tag{5.12}$$

$q(g,c) = \delta \wedge (q(g,f) \vee \gamma)$ by (5.5) and (5.11);
$$\boxed{b,c,g}\,\boxed{a,f} \wedge \begin{array}{lc} \boxed{a,b,d} & \boxed{c,f} \\ & \mid \\ & \boxed{g} \end{array}\,. \tag{5.13}$$

$e(a,f) = \delta \wedge (e(a,d) \vee e(d,f))$ by (5.5), (5.12), and (5.10);
$$\boxed{b,c,g}\,\boxed{a,f} \wedge \boxed{a,d,f}\,. \tag{5.14}$$

$q(g,a) = (q(g,f) \vee e(f,a)) \wedge (q(g,c) \vee e(c,b) \vee q(b,a))$ by (5.6), (5.11), (5.14), (5.13), (5.8), and (5.9). (5.15)

$q(g,d) = (q(g,f) \vee e(f,d)) \wedge (q(g,a) \vee e(a,d))$ by (5.6), (5.11), (5.10), (5.15), and (5.12). (5.16)

$q(b,d) = (q(b,a) \vee e(a,d)) \wedge (\delta \vee q(g,d))$ by (5.9), (5.12), and (5.16);

$$\begin{array}{c}\boxed{a,d}\\ | \\ \boxed{b}\end{array} \wedge \begin{array}{c}\boxed{d}\\ | \\ \boxed{b,c,g}\,\boxed{a,f}\end{array}. \tag{5.17}$$

$q(g,b) = (q(g,c) \vee e(c,b)) \wedge (q(g,d) \vee \gamma)$ by (5.13), (5.8), and (5.16);

$$\begin{array}{c}\boxed{c,b}\\ | \\ \boxed{g}\end{array} \wedge \begin{array}{c}\boxed{a,b,d}\,\boxed{c,f}\\ | \\ \boxed{g}\end{array}. \tag{5.18}$$

$q(b,f) = (q(b,a) \vee e(a,f)) \wedge (q(b,d) \vee e(d,f))$ by (5.6), (5.9), (5.14), (5.17), and (5.10). (5.19)

$q(c,f) = (e(c,b) \vee q(b,f)) \wedge \gamma$ by (5.8) and (5.19);

$$\begin{array}{c}\boxed{f}\\ | \\ \boxed{c,b}\end{array} \wedge \boxed{a,b,d}\,\boxed{c,f}. \tag{5.20}$$

$q(b,c) = e(b,c) \wedge (q(b,f) \vee \gamma)$ by (5.8) and (5.19);

$$\boxed{b,c} \wedge \begin{array}{c}\boxed{c,f}\\ | \\ \boxed{a,b,d}\end{array}. \tag{5.21}$$

$q(d,f) = e(d,f) \wedge (q(b,f) \vee \gamma)$ by (5.10) and (5.19);

$$\boxed{d,f} \wedge \begin{array}{c}\boxed{c,f}\\ | \\ \boxed{a,b,d}\end{array}. \tag{5.22}$$

$q(a,f) = e(a,f) \wedge (q(b,f) \vee \gamma)$ by (5.14) and (5.19);

$$\boxed{a,f} \wedge \begin{array}{c}\boxed{c,f}\\ | \\ \boxed{a,b,d}\end{array}. \tag{5.23}$$

$q(d,a) = e(d,a) \wedge (q(d,f) \vee e(f,a))$ by (5.6), (5.12), (5.22), and (5.14). (5.24)

$$q(f,a) = e(f,a) \wedge (e(f,d) \vee q(d,a)) \quad \text{by (5.6), (5.14), (5.10), and (5.24).} \tag{5.25}$$

$$q(b,g) = (q(b,f) \vee \alpha) \wedge \delta \text{ by (5.19);} \quad \begin{array}{c}\boxed{d,f,g}\\ | \\ \boxed{b}\end{array} \wedge \boxed{b,c,g}\,\boxed{a,f}\,. \tag{5.26}$$

$$q(a,d) = e(a,d) \wedge (q(a,f) \vee e(f,d)) \quad \text{by (5.6), (5.12), (5.23), and (5.10).} \tag{5.27}$$

$$q(f,d) = e(f,d) \wedge (q(f,a) \vee q(a,d)) \quad \text{by (5.6), (5.10), (5.25), and (5.27).} \tag{5.28}$$

$$q(c,g) = (e(c,b) \vee q(b,g)) \wedge (q(c,f) \vee \alpha) \text{ by (5.8), (5.26), and (5.20);} \quad \begin{array}{c}\boxed{g}\\ | \\ \boxed{c,b}\end{array} \wedge \begin{array}{c}\boxed{d,f,g}\\ | \\ \boxed{c}\end{array}\,. \tag{5.29}$$

$$q(c,b) = (q(c,g) \vee q(g,b)) \wedge e(c,b) \quad \text{by (5.6), (5.29), (5.18), and (5.8).} \tag{5.30}$$

$$q(f,g) = \alpha \wedge (\gamma \vee q(c,g)) \text{ by (5.29);} \quad \boxed{d,f,g} \wedge \begin{array}{c}\boxed{g}\\ | \\ \boxed{a,b,d}\,\boxed{c,f}\end{array}\,. \tag{5.31}$$

$$q(a,b) = (q(a,f) \vee q(f,g) \vee q(g,b)) \wedge \gamma \text{ by (5.23), (5.31), and (5.18);} \quad \begin{array}{c}\boxed{b}\\ | \\ \boxed{g}\\ | \\ \boxed{f}\\ | \\ \boxed{a}\end{array} \wedge \boxed{a,b,d}\,\boxed{c,f}\,. \tag{5.32}$$

In the rest of the proof, we only need the twelve atoms of Quo(6) =

Quo($A$) that are shown in Figure 5.2. Using that these twelve atoms belonging to $S$, we conclude using (5.6) that, for all $x, y \in A$, the quasiorder $q(x, y)$ belongs to $S$ as well. Hence, it follows from (5.2) that $\rho \in S$ for all $\rho \in \mathrm{Quo}(A)$. Consequently, $S = \mathrm{Quo}(A)$ and $\{\alpha, \beta, \gamma, \delta\}$ is a generating set of Quo($A$). Since $\{\alpha, \beta, \gamma, \delta\}$ is a $(1 + 1 + 2)$-subset of Quo($A$), the proof of Theorem 5.3.2 is complete. □

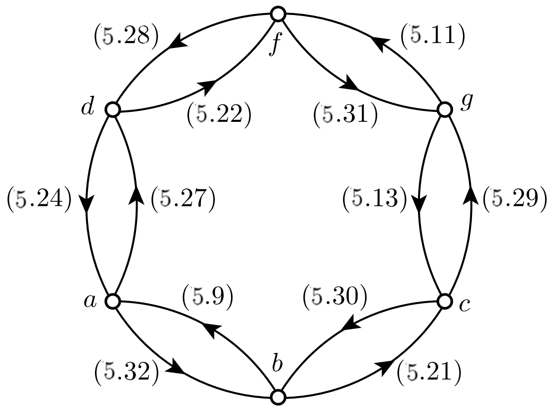


Figure 5.2: Twelve atoms of Quo($A$)

In addition to the fact that the Disjoint Paths Principle (see Observation 5.3.3) played an important role in the proof above, this principle is also useful for simplifying the proof of the following lemma. This lemma is implicit in Kulin [35]. The reader can see the proof of Part (i) of Theorem 2.1 there.

**Lemma 5.3.4** (Kulin [35]). *If $A$ is a set consisting of at least three elements and $\rho$ belongs to* Quo($A$) $\setminus$ Equ($A$)*, then* Equ($A$) $\cup \{\rho\}$ *generates the lattice* Quo($A$).

Since $\mathrm{Equ}(\{a, b\}) \cup \{q(a, b)\}$, which is a three-element chain, is a proper sublattice of $\mathrm{Quo}(\{a, b\})$, the stipulation that $A$ has at least three elements cannot be omitted from Lemma 5.3.4. For the reader's convenience and also to demonstrate the power of the Disjoint Paths Principle, we are going to present a new proof of this lemma.

*Proof of Lemma 5.3.4.* We can assume that $A$ consists of the vertices $a_0, a_1, \ldots, a_{n-1}$, listed counterclockwise, of a regular $n$-gon such that $\langle a_0, a_1 \rangle \in \rho$

but $\langle a_1, a_0\rangle \notin \rho$. This $n$-gon is non-degenerate since $n = |A| \geq 3$. If $i, j \in \{0, \ldots, n-1\}$ and $j \equiv i+1 \pmod n$, then $\{a_i, a_j\}$, $\langle a_i, a_j\rangle$, and $\langle a_j, a_i\rangle$ are called an *undirected edge*, a *counterclockwise edge*, and a *clockwise edge* of the $n$-gon, respectively.

Let $S$ denote the sublattice of $\mathrm{Quo}(A)$ generated by $\mathrm{Equ}(A) \cup \{\rho\}$. Then all the undirected edges are in $S$, which means that $e(a_i, a_j) \in S$ for all $i, j \in \{0, \ldots, n-1\}$ with $j \equiv i+1$. We say that the counterclockwise version and the clockwise version of an edge $\{a_i, a_j\}$ are in $S$ if $q(a_i, a_j) \in S$ and $q(a_j, a_i) \in S$, respectively. It follows from (5.6) that if all the counterclockwise edges and all the clockwise edges of the $n$-gon are in $S$, then all the atoms of $\mathrm{Quo}(A)$ are in $S$ and so $S = \mathrm{Quo}(A)$ by (5.2). It also follows from (5.6) that if the counterclockwise version of an (undirected) edge belongs to $S$, then the clockwise versions of all other edges are in $S$. Combining this fact with its counterpart in which the two directions are interchanged, we see that if at least one directed edge is in $S$, then all directed edges are in $S$ and $S = \mathrm{Quo}(A)$. Therefore, using the fact that $\langle a_0, a_1\rangle \in \rho$ but $\langle a_1, a_0\rangle \notin \rho$ leads to $q(a_0, a_1) = e(a_0, a_1) \wedge \rho \in S$, we obtain the statement of the lemma. $\square$

**Corollary 5.3.4.1.** $\mathrm{Quo}(3)$ *is* $(1+1+2)$*-generated.*

*Proof.* Since $\mathrm{Equ}(3) = \mathrm{Equ}(\{a, b, c\})$ is generated by the set $\{e(a, b), e(b, c), e(c, a)\}$ of its atoms, $\{q(a, b), e(a, b), e(b, c), e(c, a)\}$ is a $(1+1+2)$-generating set of the lattice $\mathrm{Quo}(3) = \mathrm{Quo}(\{a, b, c\})$ by Lemma 5.3.4. $\square$

## 5.4 Some related results

Czédli in the second part of paper [2], proved that the lattice $\mathrm{Quo}(n)$ of all quasiorders of an $n$-element set is $(1+1+2)$-generated for $n = 11$ and for all $n \geq 13$. Below, we provide an illustration of this result.

**Definition 1** (Zádori configuration)**.** *For* $2 \leq k \in \mathbb{N}^+$*, let* $a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_{k-1}$ *be pairwise distinct elements of a finite set* $B$*. Let*

$$\begin{aligned} &\alpha = \bigvee_{i=1}^{k} e(a_{i-1}, a_i) \vee \bigvee_{i=1}^{k-1} e(b_{i-1}, b_i), \quad \beta = \bigvee_{i=0}^{k-1} e(a_i, b_i) \\ &\gamma = \bigvee_{i=1}^{k} e(a_i, b_{i-1}), \quad \epsilon_0 = e(a_0, b_0), \quad \textit{and} \quad \eta = e(a_k, b_{k-1}); \end{aligned} \tag{5.33}$$

*they are members of* $\mathrm{Equ}(B)$. *The system of these* $2k+1$ *elements and five equivalences of* $B$ *is called a* Zádori configuration *of (odd) size* $2k+1$ *in* $B$. *The set*

$$A := \{a_0, \dots, a_k, b_0, \dots, b_{k-1}\} \tag{5.34}$$

*is the* support *of this configuration.*

A Zádori configuration is easy to visualize; following Zádori's original drawing, we do this with the help of a graph in the following way. We say that a path in a graph is horizontal, is of slope 1, and is of slope $-1$ if all of the edges constituting the path are such. For vertices $x$ and $y$ in the graph,

$$\begin{aligned} \langle x,y\rangle \in \alpha &\overset{\text{def}}{\iff} \text{there is a horizontal path from } x \text{ to } y;\\ \langle x,y\rangle \in \beta &\overset{\text{def}}{\iff} \text{there is a path of slope } -1 \text{ from } x \text{ to } y;\\ \langle x,y\rangle \in \gamma &\overset{\text{def}}{\iff} \text{there is a path of slope 1 from } x \text{ to } y. \end{aligned} \tag{5.35}$$

Note that a path of length 0 is simultaneously of slope 1 and of slope $-1$, and it is also horizontal. Also, note that (5.35) complies with (5.33).

For example, a Zádori configuration of size 11 is given in Figure 5.3; disregard the dashed curved edges for a while. Some of the horizontal edges are labeled by $\alpha$ but, to avoid crowdedness, not all. The same convention applies for edges of slope $-1$ and $\beta$, and edges of slope 1 and $\gamma$.

Zádori configurations played a decisive role in all papers that applied extensions of Zádori's method; see Section 5.2 for the list of these papers. Given a Zádori configuration in $B$ with support set $A$ (see (5.33)–(5.34)), we define

$$\mathrm{Equ}(B\rceil_A) := \{\theta \in \mathrm{Equ}(B) : \text{if } \langle x,y\rangle \in \theta \text{ and } \{x,y\} \not\subseteq A, \text{ then } x = y\}. \tag{5.36}$$

In Zádori [57], this configuration and the following lemma assumed that $B = A$. However, this assumption is not a real restriction since the map

$$\mathrm{Equ}(B\rceil_A) \to \mathrm{Equ}(A) \text{ defined by } \theta \mapsto \theta \cap (A \times A) \tag{5.37}$$

is clearly an isomorphism, whereby the validity of the following lemma follows from its original particular case $B = A$.

**Lemma 5.4.1** (Zádori [57]). *Assume that a Zádori configuration of size* $2k+1$ *with support* $A$ *is given in* $B$*; see* (1) *and* (5.34). *Then* $\{\alpha, \beta, \gamma, \epsilon_0, \eta_k\}$ *generates* $\mathrm{Equ}(B\rceil_A)$.

Note that this lemma is explicitly stated in Czédli [24] and Czédli and Kulin [26], and implicitly proved (hidden in long proofs) in Czédli [15], [14], [16], and [18], and Czédli and Oluoch [27]. Now we are in the position to state the following result.

**Theorem 5.4.2** (Czédli [2]). *Let* $n \in \mathbb{N}^+$ *be a natural number.*

*(i) If* $n \geq 11$ *and* $n$ *is odd, then* $\mathrm{Quo}(n)$ *is* $(1+1+2)$*-generated.*

*(ii) If* $n \geq 13$*, then* $\mathrm{Quo}(n)$ *is* $(1+1+2)$*-generated.*

For illustration, we borrow the constructions for $n = 11$ and $n = 14$ from Czédli [2] (see Figures 5.3 and 5.4). In these figures, $\alpha, \beta$, and $\gamma$ are given by convention (5.3); note that they happen to be equivalences. The dotted (oriented or non-oriented) arcs define $\delta$.

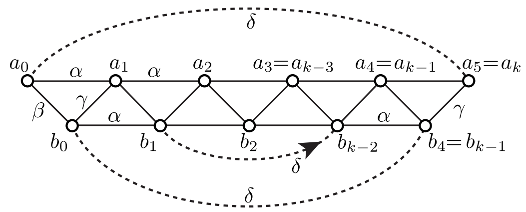


Figure 5.3: $\{\alpha, \beta, \gamma, \delta\}$ is a $(1+1+2)$-generating set of Quo(11)

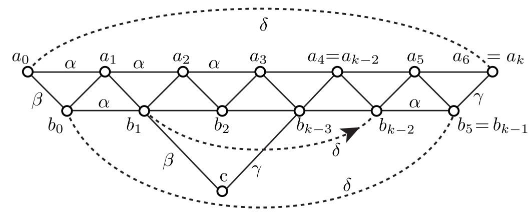


Figure 5.4: $\{\alpha, \beta, \gamma, \delta\}$ is a $(1+1+2)$-generating set of Quo(14)

# 6

# Summary

The topic of this dissertation is restricted to finite lattices and semilattices. The dissertation consists of five chapters, followed by a summary, a bibliography and appendices. In the first chapter, we give an overview of works related to the topic of the dissertation.

In the forthcoming paragraphs, we briefly summarise the main results upon which this dissertation is built on.

**Chapter 2.** Concerning the numbers of subuniverses of finite lattices, we proved that the fourth largest number of subuniverses of an $n$-element lattice is $21.5 \cdot 2^{n-5}$ for $n \geq 6$, and the fifth largest number of subuniverses of an $n$-element lattice is $21.25 \cdot 2^{n-5}$ for $n \geq 7$. Also, we described the $n$-element lattices with exactly $21.5 \cdot 2^{n-5}$ (for $n \geq 6$) and $21.25 \cdot 2^{n-5}$ (for $n \geq 7$) subuniverses, as in the following theorem:

**Theorem 6.0.1.** *The following two assertions hold.*

*(i) The fourth largest number in* $\mathrm{NS}(n)$ *is* $21.5 \cdot 2^{n-5}$ *for* $n \geq 6$*. Furthermore, for* $n \geq 6$*, an* $n$*-element lattice* $L$ *has exactly* $21.5 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} N_6 +_{\mathrm{glu}} C_1$*, where* $C_0$ *and* $C_1$ *are chains.*

*(ii) The fifth largest number in* $\mathrm{NS}(n)$ *is* $21.25 \cdot 2^{n-5}$ *for* $n \geq 7$*. Furthermore, for* $n \geq 7$*, an* $n$*-element lattice* $L$ *has exactly* $21.25 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C_1$*, where* $C_0$ *and* $C_1$ *are chains.*

**Chapter 3.** Concerning the numbers of subuniverses of finite semilattices, motivated by the results in the second chapter, we proved that the first largest number of subuniverses of an $n$-element semilattice is $2^n = 32 \cdot 2^{n-5}$, the second largest number is $28 \cdot 2^{n-5}$ and the third one is $26 \cdot 2^{n-5}$, where $n \geq 5$. Also, we described the $n$-element semilattices with exactly $32 \cdot 2^{n-5}$, $28 \cdot 2^{n-5}$, or $26 \cdot 2^{n-5}$ subuniverses, as in the following theorem:

**Theorem 6.0.2.** *If $5 \leq n \in \mathbb{N}^+$, then the following three assertions hold.*

*(i) The first largest number in $\mathrm{NS}(n)$ is $2^n = 32 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L, \vee)$ has exactly $2^n$ subuniverses if and only if $(L, \vee)$ is a chain.*

*(ii) The second largest number in $\mathrm{NS}(n)$ is $28 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L, \vee)$ has exactly $28 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_3 +_{\mathrm{glu}} C_1$ or $(L, \vee) \cong C_0 +_{\mathrm{ord}} H_3 +_{\mathrm{glu}} C_1$, where $C_0$ and $C_1$ are finite chains.*

*(iii) The third largest number in $\mathrm{NS}(n)$ is $26 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L, \vee)$ has exactly $26 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_4 +_{\mathrm{glu}} C_1$ or $(L, \vee) \cong C_0 +_{\mathrm{ord}} H_4 +_{\mathrm{glu}} C_1$, where $C_0$ and $C_1$ are finite chains.*

**Chapter 4.** It concerns the number of subuniverses, congruences, weak congruences of semilattices defined by trees. We determined the number of subuniverses of semilattices defined by arbitrary and special kinds of trees via combinatorial considerations, as follows:

**Lemma 6.0.3.** *If $(T, \vee)$ is a semilattice defined by a tree $T$, then*

$$|\mathrm{Sub}(T, \vee)| = \prod_{i=1}^{n} (|\mathrm{Sub}(T_i, \vee)|) + \sum_{i=1}^{n} (|\mathrm{Sub}(T_i, \vee)|) - (n-1),$$

*where $T_1, \ldots, T_n$ is a repetition free list of maximal subtrees of the tree $T$.*

**Corollary 6.0.3.1.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$, then*

$$|\mathrm{Sub}(B, \vee)| = |\mathrm{Sub}(B_1, \vee)| \cdot |\mathrm{Sub}(B_2, \vee)| + (|\mathrm{Sub}(B_1, \vee)| + |\mathrm{Sub}(B_1, \vee)|) - 1,$$

*where $B_1$, $B_2$ are the left and right maximal subtrees of the tree, respectively.*

**Corollary 6.0.3.2.** *If* $(B, \vee)$ *is a semilattice defined by a prickly-snake* $B$ *of height* $h$, *then*

$$|\mathrm{Sub}(B, \vee)| = 3\,|\mathrm{Sub}(B_1, \vee)| + 1 = \frac{5 \cdot 3^h - 1}{2},$$

*where* $B_1$ *is the left maximal subtree of the tree.*

Using a result of Freese and Nation [32], we gave a formula for the number of congruences of semilattices defined by arbitrary and special kinds of trees, as follows:

**Lemma 6.0.4.** *If* $(T, \vee)$ *is a semilattice defined by a tree* $T$, *then*

$$|\mathrm{Con}(T, \vee)| = 2^{|T|-1} = 2^{\sum_{i=1}^{n} |T_i|} = 2^n \cdot \prod_{i=1}^{n} |\mathrm{Con}(T_i, \vee)|\,,$$

*where* $T_1, \ldots, T_n$ *is a repetition free list of maximal subtrees of the tree* $T$.

**Corollary 6.0.4.1.** *If* $(B, \vee)$ *is a semilattice defined by a binary tree* $B$, *then*

$$|\mathrm{Con}(B, \vee)| = 2^{|B_1|+|B_2|} = 4 \cdot |\mathrm{Con}(B_1, \vee)| \cdot |\mathrm{Con}(B_2, \vee)|\,,$$

*where* $B_1, B_2$ *are the left and right maximal subtrees of the tree, respectively.*

**Corollary 6.0.4.2.** *If* $(B, \vee)$ *is a semilattice defined by a prickly-snake* $B$ *of height* $h$, *then*

$$|\mathrm{Con}(B, \vee)| = 4 \cdot |\mathrm{Con}(B_1, \vee)| = 4^h,$$

*where* $B_1$ *is the left maximal subtree of the tree.*

**Corollary 6.0.4.3.** *If* $(B, \vee)$ *is a semilattice defined by a perfect binary tree* $B$ *of height* $h$, *then*

$$|\mathrm{Con}(B, \vee)| = 4 \cdot |\mathrm{Con}(B_1, \vee)|^2 = 2^{2^{h+1}-2},$$

*where* $B_1$ *is the left maximal subtree of the tree.*

Using both results, we proved a formula for the number of weak congruences of semilattices defined by a binary tree. These are contained in the following:

**Lemma 6.0.5.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$ and $1' \notin B$, then*

$$|\mathrm{Cw}(B, \vee) +_{\mathrm{ord}} \{1'\}| = 3 \cdot |\mathrm{Cw}(B, \vee)| - 1.$$

**Theorem 6.0.6.** *If $(B, \vee)$ is a semilattice defined by a binary tree $B$, then*

$$|\mathrm{Cw}(B, \vee)| = 4(|\mathrm{Cw}(B_1, \vee)| \cdot |\mathrm{Cw}(B_2, \vee)|) - (|\mathrm{Cw}(B_1, \vee)| + |\mathrm{Cw}(B_2, \vee)|),$$

*where $B_1$, $B_2$ are the left and right maximal subtrees of the tree, respectively.*

**Corollary 6.0.6.1.** *If $(B, \vee)$ is a semilattice defined by a prickly-snake $B$ of height $h$, then*

$$|\mathrm{Cw}(B, \vee)| = 7 \cdot |\mathrm{Cw}(B_1, \vee)| - 2 = \frac{5 \cdot 7^h + 1}{3},$$

*where $B_1$ is the left maximal subtree of the tree.*

We solved two related nontrivial recurrences by applying the method of Aho and Sloane, as stated in the following theorems:

**Theorem 6.0.7.** *If $(B, \vee)$ is a semilattice defined by a perfect binary tree $B$ of height $h$, then*

$$|\mathrm{Sub}(B, \vee)| = |\mathrm{Sub}(B_1, \vee)|^2 + 2\,|\mathrm{Sub}(B_1, \vee)| - 1,$$

*where $B_1$ is the left maximal subtree of the tree.*

*Moreover,*

$$|\mathrm{Sub}(B, \vee)| = \left\lceil C^{2^{h+1}} \right\rceil - 1, \quad C = 1.6784589651254\ldots$$

*where $\lceil x \rceil$ denotes the least integer greater than or equal to $x$.*

**Theorem 6.0.8.** *If $(B, \vee)$ is a semilattice defined by a perfect binary tree $B$ of height $h$, then*

$$|\mathrm{Cw}(B, \vee)| = 4 \cdot |\mathrm{Cw}(B_1, \vee)|^2 - 2 \cdot |\mathrm{Cw}(B_1, \vee)|\,,$$

*where $B_1$ is the left maximal subtree of the tree.*

*Moreover,*

$$|\mathrm{Cw}(B, \vee)| = \left\lceil \tfrac{1}{4}\, C^{2^{h+1}} \right\rceil, \quad C = 2.61803398874989\ldots$$

*where $\lceil x \rceil$ denotes the least integer greater than or equal to $x$.*

**Chapter 5.** Concerning $(1+1+2)$-generated lattices of quasiorders, we proved that the lattice $\mathrm{Quo}(n)$ of all quasiorders (also known as preorders) of an $n$-element set is $(1+1+2)$-generated for $n = 3$, $n = 6$ (when $\mathrm{Quo}(6)$ consists of 209 527 elements), in the following way:

Let $A = \{a, b, c, d, f, g\}$. We define the following quasiorders of $A$:

$$\alpha := e(d,f) \vee e(f,g), \qquad \beta := \alpha \vee e(b,c) \vee q(b,a)$$
$$\gamma := e(a,b) \vee e(a,d) \vee e(c,f), \quad \delta := e(b,c) \vee e(c,g) \vee e(a,f). \tag{6.1}$$

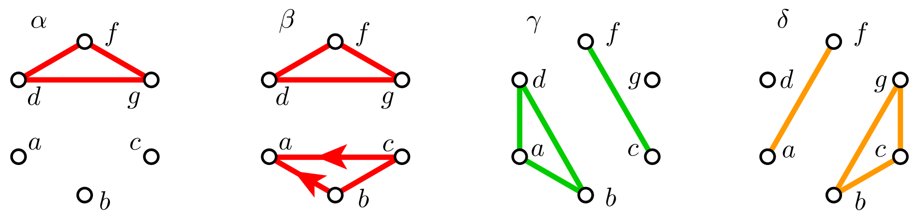


Figure 6.1: $\alpha$, $\beta$, $\gamma$, and $\delta$

**Theorem 6.0.9.** *With the quasiorders defined in* (6.1), $\{\alpha, \beta, \gamma, \delta\}$ *is a* $(1+1+2)$*-generating set of the quasiorder lattice* $\mathrm{Quo}(6) = \mathrm{Quo}(\{a, b, c, d, f, g\})$. *Hence,* $\mathrm{Quo}(6)$ *is* $(1+1+2)$*-generated.*

**Corollary 6.0.9.1.** $\mathrm{Quo}(3)$ *is* $(1+1+2)$*-generated.*

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# Appendices

# Appendix A

## .1 $N_5B_4$

**Input:**

\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*
\beginjob
\name
N_5B_4
\size
7
\elements
0docabi
\P edges
\P 0d da ac ci 0o oa ob bi
\constraints
a+b=i b+c=i b+d=i o+d=a , a*b=o c*b=o o*d=0 d*b=0
\endofjob
\enddata

**Output:**

Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 15:19:5) reports:
[ Supported by the Hungarian Research Grant KH 126581,

(C) Gabor Czedli, 2018 ]

|A|=7, A(without commas)={0docabi}. Constraints:
edges
0d da ac ci 0o oa ob bi
a+b=i b+c=i b+d=i o+d=a a*b=o c*b=o o*d=0 d*b=0
Result for A=N_5B_4: |Sub(A)| = 69, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 17.2500000000000000 .

## .2 $N'_6$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*
\beginjob
\name
$N'$_6
\size
6
\elements
0abcd1
\P edges
\P 0a ac c1 0d db b1
\constraints
a+b=1 b+c=1 a+d=1 c+d=1 , a*b=0 c*b=0 a*d=0 d*c=0
\endofjob
\enddata

**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 15:30:39) reports:

[ Supported by the Hungarian Research Grant KH 126581,


|A|=6, A(without commas)={0abcd1}. Constraints:
edges
0a ac c1 0d db b1
a+b=1 b+c=1 a+d=1 c+d=1 a*b=0 c*b=0 a*d=0 d*c=0
Result for A=$N'$_6: |Sub(A)| = 37, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 18.5000000000000000.

## .3 $H_1$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*
\beginjob
\name
H_1
\size
7
\elements
ocaivbd
\P edges
\P oc ca ai iv ob bi dv
\constraints
c+b=i a+b=i i+d=v , a*b=o c*b=o
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021

SUBSIZE version June 30, 2019 (started at 15:41:45) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=7, A(without commas)={ocaivbd}. Constraints:
edges
oc ca ai iv ob bi dv
c+b=i a+b=i i+d=v a*b=o c*b=o
Result for A=H_1: |Sub(A)| = 79, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 19.7500000000000000 .

## .4 $N_7$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*
\beginjob
\name
N_7
\size
7
\elements
oaedcib
\P edges
\P oa ae ed dc ci ob bi
\constraints
a+b=i e+b=i d+b=i c+b=i , a*b=o c*b=o e*b=o d*b=o
\endofjob
\enddata

**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 15:50:37) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=7, A(without commas)={oaedcib}. Constraints:
edges
oa ae ed dc ci ob bi
a+b=i e+b=i d+b=i c+b=i a*b=o c*b=o e*b=o d*b=o
Result for A=N_7: |Sub(A)| = 83, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 20.7500000000000000 .

## .5 $G_7$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
G_7
\size
7
\elements
oacvibd
\P edges
\P oa ac cv vi od db bi dv
\constraints
a+b=i c+b=i a+d=v v+b=i , a*b=o c*b=o

\endofjob
\enddata
Output:
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 17:48:46) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=7, A(without commas)={oacvibd}. Constraints:
edges
oa ac cv vi od db bi dv
a+b=i c+b=i a+d=v v+b=i a*b=o c*b=o
Result for A=G_7: |Sub(A)| = 78, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 19.5000000000000000 .

## .6 $G_7^-$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
$G_7^-$
\size
7
\elements
oacvibd
\P edges
\P oa ac cv vi od db bi dv

\constraints
a+b=i c+b=i a+d=v v+b=i c+d=v , a*b=o c*b=o a*d=0 c*d=0
\endofjob
\enddata
Output:
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 17:48:46) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=7, A(without commas)={oacvibd}. Constraints:
edges
oa ac cv vi ob db bi dv
a+b=i c+b=i a+d=v v+b=i c+d=v a*b=o c*b=o a*d=0 c*d=0
Result for A=$G_7^-$: |Sub(A)| = 67, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 16.75000000000000000 .

## .7 $Q_8$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
Q_8
\size
8
\elements
oadceifb

\P edges
\P oa ad dc ce ei of fb fe bi
\constraints
a+b=i d+b=i c+b=i e+b=i , c*b=o d*b=o a*b=o a*f=o c*f=o d*f=o e*b=f
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 18:39:9) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=8, A(without commas)={oadceifb}. Constraints:
edges
oa ad dc ce ei of fb fe bi
a+b=i d+b=i c+b=i e+b=i c*b=o d*b=o a*b=o a*f=o c*f=o d*f=o e*b=f
Result for A=Q_8: |Sub(A)| = 131, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 16.3750000000000000 .

# Appendix B

## .8 $H_4$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
H_4
\size
4
\elements
abc1
\P edges
\P ab b1 c1
\constraints
a+c=1 b+c=1 ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 19:7:59) reports:
[ Supported by the Hungarian Research Grant KH 126581,

(C) Gabor Czedli, 2018 ]

|A|=4, A(without commas)={abc1}. Constraints:
edges
ab b1 c1
a+c=1 b+c=1
Result for A=H_4: |Sub(A)| = 13, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 26.0000000000000000 .

## .9 $H_5$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
H_5
\size
5
\elements
abc1d
\P edges
\P ab bc c1 d1
\constraints
a+d=1 b+d=1 c+d=1 ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021

SUBSIZE version June 30, 2019 (started at 19:31:17) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=5, A(without commas)={abc1d}. Constraints:
edges
ab bc c1 d1
a+d=1 b+d=1 c+d=1
Result for A=H_5: |Sub(A)| = 25, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 25.0000000000000000 .

## .10 $K_3$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
K_3
\size
4
\elements
abc1
\P edges
\P a1 b1 c1
\constraints
a+b=1 b+c=1 c+a=1 ,
\endofjob
\enddata

**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 19:35:43) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=4, A(without commas)={abc1}. Constraints:
edges
a1 b1 c1
a+b=1 b+c=1 c+a=1
Result for A=K_3: |Sub(A)| = 12, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 24.0000000000000000 .

## .11 $B_4$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
B_4 \size
4
\elements
0ab1
\P edges
\P 0a a1 0b b1
\constraints
a+b=1 ,

\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 19:40:26) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=4, A(without commas)={0ab1}. Constraints:
edges
0a a1 0b b1
a+b=1
Result for A=B_4: |Sub(A)| = 14, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 28.0000000000000000 .

## .12 *K*

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
K
\size
5
\elements
ad1cb
\P edges
\P da a1 dc bc c1

\constraints
a+c=1 d+b=c a+b=1,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 19:46:16) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=5, A(without commas)={ad1cb}. Constraints:
edges
da a1 dc bc c1
a+c=1 d+b=c a+b=1,
Result for A=K: |Sub(A)| = 23, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 23.0000000000000000 .

## .13 *N*

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
N
\size
6
\elements
abcde1

\P edges
\P ad ac bc be e1 d1 c1
\constraints
a+b=c c+d=1 d+e=1 c+e=1 b+d=1 a+e=1 ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 21:0:39) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=6, A(without commas)={abcde1}. Constraints:
edges
ad ac bc be e1 d1 c1
a+b=c c+d=1 d+e=1 c+e=1 b+d=1 a+e=1
Result for A=N: |Sub(A)| = 39, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 19.5

## .14 $K_0$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
K_0
\size
7

\elements
abcxyz1
\P edges
\P ax az bx by cy cz x1 z1 y1
\constraints
a+b=x a+c=z b+c=y x+z=1 x+y=1 z+y=1 a+y=1 c+x=1 b+z=1 ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 21:11:18) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=7, A(without commas)=abcxyz1. Constraints:
edges
ax az bx by cy cz x1 z1 y1
a+b=x a+c=z b+c=y x+z=1 x+y=1 z+y=1 a+y=1 c+x=1 b+z=1
Result for A=K_0: |Sub(A)| = 61, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 15.2500000000000000 .

## .15 $U_1$

\textbfInput:
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
U_1

\size
6
\elements
abxcdy
\P edges
\P ax bx cy dy
\constraints
a+b=x c+d=y ,
\endofjob
\endd
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 21:19:19) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=6, A(without commas)={abxcdy}. Constraints:
edges
ax bx cy dy
a+b=x c+d=y
Result for A=U_1: |Sub(A)| = 49, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 24.5000000000000000 .

## .16 $U_2$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob

\name
U_2
\size
5
\elements
abxcd
\P edges
\P ax bx cx dx
\constraints
a+b=x c+d=x ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 21:25:19) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=5, A(without commas)={abxcd}. Constraints:
edges
ax bx cx dx
a+b=x c+d=x
Result for A=U_2: |Sub(A)| = 25, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 25.0000000000000000 .

## .17 $U_3$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
U_3
\size
5
\elements
abxcd
\P edges
\P ax bx ca da
\constraints
a+b=x c+d=a ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 21:29:26) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=5, A(without commas)=abxcd. Constraints:
edges
ax bx ca da
a+b=x c+d=a
Result for A=U_3: |Sub(A)| = 24, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 24.0000000000000000 .

## .18 $K_1$

Observe that

$$|\{S \in \mathrm{Sub}(K_1, \vee) : a \notin S\}| = 23, \qquad \text{(by Lemma 3.3.2(v))},$$

$$|\{S \in \mathrm{Sub}(K_1, \vee) : a \in S, \{b, c, x, y\} \cap S = \emptyset\}| = 2, \text{ and}$$

$$|\{S \in \mathrm{Sub}(K_1, \vee) : a \in S, \{b, c, x, y\} \cap S \neq \emptyset\}| = 12,$$

whereby $|\,\mathrm{Sub}(K_1, \vee)| = 23 + 2 + 12 = 37 = 18.5 \cdot 2^{6-5}$, which means that $\sigma_5(K_1, \vee) = 18.5$.

**Input:**

\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
K_1
\size
6
\elements
abcxy1
\P edges
\P ax bx by cy x1 y1
\constraints
a+b=x b+c=y a+c=1 x+y=1 x+c=1 a+y=1,
\endofjob
\enddata

**Output:**

Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 20:22:59) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=6, A(without commas)={abcxy1}. Constraints:
edges
ax bx by cy x1 y1
a+b=x b+c=y a+c=1 x+y=1 x+c=1 a+y=1,
Result for A=K_1: |Sub(A)| = 37, whence

sigma(A) = |Sub(A)|*2^(5-|A|) = 18.5000000000000000 .

## .19 $K_2$

**Input:**
\PVersion of the input file: Nov 29, 2021
\verbose=false
\subtrahend-in-exponent=5
\operationsymbols=+*

\beginjob
\name
K_2
\size
5
\elements
abxc1
\P edges
\P ax bx x1 c1
\constraints
a+b=x a+c=1 b+c=1 x+c=1 ,
\endofjob
\enddata
**Output:**
Version of the input file: Nov 29, 2021
SUBSIZE version June 30, 2019 (started at 21:29:26) reports:
[ Supported by the Hungarian Research Grant KH 126581,
(C) Gabor Czedli, 2018 ]

|A|=5, A(without commas)=abxc1. Constraints:
edges
ax bx x1 c1

a+b=x a+c=1 b+c=1 x+c=1
Result for A=K_2: |Sub(A)| = 22, whence
sigma(A) = |Sub(A)|*2^(5-|A|) = 22.0000000000000000 .

# Appendix C

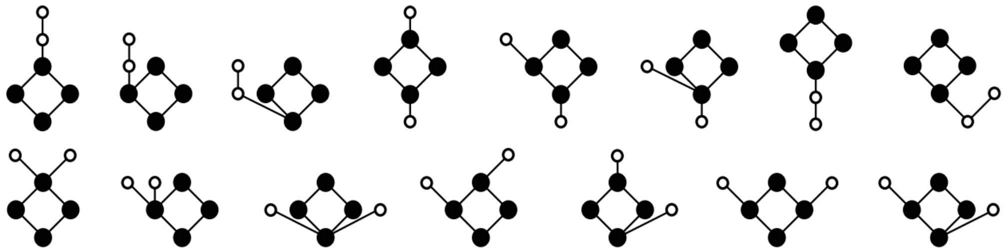

Figure 2: The full list of 6-element meet-semilattices with exactly $28 = 28 \cdot 2^{6-6}$ many congruences

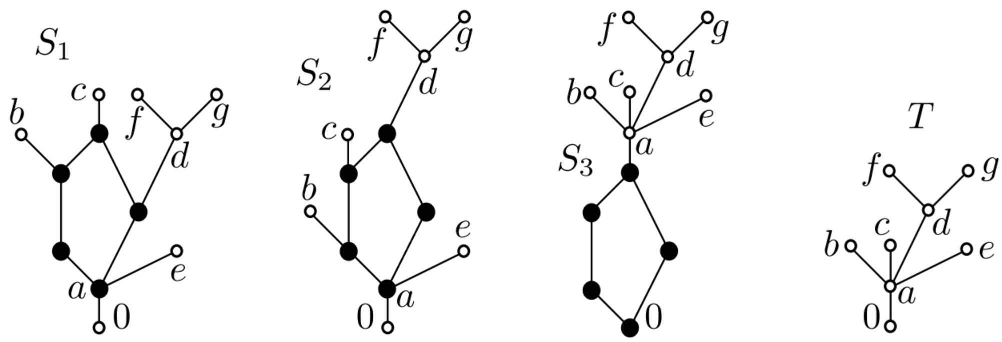


Figure 3: Three twelve-element meet-semilattices with the same skeleton $T$ and the same number, $26 \cdot 2^{12-6} = 1664$ , of congruences

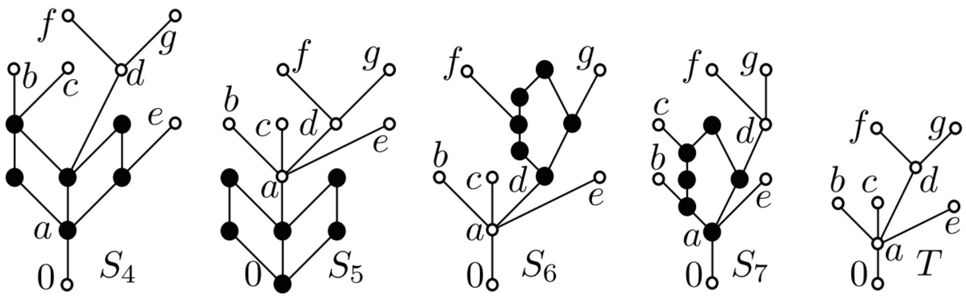


Figure 4: Four thirteen-element meet-semilattices with the same skeleton $T$ and the same number, $25 \cdot 2^{13-6} = 3200$, of congruences

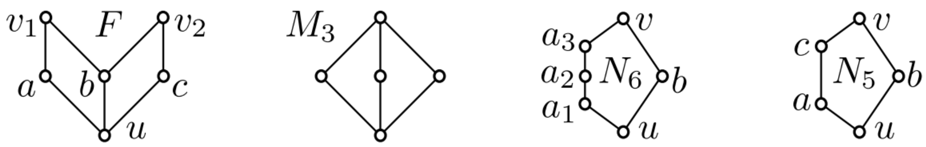


Figure 5: $F, M_3, N_6$, and the pentagon, $N_5$

# Appendix D

## .20 $TH4.2.3$

```
A = 2 ;
S = N[Log[A]] ;
For[n = 0, n < 10, n++, Print[n];
S = S + Log [1 - 1/(A^2)]/(2^(n + 1)); A = A^2 - 1; Print[A];
Print[S]];
Exp[S]
1.6784589651254 ...
```

## .21 $TH4.4.3$

```
B = 1 ;
SB = N[Log[B]] + Log[4];
For[n = 0, n < 10, n++, Print[n];
SB = SB + Log [1 - 1/(2 B)]/(2^(n + 1)); B = 4 B^2 - 2 B; Print[B];
Print[SB]];
Exp[SB]
2.61803398874989 ...
```