University of Szeged
Bolyai Institute

Ahmed Mohsin Mahdi

Conical Curves in Constant Curvature Planes

outline of the Ph.D. Dissertation

Supervisor: dr. Árpád Kurusa

Doctoral School of Mathematics and Computer Science
Faculty of Science and Informatics

Szeged, 2020

Preface

In a projective-metric space $(\mathcal{M}, d)$ we define

(D_1) a *conical curve* as the set
$$\mathcal{C}^{\varepsilon}_{F,\mathcal{H}} := \{X \in \mathbb{R}^n : \varepsilon d(X, \mathcal{H}) = d(F, X)\},$$

where $\mathcal{H}$ is a hyperplane, the *leading hyperplane* or *directrix*, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varepsilon > 0$ is a number, the *numeric eccentricity*. A conical curve is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

For given fixed points F_1, F_2, the *focuses*, and number $a \neq d(F_1, F_2)/2$, the *radius*, we define

(D_2) the *ellipsoid* (*ellipse* in dimension 2) as the set
$$\mathcal{E}^a_{d;F_1,F_2} := \{E : 2a = d(F_1, E) + d(E, F_2)\}, \text{ and}$$
(D_3) the *hyperboloid* (*hyperbola* in dimension 2) as the set
$$\mathcal{H}^a_{d;F_1,F_2} := \{X : 2a = |d(F_1, X) - d(X, F_2)|\},$$

according to $a > d(F_1, F_2)/2$ or $a < d(F_1, F_2)/2$, respectively. Value $2f := d(F_1, F_2)$ is the *eccentricity*, and if the eccentricity vanishes, then the ellipsoid (ellipse) is called *sphere (circle)*. Further, an ellipsoid (ellipse) or hyperboloid (hyperbola) is called *conical* if it is a conical curve.

According to [9], A. Moór raised the request for determining those Finsler manifolds in which the class of elliptic conical curves coincides with the class of ellipses, or the class of hyperbolic conical curves coincides with the class of hyperbolas. Tamássy and Bélteky found in [10, Theorem 2], that the only Finsler space where the class of elliptic conical curves coincides with the class of ellipses is the Euclidean space.

A similar problem was solved by Kurusa in [5, Theorem 6.1], where he proved that the only Minkowski geometry in which either a conical ellipsoid or a conical hyperboloid exists is the Euclidean one. At the end of his paper [5] Kurusa formulated the problem of determining projective-metric spaces in which

(a) some or all ellipses are conical, or
(b) some or all hyperbolas are conical.

Kurusa's main result [5, Theorem 6.1] was based on that, by [5, Theorem 4.2 and 4.3], the only Minkowski geometry in which a symmetric conical curve exists is the Euclidean one. Additionally, it is also proved in [5, Theorem 5.1] that the only Minkowski plane in which a quadratic conical curve exists is the Euclidean one. So Kurusa also raised the request to determine the projective-metric spaces in which

(c) some or all elliptic conical curves are symmetric, or
(d) some or all hyperbolic conical curves are symmetric, or
(e) some or all elliptic conical curves are quadratic, or
(f) some or all hyperbolic conical curves are quadratic.

All these problems are open for curved projective-metric spaces, so it was natural to set the goal of the research to answer Kurusa's request for curved constant curvature spaces. We reached this goal and published the results in [6–8].

Our results are as follow:

Theorem A. *If a conical curve $\mathcal{C}$ in a curved constant curvature plane $\mathcal{P}$ is symmetric, then $\mathcal{P}$ is the sphere and the focus of $\mathcal{C}$ is the pole of the directrix of $\mathcal{C}$.*

Theorem B. *If a conical curve $\mathcal{C}$ in a curved constant curvature plane $\mathcal{P}$ is quadratic, then $\mathcal{P}$ is the sphere and either the focus of $\mathcal{C}$ is the pole of the directrix of $\mathcal{C}$ or $\mathcal{C}$ is parabolic.*

Theorem C. *If $\mathcal{C}$ is a conical ellipse or a conical hyperbola in a curved constant curvature plane $\mathcal{P}$, then $\mathcal{P}$ is the sphere and the focus of $\mathcal{C}$ is the pole of the directrix of $\mathcal{C}$.*

The presentation is based on my papers [6,7] and [8], but for the sake of a broader view the dissertation gives precise definitions from the ground up, provides basic theorems for curves and surfaces, and describes thoroughly from both the projective and the differential geometric point of views the spaces used in the text to show the dual nature of the constant curvature spaces.

Acknowledgment. First and foremost, my dissertation could have been never written without the help of my great supervisor Dr. Árpád Kurusa. I would also like to extend my thanks to the Bolyai Institute of the Faculty of Sciences and Informatics and to the Stipendium Hungaricum Foundation for providing me the opportunity to join Ph.D. studies, and giving access to all research facilities. I would like to thank Dr. Béla Nagy for everything he has done for me. I cannot imagine ever coming this far without my extended family.

1. Preliminaries and preparations

In this chapter we collect definitions, theorems and some proofs which will be used as auxiliary facts for the next chapters.

Points of $\mathbb{R}^n$ are denoted as $A, B, \ldots$, vectors are $\overrightarrow{AB}$ or $\mathbf{a}, \mathbf{b}, \ldots$, but we use these latter notations also for points if the origin is fixed. The open segment with endpoints A and B is denoted by $\overline{AB} = (A, B)$, $\overline{A}B$ is the open ray starting from A passing through B, and AB denotes the line through A and B.

We denote the *affine ratio* of the collinear points A, B and C by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *cross ratio* of the collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [2, page 243].

Notations $\mathbf{u}_\varphi = (\cos\varphi, \sin\varphi)$ and $\mathbf{u}_\varphi^\perp := (\cos(\varphi+\pi/2), \sin(\varphi+\pi/2))$ are frequently used.

1.1 Basic differential geometry

In this section we provide the basic definitions and theorems of differential geometry that are necessary to understand our results in the next chapter.

1.1.1 Curves

Definition 1.1. A *parameterized differentiable curve* is a differentiable map $\mathbf{p}\colon \mathcal{I} \to \mathbb{R}^3$ of an open interval $\mathcal{I} = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

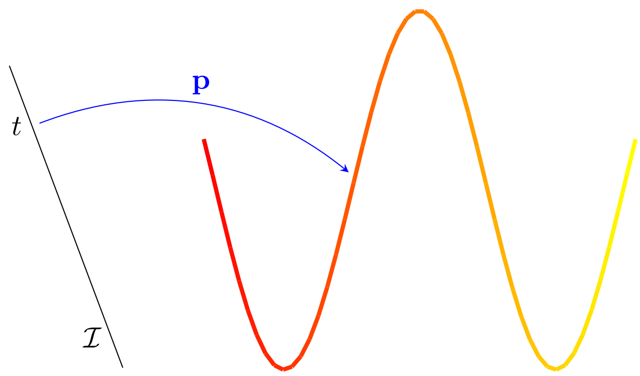


Figure 1.1: Curve and its parameterization

The differentiability means that $\mathbf{p}$ maps each $t \in \mathcal{I}$ into point $\mathbf{p}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ in such a manner that the functions $x(t), y(t), y(t)$ are differentiable. The variable t is called the *parameter* of the curve.

The vector $\mathbf{p}'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$ is the *tangent vector* of the curve $\mathbf{p}$ at t, and the image set $\mathbf{p} \subset \mathbb{R}^3$ is called the *trace* of $\mathbf{p}$.

Definition 1.2. A parameterized differentiable curve $\mathbf{p}\colon \mathcal{I} \to \mathbb{R}^3$ is said to be *regular* if $\mathbf{p}'(t) \neq 0$ for all $t \in \mathcal{I}$. Then the vector $\mathbf{p}'(t)$ is called the *tangent vector* of $\mathbf{p}$ at $\mathbf{p}(t)$ or at t.

Definition 1.3. The *arc length* of a regular parameterized curve $\mathbf{p}$ from the point $\mathbf{p}(t_0)$ to $\mathbf{p}(t_1)$ is

$$s(t) = \int_{t_0}^{t_1} \left|\mathbf{p}'(t)\right| dt, \text{ where } \left|\mathbf{p}'(t)\right| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

A regular parameterized curve $\mathbf{p}$ is said to be *arc length parameterized* if $|\mathbf{p}'(s)| = 1$.

1.1.2 Surfaces

Definition 1.4. A subset $\mathcal{S} \subset \mathbb{R}^3$ is a regular surface if for each point $S \in \mathcal{S}$ there exists a neighborhood $\mathcal{V} \subseteq \mathbb{R}^3$ and a map $\mathbf{r}\colon U \to V \cap \mathcal{S}$ of an open set $\mathcal{U} \subseteq \mathbb{R}^2$ onto $\mathcal{V} \cap \mathcal{S} \subseteq \mathbb{R}^3$ such that

(1) the coordinate functions x, y, z of $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ $((u,v) \in U)$, have continuous partial derivatives of all orders;

(2) the inverse $\overline{\mathbf{r}}\colon \mathcal{V} \cap \mathcal{S} \to U$ is well defined and is continuous;

(3) (*The regularity condition.*) the derivative $\dot{\mathbf{r}}$ is one to one.

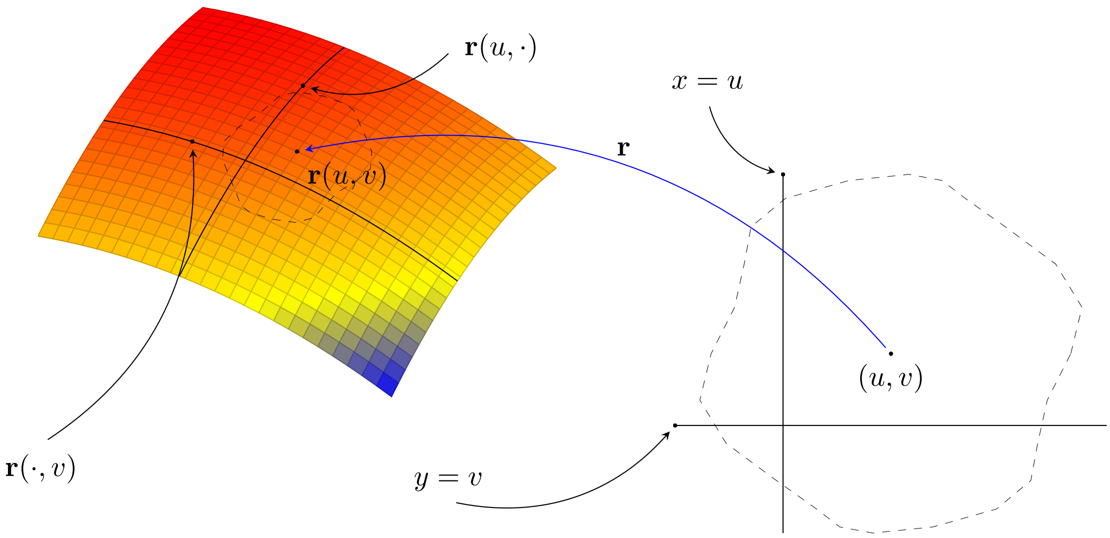


Figure 1.2: Surface and its parameterization

Proposition 1.5. *If $f\colon \mathcal{U} \to \mathbb{R}$ is a differentiable function on an open set $\mathcal{U} \subseteq \mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x,y))$ for $(x,y) \in U$, is a regular surface.*

Proposition 1.6. *If $f\colon \mathcal{U} \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(\mathcal{U})$ is a regular value of f, then $f^{-1}(\mathcal{U})$ is a regular surface in $\mathbb{R}^3$*

Proposition 1.7. *Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a regular surface and $P \in \mathcal{S}$. Then there exists a neighborhood $\mathcal{V} \subseteq \mathcal{S}$ of P such that $\mathcal{V}$ is the graph of a differentiable function which has one of the following three forms $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$.*

Definition 1.8. The set $T\mathcal{S}$ of the tangent vectors of the curves on the surface $\mathcal{S}$ is called the *tangent bundle*. The set $T_P\mathcal{S}$ of tangent vectors $\mathbf{p}'(t) \in T\mathcal{S}$, where $\mathbf{p}(t) = P$, is called the *tangent plane of $\mathcal{S}$ at $P \in \mathcal{S}$*.

Every tangent plane $T_P\mathcal{S}$ is a 2-dimensional vector space. For every tangent vector $\mathbf{v} \in T_P\mathcal{S}$ there are great many curves $\mathbf{p}$ on the surface $\mathcal{S}$ that satisfies $\mathbf{p}(0) = P$ and $\mathbf{v} = \mathbf{p}'(0)$.

Definition 1.9. A differentiable map $f : \mathcal{S} \to \mathbb{R}$ is called differentiable *scalar field* on $\mathcal{S}$. The *differential $\partial_{\mathbf{v}} f$ of the scalar field f evaluated against the tangent vector* $\mathbf{v} \in T_P\mathcal{S}$ is the derivative $(f \circ \mathbf{p})'(0)$, where $\mathbf{p}$ is a curve on the surface $\mathcal{S}$ satisfying $\mathbf{p}(0) = P$ and $\mathbf{v} = \mathbf{p}'(0)$.

We notice that the differential of a scalar field evaluated against a tangent vector does not depend on the choice of the curve chosen in the definition.

Definition 1.10. A differentiable map $X : \mathcal{S} \to T\mathcal{S}$ is called differentiable *vector field* on $\mathcal{S}$, if $X(P) \in T_P\mathcal{S}$ for every $P \in \mathcal{S}$. The vector space of the differentiable vector fields on $\mathcal{S}$ is denoted by $T_*\mathcal{S}$.

Definition 1.11. The *Lie-bracket* $[X, Y]$ of two vector fields $X, Y \in T_*\mathcal{S}$ is a linear mapping of scalar fields defined by $f \mapsto [X, Y]f = \partial_X(\partial_Y f) - \partial_Y(\partial_X f)$.

1.1.3 Riemann manifolds

We consider only Riemannian manifolds given on surfaces of the 3-dimensional space.

Definition 1.12. The pair $(\mathcal{S}, g)$ is called a *Riemannian manifold* of dimension 2, if $\mathcal{S}$ is a regular surface and $g : \mathcal{S} \ni P \mapsto g_P$ provides a Euclidean product $g_P : T_P\mathcal{S} \times T_P\mathcal{S} \to \mathbb{R}$ at every point $P \in \mathcal{S}$ on the corresponding tangent plane $T_P\mathcal{S}$ such that if X and Y are differentiable vector fields on $\mathcal{S}$, then the function $\mathcal{S} \ni P \mapsto g_P(X(P), Y(P))$ is a smooth function of P. The function g is called a *Riemannian metric* (or *Riemannian metric tensor*).

Every surface with its tangent planes equipped with the Euclidean product $g_P(\mathbf{u}, \mathbf{v}) := \langle \mathbf{u}, \mathbf{v} \rangle$ given by the restriction of the Euclidean product $\langle \cdot, \cdot \rangle$ of the space $\mathbb{R}^3$ is such a Riemannian manifold of dimension 2. The Riemannian metric given in this way called *inherited Riemannian metric*.

Definition 1.13. The *length of a differentiable curve* $\mathbf{p} : (a, b) \to \mathcal{S} \subset \mathbb{R}^3$ in a Riemannian manifold $(\mathcal{S}, g)$ is $\ell(\mathbf{p}) := \int_a^b \sqrt{g_{\mathbf{p}(t)}(\dot{\mathbf{p}}(t), \dot{\mathbf{p}}(t))}dt$.

Definition 1.14. The *Riemannian distance function* $d_g\colon \mathcal{S}\times\mathcal{S}\ni(P,Q)\mapsto d_g(P,Q)\in\mathbb{R}$ on a Riemannian manifold $(\mathcal{S},g)$ is $\inf_{\mathbf{p}\in\mathcal{C}_{P,Q}}\ell(\mathbf{p})$, where $\mathcal{C}_{P,Q}$ is the set of all the differentiable curve $\mathbf{p}$ in the Riemannian manifold $(\mathcal{S},g)$ connecting P and Q.

A Riemannian manifold with the Riemannian distance function is a metric space.

Definition 1.15. A bilinear mapping $\nabla\colon T_*\mathcal{S}\times T_*\mathcal{S}\ni(X,Y)\to\nabla_XY\in T_*\mathcal{S}$ is called *affine connection* if for all differentiable functions $f\colon\mathcal{S}\to\mathbb{R}$ and for all vector fields $X,Y\in T_*\mathcal{S}$ if $\nabla_{fX}Y=f\nabla_XY$ (*functional linearity in the first variable*) and $\nabla_X(fY)=\partial_XfY+f\nabla_XY$ (*Leibniz rule in the second variable*) hold.

An affine connection is called *torsion-free* if $[X,Y]:=\nabla_XY-\nabla_YX$ for every $X,Y\in T_*\mathcal{S}$.

Definition 1.16. An affine connection is a *Levi-Civita* connection if it is torsion-free, and *compatible with the Riemannian metric* g, i.e. $\nabla_X\big(g(Y,Z)\big)=g(\nabla_XY,Z)+g(Y,\nabla_XZ)$.

There is always a unique Levi-Civita connection that is easy to prove through the *Koszul formula* $2g(\nabla_XY,Z)=\partial_X\big(g(Y,Z)\big)+\partial_Y\big(g(Z,X)\big)-\partial_Z\big(g(X,Y)\big)$.

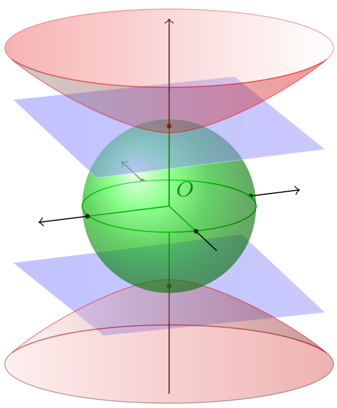
Definition 1.17. The *Riemannian curvature* is the trilinear mapping R of vector fields to vector fields defined by $\mathsf{R}(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$.

The Riemannian curvature is a tensor, because $\mathsf{R}(fX,Y)Z=\mathsf{R}(X,fY)Z=\mathsf{R}(X,Y)(fZ)=f\mathsf{R}(X,Y)Z$ for every scalar field f and vector fields X,Y,Z, hence $\mathsf{R}(X,Y)Z(P)$ depends in fact only on the vectors $X(P),Y(P),Z(P)\in T_P\mathcal{S}$. Further, the expression $\kappa(\mathbf{u},\mathbf{v})=\frac{g_P(\mathsf{R}(\mathbf{u},\mathbf{v})\mathbf{v},\mathbf{u})}{g_P(\mathbf{u},\mathbf{u})g_P(\mathbf{v},\mathbf{v})-g_P^2(\mathbf{u},\mathbf{v})}$ does not depend on the independent vectors $\mathbf{u},\mathbf{v}\in T_P\mathcal{S}$.

Definition 1.18. The value $\kappa_P=\kappa(\mathbf{u},\mathbf{v})$ is called the *(sectional) curvature* of $(\mathcal{S},g)$ at the point $P\in\mathcal{S}$.

1.1.4 Two-dimensional manifolds of constant curvature

It is easy to see that the plane and the sphere with their respective inherited Riemannian metric are surfaces of constant curvature, but there is a third example worth noting.



Let the surface $\mathcal{K}_\kappa^2\subset\mathbb{R}^3$ of points $\boldsymbol{p}=(p_1,p_2,p_3)$ satisfying

$$\kappa(p_1^2+p_2^2)+p_3^2=1, \tag{1.1}$$

where $\kappa\in\{1,0,-1\}$. Equip the surface $\mathcal{K}_\kappa^2$ with the Riemannian metric g_κ such that

$$g_{\kappa;\boldsymbol{p}}\colon T_{\boldsymbol{p}}\mathcal{K}_\kappa^2\times T_{\boldsymbol{p}}\mathcal{K}_\kappa^2\ni(\boldsymbol{x},\boldsymbol{y})\mapsto x_1y_1+x_2y_2+\kappa\,x_3y_3 \tag{1.2}$$

for every point $\boldsymbol{p}\in\mathcal{K}_\kappa^2$. Then the pairs $(\mathcal{K}_\kappa^2,g_\kappa)$ have constant curvature κ.

If $\kappa \geq 0$, then the Riemannian metric in (1.2) is the inherited metric, and we have the sphere $\mathcal{K}_1^2$ and two planes $\mathcal{K}_0^2$. The Riemannian manifold $(\mathcal{K}_{-1}^2, g_{-1})$ is a different case: both sheets of the hyperboloid $\mathcal{K}_{-1}^2$ equipped with the Riemannian metric g_{-1} model the hyperbolic plane, but g_{-1} is not the inherited metric.

Then one gets the so-called *projective model* $\bar{\mathcal{K}}_\kappa^2$ of the constant curvature space $\mathbb{K}_\kappa^2$ of curvature $\kappa \in \{1, 0, -1\}$ [3], and also the *canonical correspondence* identifying the points of $\mathcal{K}_\kappa^2 \subset \mathbb{R}^3$ that are symmetric in the origin.

1.2 Projective-metric spaces

Real projective plane $\mathbb{P}^2$ arises in several different ways.

Considering the real affine plane $\mathbb{R}^2$, we call the equivalence sets of the straight lines by parallelism *ideal* points, and add these points to the set of the usual (real) points of $\mathbb{R}^2$ so that each ideal point becomes a common point of every straight line belonging to that particular ideal point. This extended geometry is the real projective plane.

Another method to construct *real projective plane* $\mathbb{P}^2$ is to think of the straight lines passing through the origin $(0,0,0)$ in $\mathbb{R}^3$ as *projective points*, and think of the planes passing through the origin $(0,0,0)$ in $\mathbb{R}^3$ as *projective straight lines.*

A more algebraic way is to consider the equivalence classes of the non-vanishing directional vectors by the equivalency relation $\sim$ that relates two non-vanishing directional vectors equivalent $\sim$ if one of them is a scalar multiple of the other one. This leads to the *homogeneous coordinates* which is a coordinatization of the real projective plane $\mathbb{P}^2$.

Finally, an intuitive way of considering the real projective plane is to identify diametrical points of the sphere, i.e. these pairs constitute the points of the real projective plane.

A *metric space* is an ordered pair $(\mathcal{M}, d)$ such that $\mathcal{M}$ is a set, the set of *points*, and $d\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is a *metric*, i.e. for any three points $x, y, z \in \mathcal{M}$ it satisfies $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$, the *triangle inequality.*

If the metric space $(\mathcal{M}, d)$ is that $\mathcal{M}$ is a projective plane $\mathbb{P}^2$, or an affine plane $\mathbb{R}^2 \subset \mathbb{P}^2$, or a (not necessarily bounded) proper open convex subset of an affine plane $\mathbb{R}^2 \subset \mathbb{P}^2$, and the metric d is complete, continuous with respect to the usual topology of $\mathbb{P}^n$, additive on the segments, and the geodesic lines of d are exactly the non-empty intersection of $\mathcal{M}$ with the straight lines, then the pair $(\mathcal{M}, d)$ is called *projective-metric space*[1] [2, p. 115].

Such projective-metric planes are called *elliptic*, *parabolic*, or *hyperbolic*, respectively, according to whether $\mathcal{M}$ is $\mathbb{P}^2$, $\mathbb{R}^2$, or a proper convex subset of $\mathbb{R}^2$. The projective-metric planes of the latter two types are called *straight* [1, p. 1].

[1]Determining the projective-metric spaces and studying the individual ones is known as Hilbert's fourth problem.

The geodesics of a projective-metric space of elliptic type have equal lengths, so we can set their length to π by simply multiplying the metric with an appropriate positive constant. Therefore we assume from now on that projective-metric spaces of elliptic type have geodesics of length π.

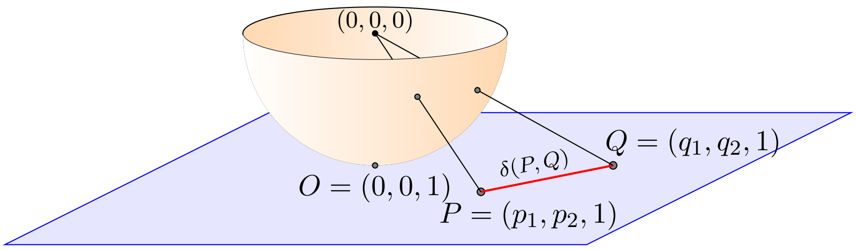
Every isometry of $(\mathcal{M}, d)$ is a restriction of a projectivity of the projective space $\mathbb{P}^n$ [1].

A set $\mathcal{S} \subset \mathcal{M}$ is called *symmetric about a point* C, if $X \in \mathcal{S}$ if and only if $Y \in \mathcal{S}$, where C is in the metric midpoints of the segment $\overline{XY}$, i.e. $2d(X, C) = 2d(C, Y) = d(X, Y)$.

1.2.1 Elliptic projective-metric planes

Every *elliptic plane* can be constructed in the following way. Take a Euclidean metric on $\mathbb{R}^2$ and let $\langle \cdot, \cdot \rangle$ be its Euclidean product. Define the function $\hat{\delta} \colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathbb{R}$ by $\hat{\delta}(\boldsymbol{x}, \boldsymbol{y}) = \arccos\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. This is a metric on $\mathcal{S}^2$, and it satisfies the strict triangle inequality, i.e. $\hat{\delta}(A, B) + \hat{\delta}(B, C) = \hat{\delta}(A, C)$ if the points A, B and C are in a hemisphere. Equality happens if and only if B is on the great circle determined by A and C. If the diametrical points are identified and the metric is inherited, then we get an elliptic plane.

To show that the constructed geometry is an elliptic projective-metric space, we use the *gnomonic projection* [11] $\Gamma_O \colon \mathcal{S}^2 \to T_O\mathcal{S}^2$ of the sphere, where $O \in \mathcal{S}^2$ and $T_O\mathcal{S}^2$ is the tangent hyperplane of $\mathcal{S}^2$ at point O with the projective extension.



Γ_O projects the spherical metric $\hat{\delta}$ to the metric

$$\delta \colon \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to [0, \pi) \qquad (P, Q) \mapsto \hat{\delta}(P, Q) = \arccos\left(\frac{\langle P, Q \rangle}{|P|\,|Q|}\right). \tag{1.3}$$

1.2.2 Parabolic projective-metric planes

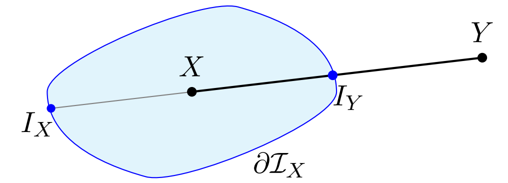
The most important parabolic projective-metric planes are the *Minkowski planes*[2]. They are constructed in the following way.

Let $\mathcal{I}$ be an open, strictly convex, bounded domain in $\mathbb{R}^2$, (centrally) symmetric to the origin. Then the function $d \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf\{\lambda > 0 : (\boldsymbol{y} - \boldsymbol{x})/\lambda \in \mathcal{I}\}$$

[2] They are also known as normed planes.

is a metric on $\mathbb{R}^2$ [2, IV.24], and is called *Minkowski metric on* $\mathbb{R}^2$. It satisfies the strict triangle inequality, i.e. $d(A,B)+d(B,C)=d(A,C)$ is valid if and only if $B\in\overline{AC}$.



The pair $(\mathbb{R}^2,d)$ is the Minkowski plane, and $\mathcal{I}$ is called the *indicatrix* of it.

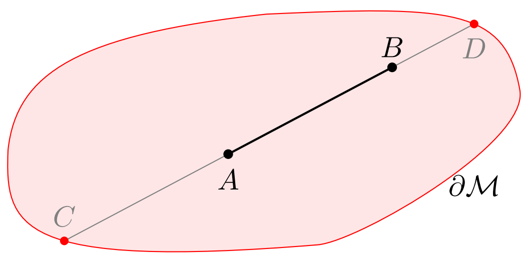
1.2.3 Hyperbolic projective-metric planes

The most important hyperbolic projective-metric planes are the *Hilbert planes*. They are constructed in the following way.

If $\mathcal{M}$ is an open, strictly convex, proper subset of $\mathbb{R}^2$, then the function $d\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}$ defined by

$$d(A,B)=\begin{cases}0, & \text{if } A=B,\\ \frac{1}{2}\big|\ln(A,B;C,D)\big|, & \text{if } A\neq B, \text{ where } \overline{CD}=\mathcal{M}\cap AB,\end{cases} \tag{1.4}$$

is a *metric* on $\mathcal{M}$ [2, page 297] which satisfies the strict triangle inequality, i.e. $d(A,B)+d(B,C)=d(A,C)$ if and only if $B\in\overline{AC}$.



The pair $(\mathcal{M},d)$ is the Hilbert plane, $\mathcal{M}$ is its *domain*, and the function d is called the *Hilbert metric* on $\mathcal{M}$.

1.2.4 Constant curvature planes

There are special elliptic, parabolic and hyperbolic projective-metric planes that make Riemannian manifolds.

It is clear that a Minkowski plane is Euclidean if and only if its indicatrix is an ellipse.

It is known [2, (29.3)] that a Hilbert plane is a model of the hyperbolic plane of Bolyai, Lobachevskii and Gauss, if and only if its domain is the interior of an ellipse. Such Hilbert planes are called *Cayley–Klein models* of the hyperbolic plane.

It happens that these have constant curvature, and can also be constructed by the gnomonic projection of the 2-dimensional manifolds $(\mathcal{K}^2_\kappa, g_\kappa)$ [4], where $\kappa \in \{0, \pm 1\}$.

The isometry groups of all these three constant curvature planes are generated by reflections in straight lines. Moreover specifically, we have

Theorem 1.19 ([2])**.** *Every isometry of each of these three constant curvature planes can be given as a product of at most three reflections in straight lines.*

1.3 Classes of curves in the Euclidean plane

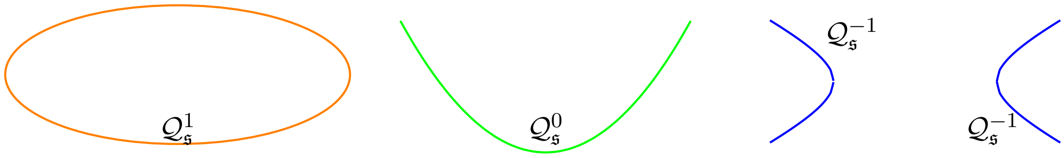
In the Euclidean plane there are four differently defined classes of curves which however coincide in most of the cases. Here we briefly describe only three of these classes to shade light over the problem considered in the main part of the dissertation.

1.3.1 Quadratic curves

The curves presented in this subsection are independent from the metric. A curve in the plane is called *quadratical*, if it is part of a *quadric*

$$\mathcal{Q}^\sigma_\mathfrak{s} := \left\{ (x,y) : \begin{cases} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{cases} \right\}, \tag{D_q}$$

where $\mathfrak{s}$ is an affine coordinate system. A quadric is called *ellipse* (*affine circle*), *parabola* and *hyperbola*, if $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$, respectively.



The *ellipse* is the only bounded conical curve. The *parabola* is a connected conical curve that has exactly one complete set of parallel lines such that its every member line intersects the parabola in exactly one point. The *hyperbola* is two connected curves (called *branches*) and it is such that exactly two complete sets of parallel lines are such that their every member line, except the one called *asymptote*, intersects the hyperbola in exactly one point.

1.3.2 Curves defined by sum or difference of distances

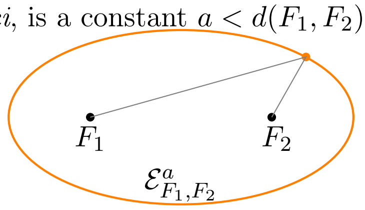
The curves presented in this subsection are bound to the metric. For now, we stay in Euclidean geometry.

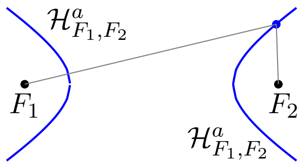
A closed segment $\overline{F_1F_2}$ of the different points F_1, F_2 is the locus of points P in the plane such that the sum of the distances from P to the two fixed points F_1 and F_2 is the constant $d(F_1, F_2)$.

Definition 1.20. An *ellipse* $\mathcal{E}^a_{F_1,F_2}$ is the locus of points P in the plane such that the sum of the distances from P to the two fixed points F_1 and F_2, the *foci*, is a constant $a > d(F_1, F_2)$. An ellipse $\mathcal{E}^r_{C,C}$ is called *circle of radius r with center C.*

The closed rays $F_1F_2 \setminus \overline{F_1F_2}$ of the different points F_1, F_2 are the locus of points P in the plane such that the absolute value of the difference of the distances from P to the two fixed points F_1 and F_2 is the constant $d(F_1, F_2)$.

Definition 1.21. A *hyperbola* $\mathcal{H}^a_{F_1,F_2}$ is the locus of points P in the plane such that the absolute value of the difference of the distances from P to the two fixed points F_1 and F_2, the *foci*, is a constant $a < d(F_1, F_2)$.





Every ellipse is an affine ellipse $\mathcal{Q}^1_{\mathfrak{s}}$, and every affine ellipse $\mathcal{Q}^1_{\mathfrak{s}}$ is the circle $\mathcal{E}^1_{(0,0),(0,0)}$ in the Euclidean metric d defined by the inner product $\langle (x, y), (z, t)\rangle = xz + yt$.

Every hyperbola $\mathcal{H}^a_{F_1,F_2}$ is an affine hyperbola $\mathcal{Q}^{-1}_{\mathfrak{s}}$, and every affine hyperbola $\mathcal{Q}^{-1}_{\mathfrak{s}}$ is the hyperbola $\mathcal{H}^2_{(2,0),(-2,0)}$ in the Euclidean metric d.

1.3.3 Curves defined by ratio of distances

In this section we consider curves which are bound to the metric, for now, it is the Euclidean metric.

Definition 1.22. Given a positive number ε, the *numerical eccentricity*, a straight line ℓ, the *directrix*, and a point $F \notin \ell$, the *foci*, in the plane, the *conical curve* $\mathcal{C}^\varepsilon_{F,\ell}$ is the locus of points P in the plane such that $d(F, P) = \varepsilon d(P, \ell)$.

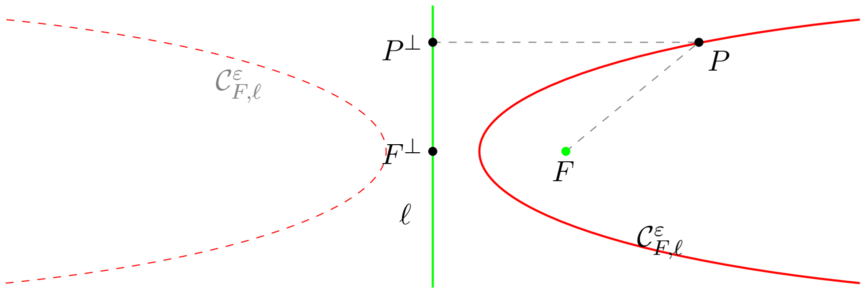


Figure 1.3: Conical curve in Euclidean plane

A conical curve is called *elliptic*, *parabolic*, and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$, and $\varepsilon > 1$, respectively.

Every elliptic conical curve is a bounded closed curve contained in one side of the directrix. The elliptic conical curves are affine ellipses $\mathcal{Q}_{\mathfrak{s}}^{1}$, metric ellipses $\mathcal{E}_{F_1,F_2}^{a}$. Further, except the circles, every metric ellipse $\mathcal{E}_{F_1,F_2}^{a}$ is an elliptic conical curve.

Every parabolic conical curve is an unbounded curve contained in one side of the directrix. The parabolic conical curves are affine parabolas $\mathcal{Q}_{\mathfrak{s}}^{0}$, and conic sections. Further, every affine parabola $\mathcal{Q}_{\mathfrak{s}}^{0}$ is a parabolic conical curve.

Every hyperbolic conical curve has two separate unbounded connected curves, the *branches*, one-one on both sides of the directrix. The hyperbolic conical curves are affine hyperbolas $\mathcal{Q}_{\mathfrak{s}}^{-1}$, metric hyperbolas $\mathcal{H}_{F_1,F_2}^{a}$, and conic sections. Further, every affine hyperbola $\mathcal{Q}_{\mathfrak{s}}^{-1}$ is a hyperbolic conical curve.

2. Conical curves with given properties

In this chapter we consider conical curves in constant curvature planes. It turns out that some of their usual properties, like symmetry and quadraticity, remains valid only in very special configurations. We prove that

(1) no conical curve in the hyperbolic plane can be quadratic;

(2) no conical curve in the hyperbolic plane can be symmetric;

(3) if the focus of a conic curve on the sphere is not the pole of the directrix, then the conic can only be quadratic if it is a parabolic, and it can not be symmetric.

2.1 Quadratic conical curves in the hyperbolic plane

As for any pair (F, ℓ) of a point F in $\mathcal{D}$ and an h-line ℓ there exists an isometry $\imath$ such that $\imath(\ell)$ goes through the center O of $\mathcal{D}$, and O is the foot of $\imath(F)$ on $\imath(\ell)$, we can restrict without loss of generality the investigation of conical curves to those conical curves $\mathcal{C}_{F,\ell}^{\varepsilon}$ in $(\mathcal{D}, \delta)$ for which the directrix ℓ is the y-axis, and the focus F is $(f, 0)$, where $f \in (0, 1)$.

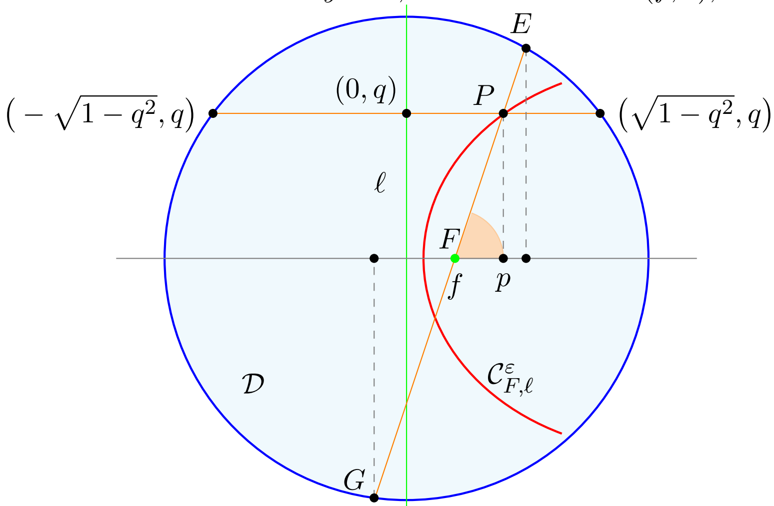


Figure 2.1: Directrix ℓ is through the center of the Cayley–Klein model, the focus F is at $(f, 0)$, where $f \in (0, 1)$.

To calculate the points $P = (p, q)$ on $\mathcal{C}^{\varepsilon}_{F,\ell}$, we have to calculate $\delta(P, \ell)$ and $\delta(F, P)$, where $P = (p, q) \in \mathcal{C}^{\varepsilon}_{F,\ell}$.

It is easy to get that

$$\delta(P, \ell) = \frac{1}{2}\left|\log\left\{\frac{p + \sqrt{1-q^2}}{p - \sqrt{1-q^2}} : \frac{0 + \sqrt{1-q^2}}{0 - \sqrt{1-q^2}}\right\}\right|. \tag{2.1}$$

To obtain $\delta(F, P)$, we firstly determine the points $\{E, G\} = \{(x_\pm, y_\pm)\}$, where line FP intersects the unit circle, the border of $\mathcal{D}$. So we get

$$\delta(F, P) = \frac{1}{2}\left|\log\left\{\frac{(fp - 1 - \sqrt{(p-f)^2 + (1-f^2)q^2})^2}{(1-f^2)(1-p^2-q^2)}\right\}\right|. \tag{2.2}$$

According to (D_1) equations (2.1) and (2.2) give

$$(1 - q^2 - p^2)\Big(1 + \frac{2p}{\sqrt{1-q^2} - p}\Big)^{\epsilon} = \frac{\big(fp - 1 - \sqrt{q^2(1-f^2) + (p-f)^2}\big)^2}{1 - f^2}, \tag{2.3}$$

where $\epsilon = \pm\varepsilon$. Figure 2.4 shows how these conical curves look like based on (2.3).

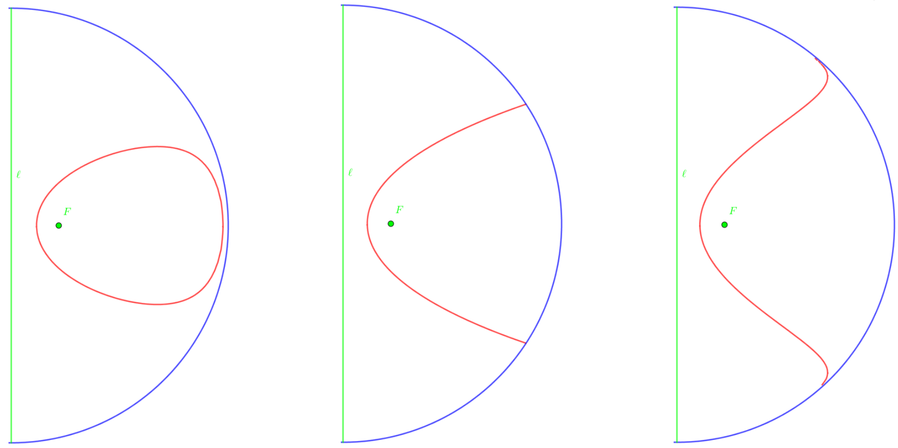


Figure 2.2: An elliptic ($\varepsilon = 0.9$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in the Cayley–Klein model of the hyperbolic geometry.

For the sake of later contradiction, we assume from now on that

$$\boxed{\text{conical curve } \mathcal{C}^{\varepsilon}_{F,\ell} \text{ is quadratic } (D_q),}$$

hence it satisfies an equation of the form $\bar{a}x^2 + \bar{b}xy + \bar{c}y^2 + \bar{d}x + \bar{e}y + \bar{f} = 0$, where the coefficients are real and $\bar{a} \geq 0$.

As the conical curves $\mathcal{C}^{\varepsilon}_{F,\ell}$ are symmetric in the x-axis, the quadratic equation should be invariant under changing y to $-y$, so $\bar{b} = \bar{e} = 0$ follows. So the equation is of the form $\bar{a}x^2 + \bar{c}y^2 + \bar{d}x + \bar{g} = 0$, hence $\bar{c} \neq 0$, because otherwise the curve will degenerate into straight lines. So the quadratic equation simplifies to

$$ax^2 + y^2 + bx + c = 0, \quad a \geq 0. \tag{2.4}$$

As conical curve $\mathcal{C}^{\varepsilon}_{F,\ell}$ is quadratical, we have $q^2 = -ap^2 - bp - c, \quad a \geq 0$. Putting this into (2.3) gives an identity for p. Differentiating this with respect to p simplifies to the identity of two polynomials:

$$\begin{aligned}
&\varepsilon^4(2(1+c)+pb)^4+\\
&\quad+\Big(\big((fb+2a-2)p+2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad\quad+\big(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)^2\times\\
&\qquad\times(1+ap^2+bp+c)^2+\\
&\quad+2\varepsilon^2(2(1+c)+pb)^2\times\\
&\qquad\times\Big(\big((fb+2a-2)p+2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad\quad+\big(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)\times\\
&\qquad\times(1+ap^2+bp+c)\\
&=4\big((fb+2a-2)p+2f(c+1)+b\big)^2(1+ap^2+bp+c)^2\times\\
&\qquad\times(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b)^2\times\\
&\qquad\times\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big).
\end{aligned}$$

Two polynomials can only be equal on a segment if their corresponding coefficients are pairwise equal.

Carefully comparing the corresponding coeffictients leads to the outcome that the conical curve $\mathcal{C}^{\varepsilon}_{F,\ell}$ is of the form $x^2 + y^2 = 1$, a clear contradiction that proves the following:

Theorem 2.1 ([6])**.** *No conical curve of the hyperbolic plane can be quadratic in Cayley–Klein models.*

2.2 Symmetric conical curves in the hyperbolic plane

Consider a conical curve $\mathcal{C}^{\varepsilon}_{F,\ell}$. Let $F^{\perp}$ be the foot of F on the h-line ℓ, and let C be a point on the h-line $FF^{\perp}$ different from $F^{\perp}$.

It is well known that there are h-isometries that maps C into the center O of $\mathcal{D}$. Thus we can restrict without loss of generality the investigation of conical curves $\mathcal{C}^{\varepsilon}_{F,\ell}$ in $(\mathcal{D}, \delta)$ to those ones for which $\overline{(m, -\sqrt{1-m^2})(m, \sqrt{1-m^2})}$ is the directrix ℓ for some $m \in (-1, 0)$, the center is $O = (0, 0)$, and the focus F is $(f, 0)$, where $f \in (-1, 1) \setminus \{m\}$.

To calculate the points $P = (p, q)$ on $\mathcal{C}^{\varepsilon}_{F,\ell}$, we have to calculate $\delta(P, \ell)$ and $\delta(F, P)$, where $P = (p, q) \in \mathcal{C}^{\varepsilon}_{F,\ell}$. Observe that the line through P orthogonal to ℓ is the one that connects P to L, the intersection of the tangents of $\mathcal{D}$ at the limit points of ℓ. We clearly have $L = (-1/m, 0)$.

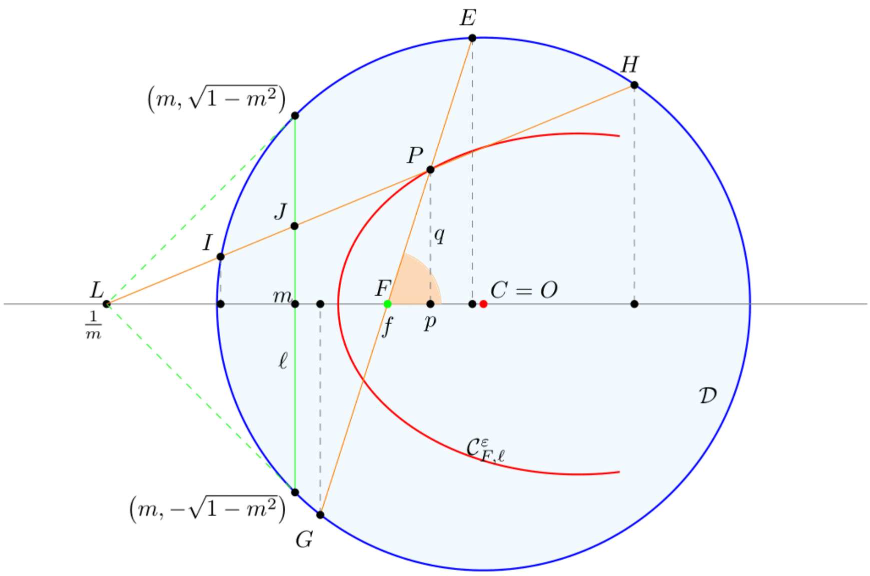


Figure 2.3: Directrix ℓ is $\overline{(m, -\sqrt{1-m^2})(m, \sqrt{1-m^2})}$, $\mathcal{C}^{\varepsilon}_{F,\ell}$ is symmetric in O, the center of the Cayley–Klein model, and the focus F is at $(f, 0)$, where $f \in (-1, 1) \setminus \{m\}$.

To obtain $\delta(P, \ell)$, we firstly determine the points where line LP intersects the unit circle. Further, we need the coordinates of point J, where PL intersects ℓ. Thus

$$\delta(P, \ell) = \frac{1}{2}\left|\log\left\{\frac{\left(\sqrt{\left(p-\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1-\frac{1}{m}p\right)\right)^2}{\left(1-p^2-q^2\right)\left(\frac{1}{m^2}-1\right)}\right\}\right|. \tag{2.5}$$

To obtain $\delta(F, P)$, we firstly determine the points $E = (x_1, y_1)$ and $G = (x_2, y_2)$, where line FP intersects the unit circle, the border of $\mathcal{D}$. Thus, we get

$$\delta(F, P) = \frac{1}{2}\left|\log\left\{\frac{(fp-1-\sqrt{(p-f)^2+(1-f^2)q^2})^2}{(1-f^2)(1-p^2-q^2)}\right\}\right|. \tag{2.6}$$

According to (D_1) equations (2.5) and (2.6) give

$$\left(\frac{\left(\sqrt{\left(p-\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1-\frac{1}{m}p\right)\right)^2}{\left(1-p^2-q^2\right)\left(\frac{1}{m^2}-1\right)}\right)^{\epsilon} = \frac{\left(fp-1-\sqrt{q^2(1-f^2)+(p-f)^2}\right)^2}{(1-f^2)(1-q^2-p^2)}, \tag{2.7}$$

where $\epsilon \in \{\varepsilon, -\varepsilon\}$. Figure 2.4 shows how these conical curves look like based on (2.7) with $\epsilon = \varepsilon$.

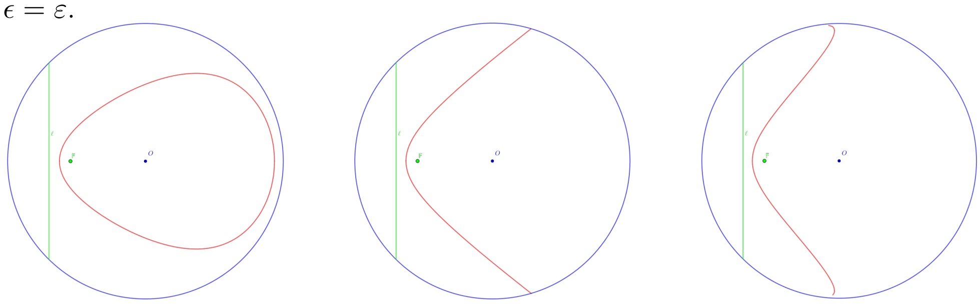


Figure 2.4: An elliptic ($\varepsilon = 0.9$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in the Cayley–Klein model of the hyperbolic geometry.

For the sake of later contradiction, we assume from now on that

$$\boxed{\text{conical curve } \mathcal{C}^{\varepsilon}_{F,\ell} \text{ is symmetric in a point } C.}$$

Such a point of symmetry C clearly is on the h-line $FF^{\perp}$, where $F^{\perp}$ is the foot of F on the h-line ℓ. So we can restrict without loss of generality the investigation of symmetric conical curves $\mathcal{C}^{\varepsilon}_{F,\ell}$ in $(\mathcal{D},\delta)$ to those ones for which directrix ℓ is $\overline{(m,-\sqrt{1-m^2})(m,\sqrt{1-m^2})}$ for some $m\in(-1,1)$, the center is $O=(0,0)$, and the focus F is $(f,0)$, where $f\in(m,1)$. Thus we can use the formulas given in the previous section.

As the conical curve is symmetric in the x-axis, and it is symmetric in point O, it is symmetric about the y-axis too, so, substituting $-p$ into p, dividing the two equations and taking the square root than restricting to $q=0$, after some rearrangement we get

$$\frac{1\pm f}{1\mp f}\left(\frac{1-m}{1+m}\right)^{\epsilon}=\left(\frac{1+p}{1-p}\right)^{-\epsilon\pm 1},\quad \text{where } \pm 1=\frac{p-f}{|p-f|}. \tag{2.8}$$

If p is a solution of these equations, then the symmetry in O implies, that $-p$ is also a solution of (2.8). Thus we have either $\varepsilon\in(0,1)$ or $\varepsilon\in(1,\infty)$. If $\varepsilon\in(0,1)$, then $p\to 0$ causes contradiction. If $\varepsilon>1$, then $p^2+q^2\to 1$ causes contradiction.

Theorem 2.2 ([7]). *No conical curve of the hyperbolic plane can be symmetric.*

2.3 Quadratic conical curves on the sphere

Let $\hat{O}$ be the polar of the great circle $\hat{\ell}$ on the $\mathcal{S}^2$. Let $\hat{F}$ be in the half sphere $\mathcal{S}^2_{\hat{O}}$ of $\hat{\ell}$ that contains $\hat{O}$. Let $\hat{P}$ be on the half circle $\mathcal{G}^2_{\hat{O}}$ of the great circle of $\hat{O}$ and $\hat{F}$ that is contained by $\mathcal{S}^2_{\hat{O}}$.

It is not hard to prove that there is exactly one $\varpi\in(-\pi/2,\varphi)$ for which $\hat{P}\in\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$.

Let $\mathcal{C}^{\varepsilon}_{F,\ell}:=\Gamma_{\hat{O}}(\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}})$, $O:=\Gamma_{\hat{O}}(\hat{O})$, $F:=\Gamma_{\hat{O}}(\hat{F})$, and $\ell:=\Gamma_{\hat{O}}(\hat{\ell})$. Choose the coordinate system so that $O=(0,0,1)$ and $F=(f,0,1)$, where $f>0$. Figure 2.5 shows what we have on the plane $\mathcal{P}:=T_{\hat{O}}\mathcal{S}^2=\{(x,y,z):z=1\}$.

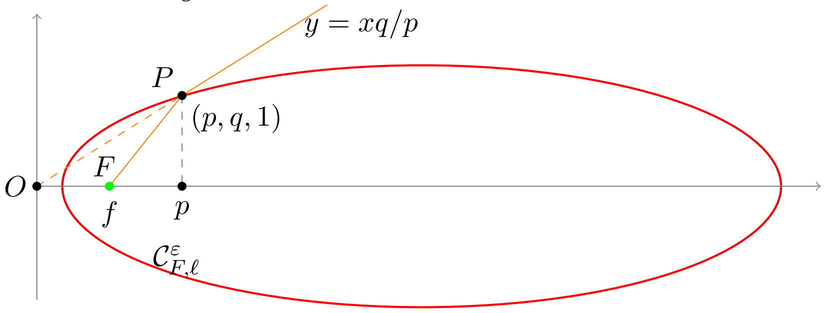


Figure 2.5: Projected conical curve $\mathcal{C}^{\varepsilon}_{F,\ell}$, if the directrix ℓ is in the infinity and the focus F is at $(f,0)$, where $f>0$.

To calculate the points $(p,q,1)=P=\Gamma_{\hat{O}}(\hat{P})$ of $\mathcal{C}^{\varepsilon}_{F,\ell}$ we have to calculate $\delta(P,\ell)$ and $\delta(F,P)$, where $P\in\mathcal{C}^{\varepsilon}_{F,\ell}$.

Thus,

$$\delta(P,\ell) = \frac{\pi}{2} - \delta(P,O) = \frac{\pi}{2} - \arccos\frac{1}{\sqrt{p^2+q^2+1}} \tag{2.9}$$

$$\delta(P,F) = \delta(P,(f,0,1)) = \arccos\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}. \tag{2.10}$$

According to (D_1) equations (2.9) and (2.10) give that

$$\varepsilon\Big(\frac{\pi}{2} - \arccos\frac{1}{\sqrt{p^2+q^2+1}}\Big) = \arccos\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}} \tag{2.11}$$

is the equation of $\mathcal{C}^{\varepsilon}_{F,\ell}$. Figure 2.6 shows how $\mathcal{C}^{\varepsilon}_{F,\ell}$ looks like for different values of ε.

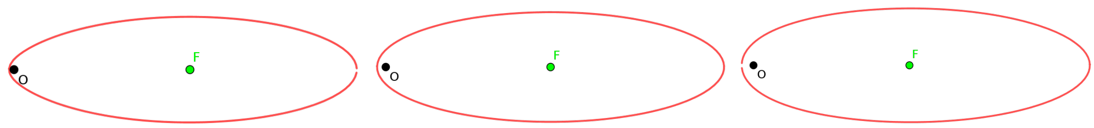


Figure 2.6: An elliptic ($\varepsilon = 0.90$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in the projected model of the sphere.

The parabolic conical curves (i.e. $\varepsilon = 1$) are quadratic because taking the cosine of (2.11) results in

$$\sqrt{1-\frac{1}{p^2+q^2+1}} = \Big|\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}\Big|,$$

the square of which is the clearly quadratic equation $(p^2+q^2)(f^2+1) = pf+1$.

To find all the quadratic conical curves,

from now on we assume that $\mathcal{C}^{\varepsilon}_{F,\ell}$ is quadratic,

hence satisfies an equation of the form $\bar{a}x^2+\bar{b}xy+\bar{c}y^2+\bar{d}x+\bar{e}y+\bar{f}=0$, where the coefficients are real and $\bar{a} \ge 0$. As every conical curve $\mathcal{C}^{\varepsilon}_{F,\ell}$ is symmetric in the x-axis, the quadratic equation should be invariant under changing y to $-y$, so $\bar{b} = \bar{e} = 0$ follows. So the quadratic equation simplifies to $q^2 = -ap^2 - bp - c$. Putting this into (2.11) then differentiating with respect to p gives

$$\begin{aligned}\varepsilon^2(2(1-a)p-b)^2((1-a(1+f^2))p^2-(2f+b(1+f^2))p+(f^2-c(1+f^2)))\\ =((fb+2(1-a))p-(b+2f(1-c)))^2((1-a)p^2-bp-c).\end{aligned} \tag{2.12}$$

This equation is valid on an interval of p, so the coefficients of the polynomials on the sides are equal, hence

(p^4) $\quad 4\varepsilon^2(1-a)^2(1-a(1+f^2)) = (1-a)(fb+2(1-a))^2$

(p^3) $\quad 4\varepsilon^2\big((1-a)^2(2f+b(1+f^2))+b(1-a)(1-a(1+f^2))\big)$
$\quad = b(fb+2(1-a))^2+2(1-a)(b+2f(1-c))(fb+2(1-a))$

(p^2) $\quad \varepsilon^2\big(b^2(1-a(1+f^2))+4b(1-a)(2f+b(1+f^2))+4(1-a)^2(f^2-c(1+f^2))\big)$
$\quad = -c(fb+2(1-a))^2+2b(b+2f(1-c))(fb+2(1-a))+$
$\quad + (1-a)(b+2f(1-c))^2$

$$(p^1) \qquad 4\varepsilon^2\big(b(1-a)(f^2-c(1+f^2))+b^2(2f+b(1+f^2))\big)$$
$$= b(b+2f(1-c))^2-2c(b+2f(1-c))(fb+2(1-a))$$
$$(p^0) \qquad \varepsilon^2b^2(f^2-c(1+f^2)) = -c(b+2f(1-c))^2,$$

where $\varepsilon, f > 0$ are fixed, and $a > 0$, $b^2 > 4ac$.

A long and very careful investigation of this system of equation reveals that the system of equations (p^0)–(p^4) does not have a solution, so the polynomials of the sides in (2.12) are different, hence the conical curves in this case are not quadratic.

Theorem 2.3 ([8]). *A conical curve on the sphere is quadratic if and only if either the focus is the pole of the directrix, or the focus is not the pole of the directrix, but the conical curve is parabolic, i.e.* $\varepsilon = 1$.

2.4 Symmetric conical curves on the sphere

Firstly we notice that the conical curve on the sphere is a hypersphere, hence symmetric if the focus is the pole of the directrix, so we assume for the sake of a later contradiction that

$$\boxed{\hat{F} \text{ is not the pole of } \hat{\ell}, \text{ and } \hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}} \text{ is symmetric in a point } \hat{C}.}$$

Such a point of symmetry $\hat{C}$ clearly is on the great circle of $\hat{F}\hat{F}^{\perp}$, where $\hat{F}^{\perp}$ is the unique foot of $\hat{F}$ on the great circle $\hat{\ell}$.

Take the gnomonic projection $\Gamma_{\hat{C}}$. Let $\mathcal{C}^{\varepsilon}_{F,\ell} := \Gamma_{\hat{C}}(\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}})$, $P := \Gamma_{\hat{C}}(\hat{P})$ and $P^{\perp} := \Gamma_{\hat{C}}(\hat{P}^{\perp})$ for any point P, and $\ell := \Gamma_{\hat{C}}(\hat{\ell})$. Choose the coordinate system so that $C = (0,0,1)$, $F = (f,0,1)$, and $\ell = \{(x,y,z) : x = m \wedge z = 1\}$. Figure 2.7 shows what we have on the plane $\mathcal{P} := T_{\hat{C}}\mathcal{S}^2 = \{(x,y,z) : z = 1\}$.

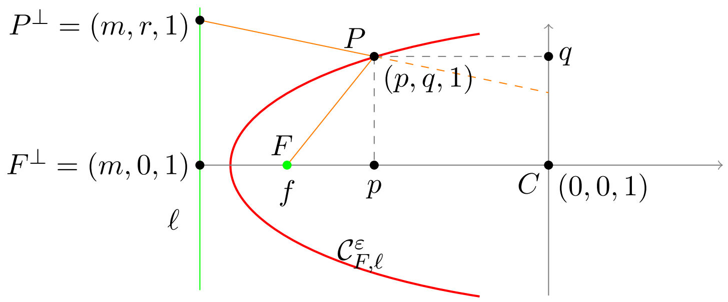


Figure 2.7: Projected conical curve $\mathcal{C}^{\varepsilon}_{F,\ell}$, if the directrix ℓ is parallel to the y-axis and the focus F is at $(f,0)$, where $f < 0$.

The advantage of taking the gnomonic projection $\Gamma_{\hat{C}}$ is that $\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$ is symmetric about $\hat{C}$ in the spherical meaning if and only if $\mathcal{C}^{\varepsilon}_{F,\ell}$ is symmetric about C in the Euclidean meaning.

By (1.3), we have

$$\delta(P,\ell)=\arccos\frac{\sqrt{(mp+1)^2+q^2(m^2+1)}}{\sqrt{m^2+1}\sqrt{p^2+q^2+1}}. \tag{2.13}$$

According to (D_1) equations (2.13) and (2.10) give

$$\varepsilon\arccos\frac{\sqrt{(mp+1)^2+q^2(m^2+1)}}{\sqrt{m^2+1}\sqrt{p^2+q^2+1}}=\arccos\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}. \tag{2.14}$$

Figure 2.8 shows how these conical curves look like by (2.11).

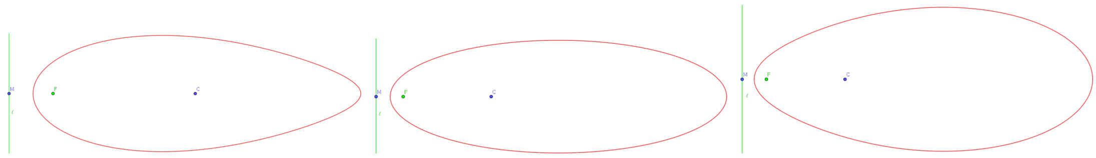


Figure 2.8: An elliptic ($\varepsilon = 0.90$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in projected model of the sphere.

We now that there exist exactly two solutions of (2.14) for $q = 0$, and by the symmetry these are $\pm p_0$. Substituting these values leads to a contradiction.

Theorem 2.4 ([8])**.** *A conical curve on the sphere is symmetric if and only if the focus is the pole of the directrix.*

2.5 Conical ellipses and conical hyperbolas

As every ellipse and every hyperbola in the hyperbolic plane is symmetric, every conical ellipse and every conical hyperbola is a symmetric conical curve, hence Theorem 2.2 implies the following.

Theorem 2.5. *There is no conical ellipse or conical hyperbola in the hyperbolic plane.*

As every ellipse and every hyperbola on the sphere is symmetric, every conical ellipse and every conical hyperbola is a symmetric conical curve, hence Theorem 2.4 implies the following.

Theorem 2.6. *Every conical ellipse and every conical hyperbola on the sphere is a circle.*

Bibliography

[1] H. Busemann, *The geometry of geodesics*, Academic Press Inc., New York, N. Y., 1955.

[2] H. Busemann, and P. J. Kelly, *Projective geometry and projective metrics*, Academic Press Inc., New York, N. Y., 1953.

[3] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry, Hyperbolic geometry, *Flavors of geometry*, Math. Sci. Res. Inst. Publ., vol. **31**, Cambridge Univ. Press, Cambridge, 1997, pp. 59–115.

[4] Á. Kurusa, Support theorems for totally geodesic Radon transforms on constant curvature spaces, *Proc. Amer. Math. Soc.* **122** (1994), no. 2, 429–435, doi: 10.2307/2161033.

[5] Á. Kurusa, Conics in Minkowski geometries, *Aequationes Math.* **92** (2018), no. 5, 949–961, doi: 10.1007/s00010-018-0592-1.

[6] A. M. Mahdi, Quadratic conics in hyperbolic geometry, *Int. J. Geom.* **8** (2019), no. 2, 60–69, available at https://ijgeometry.com/wp-content/uploads/2019/09/60-69.pdf.

[7] A. M. Mahdi, Symmetric conics in hyperbolic geometry, *Glob. J. Adv. Res. Class. Mod. Geom.* **8** (2019), no. 2, 49–56, available at http://geometry-math-journal.ro/pdf/Volume8-Issue2/1.pdf.

[8] A. M. Mahdi, Conics on the sphere, *Int. J. Geom.* **9** (2020), no. 2, 5–14, available at https://ijgeometry.com/wp-content/uploads/2020/.

[9] L. Tamássy, and K. Bélteky, On the coincidence of two kinds of ellipses in Minkowskian spaces and in Finsler planes, *Publ. Math. Debrecen* **31** (1984), no. 3-4, 157–161.

[10] L. Tamássy, and K. Bélteky, On the coincidence of two kinds of ellipses in Riemannian and in Finsler spaces, *Topics in differential geometry, Vol. I, II (Debrecen, 1984)*, Colloq. Math. Soc. János Bolyai, vol. **46**, North-Holland, Amsterdam, 1988, pp. 1193–1200.

[11] Wikipedia, Gnomonic projection, available at https://en.wikipedia.org/wiki/Gnomonic_projection.