Tic-Tac-Toe, Amoeba and other animals

Outline of Ph. D. thesis

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1 Introduction

In the thesis we investigate pairing strategies and their generalizations applied to **hypergraph games**. We can define several games on a given hypergraph $\mathcal{H} = (V, E)$. The first and second players take elements of V in turns. In the *normal* version the player who is first to take all elements of some edge (winning set) $A \in E$ wins the game. A normal game can be either a first player win or a draw but the second player cannot win if both players play in a perfect way. The concept of *strategy stealing* was introduced by John Nash for a special game, hex, and rigorously proved in the general case by Alfred Hales and Robert Jewett, see [8].

Since in the normal game the second player cannot win, we introduce the **Maker-Breaker** (M-B) games. In these, Maker wins by achieving the original goal, while Breaker wins by preventing Maker to doing so. A Maker-Breaker game can be either a Maker win or a Breaker win. Beside the M-B version, one can play *accelerated* or *biased* versions of a hypergraph game, where the players can have more than one elements in one step. Jozsef Beck introduced the **Picker-Chooser** (P-C) and **Chooser-Picker** (C-P) versions, where Picker selects a pair of elements and Chooser keeps one of these and gives the other one back to Picker. In the C-P version Chooser plays as Maker and Picker as Breaker, while the roles are swapped in the P-C game.

1.1 Definition *The vertices of the* ***k-in-a-row hypergraph*** $\mathcal{H}_k$ *are the squares of the infinite square grid, while the edges are the k-element sets of consecutive squares in a row horizontally, vertically or diagonally.*

A normal or M-B game on $\mathcal{H}_k$ is the k-in-a-row game. We know that the k-in-a-row is a first player win if $k \leq 4$, while the second player wins if $k \geq 8$. Allis solved the 5-in-a-row on the 19×19 board: the first player wins, however, the case of infinite board is still only a conjecture (in the normal version). For $k = 6, 7$, the questions are also open, but draw and Breaker wins are conjectured.

We call a function **strategy**, if it gives the next move of the player from the previous moves. The main topic of the thesis is the pairing strategy, but we mention some other possible strategies: cutting to sub-boards, the potential based Erdős-Selfridge method and also case-studies which last could give the winning strategies instantly, but studying all cases is already hopeless for small games.

2 Pairings

In a Breaker winning pairing strategy Breaker pairs the elements before the game and when Maker chooses an element Breaker chooses its pair in the next move. Picker (as Breaker) also can use a Breaker winning pairing strategy in a C-P game – giving the pairs to Chooser Picker can have one element of all pairs and win the game.

2.1 Definition *Given a hypergraph $\mathcal{H} = (V, E)$, where $V = V(\mathcal{H})$ and $E = E(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}) = \{S : S \subseteq V\}$ are the set of vertices and edges. A bijection $\rho : X \to Y$, where $X, Y \subset V(\mathcal{H})$ and $X \cap Y = \emptyset$ is a* ***pairing*** *on the hypergraph $\mathcal{H}$.*

2.2 Definition *A pair $(x, \rho(x))$* ***blocks*** *an edge $A \in E(\mathcal{H})$, if A contains both elements of the pair. If the pairs of ρ block all edges, we say* *that ρ is a* ***good pairing*** *of $\mathcal{H}$.*

We know for k-in-a-row games that Breaker wins by pairings in cases of $k \geq 9$, but not for smaller k values.

2.3 Theorem (Hales-Jewett) *[6] Breaker wins the 9-in-a-row M-B game by a pairing strategy, i.e., there exists a good pairing for the 9-in-a-row.*

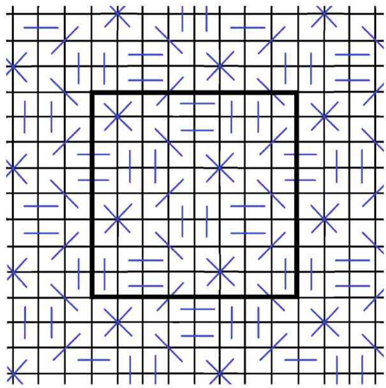

Figure 1: Hales-Jewett pairing for 9-in-a-row

For a hypergraph $\mathcal{H}$ let $d_2(\mathcal{H})$ (briefly d_2) be the greatest number of edges that can be blocked by two vertices of $\mathcal{H}$, i.e., d_2 is the maximal co-degree. We can call that value the **blocking power** of the pair, which can be at most $k-1$ in case of k-in-a-row.

2.4 Proposition *[2] If there is a good pairing ρ for the hypergraph $\mathcal{H} = (V, E)$, then $d_2|X|/2 \geq |\mathcal{G}|$ must hold for all $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

2.5 Definition *The hypergraph of the 3-direction version of k-in-a-row is h_k on the squaregrid, where the edges are the k-element sets of consecutive squares in a row along the vectors (1,0), (0,1) and (1,1).*

If $k \leq 4$, then Maker wins. If $k \geq 7$, then Breaker wins by pairings. The cases 5 and 6 are open problems here. According to Proposition 2.4 the sharp case here is $k = 7$ for which we can see a good pairings in Figure 2. The number of good pairings were also an open question.

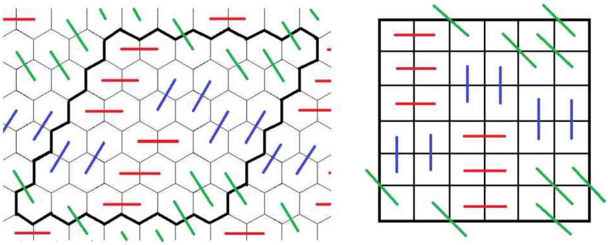

Figure 2: A good pairing for the 7-in-a-row

If we consider only the vertical and horizontal directions then Maker wins if $k \leq 4$, and Breaker wins by pairings if $k \geq 5$. In one direction the treshold is between 2 and 3.

2.6 Definition *We denote the hypergraph of the 2-direction (vertical and horizontal) version of k-in-a-row by P_k and the one direction (horizontal) version by E_k.*

3 Generalized pairings

We talk about two-colorings of a hypergraph, if we color the vertices by two colors. A two-coloring **blocks** an edge, if the edge contains both colors. A coloring is a **good coloring**, if it blocks all edges of the hypergraph.

Since two-colorings can be considered as pairing of two subsets and similarly pairings can be considered as step-by-step colorongs, we defined a bridge between the two notions. This section is based to the results of the paper [3].

3.1 Definition *Let us call a subset of the hypergraph $\mathcal{H} = (V, E)$* ***t-cake*** *if it consists of* exactly *t vertices of $\mathcal{H}$, with a previously given bipartition p, q of its elements, where $t \in \mathbb{N}, t \geq 2$ and $1 \leq p, q \in \mathbb{N}, p, q < t, p + q = t$. A t-cake is* ***balanced*** *if the two parts contain equal number of elements, i. e. $p = q$.*

3.2 Definition *A* ***t-placement*** *is a non-overlapping placement of cakes on the hypergraph, where the size of every cake is* ***at most*** *t. If a t-placement for an even t contains only balanced t-cakes, we are talking about* ***p-pairing****, where $p = t/2$.*

3.3 Definition *A t-cake* ***blocks an*** *$A \in E$, if both parts of the cake have a non-empty intersection with A. A t-placement $\mathcal{T}$ is a* ***good t-placement of*** *$\mathcal{H}$, if all edges of $\mathcal{H}$ are blocked by a cake of $\mathcal{T}$. An $A \in E$ of $\mathcal{H}$ is an* ***unblocked edge****, if there is no cake in $\mathcal{T}$ blocks A.*

We see a kind of monotonicity: if there is a good t-placement for a hypergraph $\mathcal{H}$, then there is a good $(t+1)$-placement for $\mathcal{H}$.

Let us count the number of hypergraph edges can be blocked by a given cake, that is the **blocking power of the cake**. For a given $t \in \mathbb{N}$ let d_t be the greatest blocking number among all t-cakes. A t-cake with blocking number d_t is called a **best** t-cake.

3.4 Proposition *If there is a good t-placement of $\mathcal{H} = (V, E)$ such that $d_2/2 \le d_3/3 \le \cdots \le d_t/t$, then $\frac{d_t}{t}|X| \ge |\mathcal{G}|$ for every $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

From now on we study the blocking power of the cakes for the k-in-a-row. Table 3 shows the maximal blocking power for $t \le 8$.

t	2	3	4	5	6	7	8
d_t	$k-1$	$2k-2$	$4k-4$	$5k-4$	$7k-6$	$9k-8$	$11k-9$
$\frac{d_t}{t}$	$\frac{1}{2}k-\frac{1}{2}$	$\frac{2}{3}k-\frac{2}{3}$	$k-1$	$k-\frac{4}{5}$	$\frac{7}{6}k-1$	$\frac{9}{7}k-\frac{8}{7}$	$\frac{11}{8}k-\frac{9}{8}$
$k \ge$	9	7	5	4.8	4.29	4	3.73

3.1 Results for k-in-a-row

3.5 Observation *There exists no good two-coloring for $\mathcal{H}_2$.*

3.6 Theorem *There is a unique good two-coloring for $\mathcal{H}_3$.*

3.7 Theorem *There are no good t-placement for t=7 and 8 for $\mathcal{H}_4$.*

3.8 Theorem *There is a good 8-placement for $\mathcal{H}_5$.*

3.9 Theorem *There is a good 4-placement for the 7-in-a-row.*

3.10 Theorem *There is a good 6-placement for $\mathcal{H}_6$.*

Table 3.1 summarizes the results. Columns and rows stand for the values of k and t, respectively. "Yes" designates the existence of a good placement, "No" means that there is no good placement, while the case of "?" is undecided yet.

$k \setminus t$	2	3	4	5	6	7	8	$t \geq 9$	∞
2	No	No	No	No	No	No	No	No	No
3	No	No	No	No	No	No	No	No	Yes
4	No	No	No	No	No	No	No	?	Yes
5	No	No	?	?	?	?	Yes	Yes	Yes
6	No	No	?	?	Yes	Yes	Yes	Yes	Yes
7	No	?	Yes	Yes	Yes	Yes	Yes	Yes	Yes
8	No	?	Yes	Yes	Yes	Yes	Yes	Yes	Yes
9	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Igen

3.11 Theorem *There are no good 4-placements for* h_4.

3.12 Theorem *There is a good 4-placement for* h_5.

3.13 Theorem *There exists a good 4-placement for* P_3.

3.14 Theorem *There exists a good 3-placement for* P_4.

4 The pairings of 9-in-a-row

In this section we study the sharp case of the k-in-a-row, the 9-in-a-row based on [4, 5]. We describe the conditions of the possible pairings, count them and give them a structure.

A pairing is optimal, if:

1. Every pair blocks exactly $k-1$ edges.

2. There are no overblockings, i.e., every edge is blocked once.

3. Every square is contained in a pair.

Concluding this conditions we get that an optimal good pairing of $\mathcal{H}_9$ consists of only domino pairs (neighboring cells), the dominoes are following each other by 8-periodicity in each line and all squares are covered by a pair. We call a square of a pairing **anomaly** where the 8-periodicity is violated, a non-domino type pair or an empty square appears in the pairing. Because of the $O(n)$ in Proposition 2.4, there might be anomalies in a good pairing, but we proved that a good pairing is always anomaly-free.

4.1 Theorem *A given anomaly-free pairing of a large enough square sub-board can be extended uniquely to the whole plane.*

4.2 Definition *A pairing of the infinite boeard is* ***k-toric*** *if it is a repetition of a $k \times k$ square, where k is the smallest possible value.*

4.3 Theorem *Suppose we have a good pairing of $\mathcal{H}_9$. Then that pairing is either 8-toric or 16-toric.*

4.4 Theorem *There are 8-toric good pairings of $\mathcal{H}_9$, that are not isomorphic to the Hales-Jewett pairing.*

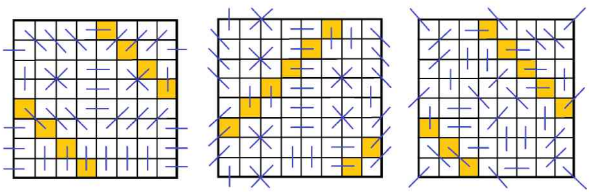

Figure 3: Some new good pairings

4.5 Theorem *An 8-toric good pairing gives a 16-toric good pairing if and only if another 8-toric good pairing exists, differing in some diagonal dominoes, such that their union gives a system of diagonal alternating cycles. There are only two possiblesystems of diagonal alternating cycles which are shownin Figure 4; the left and middle ones.*

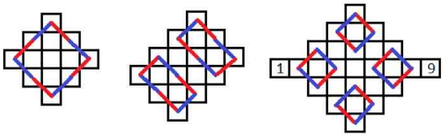

Figure 4: The alternating cycle systems

4.6 Theorem *There exists good pairings containing the first or second type of diagonal alternating systems, so, there exists 16-toric good pairings.*

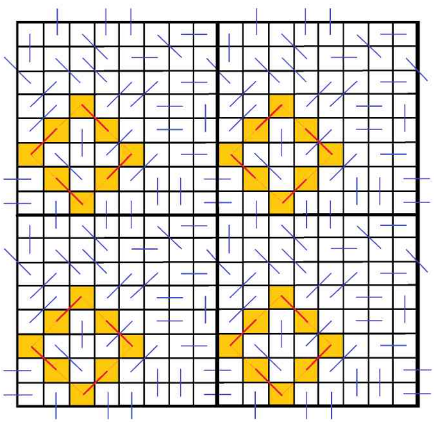

Figure 5: A 16-toric pairing

4.1 Counting the good pairings

We counted all possible essentially different 8-toric good pairings by a computer program and we found 194 543 of them. Solving the problem the difficulties were not only finding all pairings but to check the torus symmetries (shifting, rotational and reflectional) to list the non-isomorphic pairings.

Since we have many such different pairings, an obvious way of finding a structure is to store the pairings in a graph. In the thesis we present a natural method for finding connections between pairings.

1. Move the first pair on the table. This move creates a cell (say A) without a pair, and another cell (say B) with two pairs.

2. Move the pair containing cell B which was not the just moved pair so that cell B has one pair after the move. But then another cell may has two pairs.

3. Repeat step 2 as long as it creates a cell with two pairs.

4. This method will end when the last move creates a new pair for cell A, which had no pair before the move.

Two pairings are connected, if one can obtain the second pairing from the first one via this method. This relation is symmetric, meaning that this creates a graph where the vertices are the pairings and the edges are defined by this moving transition.

Studying the obtained graph we get that it has a *giant component* with 194 333 vertices of the 194 543 and there are 13 other small components. The degrees are between 1 and 11 where the average degree is 5,47. The graph has 532 107 edges.

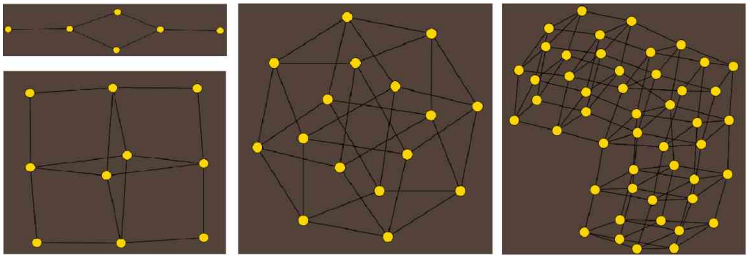

Figure 6: Some component of the graph, by Sixtep

One can construct a similar graph to the hexagonal 7-in-a-row, for which there are 26 different good pairings.

The end of the chapter we study a related problem [7], in 3-dimension and give a good pairing for the sharp case of the three direction version (paralel to the 3 axis) 7-in-a-row.

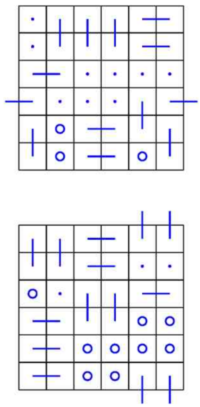

Figure 7: A good pairing for the 7-in-a-row without diagonals in 3D

5 Open questions

In the last chapter we give a sub-board division for the 7-in-a-row which could help Breaker to win the M-B version of the original 7-in-a-row. Unfortunately, giving a whole description of those tables is hopeless with recent computer capacity.

After that we examine the blocking power of sub-boards which takes easier to find possible sub-boards to hypergraph games. We study the question how to decrease the possible cases and we disprove a natural heuristic, namely, if there is a two element edge Maker has to take it necessarily.

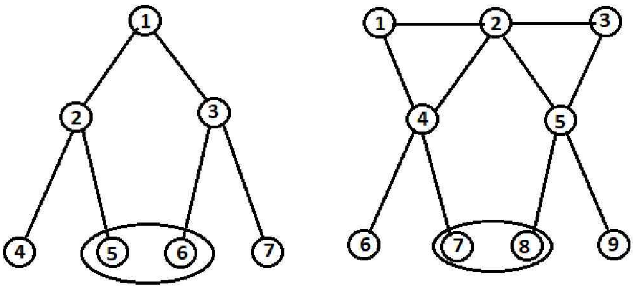

Figure 8: Should we start by a 2-edge?

At the end of the chapter we review a hypergraph classification from [1] and illustrate them by examples.

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